1. Introduction. This paper is concerned with the characterization and representation theory of topological selections of closed linear relations in Banach spaces, and with some applications to differential and integral operators.

We first introduce some definitions. Let $X$ and $Y$ be real or complex Banach spaces and let $M$ be a subspace (i.e., a vector subspace) of $X \times Y$. We can also view $M$ as the graph of a multivalued linear mapping, and call it a linear relation in $X \times Y$. We say that $R$ is an algebraic selection (or algebraic operator part) of $M$ if $R$ is the graph of a single-valued linear operator on $\text{Dom} M$ into $\text{Range} M$ such that $R \subset M$. Equivalently, $R = \text{Null} P := \{a \in M : P(a) = 0\}$ for some algebraic projector $P$ on $M$ with $\text{Range} P = \{0\} \times M(0)$. If $P$ is continuous, then $R$ is called a topological selection (or topological operator part) of $M$. If, in addition, $P(x, y) = P(z, y)$ for all $(x, y), (z, y)$ in $M$, then $R$ is called a principal topological selection of $M$. Let $M^+$ be a subspace of $Y^* \times X^*$, where $X^*$ is the dual of $X$. An algebraic selection of $M^+$ is called a $w^*$-topological selection of $M^+$ if the corresponding projector is $w^*$-continuous.

We now summarize briefly the contents of this paper. In §2 we consider a general closed subspace (linear relation) $M$ of $X \times Y$ and a $w^*$-closed subspace $M^+$ of $Y^* \times X^*$ such that $M(0)$ and $M^+(0)$ are both finite dimensional. In Theorem 2.1 and Corollary 2.2, any topological selection is expressed by an adjoint subspace, while in Theorem 2.3 and Corollary 2.4, any $w^*$-topological selection is expressed by a preadjoint subspace. These theorems and corollaries generalize and complete the corresponding theorems of Coddington and Dijksma [1]. §3 is concerned with the problem of characterizing a topological selection for a subspace of a linear relation in terms of a known topological selection of that relation, and with some related consequences. More specifically, suppose that $M_1$ is a known closed...
linear relation in $X \times Y$ with $\dim \text{Null} M_1 < \infty$. Let $M$ be any closed subspace of $M_1$ with $\dim M_1/M < \infty$. In Theorem 3.5 we give a necessary and sufficient condition for the graph of an operator to be a topological selection of $M^{-1}$. This is expressed via an arbitrary, but fixed topological selection, say $R_1$, of $M_1^{-1}$. Using this theorem we prove in Theorem 3.6 that any topological selection of $M^{-1}$ is necessarily a finite-rank "perturbation" of $R_1$ restricted to $\text{Dom} R$, and hence if $R_1$ is compact, so is any topological selection of $M^{-1}$. In §4 we are concerned with $M$ and $M_1$ when $X$ and $Y$ are $L_p$-type spaces or the space of continuous functions on a compact Hausdorff space. Using notions of general integral operators as in [2], we give in Lemmas 4.1 and 4.2 necessary conditions for certain functionals to be represented by integral operators. We prove in Theorem 4.3 that if a topological selection $R_1$ of $M_1^{-1}$ is an integral operator with a suitable kernel, then any topological selection of $M^{-1}$ is also an integral operator with a related kernel. In §5 we give an application of Theorem 3.6 to a concrete case when $M_1$ is a finite-dimensional "perturbation" of the graph of a regular ordinary differential operator in an $L_p$-type space. It is shown in Theorem 5.1 that if $M$ is a closed subspace of $M_1$ with $\dim M_1/M < \infty$, then any topological selection of $M^{-1}$ is a compact integral operator. This theorem, even in the special case when $M_1$ is the graph of a maximal ordinary differential operator, generalizes the corresponding theorems in the literature that are concerned with two-point boundary conditions.

2. Characterization of topological selections of multivalued linear operators by adjoints. When a linear ordinary differential operator is nondensely defined, its adjoint is a multivalued operator. It is sometimes desirable to describe the linear selections of its adjoint in terms of boundary conditions. In the present section we address this problem in an abstract setting. Thus we will describe all topological selections of certain linear relations via "boundary" conditions. The setting and results generalize and at the same time complete some results of Coddington and Dijksma [1]. Throughout this section $X, Y$ are Banach spaces and $M \subset X \times Y$ is a closed linear relation. For a Banach space $W$ with dual $W^\#$, the natural pairing on $W \times W^\#$ is denoted by $\langle \cdot, \cdot \rangle$. The adjoint (or adjoint subspace) of a linear relation $M \subset X \times Y$ is the linear relation $M^* \subset Y^\# \times X^\#$ defined by
\[ M^* = \{(y, x) : (x, -y) \in M^+\} \] where \( M^+ \) denotes the annihilator of \( M \). In the case when \( M^+ \subseteq Y^* \times X^* \), the preadjoint of \( M^+ \) is the linear relation \( ^* (M^+) := \{(x, y) \in X \times Y : (y, -x) \in M^+\} \), where \( \perp M^+ \) is the preannihilator of \( M^+ \).

**Theorem 2.1.** Assume that \( \dim M(0) := n < \infty \). Then \( R \) is a topological selection of \( M \) if and only if \( R = M \cap *Z^+ \) for some \( n \)-dimensional vector space

\[ Z^+ := \text{linear span} \{ (\phi_{2i}^+, \phi_{2i+1}^+) : 1 \leq i \leq n \}, \]

where

(i) \( \phi_{2i}^+ \in X^* \),

(ii) \( \phi_{2i+1}^+ \in (M(0))^\perp \),

(iii) \( \psi_{2i}^+ \in Y^* \), and \( \{ \psi_{2i}^+ : 1 \leq i \leq n \} \) is a linearly independent set on \( M(0) \).

In the case when \( Y \) is a Hilbert space, the condition (iii) is replaced by

(iii)' \( \{ \psi_{2i}^+, \ldots, \psi_{2n}^+ \} \) is a basis for \( M(0) \).

