JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 2, Number 1, Winter 1989

DISCRETE NUMERICAL SOLVABILITY OF HAMMERSTEIN INTEGRAL EQUATIONS OF MIXED TYPE

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ABSTRACT. In a recent paper using a variation of the Kumar and Sloan new collocation-type method, we studied the numerical solvability of Hammerstein integral equation of mixed type

$$x(s) + \sum_{i=1}^{m} \int_a^b k_i(s,t) f_i(t, x(t))\, dt = y(s), \quad s \in [a,b]. \tag{I}$$

In this paper a discretized version of the above method is considered. The discrete version is obtained when the integrals are evaluated using quadrature formula. Using interpolatory quadrature rules and piecewise-polynomial function spaces, the convergence of the discrete approximate solutions to the actual solution of (I) is proved. The order of convergence is obtained when the quadrature rule is of certain degree of precision.

1. Introduction. In [**3**], we studied the numerical solvability of Hammerstein integral equation of mixed type

$$x(s) + \sum_{i=1}^{m} \int_a^b k_i(s,t) f_i(t, x(t))\, dt = y(s), \quad s \in [a,b] \tag{1.1}$$

where $-\infty < a < b < \infty, y, k_i$ and f_i are known functions and x is a solution to be determined.

A discretized version of the numerical solvability discussed in [**3**] is considered in this paper. The discretized version is obtained when all the definite integrals required to be evaluated for computing the numerical solution of (1.1) using the method of [**3**], are approximated by numerical quadrature.

The convergence of the discrete approximate solutions to the actual solution of (1.1) is discussed and its rate of convergence is obtained in certain piecewise-polynomial function spaces. When the quadrature rule is of certain degree of precision, the superconvergence rates obtained in [**3**] are maintained in the discrete case. The results obtained are extensions of the results of [**5**].

2. Preliminaries. Let $C = C[a,b]$ be the Banach space of continuous real valued functions on $[a,b]$ equipped with uniform norm and let $R = R[a,b]$ be the Banach space of bounded Riemann-integrable real-valued functions on $[a,b]$. The following assumptions will be used to carry out the analysis in the spaces C and R. Let $x^* \in C$ be a solution of (1.1) and

$$B(x^*,\delta) = \{x \in C : ||x - x^*||_\infty \le \delta\}.$$

ASSUMPTIONS [A].

A1. $y \in C$; for each i, $1 \le i \le m$,

A2. the kernel $k_i(s.t)$ is continuous on $a \le s$, $t \le b$,

A3. the function $f_i(t,x)$ is defined and continuous on $[a,b] \times \mathbf{R}$,

A4. the partial derivative $f_{i,x}(t,x) = (\delta/\delta x) f_i(t,x)$ exists and is continuous on $[a,b] \times \mathbf{R}$,

A5. the function $f_{i,x}$ satisfies the Lipschitz condition; there exists a constant α_i such that

$$|f_{i,x}(t,x_1(t)) - f_{i,x}(t,x_2(t))| \le \alpha_i |x_1(t) - x_2(t)|$$

for all $t \in [a,b]$ and all $x_1, x_2 \in B(x^*,\delta)$ for some $\delta > 0$.

For $1 \le i \le m$, Assumption A2 implies that each linear integral operator K_i defined by

$$[K_i x](s) = \int_a^b k_i(s,t)x(t)\,dt, \quad s \in [a,b], \quad x \in R \tag{2.1}$$

is a compact operator from R to C and assumption A3 implies that each Nemytskii operator N_i defined as

$$[N_i x](t) = f_i(t,x(t)), \quad t \in [a,b], \quad x \in C \tag{2.2}$$

is a continuous, bounded operator from C to C.

So (1.1) can be written as an operator equation

$$x + \sum_{i=1}^{m} K_i N_i x = y. \tag{2.3}$$

Existence and uniqueness of the solution of (2.3) are studied in [**3**]. As in [**3**], we sought the approximation in piecewise-polynomial function spaces which are defined below:

For any natural number n, let

$$\prod_n : a = s_1 < s_2 \cdots < s_n < s_{n+1} = b$$

be a partition of $[a, b]$ and let

$$h = h(n) = \max_{1 \le i \le n} (s_{i+1} - s_i).$$

Assume that $h \to 0$ as $n \to \infty$ and the partition $\prod_n$ is quasi-uniform, that is there exists a constant β such that

$$h \le \beta \min_{1 \le i \le n} (s_{i+1} - s_i).$$

With r a positive integer and γ an integer satisfying $0 \le \gamma < r$, let $S^{\gamma}_{r,n}$ denote the space of piecewise-polynomial functions of order r and continuity γ. That is $\phi \in S^{\gamma}_{r,n}$, if and only if, it is a polynomial of degree $\le r - 1$ on each subinterval (s_i, s_{i+1}), $1 \le i \le n$ and has $\gamma - 1$ continuous derivatives on (a, b). If $\gamma = 0$ there is no continuity requirement at break points s_i, $1 \le i \le n+1$. In this case we arbitrarily take each $\phi \in S^{0}_{r,n}$ to be right continuous at $s_1 = a$ and left continuous at every $s_i, 2 \le i \le n + 1$.

