JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 2, Number 2, Spring 1990

DISTRIBUTIONAL SOLUTIONS OF SINGULAR INTEGRAL EQUATIONS

H. FRISCH, C.V.M. VAN DER MEE AND P.F. ZWEIFEL

ABSTRACT. It is proved that certain singular equations, which have no classical solutions because of singularities of the coefficients on the interval of integration, still have distributional solutions. The explicit form of these distributional solutions is presented.

1. Introduction. In an accompanying paper [**5**], Cauchy type singular integral equations over an interval $I \subseteq \mathbf{R}$ are solved by an orthogonality technique. The two equations considered are the direct equation (I-1) and the associated equation (I-1*). As was pointed out in [**5**], certain problems may arise in the solutions of these two equations when either of the functions defined by

$$\Lambda^{\pm}(t) = \lambda(t) \pm \pi i \eta(t)$$

has a zero. In certain cases these problems may be circumvented. In particular, if λ and η are real functions, then Λ^{+} and Λ^{-} vanish at the same point(s), and the solution of (I-1) presents no problem. The same is true of (I-1*) if $g(t)$ has zero(s) at the same point(s) as $\Lambda^{\pm}$ (of order greater than or equal to the order of the zero(s) of $\Lambda^{\pm}$).

In case this condition on $g(t)$ is not satisfied, it was shown in [**5**] that a weak solution of (I-1*) could still be obtained under the condition that $\Lambda^{\pm}$ are the boundary values of an analytic function $\Lambda(z)$. This condition, while restrictive, is often satisfied in transport theory [1, 2], so this type of solution is not without practical application.

In the present note, we see (in §2) how in certain cases the distributional solutions to (I-1*) can still be obtained when the condition on the vanishing of $f(t)$ is not satisfied, even though no such function $\Lambda(z)$, as described above, exists. In §3 we present an application of the method developed in §2.

2. Distributional solutions of (I-1*). The method followed here was suggested by the analysis of a transport equation in which $\Lambda^{\pm}(t)$

were vanishing somewhere on an interval [4]. We illustrate the method first for the simplest case, namely, when λ and η both have a single simple zero at $t = \nu_0$ in $(-1,1)$ while $\lambda(\pm 1)$ and $\eta(\pm 1)$ do not vanish.

For the sake of convenience we rewrite (I-1*) as

$$g(t) = \lambda(t)B(t) + \eta(t)\int_{-1}^{1}\frac{B(\nu)}{\nu - t}d\nu, \tag{1*}$$

and we recall [5] that λ and η are assumed to be uniformly Hölder continuous on $[-1,1]$. We can now state

PROPOSITION 1. *Let λ and η be non-vanishing except for a common simple zero ν_0. Then a solution to (1*) is given by*

$$B(t) = \hat{B}(t) - \frac{f(\nu_0)}{\eta'(\nu_0)}\delta(t - \nu_0), \tag{2}$$

where $\hat{B}(t)$ is a solution to the equation

$$h(t) = \lambda(t)\hat{B}(t) + \eta(t)\int_{-1}^{1}\frac{\hat{B}(t)}{\nu - t}d\nu \tag{3a}$$

with

$$h(t) = g(t) + \frac{\eta(t)f(\nu_0)}{\eta^{(t)}(\nu_0 - t)}. \tag{3b}$$

PROOF. We observe that the solution to (2a) can be obtained by the methods of [5] (see Equation (I-24)), because $h(\nu_0) = 0$ (cf. (3b)). Substituting (2) into (1*) and noting that $\lambda(t)\delta(t - \nu_0) = 0$ yields (3a) immediately. □

More generally, we can deal with several higher order zeros.