Moreover, when \( R, Z^+ \) are as above, the adjoint subspace of \( R \) is represented by

\[ R^* = M^* \hat{+} Z^+ \quad \text{(direct sum)}. \]

**Proof.** Suppose that \( R \) is a topological selection of \( M \). Then \( R = M \cap \text{Null } P \) for some continuous projector \( P : M \to \{0\} \times M(0) \) with \( \text{Range } P = \{0\} \times M(0) \). Let \( \{ \phi_1, \ldots, \phi_n \} \) be a basis for \( M(0) \) and let \( \psi_{2i}^+ \in (M(0))^\# \), the dual of \( M(0) \), such that \( \langle \phi_i, \psi_{2i}^+ j \rangle = \delta_{ij} (1 \leq i, j \leq n) \), where \( \delta_{ij} \) is the Kronecker delta. Define \( P_0 : \text{Range } M \to M(0) \) by

\[ P_0(y) = \sum_{i=1}^n \langle y, \psi_{2i}^+ \rangle \phi_i, \quad y \in \text{Range } M. \]

Then \( P_0 \) is a continuous projector of \( \text{Range } M \) onto \( M(0) \). Thus by Sobczyk's lemma, which characterizes all continuous projectors with a prescribed range (see [7]), there exists a continuous linear operator \( A : M \to M(0) \) such that

\[ P(x, y) = (0, A(x, y) + P_0(y)), \quad (x, y) \in M, \]

\[ P(x, y) = (0, A(x, y) + P_0(y)), \quad (x, y) \in M, \]
and

\[(2.3) \quad A(0, y) = 0 \quad \text{for all} \quad y \in M(0).\]

In our setting we can write

\[(2.4) \quad A(x, y) = \sum_{i=1}^{n} \langle A(x, y), \psi_{2i}^+ \rangle \phi_i.\]

Since \( M \) is closed, \( M \times M(0) \) is Banach space. Let \( A_* \) denote the graph of the adjoint operator of \( A \) in the Banach space \( M \times M(0) \). Thus \( \text{Dom} \ A_* = (M(0))\# \times M\# \) and

\[A_* = \{(y^+, k^+) : y^+ \in (M(0))\#, k^+ \in M\#, \langle Ak^+, y^+ \rangle = \langle k, k^+ \rangle, \quad k \in \text{Dom} \ A\}.\]

Now \( A_*(\psi_{2i}^+) \in M\# \). By the Hahn-Banach extension theorem, we can extend \( A_*(\psi_{2i}^+) \) to \( Y\# \times X\# \). Denote this extension again by \( A_*(\psi_{2i}^+) \). Thus \( A_*(\psi_{2i}^+) \in Y\# \times X\# \). Let us write \( A_*(\psi_{2i}^+) = (-\phi_{1i}^+, \phi_{2i}^+) \). Then (2.4) becomes

\[(2.5) \quad A(x, y) = \sum_{i=1}^{n} [\langle x, -\phi_{1i}^+ \rangle + \langle y, \phi_{2i}^+ \rangle] \phi_i.\]

Let

\[(2.6) \quad P_2(x, y) := A(x, y) + P_0(y), \quad (x, y) \in M.\]

Then substituting \( A \) in (2.5) and \( P_0 \) in (2.1) into (2.6) we obtain

\[(2.7) \quad P_2(x, y) = \sum_{i=1}^{n} [\langle x, -\phi_{1i}^+ \rangle + \langle y, \phi_{2i}^+ \rangle + \psi_{2i}^+] \phi_i.\]

Let \( Z^+ \) be the linear span of the set \( \{\phi_{2i}^+, \psi_{2i}^+, \phi_{1i}^+ : 1 \leq i \leq n\} \). Then it follows from (2.7) that \( (x, y) \in M, P_2(x, y) = 0 \) if and only if \( (x, y) \in M \cap *Z^+ \). Now the condition (2.3) implies, using (2.5), that \( \phi_{2i}^+ \in (M(0))^\perp \) for all \( i \leq n \). Since \( \langle \phi_i, \psi_{2j}^+ \rangle = \delta_{ij} \), we see easily that \( \{\psi_{2i}^+ : 1 \leq i \leq n\} \) is linearly independent. Moreover, since
\[ \psi_{2i}^+ \in (M(0))^\# \], we can treat \( \psi_{2i}^+ \) as an element of \( Y^\# \). Thus \( Z^+ \) satisfies the required conditions.

Conversely, suppose that \( Z^+ \) is defined as in the theorem. Let \( P_2 \) be as in (2.7) and define \( P : M \to \{0\} \times M(0) \) by

\[ P(x, y) = (0, P_2(x, y)). \]

Then it is easy to check that \( P \) is a continuous projector on \( M \) onto \( \{0\} \times M(0) \). We now show that \( \text{Range } P_2 = M(0) \). This would then imply that \( \text{Range } P = \{0\} \times M(0) \). Since \( \{\phi_1, \ldots, \phi_n\} \) is a basis for \( M(0) \), it is sufficient to show that the map \( \pi : M \to \ell^m \) defined by

\[ (\pi(x, y))_i = (x, -\phi^+_{1i}) + (y, \phi^+_{2i} + \psi^+_{2i}) \]

is surjective, where \((y)_i\) denotes the \( i \)th component of \( y \). It is, in turn, sufficient to show that \( (\text{Range } \pi)^\perp = \{0\} \) since \( \pi \) is of finite-dimensional range. Suppose that \( \alpha_i \in \ell^m \) satisfy

\[ \sum_{i=1}^n \alpha_i [(x, -\phi^+_{1i}) + (y, \phi^+_{2i} + \psi^+_{2i})] = 0 \]

for all \((x, y) \in M\). This is true, in particular, for all \((x, y) \in \{0\} \times M(0)\). Thus \( \sum_{i=1}^n \alpha_i (y, \phi^+_{2i} + \psi^+_{2i}) = 0 \) for all \( y \in M(0) \). Since \( \phi^+_{2i} \in (M(0))^\perp \) and \( \{\psi^+_{2i} : M(0) \} : 1 \leq i \leq n \} \) is linearly independent, \( \alpha_i = 0 \) for all \( i \). Thus \( (\text{Range } \pi)^\perp = \{0\} \) and the claim is proved. Therefore \( P \) is a continuous projector of \( M \) onto \( \{0\} \times M(0) \) and so \( M \cap \text{Null } P \) is a topological selection of \( M \). Now \( M \cap \text{Null } P = M \cap \text{Null } R^* \). Finally since \( M^* + Z^+ \) is \( w^* \)-closed, \( R^* \) is the direct sum of \( M^* \) and \( Z^+ \).

**COROLLARY 2.2.** Assume that \( \dim M(0) =: n < \infty \). Then \( R \) is a principal topological selection of \( M \) if and only if \( R = M \cap ^* Z^+ \) for some \( n \)-dimensional vector subspace \( Z^+ \) of \( Y^\# \times X^\# \) which is the same as in the above theorem with the additional condition: \( \phi^+_{1i} = 0 \) for all \( 1 \leq i \leq n \).