For $N = (n - 1)(r - \gamma) + r$ (the dimension of $S^{\gamma}_{r,n}$), let $\{u_j\}_{j=1}^{n}$ be a basis for $S^{\gamma}_{r,n}$ and let $\{\tau_i\}_{i=1}^{N}$ be a set of distinct points in $[a, b]$ such that $u_j(\tau_i) = \delta_{ij}$.

3. Numerical Solvability. In [**3**], the numerical solution of (1.1), using a variation of new collocation-type method of [**6**], is computed as follows.

First find collocation approximation to the functions $z_i, 1 \le i \le m$, defined by

$$z_i(s) = f_i(s, x(s)), \quad s \in [a,b]. \tag{3.1}$$

Substituting (3.1) in (1.1), we have

$$x(s) = y(s) - \sum_{j=1}^{m} \int_a^b k_j(s,t) z_j(t)\, dt, \quad s \in [a,b] \tag{3.2}$$

and (3.1) can be written as

$$z_i(s) = f_i\Big(s, y(s) - \sum_{j=1}^{m} \int_a^b k_j(s,t) z_j(t)\, dt\Big), \quad 1 \le i \le m. \tag{3.3}$$

The collocation approximation to z_i is of the form

$$z_{i,n}(t) = \sum_{j=1}^{N} a_{i,j} u_j(t), \quad 1 \le i \le m \tag{3.4}$$

where $\{u_j\}_{j=1}^N$ is a basis for $S^\gamma_{r,n}$ and the coefficients $a_i, \ldots, a_{i,N}$, $1 \le i \le m$ are determined by collocating (3.3) at the (collocation) points $\tau_1, \ldots, \tau_N$:

$$\begin{gathered} z_{i,n}(\tau_l) = f_i\Big(\tau_l, y(\tau_l) - \sum_{j=1}^{m} \int_a^b k_j(\tau_l,t) z_{j,n}(t)\, dt\Big), \\ 1 \le i \le m, \quad 1 \le l \le N. \end{gathered} \tag{3.5}$$

That is

$$\begin{gathered} z_{i,n}(\tau_l) = f_i\Big(\tau_l, y(\tau_l) - \sum_{j=1}^{m} \sum_{q=1}^{N} a_{j,q} \int_a^b k_j(\tau_l,t) u_q(t)\, dt\Big), \\ 1 \le i \le m, \quad 1 \le l \le N. \end{gathered} \tag{3.6}$$

The approximation x_n to the solution x^* of (1.1) is given by

$$\begin{aligned} x_n(s) &= y(s) - \sum_{j=1}^{m} \int_a^b k_j(s,t) z_j(t)\, dt \\ &= y(s) - \sum_{j=1}^{m} \sum_{q=1}^{N} \Big(\int_a^b k_j(s,t) u_q(t)\, dt \Big) a_{j,q}. \end{aligned} \tag{3.7}$$

Note that for $1 \le i \le m$, the calculation of $z_{i,n}$ requires the evaluation of the definite integrals $K_j u_q(\tau_l), 1 \le j \le m,\ 1 \le q \le N,\ 1 \le l \le N$. Also the calculation for x_n requires the evaluation of the definite integrals $K_j u_q(s),\ 1 \le j \le m,\ 1 \le q \le N$, for $s \in [a,b]$. When these integrals are approximated by numerical quadrature, we get the discretized version of the above method, which is discussed in the next section.

Let $P_n : C + S^{\gamma}_{r,n} \to S^{\gamma}_{r,n}$ be the interpolatory operator defined by

$$[P_n x](t) = \sum_{j=1}^{N} x(\tau_j) u_j(t), \quad t \in [a,b], \quad x \in C + S_{r,n}.$$

Assume that the collocation points $\{\tau_i\}_{i=1}^N$ are chosen in such a way that P_n is uniformly bounded as an operator from $C + S^{\gamma}_{r,n}$ to $S^{\gamma}_{r,n}$, that is

$$||P_n|| \le M \tag{3.8}$$

where $M > 0$ is independent of n and

$$\lim_{n\to\infty} ||P_n x - x||_\infty = 0 \ \text{ for all } x \in C. \tag{3.9}$$

Using operator theoretic representations, (3.3) and (3.4), together with (3.5), and (3.7) can be written as

$$z_i = N_i\Big(y - \sum_{j=1}^{m} K_j z_j\Big), \quad 1 \le i \le m, \tag{3.10}$$

$$z_{i,n} = P_n N_i\Big(y - \sum_{j=1}^{m} K_j z_{j,n}\Big), \quad 1 \le i \le m, \tag{3.11}$$

and

$$x_n = y - \sum_{j=1}^{m} K_j z_{j,n} \tag{3.12}$$

respectively.

