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UNIFORM ASYMPTOTIC STABILITY OF A CLASS OF INTEGRODIFFERENTIAL SYSTEMS

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1. Introduction. Consider the linear Volterra integrodifferential system

$$x'(t) = \mathbf{A}x(t) + \int_0^t \mathbf{B}(t-s)x(s)\,ds \tag{1.1}$$

where $x(t)$ is a vector function with n components and

$$(\mathrm{H}_1) \qquad \begin{aligned} \mathbf{A} &= [a_{ij}] \text{ is a real constant } n \times n \text{ matrix,} \\ \mathbf{B}(t) &= [b_{ij}(t)] \text{ is a real } n \times n \text{ matrix, } \mathbf{B}(t) \in L^1[0,\infty). \end{aligned}$$

It is known [**9, 13**] that under assumption (H_1) the solution $x = 0$ of (1.1) is uniformly asymptotically stable in the sense of [**13**] if and only if

$$\det[z\mathbf{I} - \mathbf{A} - \mathbf{B}^*(z)] \neq 0 \quad (\operatorname{Re} z \geq 0), \tag{1.2}$$

where

$$\mathbf{B}^*(z) = \int_0^\infty e^{-zt}\mathbf{B}(t)\,dt$$

is the Laplace transform of $\mathbf{B}(t)$.

If particular $\mathbf{A}$ and $\mathbf{B}(t)$ are given then, in most of the cases, the necessary and sufficient condition (1.2) can be directly verified by using a polar plot (see e.g., [**7, 12**]). In many circumstances the problem of uniform asymptotic stability arises in a different manner. For example, we may be interested in the stability properties of the equilibrium states not only for specific values of the parameters but also for certain region of the parameters, or we would like to maximize a parameter under the condition that the equilibrium state remain asymptotically stable.

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This generally requires a real hard analysis even for relatively simple equations (see e.g., [**1**]). Easily verifiable explicit conditions for uniform asymptotic stability in terms of the entries of $\mathbf{A}$ and $\mathbf{B}(t)$ have great importance.

In [**3**] F. Brauer used (1.2) to get the following result for the scalar case of (1.1).

THEOREM A. *Let $n = 1$ and assume* (H_1) *holds. In addition, assume* $\mathbf{B}(t)$ *is continuous and of one sign on* $[0, \infty)$ *and that*

$$\mathbf{T} \equiv \int_0^\infty t|\mathbf{B}(t)|\,dt < \infty. \tag{1.3}$$

If $\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt \geq 0$, then the zero solution of (1.1) is not uniformly asymptotically stable. If $\int_0^\infty \mathbf{B}(t)\,dt > 0$ and $\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt < 0$, then the zero solution of (1.1) is uniformly asymptotically stable. If $\int_0^\infty \mathbf{B}(t)\,dt < 0$ and $\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt < 0$, then the zero solution of (1.1) is uniformly asymptotically stable provided

$$\mathbf{T} \leq 1. \tag{1.4}$$

Note that in the last statement of Theorem A, F. Brauer [**3**] stated that "$\mathbf{T}$ is sufficiently small" instead of (1.4). But (1.4) can be obtained easily from his proof.

Jordan [**11**] studied the n-dimensional case of (1.1). His result, when specialized to the scalar case, is the following:

THEOREM B. *Let $n = 1$ and assume* (H_1) *holds. If* $\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt \geq 0$, *then the zero solution of* (1.1) *is not uniformly asymptotically stable. If* $\mathbf{A} + \int_0^\infty |\mathbf{B}(t)|\,dt \leq 0$ *and* $\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt < 0$, *then the zero solution of* (1.1) *is uniformly asymptotically stable. If* $\int_0^\infty \mathbf{B}(t)\,dt < \mathbf{A} < -\int_0^\infty \mathbf{B}(t)\,dt$, *then the zero solution of* (1.1) *is uniformly asymptotically stable provided that the moment* $\mathbf{T}$ *defined in* (1.3) *satisfies*

$$\mathbf{T} \leq \left|\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt\right| \Big/ \left(|\mathbf{A}| + \int_0^\infty |\mathbf{B}(t)|\,dt\right). \tag{1.5}$$

Burton and Mahfoud [**6**] obtained criteria for the uniform asymptotic stability of the zero solution of (1.1) by decomposing the kernel $\mathbf{B}(t)$. Although their method is quite effective in many special cases, it is hard to apply because they do not give a procedure to choose the decomposition.

Many examples [**3, 6**] suggest that conditions (1.4) and (1.5) in Theorems A and B cannot be changed by $\mathbf{T} < \infty$ (that is, large $\mathbf{T}$ may imply instability) and that the moment conditions (1.4) and (1.5) (and the conditions which correspond to (1.5) in the n-dimensional case in [**11**]) are too strong.

The purpose of this paper is to weaken the moment conditions in Theorems A and B and in the result of Jordan [**11**] for systems. We present explicit conditions which give a larger region of uniform asymptotic stability than those of [**3, 11**]. The scalar version of our main result will be given now in order to illustrate that the improvement over the estimates obtained in [**3, 11**] is significant.