**PROOF.** Let \( P_0 \) be the projector defined as in (2.1). It follows from the definition of a principal selection of \( M \) that \( R \) is a principal
selection of $M$ if and only if there exists a continuous projector $P$ on \( \text{Range } M \) onto $M(0)$ such that $R = \{(x, y) \in M : P(y) = 0\}$ (see [5]). Thus $R$ is a principal topological selection of $M$ if and only if $R = \{(x, y) \in M : B(y) + P_0(y) = 0\}$ for some continuous linear operator $B : \text{Range } M \to M(0)$ such that $B(y) = 0$ for all $y \in M(0)$. We now argue similarly as we did in the proof of Theorem 2.1. □

REMARK 2.1. Suppose that $X, Y$ are Hilbert spaces and $M$ a closed linear relation in $X \times Y$. Then $M \cap \{(0) \times M(0)\}^\perp =: R_0$ is called the orthogonal selection of $M$. This can be written as

$$R_0 = \{(x, y) \in M : y \in (M(0))\perp\}.$$ 

Thus if $\text{dim } M(0) =: n < \infty$, then we can write $R_0$ as

$$R_0 = M \cap Z^*$$

for some $Z =: \text{linear span } \{(\phi_i, 0) : 1 \leq i \leq n\}$ where $\{\phi_1, \ldots, \phi_n\}$ is a basis for $M(0)$. □

The following is the dual of Theorem 2.1.

THEOREM 2.3. Let $M^+$ be a $w^*$-closed linear relation in $Y^* \times X^*$ with $\text{dim } M^+(0) =: n^+ < \infty$. Then $R^+$ is a $w^*$-topological selection of $M^+$ if and only if $R^+ = M^+ \cap Z^*$ for some $n^+$-dimensional space $Z := \text{linear span } \{(\phi_{2i} + \psi_{2i}, \phi_{1i}) : 1 \leq i \leq n^+\}$

where

(i) $\phi_{1i} \in X$, $1 \leq i \leq n^+$,

(ii) $\phi_{2i} \in \perp (M^+(0))$,

(iii) $\{\psi_{2i} : 1 \leq i \leq n^+\}$ is a subset of $Y$ which is linearly independent mod $\perp (M^+(0))$.

Moreover, when $R^+$ and $Z$ are as above,

$$^* R^+ = ^* M^+ \oplus Z \quad (\text{direct sum}).$$

The following is a dual of Corollary 2.2.
Corollary 2.4. Let $M^+$ be as Theorem 2.3. Then $R^+$ is a $w^*$-principal topological selection of $M^+$ if and only if

$$R^+ = M^+ \cap Z^*$$

for some $n^+$-dimensional space

$$Z := \text{linear span} \left\{ (\phi_{2i} + \psi_{2i}, 0) : 1 \leq i \leq n^+ \right\}$$

where

1. $\phi_{2i} \in \perp (M^+(0))$,
2. $\{\psi_{2i} : 1 \leq i \leq n^+\}$ is linearly independent mod $\perp (M^+(0))$.

Remark 2.1. Theorem 2.1 and Corollary 2.2 generalize and complete Theorem 3.1 of Coddington and Dijksma [1], while Theorem 2.3 and Corollary 2.4 generalize and complete Theorem 3.4 of [1].

3. Representation theory of topological selections of inverses of linear relations. Let $X, Y$ be Banach spaces and let $M_1$ be a closed subspace of $X \times Y$. Suppose that $M$ is a closed subspace of $M_1$ such that $\dim M_1/M < \infty$. Let $R_1$ be a given topological selection of $M_1^{-1}$. In this section we consider the following problems:

(i) Is it possible to express any topological selection of $M_1^{-1}$ in terms of $R_1$ and possibly some other known quantities?

(ii) If $R_1$ is compact, then is every topological selection of $M_1^{-1}$ also compact?

(iii) If $R_1$ is an “integral” operator, then is every topological selection of $M_1^{-1}$ also an “integral” operator?

In the case when $M_1$ is single-valued, the problems (i), (ii) arise, for example, in inverting ordinary differential operators (see also §5 below). To address the above problems it is convenient to introduce a “minimal” closed subspace $M_0$ such that $M_0 \subset M \subset M_1$. In a concrete case $M_0$ and $M_1$ may be considered as the graphs of the minimal and maximal ordinary differential operators. Throughout this section unless otherwise mentioned $M_0$ is an arbitrary but fixed closed subspace of $M_1$, where $M_0$ and $M_1$ satisfy the conditions:

$$\begin{align*}
\dim \text{Null } M_1 &= n_1 < \infty \\
M_0 \subset M_1, \quad \dim M_1/M_0 &= d < \infty.
\end{align*}$$
For the rest of this section, let $B$ be an arbitrary, but fixed bounded linear operator on $M_1$ onto $\mathcal{H}^d$ with $\text{Null } B = M_0$. Such an operator clearly exists since $M_1/M_0$ is isometrically isomorphic to $\mathcal{H}^d$.

**Lemma 3.1.** Let $M$ be a closed linear relation such that $M_0 \subset M \subset M_1$. Then $\dim M_1/M = m$ if and only if there exists a $m \times d$ ($m \leq d$) constant matrix $\Gamma$ such that

\begin{equation}
M = \{ a \in M_1 : \Gamma B(a) = 0_{m \times 1} \}.
\end{equation}

**Proof.** See [4]. □

Let $\{\phi_1, \ldots, \phi_{n_1}\}$ be a basis for $\text{Null } M_1$ and let $\Gamma$ be the $d \times n_1$ constant matrix defined by

\[ \Gamma = [B(\phi_1, 0), \ldots, B(\phi_{n_1}, 0)]. \]

Let $R_1$ be an arbitrary but fixed topological selection of $M^{-1}_1$. Let $\Gamma$ be a $m \times d$ constant matrix. Let $(\Gamma D)^\dagger$ denote the Moore-Penrose generalized inverse of $\Gamma D$. For $y \in \text{Range } M$, define $\eta_\Gamma(y)$ by

\[ \eta_\Gamma(y) := R_1(y) - \sum_{i=1}^{n_1} \phi_i[(\Gamma D)^\dagger \Gamma B(R_1(y), y)]_i. \]

**Lemma 3.2.** Let $M$ be as in (3.2). Then

\[ M = \left\{ (\eta_\Gamma(y) + \sum_{i=1}^{n_1} (\alpha_0)_i \phi_i, y) : y \in \text{Range}(M_1), \Gamma D\alpha_0 = 0, \right\}
\]

In particular,

\[ \text{Null } M = \left\{ \sum_{i=1}^{n_1} (\phi_i(\alpha_0)_i : \Gamma D\alpha_0 = 0 \right\}. \]