THEOREM 3.1. [**3**]. *Let Assumptions* A1 through A4 *hold. Let* $x^* \in C$ *be a solution of* (1.1) *and the interpolatory operator* P_n *satisfy* (3.8) *and* (3.9). *Let* $\sum_{i=1}^{m}(K_i, N_i)'(x^*)$ *not have* -1 *as an eigenvalue. Then, for sufficiently large* n, (3.12) *has a unique solution* x_n *in* C *such that* $x_n \to x^*$ *in supremum norm and satisfies the error estimate*

$$M_2 \Big|\Big| \sum_{i=1}^{m} K_i(P_n N_i x^* - N_i x^*) \Big|\Big|_\infty \le ||x_n - x^*||_\infty$$

$$\le M_1 \sum_{i=1}^{m} ||K_i(P_n N_i x^* - N_i x^*)||_\infty$$

4.1 *where* $M_1, M_2 > 0$ *are independent of* n.

4. Discrete solvability. In the discrete method, approximate the integrals $K_j u_q(\tau_l), 1 \le j \le m,\ 1 \le q,\ l \le N$ in (3.6) and the integrals $K_j u_q(s), 1 \le j \le m,\ 1 \le q \le N, s \in [a, b]$ in (3.7) by numerical quadrature process of [**4**].

For $x \in R[0,1]$, let the points $p_1, \ldots, P_L \in [0,1]$ and the weights $w_1, \ldots, w_L$ generate the quadrature rule

$$\int_0^1 x(t)\,dt \simeq \sum_{j=1}^{L} w_j x(p_j) \tag{4.1}$$

which is exact for polynomials of degree $\le \rho$, but not exact for polynomials of degree $\rho + 1$ (that is, the quadrature rule has degree of precision ρ). For $1 \le i \le m$, let $K_{i,n} : R \to C$ be the discrete integral operator defined by

$$[K_{i,n}x](t) = \sum_{j=1}^{n}\sum_{q=1}^{L} w_q h_j k_i(t, t_{jq}) x(t_{jq}), \tag{4.2}$$

where $h_j = s_{j+1} - s_j$ and $t_{jq} = s_j + h_j p_q$. For $x \in R$, $K_i x$ is approximated by $K_{i,n}x$ by shifting the quadrature rule (4.1) to each subinterval (s_j, s_{j+1}), $1 \le j \le n$ of the partition $\prod_n$. Note that the integrals to be approximated in (3.6) and (3.7) are of the form $K_i \phi_n$ $1 \le i \le m$ where $\phi_n \in S^{\gamma}_{r,n}$. The following theorem gives an estimate in the quadrature error, for this case.

For $1 \leq i \leq m$, let $k_i^s(t) = k_i(s,t)$ for $a \leq s,t \leq b$. For $1 \leq p \leq \infty$, with l a non-negative integer, let $W_l^p = W_l^p(a,b)$ be the usual Sobolev space with norm $||\cdot||_{l,p}$ ([**1**]).

THEOREM 4.1. [**4**] *For* $1 \leq i \leq m$, *let* $k_i^s \in W_\mu^1(\mu \geq 1)$ *and* $||k_i^s||_{\mu,1}$ *be bounded independently of* s. *Then for all* $\phi_n \in S_{r,n}^\gamma$

$$||(K_i - K_{i,n})\phi_n|| \leq c_i h^\beta \max_{1\leq q\leq n} \Big[\sum_{j=0}^{\beta}[\sup_{s_q<t<s_{q+1}} |\phi_n^{(j)}(t)|]\Big]$$

where $\beta = \min(\mu, \rho+1)$ *and each* c_i *is a constant independent of* n.

The following lemma gives the properties of $K_{i,n}, 1 \leq i \leq m$.

LEMMA 4.1 [**2**]. *For* $1 \leq i \leq m$, *let Assumption A2 hold. Then for each* i

(i) $K_{i,n}x \to K_i x$ *for all* $x \in R$,

(ii) $\{K_{i,n} : n \geq 1\}$ *is collectively compact.*

The discrete analogs corresponding to $z_{i,n}$ and x_n can be written as operator equations

$$\tilde{z}_{i,n} = P_n N_i\Big(y - \sum_{j=1}^{m} K_{j,n}\tilde{z}_{j,n}\Big), \quad 1 \leq i \leq m \tag{4.3}$$

and

$$\tilde{x}_n = y - \sum_{j=1}^{m} K_{j,n}\tilde{z}_{j,n} \tag{4.4}$$

respectively.

Let

$$C_m = C_m[a,b] = C \times \cdots \times C(m \text{ times}),$$

for $\mathbf{u} = (u_1, \ldots, u_m) \in C_m$, $|||\mathbf{u}|||_\infty = \Big(\sum_{i=1}^m ||u_i||_\infty^2\Big)^{1/2}$ be the norm in C_m,

$$R_m = R_m[a,b] = R \times \cdots \times R(m \text{ times})$$

and norm in R_m be $|||\cdot|||_\infty$ defined in a similar way.