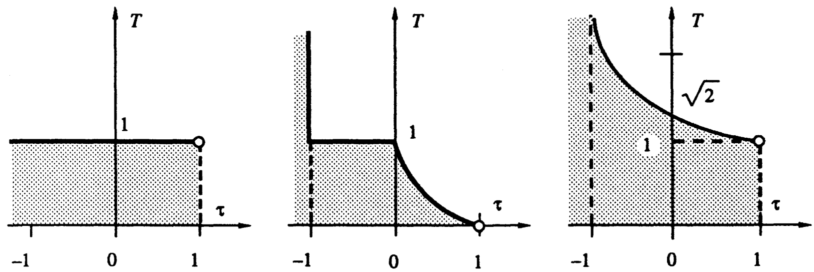
THEOREM 1. *Let* $n = 1$ *and assume* (H_1) *holds. If* $\mathbf{A} + \int_0^\infty |\mathbf{B}(t)|\,dt \le 0$ *and* $\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt < 0$, *then the zero solution of* (1.1) *is uniformly asymptotically stable. If* $\mathbf{A} \le 0$, $\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt < 0$, $\mathbf{A} + \int_0^\infty |\mathbf{B}(t)|\,dt > 0$ *and*

$$\mathbf{T} \le \left(1 + \left(\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt\right)^2 \Big/ \left(\left\{\int_0^\infty |\mathbf{B}(t)|\,dt\right\}^2 - \mathbf{A}^2\right)\right)^{\frac{1}{2}},$$

then the zero solution of (1.1) *is uniformly asymptotically stable. If* $\mathbf{A} \ge 0$, $\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt < 0$ *and*

$$\mathbf{T} \le \left(1 + \left(\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt\right)^2 \Big/ \left(\mathbf{A} + \int_0^\infty |\mathbf{B}(t)|\,dt\right)^2\right)^{\frac{1}{2}},$$

then the zero solution of (1.1) *is uniformly asymptotically stable.*



Theorem A Theorem B Theorem 1

$$\Gamma = -\frac{\mathbf{A}}{\int_0^\infty \mathbf{B}(t)\,dt}$$

FIGURE

The shaded regions in the Figure above show the region of uniform asymptotic stability obtained from Theorems A, B and 1, respectively, for the special case $n = 1$, $\mathbf{B}(t) \leq 0$, $\int_0^\infty \mathbf{B}(t)\,dt < 0$, $\mathbf{A} + \int_0^\infty \mathbf{B}(t)\,dt < 0$. These conditions are natural in the context of population models [**3, 7**].

The main result of this paper enlarges the region of uniform asymptotic stability given by [**3, 11**] not only in the scalar case but also in the n-dimensional case.

§2 contains the notation and the statement of the main result. The proof is given in §3.

In §4, in order to illustrate the usefulness of our result, a predator-prey model from mathematical ecology is considered such that the predator population is harvested at a constant time rate. First, sufficient conditions are given for the asymptotic stability of an equilibrium state at zero harvest rate. Then we try to choose the harvest rate as large as possible such that the equilibrium state, which varies as a function of the harvest rate, remains asymptotically stable.

2. Notation and statement of the result. We follow the notation of [**11**]. Let $|\cdot|$ denote a norm in $\mathbf{R}^n$. $|\mathbf{A}|$ denotes the norm of $\mathbf{A}$ deduced from $|\cdot|$. $|\mathbf{B}|$ is defined by $|\mathbf{B}| = \int_0^\infty |\mathbf{B}(t)|\,dt$, where $|\mathbf{B}(t)|$ is the norm of $\mathbf{B}(t)$. Let

$$
\begin{aligned}
\alpha_{ij} &= \int_0^\infty b_{ij}(t)\,dt && (i,j=1,\dots,n),\\
\beta_{ij} &= \int_0^\infty |b_{ij}(t)|\,dt && (i,j=1,\dots,n),\\
\mathbf{R}_i &= \sum_{j=1}^n (|a_{ij}|+\beta_{ij}) && (i=1,\dots,n),\\
\mathbf{Q}_i &= \mathbf{R}_i - (|a_{ii}|+\beta_{ii}) && (i=1,\dots,n),\\
\mathbf{T}_i &= \int_0^\infty t|b_{ii}(t)|\,dt && (i=1,\dots,n)
\end{aligned}
\tag{2.1}
$$

We need the inequalities
(H$_2$)

$$
\begin{aligned}
&|a_{ii}+\alpha_{ii}|\,|a_{kk}+\alpha_{kk}|\\
&\quad > \sum_{\substack{j=1\\ j\neq i}}^n |a_{ij}+\alpha_{ij}| \sum_{\substack{j=1\\ j\neq k}}^n |a_{kj}+\alpha_{kj}| \qquad (i\neq k;\quad i,k=1,\dots,n).
\end{aligned}
$$

If $n=1$ then (H$_2$) is interpreted as $|a_{11}+\alpha_{11}|>0$.

The following result was proved by Jordan [**11**].

THEOREM C. *Let* (H$_1$) *and* (H$_2$) *hold. The zero solution of* (1.1) *is uniformly asymptotically stable if for each* $i=1,\dots,n$ *either of the conditions*

$$a_{ii}<0,\ |a_{ii}|\geq \mathbf{Q}_i+\beta_{ii}, \tag{H$_3$}$$

$$a_{ii}+\alpha_{ii}<0,\ \mathbf{T}_i<\infty,\ \mathbf{T}_i\leq (|a_{ii}+\alpha_{ii}|-\mathbf{Q}_i)/\mathbf{R}_i \tag{H$_4$}$$

holds.

We refer to [**11**] for a discussion of conditions (H$_2$), (H$_3$) and (H_4).

Our main result can be obtained by weakening (H_4).