**Proof.** $(x, y) \in M$ if and only if $(x, y) \in M_1$ and $\Gamma B(x, y) = 0$. Now, since $R_1$ is an algebraic selection of $M^{-1}_1$, $(x, y) \in M_1$ if and only
if \( y \in \text{Range} \, M_1 \) and \( x = R_1(y) + k \) for some \( k \in \text{Null} \, M_1 \). Let us write \( k = \sum_{i=1}^{n_1} \phi_i(\alpha_i) \) for some \( \alpha_i \in \psi^{n_1} \). Then

\[
M = \left\{ (R_1(y) + \sum_{i=1}^{n_1} \phi_i(\alpha_i), y) : y \in \text{Range} \, M_1, \, \alpha_i \in \psi^{n_1}, \right\}
\]

\[
\Gamma B(R_1(y), y) + \Gamma D \alpha = 0
\]

\[
= \left\{ (\eta_{\Gamma}(y) + \sum_{i=1}^{n_1} \phi_i(\alpha_0_i), y) : y \in \text{Range} \, M_1, \, \alpha_0_i \in \psi^{n_1}, \, \Gamma D \alpha_0 = 0, \right\}
\]

and \( \Gamma P B(R_1(y), y) \in \text{Range} \left( \Gamma D \right) \).

The last assertion of the theorem follows immediately from this characterization of \( M \).

In the following we express an arbitrary principal topological selection of \( M^{-1} \) in terms of a given selection of \( M^{-1}_1 \).

**Lemma 3.3.** Let \( M \) be as in (3.2). Let \( R_1 \) be a topological selection (principal or nonprincipal) of \( M^{-1}_1 \). If \( R_0 \) is a principal topological selection of \( M^{-1} \) given by

\[
R_0 = \{(y, x) : (y, x) \in M^{-1}, \, P_0(x) = 0\}
\]

for some continuous projector \( P_0 : \text{Dom} \, M \rightarrow \text{Null} \, M \) with \( \text{Range} \, P_0 = \text{Null} \, M \), then

\[
(3.3) \quad \text{Dom} \, R_0 = \{ y : y \in \text{Range} \, M_1, \, \Gamma B(R_1(y), y) \in \text{Range} \left( \Gamma D \right) \},
\]

\[
R_0(y) = (I - P_0)\eta_{\Gamma}(y), \quad y \in \text{Dom} \, R_0.
\]

Conversely, if \( R_0 \) is defined as in (3.3) for some continuous projector \( P_0 \) on \( \text{Dom} \, M \) onto \( \text{Null} \, M \), then \( R_0 \) is a principal topological selection of \( M^{-1} \).

**Proof.** Assume that \( R_0 \) is given as in (3.3). Then \( (y, x) \in R_0 \) if and only if (i) \( (x, y) \in M \), and (ii) \( (x, y) \in M, \, P_0(x) = 0 \). By Lemma 3.2,
(i) holds if and only if

\[
\begin{align*}
(i)' & \quad \begin{cases} 
y \in \text{Range } M_1, 
\quad y = \eta(y) + \sum_{i=1}^{n_1} \phi_i(\alpha_0)_i, 
\quad F^{-1}(R_1(y), y) \in \text{Range } (FD), 
\quad FD \alpha_0 = 0. 
\end{cases}
\end{align*}
\]

Thus (ii) holds if and only if (i)' holds in addition to

\[
(ii)' \quad \mathcal{P}_0(\eta(y)) + \mathcal{P}_0 \left( \sum_{i=1}^{n_1} \phi_i(\alpha_0)_i \right) = 0.
\]

Since \( FD \alpha_0 = 0 \), we see that \( \sum_{i=1}^{n_1} \phi_i(\alpha_0)_i \in \text{Null } M \), and so \( \mathcal{P}_0 (\sum_{i=1}^{n_1} \phi_i(\alpha_0)_i) = \sum_{i=1}^{n_1} \phi_i(\alpha_0)_i \). Then, using (ii)', we obtain

\[
x = \eta(y) - \mathcal{P}_0(\eta(y)) = (I - \mathcal{P}_0) \eta(y).
\]

Thus (3.4) holds. Suppose now that (3.4) holds for some continuous projector \( \mathcal{P}_0 \) on \( \text{Dom } M \) onto \( \text{Null } M \). We will show that

\[
R_0 = \{(y, x) \in M^{-1} : \mathcal{P}_0(x) = 0\}.
\]

First we will show that "\( \subset \)" holds. Note that, by Lemma 3.2, \( \text{Dom } R_0 = \text{Range } M \). Thus \( \eta(y) \) is well-defined for \( y \in \text{Dom } R_0 \). Take any \( (y, x) \in R_0 \). Then \( y \in \text{Dom } M \) and

\[
(y, x) = (y, (I - \mathcal{P}_0) \eta(y)) = (y, \eta(y)) + (0, -\mathcal{P}_0 \eta(y)).
\]

By Lemma 3.2, \( (y, \eta(y)) \in M^{-1} \). Since \( \text{Range } \mathcal{P}_0 \subset \text{Null } M \), we see that \( (y, x) \in M^{-1} \). Now \( \mathcal{P}_0(x) = \mathcal{P}_0 R_0(y) = \mathcal{P}_0[I - \mathcal{P}_0] \eta(y) = 0 \). Thus

\[
R_0 \subset \{(y, x) \in M^{-1} : \mathcal{P}_0(x) = 0\}.
\]

If \( (y, x) \in M^{-1} \) and \( \mathcal{P}_0(x) = 0 \), then as in the first part of this proof, \( (y, x) \in R \). Hence

\[
R_0 = \{(y, x) \in M^{-1} : \mathcal{P}_0(x) = 0\}. \quad \square
\]

**Lemma 3.4.** (See [5]) Suppose that \( M \) is a linear relation in \( X \times Y \). Assume that \( R_0 \) is a principal topological selection of \( M^{-1} \). Then \( R \) is a topological selection of \( M^{-1} \) if and only if

\[
R(y) = R_0(y) - A(y, R_0(y)), \quad \text{for all } y \in \text{Range } M.
\]
for some continuous linear operator $A$ on $M^{-1}$ into $\text{Null } M$ such that $A(0, x) = 0$ for all $x \in \text{Null } M$.

We now express an arbitrary (nonprincipal) topological selection of $M^{-1}$ in terms of a given topological selection of $M_1^{-1}$.