Let $K : R_m \to C_m, K_n : R_m \to C_m$ and $N : C_m \to C_m$ be matrix operators defined as follows:

$$K = \begin{bmatrix} K_1 & K_2 & \cdots & K_m \\ K_1 & K_2 & \cdots & K_m \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ K_1 & K_2 & \cdots & K_m \end{bmatrix} \tag{4.5}$$

$$K_n = \begin{bmatrix} K_{1,n} & K_{2,n} & \cdots & K_{m,n} \\ K_{1,n} & K_{2,n} & \cdots & K_{m,n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ K_{1,n} & K_{2,n} & \cdots & K_{m,n} \end{bmatrix} \tag{4.6}$$

$$N = \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_m \end{bmatrix} \tag{4.7}$$

Since all K_i's, $1 \le i \le m$, are compact linear operators from R to C, we have that K is also a compact linear and completely continuous operator from R_m to C_m. The properties of $K_{i,n}$, $1 \le i \le m$, are also carried over to K_n and the following lemma gives this.

LEMMA 4.2. *For $1 \le i \le m$, let Assumption A2 hold. Then*

(i) $K_n\mathbf{u} \to K\mathbf{u}$ *for all* $\mathbf{u} \in R_m$,

(ii) $\{K_n : n \ge 1\}$ *is collectively compact.*

Let $T : R_m \to C_m$ be defined as

$$T\mathbf{z} = \mathbf{v} - K\mathbf{z}, \quad \mathbf{z} \in R_m$$

where $\mathbf{v} = (y, \ldots, y) \in C_m$ and K is as defined in (4.5). Since K is completely continuous, T is also completely continuous.

Let $T_n : R_m \to C_m$ be defined as

$$T_n \mathbf{z} = \mathbf{v} - K_n \mathbf{z}, \quad \mathbf{z} \in R_m.$$

From Lemma 4.2, it follows that $T_n \mathbf{z} \to T\mathbf{z}$ for all $\mathbf{z} \in \mathbf{R}_m$ and $\{T_n : n \geq 1\}$ is collectively compact. Since N is continuous, we have $\{NT_n : n \geq 1\}$ is collectively compact. For $1 \leq i \leq m$ Assumptions $A3$ and $A4$ imply that N_i is continuously Frechet differentiable on C; its Frechet derivative at $x_0 \in C$ is a bounded linear operator given by

$$[N_i'(x_0)u](t) = f_{i,x}(t, x_0(t))u(t), \quad t \in [a, b], \quad u \in C.$$

Let $\delta_1 = m^{1/2}\delta$ and $\mathbf{u} = (x^*, \ldots, x^*)$ where x^* is a solution of (1.1). Let

$$B(\mathbf{u}, \delta_1) = \{\mathbf{w} \in C_m : |||_\infty \mathbf{u} - \mathbf{w}\,|||_\infty \leq \delta_1\}.$$

LEMMA 4.3. *Let assumptions $[A]$ hold. Then for all $\boldsymbol{\phi}, \boldsymbol{\psi} \in R_m$ with $T\boldsymbol{\phi}, T\boldsymbol{\psi} \in B(\mathbf{u}, \delta)$,*

$$|||(NT)'(\boldsymbol{\phi}) - (NT)'(\boldsymbol{\psi})|||_\infty \leq \alpha|||\boldsymbol{\phi} - \boldsymbol{\psi}|||_\infty$$

for some constant $\alpha > 0$.

PROOF. For each $\mathbf{w} = (w_1, \ldots, w_m) \in C_m$, the Frechet derivative $N'(\mathbf{w})$ is given by

$$N'(\mathbf{w}) = \begin{bmatrix} N_1'(w_1) & & \\ & \ddots & \\ & & N_m'(w_m) \end{bmatrix} \tag{4.7}$$

Let $\boldsymbol{\phi}, \boldsymbol{\psi} \in R_m$ be given by $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m)$ and $\boldsymbol{\psi} = (\psi, \ldots, \psi_m)$.

Also let $T\boldsymbol{\phi}, T\boldsymbol{\psi} \in B(\mathbf{u}, \delta_1)$. Now

$$\begin{aligned}
|||(NT)'(\boldsymbol{\phi}) - (NT)'(\boldsymbol{\psi})|||_\infty^2 \\
&= |||[N'(T\boldsymbol{\phi}) - N'(T\boldsymbol{\psi})]T|||_\infty^2 \\
&\leq |||T|||^2 \sum_{i=1}^{m} ||N_1'\Big(y - \sum_{j=1}^{m} K_j\phi_j\Big) - N_i'\Big(y - \sum_{j=1}^{m} K_j\psi_j\Big)||_\infty^2 \\
&\leq |||T|||^2 \sum_{i=1}^{m} \sup_{a\leq t\leq b} |f_{i,x}\Big(t, y(t) - \sum_{j=1}^{m} K_j\phi_j(t)\Big) \\
&\qquad - f_{i,x}\Big(t, y(t) - \sum_{j=1}^{m} k_j\psi_j(t)\Big)|^2.
\end{aligned}$$