THEOREM 2. *Let* (H_1) *and* (H_2) *hold. The zero solution of* (1.1) *is uniformly asymptotically stable provided that for each* $i = 1, \dots, n$ *either* (H_3) *or one of the following conditions holds*:

$$(H_5) \qquad a_{ii} + \alpha_{ii} < 0, \quad \mathbf{T}_i^2 \le 1 - \mathbf{Q}_i^2/(a_{ii} + \alpha_{ii})^2,$$

$$(H_6) \qquad \begin{aligned} &a_{ii} \le 0, \quad a_{ii} + \alpha_{ii} < 0, \quad |a_{ii}| < \mathbf{Q}_i + \beta_{ii}, \quad \mathbf{Q}_i < |a_{ii} + \alpha_{ii}|, \\ &\mathbf{Q}_i \mathbf{T}_i \le \Big((a_{ii} + \alpha_{ii})^2 - \mathbf{Q}_i^2\Big) \Big/ \Big((\mathbf{Q}_i + \beta_{ii})^2 - a_{ii}^2\Big)^{\frac{1}{2}}, \\ &\mathbf{T}_i \le \Big(\Big((a_{ii} + \alpha_{ii})^2 + (\mathbf{Q}_i + \beta_{ii})^2 - a_{ii}^2\Big)^{\frac{1}{2}} - \mathbf{Q}_i\Big) \Big/ \\ &\qquad \Big((\mathbf{Q}_i + \beta_{ii})^2 - a_{ii}^2\Big)^{\frac{1}{2}}, \end{aligned}$$

$$(H_7) \qquad \begin{aligned} &a_{ii} \ge 0, \quad a_{ii} + \alpha_{ii} < 0, \quad \mathbf{Q}_i < |a_{ii} + \alpha_{ii}|, \\ &\mathbf{T}_i^2 > 1 - \mathbf{Q}_i^2/(a_{ii} + \alpha_{ii})^2, \\ &\Big(\mathbf{Q}_i \mathbf{T}_i + \Big(\mathbf{Q}_i^2 + (\mathbf{T}_i^2 - 1)(a_{ii} + \alpha_{ii})^2\Big)^{\frac{1}{2}}\Big) \\ &\times \Big(\Big(\mathbf{Q}_i + \mathbf{T}_i(|\mathbf{A}| + |\mathbf{B}|)\Big)^2 - (a_{ii} + \alpha_{ii})^2\Big)^{\frac{1}{2}} \\ &\le (a_{ii} + \alpha_{ii})^2 - \mathbf{Q}_i^2. \end{aligned}$$

REMARK 1. (i) It is easy to see from the assumptions that in (H_6) and (H_7) the quantities under the square roots are nonnegative.

(ii) Condition (H_4) implies (H_5) since $|a_{ii} + \alpha_{ii}| \le |a_{ii}| + |\alpha_{ii}| \le |a_{ii}| + \beta_{ii} \le \mathbf{R}_i$ and $(|a_{ii} + \alpha_{ii}| - \mathbf{Q}_i)^2/\mathbf{R}_i^2 \le (|a_{ii} + \alpha_{ii}| - \mathbf{Q}_i)(|a_{ii} + \alpha_{ii}| + \mathbf{Q}_i)/(a_{ii} + \alpha_{ii})^2$. ($H_4$) and ($H_5$) are equivalent if and only if $\mathbf{Q}_i = 0$ and $b_{ii}(t) \le 0$ a.e. on $[0, \infty)$. If (H_3) and (H_4) are not satisfied, then one of (H_5), (H_6) or (H_7) may hold. Thus, Theorem 2 is a generalization of Theorem C.

(iii) In the case $\mathbf{Q}_i = 0$ conditions (H_5), (H_6), and (H_7) have the following simple forms:

$$(H_5') \qquad a_{ii} + \alpha_{ii} < 0, \quad \mathbf{T}_i \le 1,$$

(H_6') $a_{ii} \leq 0,\ a_{ii}+\alpha_{ii} < 0,\ |a_{ii}| < \beta_{ii},\ T_i^2 \leq 1+(a_{ii}+\alpha_{ii})^2/(\beta_{ii}^2-a_{ii}^2)$,

(H_7') $a_{ii} \geq 0,\ a_{ii}+\alpha_{ii} < 0,\ 1 < \mathbf{T}_i^2 \leq 1+(a_{ii}+\alpha_{ii})^2/(|\mathbf{A}|+|\mathbf{B}|)^2$.

Consequently, Theorem 1 is a corollary of Theorem 2, because $\mathbf{Q}_1 = 0$ when $n = 1$. $\mathbf{Q}_i = 0$ means that $\mathbf{A}$ and $\mathbf{B}(t)$ are diagonal matrices.

(iv) If $n = 1$, $\mathbf{B}(t) \leq 0$ a.e. on $[0,\infty)$ and $\int_0^\infty \mathbf{B}(t)\,dt < 0$, then Theorem 1 reminds us of a result for the equation

$$x'(t) = ax(t) + bx(t-r) \tag{2.2}$$

with a single delay. The exact region of uniform asymptotic stability of (2.2) can be calculated by using a method of Pontryagin (see e.g. [**10**, p. 337]). The region given by Theorem 1 for (1.1) is close to the exact region obtained in [**10**] for (2.2)

(v) Theorem 2 remains true if $\mathbf{R}_i$ is defined by $\mathbf{R}_i = \sum_{j=1}^n(|a_{ji}| + \beta_{ji})$ instead of (2.1) and the order of subscripts is reversed in (H_2). The proof of this alternate theorem is similar to that of Theorem 2.

(vi) Conditions (H_5), (H_6) and (H_7) give upper bounds for $\mathbf{T}_i$, which measure the delay kernels $|b_{ii}(t)|$ in a certain sense. The upper bounds are functions of the entries of the matrices $\mathbf{A}, (\alpha_{ij})$ and (β_{ij}).