THEOREM 2. *Let η be non-vanishing except for the k zeros ν_{0j} of order m_j, let λ be non-vanishing except for the same k zeros of order*

n_j, and let $n_j \geq m_j$, where $j = 1, 2, \ldots, k$. Then there exists a unique set of N constants $C^j_{l_j}$, $N = \sum_{i=1}^{k} m_i$, such that a solution to (1^*) is given by

$$B(t) = \hat{B}(t) + \sum_{j=1}^{k} \sum_{l_j=1}^{m_j} C^j_{l_j} \delta^{(l_j-1)}(t - \nu_0), \tag{4}$$

where, by definition, $\int_{-1}^{1} f(t)\delta^{(r)}(t-\nu_0)dt = (-1)^r f^{(r)}(\nu_0)$ and $\hat{B}(t)$ is a solution of the equation

$$g(t) - \eta(t) \sum_{j=1}^{k} \sum_{l_j=1}^{m_j} \frac{C^j_{l_j}(l_j - 1)!}{(\nu_j - t)^{l_j}} = \lambda(t)\hat{B}(t) + \eta(t) \int_{-1}^{1} \frac{\hat{B}(\nu)}{\nu - t} d\nu. \tag{5}$$

PROOF. Substitution of the putative solution (4) into (1^*) yields (5), since $\lambda(t)$ has a zero at ν_j of order n_j with $n_j \geq m_j$, $j = 1, 2, \ldots, k$. Then (5) can be solved by the methods of [**5**], provided the left-hand side of (5) has zeros of order at least m_j at ν_j, $j = 1, 2, \ldots, k$. To prove the existence of a unique set of N constants C_{l_j} we note that the Taylor expansion of $\eta(t)$ at ν_j has the form

$$\eta(t) = \frac{\eta^{(m_j)}(\nu_j)}{(m_j)!}(t - \nu_j)^{m_j} + O((t - \nu_j)^{m_j+1}),$$

where $\eta^{(m_j)}(\nu_j) \neq 0$. By differentiation of the left-hand side of (5), we then obtain

$$g^{(r)}(\nu_j) - \lim_{t \to \nu_j} \sum_{j=1}^{k} \sum_{l_j=1}^{m_j} C^j_{l_j}(l_j - 1)!$$

$$\times \sum_{s=0}^{r} \binom{r}{s} \eta^{(s)}(t)(-1)^{r-s} \frac{l(l+1)\ldots(l+r-s-1)}{(\nu_j - t)^{l_j+r-s}} = 0,$$

where $r = 0, 1, 2, \ldots, m_j - 1$. This leads to a decoupled set of k lower triangular linear systems of order m_j where the diagonal entries are of the form $\eta^{(m_j)}(\nu_j)$ multiplied by a non-zero constant. Such a system has a unique solution, which completes the proof. □

REMARK. We could give an algorithm for the computation of the $C^j_{l_j}$'s but it is tedious. Instead, consider the following

EXAMPLE. Let $k = 1$ and $m = 2$. Then a simple computation shows

$$C_2 = \frac{2g(\nu_0)}{\eta''(\nu_0)}, \tag{6a}$$

$$C_1 = \frac{2}{\eta''(\nu_0)}\left(-g'(\nu_0) + \frac{1}{3}g(\nu_0)\frac{\eta'''(\nu_0)}{\eta''(\nu_0)}\right). \tag{6b}$$

3. An application. We consider a generalization of the transport equation considered by Paveri-Fontana and Zweifel [**6**],

$$\mu\frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) = \frac{1}{2}\left(1 - \frac{\mu^2}{a^2}\right)\int_{-1}^{1}\psi(x,\mu')d\mu', \tag{7}$$

where $0 < a \leq 1$. The case $a = 1$ was treated in [**6**]. In either case we get the singular integral equation for the "expansion coefficient" $A(\nu)$:

$$\psi_0(\mu) = \lambda(\mu)A(\mu) + \frac{1}{2}\left(1 - \frac{\mu^2}{a^2}\right)\int_{-1}^{1}\frac{\nu A(\nu)}{\nu - \mu}d\nu. \tag{8}$$

(We refer to [**1, 2, 6**] for more information on such expansions.) In (8) we have

$$\lambda(\mu) = \frac{1}{2}\{\Lambda^+(\mu) + \Lambda^-(\mu)\},\ \Lambda^\pm(\mu) = \operatorname*{Lim}_{\varepsilon\downarrow 0}\Lambda(\mu \pm i\varepsilon), \tag{9a}$$

and

$$\Lambda(z) = \left(1 - \frac{z^2}{a^2}\right)\left(1 - \frac{z}{2}\int_{-1}^{1}\frac{d\mu}{z-\mu}\right). \tag{9b}$$