Since $\mathbf{u} = (x^*, \ldots, x^*)$ and $T\mathbf{w} = (y - \sum_{j=1}^{m} K_j w_j, \ldots, y - \sum_{j=1}^{m} k_j w_j)$, for any $\mathbf{w} \in R_m$ given by $\mathbf{w} = (w_1, \ldots, w_m)$, we have $(y - \sum_{j=1}^{m} K_j\phi_j)$ and $(y - \sum_{j=1}^{m} K_j\psi_j) \in B(x^*, \delta)$ and using Assumption A5, we get

$$\begin{aligned}
|||(NT)'(\boldsymbol{\phi}) - (NT)'(\boldsymbol{\psi})|||_\infty^2 \\
&\leq |||T|||_\infty^2 \Big(\sum_{i=1}^{m} \alpha_i^2\Big) \sup_{a\leq t\leq b} |\sum_{j=1}^{m} K_j(\phi_j(t) - \psi_j(t))|^2 \\
&\leq |||T|||_\infty^2 \Big(\sum_{i=1}^{m} \alpha_i^2\Big)(\max_{1\leq j\leq m} ||K_j||^2) \sum_{j=1}^{m} \sup_{a\leq t\leq b} |\phi_j(t) - \psi_j(t)|^2.
\end{aligned}$$

Assumption A2 implies $|||T||| < \infty$, $||K_j|| < \infty$ for $1 \leq j \leq m$, let $\alpha^2 = |||T|||^2 \Big(\sum_{i=1}^{m} \alpha_i^2\Big)(\max_{1\leq j\leq m} ||K_j||^2)$. So we have

$$|||(NT)'(\boldsymbol{\phi}) - (NT)'(\boldsymbol{\psi})|||_\infty^2 \leq \alpha^2 \sum_{j=1}^{m} ||\phi_j - \psi_j||_\infty^2.$$

This implies $|||(NT)'(\boldsymbol{\phi}) - (NT)'(\boldsymbol{\psi})|||_\infty \leq \alpha |||\boldsymbol{\phi} - \boldsymbol{\psi}|||_\infty$ and the lemma is proved. □

REMARK 4.1. Lemma 4.3 holds, even if we replace T by T_n.

LEMMA 4.4. [**3**]. *Let Assumptions A1 through A4 hold. Let $x^* \in C$ be a solution of* (1.1). *Let* -1 *be not an eigenvalue of* $\sum_{i=1}^m (K_i N_i)'(x^*)$. *Then* $(NT)'(N\mathbf{u})(\mathbf{u} = (x^*, \dots, x^*))$ *exists and does not have* 1 *as an eigenvalue.*

The following theorem gives the convergence of $\tilde{x}_n$ to x^*.

THEOREM 4.2. *Let Assumptions* [*A*] *hold. Let $x^* \in C$ be a solution of* (1.1) *and the interpolatory operator P_n satisfy* (3.8) *and* (3.9). *Let* $\sum_{i=1}^m (K_i N_i)'(x^*)$ *not have* -1 *as an eigenvalue. Then for sufficiently large n,* (4.4) *has a unique solution $\tilde{x}_n$ in C such that*

$$||\tilde{x}_n - x^*||_\infty \to 0 \quad \text{as } n \to \infty.$$

PROOF. It is easy to see that $\mathbf{u} = (x^*, \dots, x^*)$ is a solution of the equation

$$\mathbf{u} = TN\mathbf{u} \tag{4.8}$$

The system of equations (3.10) can be written as

$$\mathbf{z} = NT\mathbf{z} \tag{4.9}$$

in the product space. Since $\mathbf{u}$ is a solution of (4.8), we have that $\mathbf{z} = N\mathbf{u}\,(N_1 x^*, \dots, N_m x^*)$ is the solution of (4.9). Let

$$\chi_1 = (C + S_{r,n}^\gamma) \times \cdots \times (C + S_{r,n}^\gamma) \quad (m \text{ times}),$$
$$\chi_2 = S_{r,n}^\gamma \times \cdots \times S_{r,n}^\gamma \quad (m \text{ times}).$$

Let $\mathbf{P}_n : \chi_1 \to \chi_2$ be defined by

$$\mathbf{P}_n = \begin{bmatrix} P_n & & & \\ & P_n & & \\ & & \ddots & \\ & & & P_n \end{bmatrix}$$

Since P_n is a projection, we have that $\mathbf{P}_n$ is also a projection. (3.8) implies that $\mathbf{P}_n$ is uniformly bounded as an operator from χ_1 to χ_2, that is

$$|||\mathbf{P}_n||| \le M_3. \tag{4.10}$$

(3.9) implies that

$$\lim_{n\to\infty} |||\mathbf{P}_n\mathbf{w} - \mathbf{w}|||_\infty = 0 \quad \text{for all } \mathbf{w} \in C_m. \tag{4.11}$$