3. Proof of Theorem 2. It will be shown that condition (1.2) holds. Assume the contrary, i.e. there is a z_0 with $\operatorname{Re} z_0 \geq 0$ and $\det[z_0\mathbf{I} - \mathbf{A} - \mathbf{B}^*(z_0)] = 0$. Then z_0 is a characteristic root of the matrix $\mathbf{A} + \mathbf{B}^*(z_0)$, and a result of A. Brauer [**2**, Theorem 11] tells us that z_0 must satisfy at least one of the inequalities

$$\begin{aligned} &|z_0 - a_{ii} - b_{ii}^*(z_0)|\,|z_0 - a_{kk} - b_{kk}^*(z_0)| \\ &\leq \sum_{\substack{j=1\\ j\neq 1}}^n |a_{ij} + b_{ij}^*(z_0)| \sum_{\substack{j=1\\ j\neq k}}^n |a_{kj} + b_{kj}^*(z_0)| \\ &(i \neq k;\quad i,k = 1,\ldots,n). \end{aligned} \tag{3.1}$$

If $n = 1$ then (3.1) is interpreted as $|z_0 - a_{11} - b^*_{11}(z_0)| \leq 0$. Thus, there exists at least one $i \in \{1, \dots, n\}$ such that

$$|z_0 - a_{ii} - b^*_{ii}(z_0)| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij} + b^*_{ij}(z_0)| \leq \mathbf{Q}_i. \tag{3.2}$$

Condition (H_2) means that (3.1) fails for $z_0 = 0$. Consequently $z_0 \neq 0$. Therefore, it suffices to prove that (3.2) for $z_0 \neq 0$ leads to a contradiction for any $i \in \{1, \dots, n\}$.

Suppose that (H_3) holds for some i. Then we have

$$\begin{aligned} |z_0 - a_{ii} - b^*_{ii}(z_0)| &\geq |z_0 - a_{ii}| - |b^*_{ii}(z_0)| \\ &> |a_{ii}| - \beta_{ii} \geq \mathbf{Q}_i \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij} + b^*_{ij}(z_0)|, \end{aligned}$$

where the strict inequality occurs because $\mathrm{Re}\, z_0 \geq 0$, $z_0 \neq 0$, $a_{ii} < 0$. Thus, (3.2) fails when (H_3) holds.

Now we need two estimates for $|1 - \exp(-tz_0)|$. The first one can be obtained from $\mathrm{Re}\, z_0 \geq 0$, $z_0 \neq 0$, as follows:

$$\begin{aligned} |1 - \exp(-tz_0)| &= \left| \int_{-t}^{0} (d/du) \exp(uz_0)\, du \right| \\ &= \left| \int_{-t}^{0} z_0 \exp(uz_0)\, du \right| \\ &\leq |z_0| \left| \int_{-t}^{0} \exp(uz_0)\, du \right| < |z_0| t \quad (t > 0), \end{aligned} \tag{3.3}$$

where the strict inequality is obvious for $\mathrm{Re}\, z_0 > 0$, and it can be calculated easily for $\mathrm{Re}\, z_0 = 0$.

Since z_0 is an eigenvalue of $\mathbf{A} + \mathbf{B}^*(z_0)$, there is a vector v_0 such that $|v_0| = 1$ and $[\mathbf{A} + \mathbf{B}^*(z_0)]v_0 = z_0 v_0$. Therefore, (3.3) implies that

$$\begin{aligned} |1 - \exp(-tz_0)| &< |z_0|t = |z_0 v_0| t = \left| \Big(\mathbf{A} + \mathbf{B}^*(z_0)\Big) v_0 \right| t \\ &\leq (|\mathbf{A}| + |\mathbf{B}|)t \quad (t > 0). \end{aligned} \tag{3.4}$$

In the sequel we assume that (3.2), $\mathbf{T}_i < \infty$ and $a_{ii} + \alpha_{ii} < 0$ hold for some i. Then it follows from (3.3) that

$$
\begin{aligned}
&|z_0 - a_{ii} - b^*_{ii}(z_0)| \\
&\quad = \left|z_0 - a_{ii} - \alpha_{ii} - \int_0^\infty b_{ii}(t)\Big(\exp(-tz_0) - 1\Big)\, dt\right. \\
&\quad \geq |z_0 - a_{ii} - \alpha_{ii}| - \int_0^\infty |b_{ii}(t)|\, |1 - \exp(-tz_0)|\, dt \\
&\quad \geq |z_0 - a_{ii} - \alpha_{ii}| - \mathbf{T}_i |z_0|,
\end{aligned}
\tag{3.5}
$$

where the last inequality is strict whenever $\mathbf{T}_i \neq 0$.

On the other hand, from $\operatorname{Re} z_0 \geq 0$, $a_{ii} + \alpha_{ii} < 0$, we have

$$
|z_0 - a_{ii} - \alpha_{ii}|^2 \geq |z_0|^2 + (a_{ii} + \alpha_{ii})^2. \tag{3.6}
$$

Consequently, from (3.2), (3.5) and (3.6) we obtain for $\mathbf{T}_i \neq 0$ that

$$
(1 - \mathbf{T}_i^2)|z_0|^2 - 2\mathbf{T}_i\mathbf{Q}_i|z_0| + (a_{ii} + \alpha_{ii})^2 - \mathbf{Q}_i^2 < 0. \tag{3.7}
$$

Suppose (H_5) holds. Notice that $\mathbf{T}_i = 1$, (H_5) and (3.7) are incompatible.

If (H_5) and $\mathbf{T}_i \neq 0, 1$ are satisfied, then the discriminant $\mathbf{D}_i = \mathbf{Q}_i^2 - (1 - \mathbf{T}_i^2)(a_{ii} + \alpha_{ii})^2$ of the quadratic equation

$$
(1 - \mathbf{T}_i^2)\lambda^2 - 2\mathbf{T}_i\mathbf{Q}_i\lambda + (a_{ii} + \alpha_{ii})^2 - \mathbf{Q}_i^2 = 0 \tag{3.8}
$$

is nonpositive and (3.7) leads to a contradiction. If $T_i = 0$, then $\alpha_{ii} = 0$ and equality may occur in (3.7). In this case (3.7) and (H_5) give the contradiction $|z_0| \leq 0$.