**THEOREM 3.5.** Let $R_1$ be a given topological selection of $M_1^{-1}$. Let $M$ be as in (3.2) for some $m \times d$ constant matrix $\Gamma$. Then $R$ is a topological selection of $M^{-1}$ if and only if

$$\text{Dom } R = \{ y : y \in \text{Range } M_1, \text{ } \text{ } \Gamma B(R_1(y), y) \in \text{Range}(\Gamma D) \},$$

$$R(y) = (I - \mathcal{P}_0)\eta_1(y) - A(y, (I - \mathcal{P}_0)\eta_\mathcal{P}(y)), \ y \in \text{Dom } R$$

for some continuous projector $\mathcal{P}_0$ on $\text{Dom } M$ onto $\text{Null } M$ and a continuous linear operator $A : M^{-1} \rightarrow \text{Null } M$ such that $A(0, x) = 0$ for all $x \in \text{Null } M$.

**PROOF.** This follows Lemma 3.4 together with Lemma 3.3. $\square$

In the following we give a simpler necessary form for the topological selections of $M^{-1}$.

**THEOREM 3.6.** Suppose that $M_1$ is a closed linear relation in $X \times Y$ with $\dim \text{Null } M_1 =: n_1 < \infty$. Let $R_1$ be an arbitrary, but fixed topological operator of $M_1^{-1}$. If $M$ is a closed linear relation in $X \times Y$ with $M \subset M_1$, $\dim M_1/M < \infty$ and if $R$ is any topological selection of $M^{-1}$, then there exist $x_j^+ \in X^*$ and $y_j^+ \in Y^*(1 \leq j \leq n_1)$ such that

$$R(y) = R_1(y) + \sum_{j=1}^{n_1} [(R_1(y), x_j^+) + (y, y_j^+)]\phi_j, \ y \in \text{Dom } R.$$

**PROOF.** Let $M$ be as in the theorem. Since $\dim M_1/M < \infty$, without loss of a generality we may assume that there exists a closed linear relation $M_0 \subset X \times Y$ such that $M_0 \subset M \subset M_1$. We may also assume that

$$\dim M_1/M =: m \leq \dim M_1/M_0 =: d.$$
Let $B$ be a continuous linear operator on $M_1$ onto $\mathcal{Q}^d$ annihilating $M_0$. Using Lemma 3.1, we may also assume that

$$M = \{ a \in M_1 : \Gamma B(a) = 0_{m \times 1} \},$$

for some $m \times d$ constant matrix $\Gamma$. This means that $M$ fits into the setting described at the beginning of this section. Thus we assume that $\phi_i, D$ are as before. Take any topological selection $R$ of $M^{-1}$. Then by Theorem 3.5,

$$\text{Dom } R = \{ y \in \text{Range } M_1 : \Gamma B(R_1(y), y) \in \text{Range}(\Gamma D) \},$$

$$R(y) = (I - P_0)\eta(y) - A(y, (I - P_0)\eta(y)), \quad y \in \text{Dom } R,$$

for some continuous projector $P_0$ on Dom $M$ onto Null $M$ and a continuous linear operator $A : M^{-1} \rightarrow \text{Null } M$ such that $A(0, x) = 0$ for all $x \in \text{Null } M$, where

$$\eta(y) = R_1(y) - \sum_{i=1}^{n_1} \phi_i[(\Gamma D)\Gamma B(R_1(y), y)]_i.$$

Since $B$ is a bounded linear operator there exist $h_{j_1}^+ \in X^\#, \ h_{j_2}^+ \in Y^\#$ such that

$$(B(x, y))_j = \langle x, h_{j_1}^+ \rangle + \langle y, h_{j_2}^+ \rangle, \quad (x, y) \in M_1.$$  

Since $P_0 : \text{Dom } M \rightarrow \text{Null } M$ and $A : M^{-1} \rightarrow \text{Null } M$ are continuous and Null $M \subset \text{Null } M_1$, there exist $g_{i_1}^+ \in X^\#, \ f_{i_1}^+ \in X^\#, \ f_{i_2}^+ \in Y^\#$ such that

$$P_0(x) = \sum_{i=1}^{n_1} \langle x, g_{i_1}^+ \rangle \phi_i, \quad x \in \text{Dom } M,$$

$$A(y, x) = \sum_{i=1}^{n_1} [(\langle y, f_{i_2}^+ \rangle + \langle x, f_{i_1}^+ \rangle)\phi_i$$

for some $f_{i_2}^+ \in Y^\#, \ f_{i_1}^+ \in X^\#$. Substituting $B$ in (3.6), $P_0$ in (3.8) and $A$ in (3.9) into (3.6) and then into (3.5), it follows, after a tedious calculation, that $R(y)$ has the form claimed in the above corollary. \[ \square \]
COROLLARY 3.7. Suppose that $M_1$ is a closed linear relation in $X \times Y$ with $\dim \text{Null } M_1 < \infty$. Suppose that $M$ is a closed subspace of $M_1$ such that $\dim M_1/M < \infty$. If some topological selection of $M_1^{-1}$ is compact, then all topological selections of $M^{-1}$ are also compact.

PROOF. The result follows immediately from Theorem 3.6. \(\square\)

REMARK 3.1. By the above corollary, if $R_1$ is a compact topological selection of $M_1^{-1}$ and if Null $M_1$ is finite dimensional, then any topological selection of $M_1^{-1}$ is also compact. It is proved in [5] that if $M$ is a closed linear relation in $X \times Y$, then the continuity of a particular topological selection of $M^{-1}$ implies the continuity of any topological selection of $M^{-1}$.

4. Applications to integral operators. In this section we consider the following question: Suppose that $L$ and $L_1$ are closed linear operators from a Banach space $X$ into a Banach space $Y$ such that $\mathcal{G}r L \subset \mathcal{G}r L_1$, where $\mathcal{G}r L$ denotes the graph of $L$. Assume that $(\mathcal{G}r L_1)^{-1}$, as a linear relation, has a topological selection which is an "integral" operator. Is every topological selection of $(\mathcal{G}r L)^{-1}$ also an "integral" operator? The question is, of course, trivially true if $L_1$ is one-to-one. The above question arises naturally if $L_1$ is an ordinary differential operator. We will answer this question under a certain finiteness condition. We will consider the problem when $X$ and $Y$ are $L^p$-type spaces or the space of continuous functions. Let $(S, \Sigma_S, \mu)$ and $(T, \Sigma_T, \nu)$ be $\sigma$-finite nonnegative measure spaces, and for $1 < p < +\infty$, let $X^p := X^p(S, \Sigma_S, \mu)$ be the Banach space of all complex-valued $\mu$-measurable functions $x$ on $S$ such that $\|x\|_p < \infty$, where

$$
\|x\|_p := \left(\int_S |x(s)|^p d\mu(s)\right)^{1/p} \quad \text{if } 1 \leq p < \infty, \\
:= \text{ess sup}_{s \in S} |x(s)| \quad \text{if } p = \infty.
$$

For $1 \leq q \leq \infty$, we define $Y^q := Y^q(T, \Sigma_T, \nu)$ in a similar way.