Using matrix operators, the system of equations (4.3) can be written as

$$\tilde{\mathbf{z}}_n = \mathbf{P}_n NT_n\tilde{\mathbf{z}}_n. \tag{4.12}$$

For the solvability of (4.4), first investigate the solvability of (4.12). For this verify the conditions of Theorem 2 of [**8**], which is a modification of Theorem 1 of [**10**]. Lemma 4.4 gives that $(NT)'(N\mathbf{u})$ does not have 1 as an eigenvalue. As $(NT)'(Nu)$ is compact linear, we have that $[I - (NT)'(N\mathbf{u})]$ is nonsingular. Since $\{NT_n : n \geq 1\}$ is collectively compact on R, using (4.11) it can be proved (on the lines of Lemma 4 of [**5**]) that for all $\sigma > 0, \sigma \in \mathbf{R}$,

$$\sup_{w\in B_\sigma} |||NT_n(\mathbf{w}) - \mathbf{P}_n NT_n(\mathbf{w})|||_\infty \to 0 \quad \text{as } n\to\infty$$

where $B_\sigma = \{\mathbf{w} : \mathbf{w} \in R_m, \ |||\mathbf{w}||| \leq \sigma\}$. Hence from the Appendix of [**5**], it follows that $\{\mathbf{P}_n NT_n : n \geq 1\}$ is collectively compact on R. Moreover for $\mathbf{w} \in R_m$,

$$\begin{aligned} &|||\mathbf{P}_n NT_n\mathbf{w} - NT\mathbf{w}|||_\infty \\ &\leq |||\mathbf{P}_n NT_n\mathbf{w} - \mathbf{P}_n NT\mathbf{w}|||_\infty + |||\mathbf{P}_n NT\mathbf{w} - NT\mathbf{w}|||_\infty. \end{aligned} \tag{4.13}$$

Since $T_n\mathbf{w} \to T\mathbf{w}$, N is continuous, $NT_n\mathbf{w} \to NT\mathbf{w}$. Using this, (4.10) and (4.11) in (4.13), $\mathbf{P}_n NT_n\mathbf{w} \to NT\mathbf{w}$.

Since for all $\phi \in R_m$, $T_n\phi \to T\phi$ and since $\mathbf{u}$ is a solution of (4.8), there exists $\delta^* > 0$ such that for all $\boldsymbol{\phi}, \boldsymbol{\psi} \in B(\mathbf{z}, \delta^*)$, $T_n\boldsymbol{\phi}, T_n\boldsymbol{\psi} \in B(\mathbf{u}, \delta_1)$. Using Lemma 3.1 (with T replaced by T_n) and (4.10), for all $\boldsymbol{\phi}, \boldsymbol{\psi} \in B(\mathbf{z}, \delta^*)$,

$$|||(\mathbf{P}_n NT_n)'(\phi) - (\mathbf{P}_n NT_n)'(\psi)|||_\infty \leq \alpha^* |||\phi - \psi|||_\infty$$

where $\alpha^* = M_3\alpha$. Hence all the conditions of Theorem 2 of [**8**] are satisfied and for sufficiently large n, (4.12) has a unique solution $\tilde{\mathbf{z}}_n$ such that $|||\tilde{\mathbf{z}}_n - \mathbf{z}|||_\infty \to 0$ as $n \to \infty$.

Let $\tilde{\mathbf{z}}_n = (\tilde{z}_{1,n}, \dots, \tilde{z}_{m,n})$ and $\mathbf{z} = (z_1, \dots, z_m)$. Now (4.4) is equivalent to the equation

$$\tilde{\mathbf{u}}_n = T_n \tilde{\mathbf{z}}_n \tag{4.14}$$

where $\tilde{\mathbf{z}}_n$ is a solution of (4.12). For if $\tilde{\mathbf{u}}_n = (\tilde{x}_{1,n}, \dots, \tilde{x}_{m,n})$ is a solution of (4.14) then we have $\tilde{x}_{1,n} = \cdots = \tilde{x}_{m,n} = \tilde{x}_n$ and $\tilde{x}_n$ is a solution of (4.4). Similarly if $\tilde{x}_n$ is a solution of (4.4) then $\tilde{\mathbf{u}}_n = (\tilde{x}_n, \dots, \tilde{x}_n)$ is a solution of (4.14). Since $\mathbf{z}_n$ is a unique solution of (4.12), $\tilde{\mathbf{u}}_n = T_n \tilde{\mathbf{z}}_n$ is a unique solution of (4.14). So $\tilde{\mathbf{u}}_n$ will be of the form $\tilde{\mathbf{u}}_n = (\tilde{x}_n, \dots, \tilde{x}_n)$ and $\tilde{x}_n$ is a unique solution of (4.4). Now