Suppose that $\mathbf{T}_i^2 > 1 - \mathbf{Q}_i^2/(a_{ii} + \alpha_{ii})^2$. Then $\mathbf{D}_i > 0$. Consequently, from (3.7) one obtains that $|z_0| > \lambda_1$, where λ_1 is the smaller (larger) zero of (3.8) whenever $\mathbf{T}_i < 1$. ($\mathbf{T}_i > 1$) and λ_1 is the unique solution of (3.8) when $\mathbf{T}_i = 1$. That is

$$
|z_0| > \Big((a_{ii}+\alpha_{ii})^2 - \mathbf{Q}_i^2\Big) \Big/ \Big(\Big(\mathbf{Q}_i^2 + (\mathbf{T}_i^2 - 1)(a_{ii}+\alpha_{ii})^2\Big)^{\frac{1}{2}} + \mathbf{Q}_i\mathbf{T}_i\Big). \tag{3.9}
$$

If $a_{ii} \leq 0$, then from (3.2) and Re $z_0 \geq 0$ one concludes that

$$(|z_0|^2 + a_{ii}^2)^{\frac{1}{2}} - \beta_{ii} \leq |z_0 - a_{ii}| - \beta_{ii} \leq |z_0 - a_{ii} - b_{ii}^*(z_0)| \leq \mathbf{Q}_i,$$

and therefore

$$|z_0| \leq \Big((\mathbf{Q}_i + \beta_{ii})^2 - a_{ii}^2\Big)^{\frac{1}{2}} \tag{3.10}$$

whenever $\mathbf{Q}_i + \beta_{ii} \geq |a_{ii}|$.

From (3.4) and $a_{ii} + \alpha_{ii} < 0$ one obtains

$$\begin{aligned} |z_0 - a_{ii} - b_{ii}^*(z_0)| &= |z_0 - a_{ii} - \alpha_{ii} - \int_0^\infty b_{ii}(t)\Big(\exp(-tz_0) - 1\Big)\,dt| \\ &\geq |z_0 - a_{ii} - \alpha_{ii}| - \int_0^\infty |b_{ii}(t)|\,|\exp(-tz_0) - 1|\,dt \\ &\geq \Big(|z_0|^2 + (a_{ii} + \alpha_{ii})^2\Big)^{\frac{1}{2}} - \mathbf{T}_i(|\mathbf{A}| + |\mathbf{B}|). \end{aligned}$$

From (3.2) we get that

$$|z_0| \leq \bigg(\Big(\mathbf{Q}_i + \mathbf{T}_i(|\mathbf{A}| + |\mathbf{B}|)\Big)^2 - (a_{ii} + \alpha_{ii})^2\bigg)^{\frac{1}{2}} \tag{3.11}$$

whenever $\mathbf{Q}_i + \mathbf{T}_i(|\mathbf{A}| + |\mathbf{B}|) \geq |a_{ii} + \alpha_{ii}|$.

If (H_6) holds for i and $\mathbf{T}_i^2 \leq 1 - \mathbf{Q}_i^2/(a_{ii} + \alpha_{ii})^2$, then (H_5) is also satisfied and we obtain a contradiction as above. If $\mathbf{T}_i^2 > 1 - \mathbf{Q}_i^2/(a_{ii} + \alpha_{ii})^2$, then one easily concludes from (3.9) and (3.10) that

$$\begin{aligned} &\Big((a_{ii} + \alpha_{ii})^2 - \mathbf{Q}_i^2\Big)\Big/\Big((\mathbf{Q}_i + \beta_{ii})^2 - a_{ii}^2\Big)^{\frac{1}{2}} - \mathbf{Q}_i\mathbf{T}_i \\ &\qquad < \Big(\mathbf{Q}_i^2 + (\mathbf{T}_i^2 - 1)(a_{ii} + \alpha_{ii})^2\Big)^{\frac{1}{2}}. \end{aligned}$$

The left hand side of this inequality is nonnegative by (H_6). By squaring we get

$$-\Big((a_{ii} + \alpha_{ii})^2 - \mathbf{Q}_i^2\Big)\Big(\mathbf{T}_i^2 + 2\mathbf{Q}_i\mathbf{T}_i\Big/\Big((\mathbf{Q}_i + \beta_{ii})^2 - a_{ii}^2\Big)^{\frac{1}{2}}$$

$$-\Big((a_{ii}+\alpha_{ii})^2-\mathbf{Q}_i^2\Big)\Big/\Big(((\mathbf{Q}_i+\beta_{ii})^2-a_{ii}^2\Big)-1\Big)<0.$$

Since the left hand side of the last inequality is positive at $\mathbf{T}_i = 0$, $\mathbf{T}_i$ must be greater than the larger zero of the quadratic equation

$$\lambda^2+\frac{2\mathbf{Q}_i\lambda}{((\mathbf{Q}_i+\beta_{ii})^2-a_{ii}^2)^{1/2}}-\frac{(a_{ii}+\alpha_{ii})^2-\mathbf{Q}_i^2}{((\mathbf{Q}_i+\beta_{ii})^2-a_{ii}^2)-1}=0.$$

This contradicts the last assumption of (H_6).

If (H_7) holds for some i, then (3.9) and (3.11) imply

$$\Big((a_{ii}+\alpha_{ii})^2-\mathbf{Q}_i^2\Big)\Big/\Big(\Big(\mathbf{Q}_i^2+(\mathbf{T}_i^2-1)(a_{ii}+\alpha_{ii})^2\Big)^{\frac{1}{2}}+\mathbf{Q}_i\mathbf{T}_i\Big)$$
$$<\Big(\Big(\mathbf{Q}_i+\mathbf{T}_i(|\mathbf{A}|+|\mathbf{B}|)\Big)^2-(a_{ii}+\alpha_{ii})^2\Big)^{\frac{1}{2}}$$

which obviously contradicts (H_7). □

4. An example. In order to illustrate the usefulness of Theorem 2 we consider a predator-prey model where the predator population is harvested at a constant time rate. The model equations are