Let $K$ be a compact Hausdorff space and let $C(K)$ denote the Banach space of all complex-valued functions $f$ on $K$ with $\|f\|_\infty := \sup\{|f(k)| : k \in K\}$:

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As in §3, for a Banach space Z, the natural pairing on \( Z \times Z^\# \) is denoted by \( \langle \cdot, \cdot \rangle \).

**Definition 4.1.** \( U \) is an integral operator from \( X^p \) into \( Y^q \) if \( \text{Dom } U \) is a vector subspace of \( X^p \) and there exists a complex-valued function \( \mathcal{K}(s,t) \) \((s \in S, t \in T)\) such that

(i) \( \mathcal{K}(s,t) \) is \( \mu \times \nu \)-measurable, and

(ii) for all \( x \in \text{Dom } U \) and for \( \nu \)-almost all \( t \in T \), \( \mathcal{K}(s,t)x(s) \), considered as a function of \( s \), is \( \mu \)-integrable, and

\[
(Ux)(t) := \int_S \mathcal{K}(s,t)x(s) \, d\mu(s) \in Y^q.
\]

**Definition 4.2.** \( U \) is an integral operator from \( X^p \) into \( C(K) \) if \( \text{Dom } U \) is a vector subspace of \( X^p \) and there exists a complex-valued function \( \mathcal{K}(s,k) \) \((s \in S, k \in K)\) such that

(i) \( \mathcal{K}(s,k) \) is \( \Sigma_S \times \mathcal{B} \)-measurable, where \( \mathcal{B} \) is the \( \sigma \)-algebra of all Borel subset of \( K \), and

(ii) for all \( x \in \text{Dom } U \) and all \( k \in K \), \( \mathcal{K}(s,k)x(s) \) as a function of \( s \) is \( \mu \)-integrable, and

\[
(Ux)(k) := \int_S \mathcal{K}(s,k)x(s) \, d\mu(s) \in C(K).
\]

The function \( \mathcal{K} \) is called the kernel of \( U \).

We recall that the natural pairing \( \langle \cdot, \cdot \rangle \) on \( X^p \times X^{p'} \), where \( p' \) is the conjugate number to \( p > 1 \) \( (p^{-1} + (p')^{-1} = 1) \), is given by

\[
\langle x, y \rangle := \int_S x(s)\overline{y}(s) \, d\mu(s),
\]

and that the natural pairing on \( C(K) \times (C(K))^\# \) is given by

\[
\langle x, \phi \rangle := \int_K x(k) \, d\phi(k), \ x \in C(K), \ \phi \in (C(K))^\#.
\]

**Lemma 4.1.** Let \( U \) be an integral operator from \( X^p \) into \( Y^q \) with kernel \( \mathcal{K}(x,t) \) \((s \in S, t \in T)\). Suppose that

\[
(4.1) \quad \int_S |\mathcal{K}(s,t)x(s)| \, d\mu(s) \in Y^q
\]
for all \( x \in \text{Dom} U \). Then for any \( y^+ \in (Y^q)^\# \), the functional \( \langle U(x), y^+ \rangle \), \( x \in \text{Dom} U \), is an integral operator from \( X^p \) into \( \psi' \). In particular, (4.1) is satisfied if

\[
\begin{cases}
h(t) := \|K(\cdot, t)\|_{p'} < \infty, & \text{and} \\
h \in Y^q
\end{cases}
\]

where \( p' \) is the conjugate number to \( p \).

**Lemma 4.2.** Let \( U \) be an integral operator from \( X^p \) into \( C(K) \) with kernel \( K(s, k) \) \((s \in S, k \in K)\). Suppose that

\[
\int_S |K(s, k)x(s)| \, d\mu(s) \in C(K).
\]

Then for any \( \phi \in (C(K))^\# \), the functional

\[
\langle U(x), \phi \rangle, \ x \in \text{Dom} U
\]

is an integral operator from \( X^p \) into \( \psi' \). In particular, (4.2) is satisfied if

\[
\begin{cases}
h(k) := \|K(\cdot, k)\|_{p'} < \infty & \text{and} \\
h \in C(K)
\end{cases}
\]

**Proof of Lemma 4.1.**

**Case (i).** \( 1 \leq q < +\infty \). Assume (4.1) and take \( x \in X^p \) and \( y^+ \in (Y^q)^* \). Let \( q' \) be the conjugate to \( q \). Let

\[
f(t) = \int_S |K(s, t)x(s)| \, d\mu(s).
\]

Then

\[
\int_T \left[ \int_S \left| K(s, t) \right| \left| x(s) \right| \, d\mu(s) \right] |y^+(t)| \, d\nu(t)
\]

\[
= \int_T f(t) |y^+(t)| \, d\nu(t) \leq \|f\|_{Y^q} \|y^+\|_{Y^{q'}} < \infty.
\]

Thus by Tonelli's theorem

\[
\langle U(x), y^+ \rangle = \int_S \left[ \int_T K(s, t)y^+(t) \, d\nu(t) \right] x(s) \, d\mu(s).
\]
Hence $\langle U(x), y^+ \rangle$ is an integral operator.

*Case (ii).* $q = \infty$.

It is known (see Kantorovich & Akilov [2, p. 192]) that $(Y^\infty)^\#$ is the Banach space of all “bounded additive” functions on $\Sigma_T$, that is, the set of all complex-valued additive functions $\phi$ on $\Sigma_T$ such that (i) and (ii) hold:

(i) If $A \in \Sigma_T$ and $\nu(A) = 0$, then $\phi(A) = 0$.

(ii) The total variation of $\phi$, $|\phi|(T)$, is finite.