We see that $\Lambda^\pm(\pm a) = 0$ and $\lambda(\pm a) = 0$. In order to apply Theorem 2 for the case $k = 2$, $m_1 = m_2 = 1$, we put $B(\nu) = \nu A(\nu)$ and $\eta(\nu) = \frac{1}{2}(1 - (\nu^2/a^2))$ and write

$$\mu\psi_0(\mu) = \lambda(\mu)B(\mu) + \mu\eta(\mu)\int_{-1}^{1}\frac{B(\nu)}{\nu-\mu}d\nu.$$

Using Theorem 2 we get

$$B(t) = \hat{B}(t) + C_1\delta(t-a) + C_2\delta(t+a)$$

and

$$\psi_0(t) - C_1\frac{a+t}{2a^2} + C_2\frac{a-t}{2a^2} = \lambda(t)\hat{A}(t) + \frac{1}{2}\left(1-\frac{t^2}{a^2}\right)\int_{-1}^{1}\frac{\nu\hat{A}(\nu)}{\nu-t}d\nu,$$

writing $\hat{B}(\nu) = \nu\hat{A}(\nu)$. Since the left-hand side must vanish for $t = \pm a$, we find $C_1 = a\psi_0(a)$ and $C_2 = -a\psi_0(-a)$, so that

$$\lambda(t)\hat{A}(t)+\frac{1}{2}\left(1-\frac{t^2}{a^2}\right)\int_{-1}^{1}\frac{\nu\hat{A}(\nu)}{\nu-t}d\nu = \psi_0(t)-\frac{a+t}{2a}\psi_0(a)-\frac{a-t}{2a}\psi_0(-a).$$

For $a = 1$ this result agrees with the result of [**3**], which was obtained with considerably more difficulty. For $a > 1, \Lambda^{\pm}(t) \neq 0$ for $t \in [-1,1]$, so that methods of [**5**] apply directly.

4. Comment. In case λ and/or η are not real functions, so that $\Lambda^{\pm}$ need not vanish at the same point, the procedure described in §2 will in general not work. However, if the integral in (1^*) is taken along a closed contour in the complex plane, a solution can be found. One method has been described by Estrada and Kanwal [**3**]; a simpler method, based on analytic continuation, will be published [**7**].

Acknowledgements. One of the authors (PFZ) acknowledges a useful discussion with Professor Rhonda Hughes. The research described here was supported by the U.S. Department of Energy Grant DE-FG-05-87ER25033 and National Science Foundation Grants DMS 8701050 and DMS 8823102.

REFERENCES

1. K.M. Case and P.F. Zweifel, *Linear Transport Theory,* Addison-Wesley, Reading, MA., 1967.

2. C. Cercignani, *The Boltzmann Equation and its Applications,* Springer, New York, 1988.

3. R. Estrada and R.P. Kanwal, *The Carleman type singular integral equations,* SIAM Review **29** (1987), 263-290.

4. H. Frisch, *A Cauchy integral equation method for analytic solutions of half-space convolution equations,* J. Quant. Spectrosc. Radiat. Transfer **39** (1988), 149-162.

5. C.V.M. van der Mee and P.F. Zweifel, *Application of Orthogonality Relations to Singular Integral Equations,* Preceding paper in this journal. We refer to the n-th equation in this paper as (I-n).

6. S.L. Paveri-Fontana and P.F. Zweifel, *Application of the closure relation in transport theory,* Transp. Theory Stat. Phys. **16** (8) (1987), 1095-1127.

7. P.F. Zweifel, *Singular integral equations on contours: The orthogonality method,* Preprint.

OBSERVATOIRE DE NICE, B.P.139, 06003 NICE, FRANCE

CENTER FOR TRANSPORT THEORY AND MATHEMATICAL PHYSICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VA 24061

CENTER FOR TRANSPORT THEORY AND MATHEMATICAL PHYSICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VA 24061