(4.1)
$$\mathbf{N}_1'(t)=r_1\mathbf{N}_1(t)\Big(1-p_1'\mathbf{N}_1(t)-p_1''\int_0^\infty k_1(s)\mathbf{N}_1(t-s)\,ds-q_1\mathbf{N}_2(t)\Big)$$
$$\mathbf{N}_2'(t)\doteq-r_2\mathbf{N}_2(t)\Big(1+p_2\int_0^\infty k_2(s)\mathbf{N}_2(t-s)\,ds-q_2\mathbf{N}_1(t)\Big)-h$$

where $\mathbf{N}_1(t)$, $\mathbf{N}_2(t)$ represent the prey, predator population densities, respectively: $r_1>0$ and $-r_2<0$ are the inherent net per unit growth rates which the populations would have in the absence of density restraints, the other population and harvest. $p_1'\mathbf{N}_1(t)+p_1''\int_0^\infty k_1(s)\mathbf{N}_1(t-s)ds$ and $p_2\int_0^\infty k_2(s)\mathbf{N}_2(t-s)ds$ are the delayed self-inhibition terms with $p_1'\ge 0$, $p_1''\ge 0$, $p_1=p_1'+p_1''>0$, $p_2>0$, $k_i\in C(R_+,R_+)$, $\int_0^\infty k_i(s)ds=1$, $i=1,2$; $q_1>0$ and $q_2>0$ so that predators inhibit prey growth and prey enhance predation growth and $h\ge 0$ is the harvest rate of the predator population. (For example, [**4**] considers similar problems.)

In the sequel we assume that

$$p_2 > q_2 > p_1 > q_1. \tag{4.2}$$

It is easy to see that the equilibrium state

$$e_1 = (1 - q_1 e_2)/p_1,$$
$$e_2 = \frac{q_2 - p_1 + ((q_2 - p_1)^2 - 4hp_1(p_1p_2 + q_1q_2)/r_2)^{\frac{1}{2}}}{2(p_1p_2 + q_1q_2)}$$

is positive, i.e., $e_i > 0$, $i = 1, 2$, whenever $0 \le h \le h_1 \overset{\text{def}}{=} r_2(q_2 - p_1)^2/(4p_1(p_1p_2 + q_1q_2))$. Clearly, e_1 is increasing and e_2 is decreasing in h on $[0, h_1]$. The linear variational system of (4.1) in the perturbations $x_i(t) = \mathbf{N}_i(t) - e_i$, $i = 1, 2$, is

(4.3)
$$x_1'(t) = r_1e_1\Big(- p_1' x_1(t) - p_1'' \int_0^\infty k_1(s)x_1(t-s)ds - q_1x_2(t)\Big)$$
$$x_2'(t) = r_2e_2\Big(q_2x_1(t) + \Big(h/(r_2e_2^2)\Big)x_2(t) - p_2\int_0^\infty k_2(s)x_2(t-s)\,ds\Big).$$

The equilibrium state (e_1, e_2) of (4.1) is (locally) asymptotically stable if and only if the zero solution of (4.3) is asymptotically stable [**9**]. The asymptotic stability of the zero solution of (4.3) is equivalent to that of the Volterra equation associated with (4.3) so that the integrals from 0 to ∞ in (4.3) are changed by integrals from 0 to t. The obtained Volterra equation is a particular case of (1.1) with $n = 2$,

$$\mathbf{A} = \begin{bmatrix} -r_1e_1p_1' & -r_1e_1q_1 \\ r_2e_2q_2 & h/e_2 \end{bmatrix}, \quad \mathbf{B}(t) = \begin{bmatrix} -r_1e_1p_1''k_1(t) & 0 \\ 0 & -r_2e_2p_2k_2(t) \end{bmatrix}.$$

By using the notation of §2, $\beta_{11} = -\alpha_{11} = r_1e_1p_1''$, $\beta_{22} = -\alpha_{22} = r_2e_2p_2$, $\alpha_{ij} = \beta_{ij} = 0$ for $i \ne j$, $\mathbf{Q}_i = r_ie_iq_i$ for $i = 1, 2$, $\mathbf{T}_1 = r_1e_1p_1''\mathbf{K}_1$, $\mathbf{T}_2 = r_2e_2p_2\mathbf{K}_2$, where $\mathbf{K}_i = \int_0^\infty sk_i(s)\,ds$, $i = 1, 2$.

First, we study the case $h = 0$. It is obvious from the assumptions on k_i and from (4.2) that (H_1) and (H_2) hold. We can obtain that the following conditions (C_1), (C_2), (C_3) imply (H_3), (H_5), (H_6), respectively, for $i = 1$:

$$p_1' \ge p_1'' + q_1, \tag{C$_1$}$$

$$(\mathrm{C}_2) \qquad p_1' < p_1'' + q_1, \quad K_1 < \left(1 - (q_1/p_1)^2\right)^{\frac{1}{2}} / (r_1 e_1 p_1''),$$

$$(\mathrm{C}_3) \quad p_1' < p_1'' + q_1, \quad \mathbf{K}_1 < \min\{m_1, m_2\}\left(r_1 e_1 p_1''\left((p_1'' + q_1)^2 - p_1'^2\right)^{\frac{1}{2}}\right)$$

$$\text{where } m_1 = (p_1^2 - q_1^2)/q_1, \; m_2 = \left(p_1^2 + (p_1'' + q_1)^2 - p_1'^2\right)^{\frac{1}{2}} - q_1.$$

Similarly, from the conditions (C_4) and (C_5) one can get (H_5) and (H_7), respectively, for $i = 2$:

$$(\mathrm{C}_4) \qquad \mathbf{K}_2 < \left(1 - (q_2/p_2)^2\right)^{\frac{1}{2}} / (r_2 e_2 p_2),$$

$$\mathbf{K}_2 > \left(1 - (q_2/p_2)^2\right)^{\frac{1}{2}} / (r_2 e_2 p_2),$$

$$(\mathrm{C}_5) \qquad \left(r_2 e_2 p_2 q_2 \mathbf{K}_2 + \left(q_2^2 + (r_2^2 e_2^2 p_2^2 K_2^2 - 1)p_2^2\right)^{\frac{1}{2}}\right) m_3^{\frac{1}{2}} < p_2^2 - q_2^2,$$

$$\text{where } m_3 = \left(q_2 + p_2 \mathbf{K}_2(|\mathbf{A}| + |\mathbf{B}|)\right)^2 - p_2^2.$$