The norm $\|\phi\|$ of $\phi$ is given by $\|\phi\| := |\phi|(T)$. Thus for any $y \in Y^\infty$ and $\phi \in (Y^\infty)^\#$,

$$\langle y, \phi \rangle := \int_T y(t) d\phi(t),$$

where the integral is a Stieltjes integral. Take any $x \in X^p$ and $\phi \in Y^\infty$. Then

$$\langle Ux, \phi \rangle = \int_T (Ux)(t) d\phi(t) = \int_T \int_S K(s,t)x(s) d\mu(s) d\phi(t),$$

and

$$\int_T \int_S \big| K(s,t) \big| x(s) d\mu(s) d|\phi|(t)$$

$$\leq \left( \text{ess sup}_{t \in T} \int_S \big| K(s,t) \big| x(s) d\mu(s) \right) \|\phi\| < \infty.$$

Thus the order of integration in the above expression for $\langle Ux, \phi \rangle$ may be interchanged, and so

$$\langle Ux, \phi \rangle = \int_S \left[ \int_T K(s,t) d\phi(t) \right] x(s) d\mu(s).$$

Hence $\langle Ux, \phi \rangle$ is an integral operator. Combining the cases (i) and (ii), we see that when (4.1) holds, $\langle Ux, \phi \rangle$, $\phi \in (Y^q)^\#$, is an integral operator.

Suppose now that (4.2) holds. Take any $x \in X^p$. Suppose $1 \leq q < \infty$. Then

$$\int_T \left( \int_S \big| K(s,t) \big| x(s) d\mu(s) \right)^q d\nu(t)$$

$$\leq \int_T (h(t) \|x\|_p)^q d\nu(t) = \|x\|_p^q \|h\|_q^q < \infty.$$
Suppose $q = +\infty$, Then
\[
\text{ess sup}_{t \in T} \left( \int_S |\mathcal{K}(s, t)| \, |x(s)| \, d\mu(s) \right) 
\leq \text{ess sup}_{t \in T} (h(t)) \|x\|_p = \|x\|_p \|h\|_{\infty} < \infty.
\]

Thus if (4.2) holds, then (4.1) holds. This proves Lemma 4.1. □

PROOF OF LEMMA 4.2. It is known that $(C(K))^\#$ is the Banach space of all regular countably additive complex-valued functions $\phi$ on the $\sigma$-algebra $\mathcal{B}$ of the Borel sets in $K$ such that $|\phi|(K) < \infty$. In particular, for any $x \in C(K)$, and $\phi \in (C(K))^\#$, $x$ is $\phi$-integrable ($\phi$ is a $\sigma$-finite signed measure on $\mathcal{B}$). Assume (4.3). Take any $x \in C(K)$ and $\phi \in (C(K))^\#$. Then

\[
\int_T \int_S |\mathcal{K}(s, t)x(s)| \, d\mu(s) \, d|\phi|(t) 
\leq \int_K \left( \sup_{t \in K} \int_S |\mathcal{K}(s, t)x(s)| \, d\mu(s) \right) \, d|\phi|(t) 
\leq \|\phi\| \sup_{t \in K} \int_S |\mathcal{K}(s, t)x(s)| \, d\mu(s) < +\infty.
\]

Hence,
\[
\langle Ux, \phi \rangle = \int_K \int_S \mathcal{K}(s, t)x(s) \, d\mu(s) \, d\phi(s) 
= \int_S \left[ \int_K \mathcal{K}(s, t) \, d\phi(s) \right] x(s) \, d\mu(s).
\]

Thus $\langle Ux, \phi \rangle$ is an integral operator in $x$ from $X^p$ into $C(K)$.

Assume now that (4.4) holds. Take any $x \in X^p$. Then

\[
\int_S |\mathcal{K}(s, t)x(s)| \, d\mu(s) \leq h(t) \|x\|_p
\]

Thus

\[
\sup_{t \in K} \int_S |\mathcal{K}(s, t)x(s)| \, d\mu(s) \leq \sup_{t \in K} h(t) \|x\|_p < \infty.
\]

Thus (4.3) holds. This completes the proof for Lemma 4.2. □
THEOREM 4.3 Let $Z$ be either $Y'$ or $C(K)$, as defined earlier. Let $M_1$ be a closed linear relation in $Z \times X_p$ such that $\dim \text{Null } M_1 < \infty$. Let $M$ be a closed subspace of $M_1$ such that $\dim M_1/M < \infty$. Suppose that there exists a topological selection $U_1$ of $M_1$ such that $U_1$ is an integral operator with kernel $\mathcal{K}(s,t)$ satisfying

\begin{equation}
\int_S |\mathcal{K}(s,t)x(s)|d\mu(s) \in Z
\end{equation}

for all $x \in \text{Dom } U_1$. Then any topological selection $U$ of $M_1^{-1}$ is also an integral operator. Moreover, if $U_1$ is continuous, so is $U$.

PROOF. Let $n_1 = \dim \text{Null } M_1$ and let $\{\phi_1, \ldots, \phi_{n_1}\}$ be a basis for $\text{Null } M_1$. Take any topological selection $U$ of $M_1$. Then by Theorem 3.6, $U$ has the form

\begin{equation}
U(x) = U_1(x) + \sum_{j=1}^{n_1} \left[ \langle U_1(x), z_j^+ \rangle + \langle x, x_j^+ \rangle \right] \phi_j, \quad x \in \text{Dom } U
\end{equation}

for some elements $x_j^+ \in X_p'$, $z_j^+ \in Z^\#$. By Lemmas 4.1 and 4.2, $\langle U_1(x), z_j^+ \rangle$ as a function of $x$ is an integral operator. Hence by (4.6), $U$ is also an integral operator from $X_p$ into $Z$. The rest follows easily. □

5. Applications to regular ordinary differential subspaces in $L_p[a, b] \times L_q[a, b]$. In this section we consider concrete applications of Theorem 3.6 and Corollary 3.7 to ordinary differential subspaces. Let $[a, b]$ be a compact interval. Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Let $X_p(1 \leq p \leq \infty)$ be the Banach space of all functions $f : [a, b] \to \mathbb{R}^n$ which are Lebesgue measurable on $[a, b]$ and $\|f\|_p < \infty$, where $\|f\|_p = \int_a^b (f^*(t)f(t))^p dt$ when $p < \infty$; $\|f\|_\infty := \text{ess sup}\{f^*(t)f(t) : t \in [a, b]\}$. Let $n$ be a natural integer. Let $T_1$ be the “maximal” differential operator from $\text{Dom } T_1 \subset X_p$ into $X_q$ defined by

$x \in \text{Dom } T_1$ if and only if $x \in X_p$, $x^{(n-1)} \in AC[a, b]$, $x^{(n)} \in X_q$,

$T_1x = Q_n(t)x^{(n)} + Q_{n-1}(t)x^{(n-1)} + \cdots + Q_1(t)x^{(1)} + Q_0(t)x$. 