(4.15)

$$\begin{aligned} |||\tilde{\mathbf{u}}_n - \mathbf{u}|||_\infty &= |||T_n \tilde{\mathbf{z}}_n - T\mathbf{z}|||_\infty \\ &= m^{1/2} \sum_{i=1}^{m} ||K_{n,i}\tilde{z}_{i,n} - K_i z_i||_\infty \\ &\le m^{1/2} \sum_{i=1}^{m} \Big[||K_{n,i}\tilde{z}_{i,n} - K_{n,i} z_i||_\infty + ||K_{n,i} z_i - K_i z_i||_\infty \Big] \\ &\le m^{1/2} \sum_{i=1}^{m} \Big[||K_{n,i}|| \, ||\tilde{z}_{i,n} - z_i||_\infty + ||K_{n,i} z_i - K_i z_i||_\infty \Big]. \end{aligned}$$

$|||\tilde{\mathbf{z}}_n - \mathbf{z}|||_\infty \to 0$ as $n \to \infty$ implies that $||\tilde{z}_{i,n} - z_i||_\infty \to 0$ as $n \to \infty$, $1 \le i \le m$. From Lemma 4.1, $||K_{n,i} z_i - K_i z_i||_\infty \to 0$ as $n \to \infty$, $1 \le i \le m$. Using these and uniform boundedness principle in (4.15), $|||\tilde{\mathbf{u}}_n - \mathbf{u}|||_\infty \to 0$ as $n \to \infty$. Since

$$|||\tilde{\mathbf{u}}_n - \mathbf{u}|||_\infty = m^{1/2} ||\tilde{x}_n - x||_\infty,$$

we have $||\tilde{x}_n - x|| \to 0$ as $n \to \infty$. □

As in [**3**], for $\gamma = 0$ and $\gamma = 1$, we give the order of convergence (referred as superconvergence) of $\tilde{x}_n$ to x. Also, it can be shown that when the kernels are sufficiently smooth, $\tilde{z}_{i,n}$, $1 \le i \le m$, will exhibit up to $\mathrm{O}(h^r)$ convergence. But for $\gamma = 0$ and $\gamma = 1$ and for particular sets of collocation points, $\tilde{x}_n$ will exhibit a better rate of convergence, known as superconvergence. The sets of collocation points chosen for this purpose are as follows.

Case $\gamma = 0$. The dimension of $S^0_{r,n}$ is $N = nr$ and we need nr collocation points. Let $\eta_1, \dots, \eta_r$ be the zeros of the rth degree Legendre polynomial $\phi_r(s), s \in [-1,1]$, which are known as Gauss-Legendre points. The collocation points $\{\tau_j\}_{j=1}^N$ are the points $\eta_1, \dots, \eta_r$ linearly shifted to each subinterval $(s_i, s_{i+1}), 1 \le i \le n$:

$$\tau_{(i-1)r+j} = [s_i + s_{i+1} + (s_{i+1} - s_i)\eta_j]/2, \quad 1 \le j \le r, \quad 1 \le i \le n.$$

Case $\gamma = 1$. In this case r is necessarily ≥ 2 as the collocation approximation is sought in $S^1_{r,n}$, the space of continuous piecewise-polynomial functions. As the dimension of $S^1_{r,n}$ is $N = nr - n + 1$, we need $nr - n + 1$ collocation points. Let $\eta_1, \dots, \eta_{r-2}$ be the zeros of $\phi^{(1)}_{r-1})(s)$, $r \ge 3$ (the first derivative of $\phi_{r-1}(s)$), which are known as Lobatto points. Let $\eta_{r-1} = 1$. The collocation points $\{\tau_j\}_{j=1}^N$ are the break points $s_i, 1 \le i \le n+1$, plus $\eta_1, \dots, \eta_{r-1}$ shifted linearly to each subinterval $(s_i, s_{i+1}), 1 \le i \le n$:

$$\tau_{(i-1)(r-1)+j+1} = [s_i + s_{i+1} + (s_{i+1} - s_i)\eta_j]/2$$
$$1 \le j \le r-1, \quad 1 \le i \le n$$

with $\tau_1 = s_1 = a$.

For the above sets of collocation points, P_n will satisfy (3.8) and (3.9) ([**9**]). Using the above sets of collocation points, we have the following rate of convergence of $\tilde{x}_n$ to x.

THEOREM 4.3. *Let $x^* \in C$ be a solution of* (1.1), *let Assumptions [A] hold and let -1 not be an eigenvalue of $\sum_{i=1}^m (K_i N_i)'(x^*)$. For $1 \le i \le m$, let $N_i x^* W^1_\mu$, $\mu \ge 1$ and let $k_i^s \in W^1_\beta \ge 1$ with $||k_i^s||_{\beta,1}$ bounded independently of s. Let $\xi_\gamma = \min(\mu, \beta, 2r - 2\gamma)$, $\gamma = 0$ or 1 and let $\rho \ge \xi_\gamma - 1$. Then for sufficiently large n, the discrete approximation satisfies*

$$||\tilde{x}_n - x^*||_\infty = \mathrm{O}(h\xi_\gamma).$$

PROOF.