Therefore, if $h = 0$, (4.2), one of (C_1), (C_2), (C_3) and one of (C_4), (C_5) hold, then the equilibrium state (e_1, e_2) of (4.1) is (locally) asymptotically stable. Equality can be allowed in the inequalities of (C_1)through (C_5). From (H_6) and (H_7) additional sufficient conditions can be obtained for the asymptotic stability of (e_1, e_2), whenever $i = 2$ and $i = 1$, $p_1' = 0$, respectively.

Notice that (C_2)through (C_5) guarantee the asymptotic stability of (e_1, e_2) whenever $\mathbf{K}_1, \mathbf{K}_2$ are smaller than certain numbers which are functions of the additional parameters. (C_2), (C_3) and (C_4) give explicit upper bounds. There is no restriction on the size of the delay kernels $k_i(\cdot)$ in (C_1).

Assume that the equilibrium state (e_1, e_2) of (4.1) is asymptotically stable at zero harvest rate (i.e., $h = 0$) such that one of the above sufficient conditions is satisfied. In the following we look for a bound $H > 0$ such that the asymptotic stability property of (e_1, e_2) is not destroyed for any harvest rate $h \in [0, H]$. Up until now the only restriction on h is $h \in [0, h_1]$. We need additional restrictions on h since $\det[\mathbf{A} + \mathbf{B}^*(0)] = 0$ at $h = h_1$, that is $(e_1(h_1), e_2(h_1))$ is not asymptotically stable.

From both of the inequalities $e_2(h) \geq d_1$, $h/e_2^2(h) \leq d_2$, where $d_1 < e_2(0)$, $d_2 > 0$, h can be easily expressed to get explicit upper bounds for h. In the sequel we will use these elementary facts.

There is already one restriction on h in order to have $e_1 > 0$, $e_2 > 0$. Condition (H_2) requires $r_1 e_1 p_1 |h/e_2 - r_2 e_2 p_2| > r_1 e_1 q_1 r_2 e_2 q_2$. Since $p_1 > q_1$ by (4.2), it is sufficient to assume that $p_2 \geq q_2 + h/(r_2 e_2^2)$, from which we get the bound $h_2 > 0$ for h. It is not difficult to check that $p_2 \geq q_2 + h/(r_2 e_2^2)$ fails for $h = h_1$, that is $h_2 < h_1$. Therefore, (H_1) and (H_2) hold for all $h \in [0, h_2]$.

If (C_1) is satisfied, then (H_3) follows for $i = 1$ and $h \in [0, h_2]$. Assuming $p_1' < p_1'' + q_1$, (H_5) requires for $i = 1$ that

$$e_2(h) \geq \left(1 - (p_1^2 - q_1^2)^{\frac{1}{2}} / (r_1 p_1'' \mathbf{K}_1)\right) / q_1. \tag{4.4}$$

Condition (C_2) implies (4.4) at $h = 0$ with strict inequality. From (4.4) the positive upper bound h_3 for h can be given, provided we have (C_2). Thus, (H_5) for $i = 1$ follows from (C_2) and $0 \leq h \leq \min\{h_2, h_3\}$.

If $p_1' < p_1'' + q_1$ again, then the inequality

$$e_1(h) \leq \min\{m_1, m_2\} / \left(r_1 p_1'' \mathbf{K}_1 \left((p_1'' + q_1)^2 - p_1^{12}\right)^{\frac{1}{2}}\right) \stackrel{\text{def}}{=} c_1 \tag{4.5}$$

is sufficient for (H_6) with $i = 1$, where m_1 and m_2 are defined in (C_3). Using $e_1 = (1 - q_1 e_2)/p_1$, we obtain from (4.5) that

$$e_2(h) \geq (1 - p_1 c_1)/q_1 \tag{4.6}$$

is also sufficient for (H_6) with $i = 1$. If (C_3) is assumed, then the inequality is strict in (4.6) at $h = 0$. Therefore, we can get $h_4 > 0$ such that (C_3) and $0 \leq h \leq \min\{h_2, h_4\}$ imply (H_6) for $i = 1$.

For $i = 2$ condition (H_5) requires

$$r_2^2 e_2^2(h) p_2^2 \mathbf{K}_2^2 \leq 1 - q_2^2 \Big/ \left(p_2 - h/(r_2 e_2^2(h)\right)^2. \tag{4.7}$$

Since $e_2(h)$ is decreasing in h, $0 \le h \le h_2$, (4.7) follows from

$$c_2 \le 1 - q_2^2/\Big(p_2 - h/(r_2 e_2^2(h))\Big)^2, \tag{4.8}$$

where $c_2 = r_2^2 e_2^2(0) p_2^2 \mathbf{K}_2^2$. If (C_4) is satisfied then (4.8) is valid at $h = 0$ with strict inequality. So, (4.8) is equivalent to

$$h/e_2^2(h) \le r_2\Big(p_2 - q_2/(1-c_2)^{\frac{1}{2}}\Big),$$

for $0 \le h \le h_2$, which produces the positive bound h_5 for h. From (C_4) and $0 \le h \le \min\{h_2, h_5\}$ we have (H_5) for $i = 2$.