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Here each $Q_i(t)$ is a $m \times m$ matrix-valued function of $t \in [a, b]$ which is $i$-times continuously differentiable on $[a, b]$ and $Q_{\eta}(t)$ is invertible for all $t \in [a, b]$, and $x^{(i)}$ denotes the $i^{th}$ derivative of $x$. Then it is well-known that $T_1$ is closed, i.e., $\mathcal{Gr}(T_1)$ is closed in $X_p \times X_q$. Let

$$M_1 := \{a + (x, T_1 x) : a \in A, \ x \in \text{Dom} \ T_1\},$$

where $A$ is a finite dimensional subspace of $X_p \times X_q$. Then it is not difficult to see that $\text{Null} \ M_1$ is finite dimensional and $M_1$ is closed. Thus, in particular, there exist topological selections of $M_1^{-1}$.

**Theorem 5.1.** Let $T_1$ and $M_1$ be as above. Let $M$ be an arbitrary, but fixed closed subspace of $M_1$ with $\dim M_1/M < \infty$. Let $R$ be any topological selection of $M^{-1}$. Then $R$ is the graph of a compact integral operator on $\text{Range} \ M$. Moreover, $\text{Range} \ M$ is closed.

**Proof.** Since the differential equation $T_1 x = g$ has a solution for every $g \in X_q$, we see that $\text{Range} \ T_1 = X_q$. Since $T_1$ is closed and $\dim \text{Null} \ T_1 < \infty$, it is easy to see that $(\mathcal{Gr} \ T_1)^{-1}$ has a topological selection which is a compact integral operator. Pick one and call it $R_0$. We will show that $M_1^{-1}$ has a topological selection which is a compact integral operator defined on $X_q$. It is clear that $M_1^{-1}$ has a principal topological selection. Call this $R_1$. Then $\text{Dom} \ R_1 = X_q$ and

$$R_1(y) = (I - \mathcal{P})(x), \ x \in M_1^{-1}(y), \ y \in X_q$$

where $\mathcal{P}$ is a continuous projector on $\text{Dom} \ M_1$ onto $\text{Null} \ M_1$. Take any $y \in \text{Dom} \ R_1$. Then $R_1(y)$ is given as in (5.1) for some $(x, y) \in M_1$. Write

$$x = z + f, \ y = T_1 z + g$$

for some $(f, g) \in A$. Then, in particular, $z = R_0(y - g) + k$ for some $k \in \text{Null} \ T$. It then follows that $R_1(y) = (I - \mathcal{P})[R_0(y) + k - R_0(g) + f]$. Since

$$\text{Null} \ M_1 = \{u + h : u \in \text{Dom} \ T_1, \ T_1 u + g = 0 \text{ for some } (h, g) \in A\},$$

it follows that

$$k - R_0(g) + f \in \text{Null} \ M_1.$$
Thus

\[ R_1(y) = (I - \mathcal{P})R_0(y). \]

This shows that \( R_1 \) is a compact operator. Applying Fubini's theorem, if necessary, it follows that \( R_1 \) is also an integral operator since \( \mathcal{P} \) has a finite-dimensional range. Hence we have shown that \( M_1^{-1} \) has a topological operator which is also a compact integral operator. Therefore using Theorems 3.6 and 4.1 we see that any topological selection \( R \) of \( M^{-1} \) is a compact integral operator. Finally \( \text{Dom} R := \text{Range} M \) must be closed since \( R \) is a closed bounded linear operator. This completes the proof. \( \square \)

**REMARK 5.1.** For a nonnegative integer \( m \) and complex constants \( \alpha_{ij}, \beta_{ij} \) define a differential operator \( T \) by

\[
\text{Dom} \, T = \left\{ x \in \text{Dom} \, T_1 : \sum_{i=0}^{n-1} (\alpha_{ij}x^{(i)}(a) + \beta_{ij}x^{(i)}(b)) = 0, \; 0 \leq j \leq m \right\},
\]

\( Tx = T_1x, \; x \in \text{Dom} \, T, \) where \( T_1 \) is as defined earlier in this section. Then \( T \) is a closed operator, but is not necessarily one-to-one. Using Theorem 3.1 of [5] a linear operator \( R : \text{Range} \, T \to \text{Dom} \, T \) is a topological selection of \( (\mathcal{G}r \, T)^{-1} \) if and only if \( TR = I \) on \( \text{Range} \, M \), and \( RT \) is continuous on \( \text{Dom} \, T \). Since \( \dim (\mathcal{G}r \, T_1/\mathcal{G}r \, T) = \dim (\text{Dom} \, T_1/\text{Dom} \, T) < +\infty \), it follows from the above theorem that any topological selection of \( (\mathcal{G}r \, T)^{-1} \) is a compact integral operator. The kernel of this selection is called a generalized Green function when \( T \) is not invertible, and the Green function when \( T \) is invertible. This has been extensively studied in the literature.

**REMARK 5.2.** Let \( f \) and \( g \) be \( m \times n \) matrix-valued functions on \( [a, b] \) such that the columns of \( f \) are in \( X_{p'} \) and the columns of \( g \) are in \( X_{q'} \). Define an operator \( T \) by

\[
\text{Dom} \, T := \left\{ x \in \text{Dom} \, T_1 : \int_a^b [x^*(t)f(t) + (T_1x)^*g(t)] \, dt = 0_{1 \times m} \right\},
\]

\( Tx := T_1x, \; x \in \text{Dom} \, T. \)
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Then $\mathcal{G}r T \subset \mathcal{G}r T_1$, $T$ is closed. Moreover $\dim(\mathcal{G}r T_1/\mathcal{G}r T) \leq m < \infty$. Thus by the above theorem any topological selection of $(\mathcal{G}r T)^{-1}$ is a compact integral operator.

REMARK 5.3. Let $\mu(1 \leq i \leq m < \infty)$ be $m \times n$ vector-valued functions of bounded variation on $[a, b]$. Define $T$ by

$$\text{Dom } T := \left\{ x \in \text{Dom } T_1 : \int_a^b (d\mu(t))x(t) = 0_{m \times 1} \right\},$$

$$Tx := T_1x, \quad x \in \text{Dom } T$$

The integral is taken as a Riemann-Stieltjes integral. Clearly $\mathcal{G}r T \subset \mathcal{G}r T_1$. Assume that $T$ is closed. Then one can see easily that $\dim(\mathcal{G}r T_1/\mathcal{G}r T) < \infty$ and $\text{Dom } T$ can be written as in Remark 5.2, and so, by the above theorem, any topological selection of $(\mathcal{G}r T)^{-1}$ is a compact integral operator. Special cases of this problem have been considered in the literature. See the survey paper by Krall [3].

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