$$\begin{aligned}
\|x^* - \tilde{x}_n\|_\infty &\le \|x^* - x_n\|_\infty + \|x_n - \tilde{x}_n\|_\infty \\
&\le \|x^* - x_n\|_\infty + \sum_{i=1}^{m} \|K_i z_{i,n} - K_{i,n}\tilde{z}_{i,n}\|_\infty \\
&\le \|x^* - x_n\|_\infty + \sum_{i=1}^{m} \|K_i(z_{i,n} - \tilde{z}_{i,n})\|_\infty \\
&\quad + \sum_{i=1}^{m} \|(K_i - K_{i,n})\tilde{z}_{i,n}\|_\infty \\
&\le \|x^* - x_n\|_\infty + \sum_{i=1}^{m} \|K_i\| \, \|(K_i - K_{i,n}) z_{i,n}\|_\infty \\
&\quad + \sum_{i=1}^{m} \|(K_i - K_{i,n})\tilde{z}_{i,n}\|_\infty \\
&\le \mathrm{O}(h\xi_\gamma)
\end{aligned}$$

where the penultimate step follows from [**5**] and the last step follows from [**3**] and [**4**].

REMARK 4.1. From the above theorem, if $\mu > r$ and $\beta > r$ then $\tilde{x}_n$ will exhibit a better rate of convergence than $\tilde{z}_n$. We do not have a better rate of convergence for the case $\gamma = 1$ and $r = 2$. If $\xi_\gamma = 2r - 2\gamma$ then $\tilde{x}_n$ exhibit up to $\mathrm{O}(h^{2r-2\gamma})$ (super) convergence while $\tilde{z}_n$ exhibit up to $\mathrm{O}(h^r)$ convergence. When $\beta > \min(\mu, 2r - 2\gamma)$, the rate of convergence of $\tilde{x}_n$ to x is the same as that of x_n to x given in [**3**].

REMARK 4.2. Results of [**5**] can be obtained as corollaries by taking $m = 1$.

5. Numerical Example. Consider the mixed Hammerstein integral equation

$$x(s) + \int_0^1 e^{-(s+t)}[x(t)]^2\,dt + \int_0^1 st\sin x(t)\,dt = y(s) \tag{5.1}$$

where y is chosen in such as way that $x^*(t) = t$ is a solution of (5.1). In this case y is given by

$$y(s) = s + c_1 e^{-s} + c_2 s \tag{5.2}$$

where

$$c_1 = \int_0^1 e^{-t} t^2 \, dt, \quad c_2 = \int_0^1 t \sin t \, dt.$$

Discrete approximate solutions for (5.1) with y given by (5.2) are computed using piecewise-quadratic functions with break points $s_i = (i-1)/n$, $i = 1, \dots, n+1$.

According to Theorem 4.3, we can obtain $O(h^4)$ convergence rate by using an interpolatory quadrature rule with degree of precision $\rho \geq 3$. To see the importance of the degree of precision of quadrature required in Theorem 4.3, discrete approximate solutions are computed with $\rho = 1, 2$ and 3. Tables 1, 2 and 3 give the estimated order of convergence (EOC) obtained by using Gauss 1-point rule ($\rho = 1$), Ralston 1-point rule ($\rho = 2$) and Gauss 2-point rule ($\rho = 3$) respectively. Tables 1 and 2 clearly indicate that the order of convergence obtained in Theorem 4.3 is not maintained once we use quadrature rules of less degree of precisions than required in Theorem 4.3.

The nonlinear algebraic system of equations are solved by using the subroutine BRENTM [7]. All computations are carried out in double precision on CYBER 180/840 computer. $||x^* - \tilde{x}_n||$ is estimated by taking the largest of the computed errors at $t_i = (i-1)/250$, $i = 1, \dots, 251$.

TABLE 1.
($\rho = 1$)

n	$\|\|x^* - \tilde{x}_n\|\|$	EOC
5	1.8650 E-03	2.0022747
10	4.6551 E-04	2.0007134
15	2.0683 E-04	2.0003512
20	1.1633 E-04	2.0002094
25	7.4450 E-05	2.0001392
30	5.1700 E-05	2.0000992
35	3.7983 E-05	

TABLE 2.
($\rho = 2$)

n	$\|\|x^* - \tilde{x}_n\|\|$	EOC
5	1.0196 E-04	2.9853579
10	1.2875 E-05	2.9929956
15	3.8257 E-06	2.9954295
20	1.6161 E-06	2.9966173
25	8.2807 E-07	2.9973184
30	4.7944 E-07	2.9977802
35	3.0202 E-07	

TABLE 3.
($\rho = 3$)

n	$\|\|x^* - \tilde{x}_n\|\|$	EOC
5	1.6376 E-06	4.0008060
10	1.0229 E-07	4.0002534
15	2.0204 E-08	4.0001248
20	6.3927 E-09	4.0000745
25	2.6184 E-09	4.0000495
30	1.2627 E-09	4.0000354
35	6.8158 E-10	

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