We were unable to get the above results when (C_5) holds because of the difficult dependence of (H_7) on h. Instead we can do the following. For $i = 2$ condition (H_7) requires (4.7) with reversed strict inequality and

$$\begin{aligned}&\Big(r_2 e_2(h) p_2 q_2 \mathbf{K}_2 + \Big(q_2^2 + (r_2^2 e_2^2(h) p_2^2 \mathbf{K}_2^2 - 1) m_4^2(h)\Big)^{\frac{1}{2}}\Big)\\ &\qquad \cdot \Big(m_3 + p_2^2 - m_4^2(h)\Big)^{\frac{1}{2}} \le m_4^2(h) - q_2^2,\end{aligned} \tag{4.9}$$

where m_3 is defined in (C_5) and $m_4(h) = p_2 - h/(r_2 e_2^2(h))$. We specify a norm in $\mathbf{R}^2$. Let $|(v_1, v_2)| \overset{\mathrm{def}}{=} \max\{|v_1|, |v_2|\}$. Then it is not difficult to see that

$$\begin{aligned}|\mathbf{A}| + |\mathbf{B}| &\le \max\Big\{r_1 e_1(h_2)(p_1' + q_1), r_2 e_2(0) p_2\Big\}\\ &\qquad + \max\Big\{r_1 e_1(h_2) p_1'', r_2 e_2(0) p_2\Big\} \overset{\mathrm{def}}{=} c_3\end{aligned}$$

for $h \in [0, h_2]$. We assume a stronger inequality than (C_5). Namely,

$$\begin{aligned}&\Big(r_2 e_2(0) p_2 q_2 \mathbf{K}_2 + \Big(q_2^2 + (r_2^2 e_2^2(0) p_2^2 \mathbf{K}_2^2 - 1) m_4^2(0)\Big)^{\frac{1}{2}}\Big)\\ &\qquad \Big(\Big(q_2 + p_2 \mathbf{K}_2 c_3\Big)^2 - m_4^2(h_2)\Big)^{\frac{1}{2}} < m_4^2(0) - q_2^2.\end{aligned} \tag{4.10}$$

If (4.10) holds, then

$$\begin{aligned}&\Bigg(r_2 e_2(0) p_2 q_2 \mathbf{K}_2 + \Big(q_2^2 + (r_2^2 e_2^2(0) p_2^2 \mathbf{K}_2^2 - 1) m_4^2(h)\Big)^{\frac{1}{2}}\Bigg)\\ &\qquad \times \Bigg(\Big(q_2 + p_2 \mathbf{K}_2 c_3\Big)^2 - m_4^2(h_2)\Bigg)^{\frac{1}{2}} \le m_4^2(h) - q_2^2\end{aligned} \tag{4.11}$$

holds at least for small $h \geq 0$. Solving (4.11) for $m_4(h)$ a lower bound is obtained for $m_4^2(h)$, and the inequalities $h/e_2^2(h) \leq c_4$, $c_4 > 0$ follow, which gives the bound $h_6 > 0$ for h. Then (4.9) follow for $0 \leq h \leq \min\{h_2, h_6\}$. (4.9) implies ($H_7$) for $i = 2$ whenever (4.7) is not satisfied. If (4.7) is satisfied then (H_5) holds for $i = 2$. Therefore, (4.10) and $0 \leq h \leq \min\{h_2, h_6\}$ give either (H_5) or (H_7) for $i = 2$.

REMARK 2. (i) The assumption $q_2 > p_1$ is necessary to have positive equilibrium state of (4.1) at zero harvest rate.

(ii) Biologically, the conditions $p_1 > q_1$ and $p_2 > q_2$ mean that the self-regulating effects are dominant in both populations, i.e., the intraspecies competition is more significant than the interspecies interaction. See e.g., [**8**] for biological comments in related problems.

(iii) The obtained results are valid for a large class of delay kernels $k_i(t)$, since the only restriction is that $\int_0^\infty sk_i(s)ds$ be smaller than a given number, which depends on the additional parameters.

(iv) The local asymptotic stability of equilibrium states of a large class of nonlinear systems can be studied by applying Theorem 2 similarly to (4.1). Refer to [**7**] for further examples in population ecology. The specific form of the nonlinear system is not essential in the local stability analysis. If the linear variational system has the form (1.1), then Theorem 2 can be used to obtain sufficient conditions for asymptotic stability.

(v) On the basis of Theorem 2 it was easy to find upper bounds for the harvest rate in most of the cases in system (4.1). In fact, we had to solve first and second order polynomial inequalities.

(vi) The direct application of condition (1.2) requires the location of the zeros of

$$\mathbf{D}(z) = z^2 + d_1 z + d_2$$

where $d_1 = r_1 e_1 p_1' - h/e_2 + r_1 e_1 p_1'' k_1^*(z) + r_2 e_2 p_2 k_2^*(z)$ and $d_2 = (r_1 e_1 p_1 + k_1^*(z))\,(r_2 e_2 p_2 k_2^*(z) - h/e_2) - r_1 r_2 e_1 e_2 q_1 q_2$. It is a difficult task to determine $H > 0$ such that $D(z) \neq 0$ for $\operatorname{Re} z \geq 0$, $0 \leq h \leq H$, because of the dependence of d_1 and d_2 on z and h. Applying the argument principle from complex function theory, two additional conditions, which are equivalent to (1.2), for the asymptotic stability of the equilibrium state (e_1, e_2) of (4.1), are the following (see e.g., [**7**,

12]):

(a) the variation of $\operatorname{Arg} \mathbf{D}(i\theta)$ is equal to π as θ varies from 0 to $+\infty$;

(b) $\int_0^\infty [(u(\theta)v'(\theta) - v(\theta)u'(\theta))/(u^2(\theta) + v^2(\theta))]\, d\theta = \pi$ where the real functions u and v are given by $\mathbf{D}(i\theta) = u(\theta) + iv(\theta)$.

For specific values of the parameters both conditions can be checked without difficulty, e.g., by using a computer. But an estimation for the region of parameter h by using either (a) or (b) does not seem to be as simple as the application of Theorem 2.

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