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HÖLDER ESTIMATES FOR THE CAUCHY INTEGRAL ON A LIPSCHITZ CONTOUR

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ABSTRACT. Some classical results of J. Plemelj and I. Privalov, concerning Hölder continuity of the Cauchy integral, are generalised by relaxing the smoothness assumptions on the contour of integration.

Introduction. In order to develop the theory of one-dimensional singular integral equations, it is first necessary to establish certain basic properties of the Cauchy integral. Chief among these are the Plemelj-Sokhotski formulae, and estimates in L_p or Hölder norms. Modern treatments of the theory can be found in the books of Gohberg and Krupnik [**3**], Prö𝔅dorf [**11**] and Mikhlin and Prö𝔅dorf [**9**]; the standard classical text is of course Muskhelishvili [**10**]. These authors all assume that the contour of integration Γ is reasonably smooth - more precisely, Γ must consist of a finite number of non-intersecting Lyapunov curves, and must not possess any cusps. (A curve satisfies the Lyapunov condition if and only if it is locally the graph of a function with a Hölder continuous derivative, see [**7**, Appendix A]).

The aim of this paper is to show that the basic Hölder estimates remain valid when Γ is assumed only to be a Lipschitz contour. It is then a relatively straight forward matter to generalise most of the remaining theory of one-dimensional singular integral equations with Hölder continuous coefficients - the details have been worked out in [**8**]. Happily, the Hölder estimates can be proved using only elementary methods, in stark contrast to the L_p theory for Lipschitz contours, which relies on the Coifman-McIntosh-Meyer Theorem [**1**].

The paper is organized as follows. In §1, some properties of Lipschitz contours are discussed, along with the important idea of a nontangential limit. The fundamental result, that the Cauchy integral determines a bounded linear operator on spaces of Hölder continuous functions, is proved in §2. Finally, in §3 we establish the Plemelj-Sokhotski formulae,

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and show that the Cauchy integral is Hölder continuous on the closures of the regions bounded by Γ.

1. Lipschitz contours. For notational convenience, the field of complex numbers $\mathbf{C}$ will be freely identified with the vector space $\mathbf{R}^2$. Thus, if $y \in \mathbf{C}$, then the real part of y is denoted by $y_1 = \Re y$, the imaginary part of y is denoted by $y_2 = \Im y$, and we write $y = y_1 + iy_2 = (y_1, y_2)$.

Suppose Γ is a (closed) Jordan curve, then the complex plane decomposes into the disjoint union

$$\mathbf{C} = \Omega_+ \cup \Gamma \cup \Omega_-,$$

where Ω_+ is a bounded, simply-connected open set, and Ω_- is an unbounded, connected open set. The components Ω_+ and $\Omega_-$are uniquely determined by Γ, and their boundries and closures are given by

$$\partial\Omega_+ = \Gamma = \partial\Omega_-, \quad \overline{\Omega}_+ = \Omega_+ \cup \Gamma, \quad \overline{\Omega}_- = \Omega_- \cup \Gamma.$$

These topological facts are discussed in Hille [**4**, p. 34]. (The theory which follows could be generalised by allowing Γ to consist of several nonintersecting Jordan curves, as in [**10**, p. 86]; see also [**9**, pp. 43-44].)

In the context of singular integral equations, a simple closed curve is usually called a *contour*, whereas a simple, open-ended curve is usually called an arc. Thus, if Ω_+ is a Lipschitz domain, then we say that Γ is a *Lipschitz contour.* The definition of a Lipschitz domain can be found in many texts on partial differential equations and function spaces, nevertheless, we repeat it here because the notation is needed later.

For $x \in \mathbf{C}$ and $\theta \in \mathbf{R}$, let $\mathcal{A}_{x,\theta} : \mathbf{C} \to \mathbf{C}$ be the affine transformation which first translates by $-x$, and then rotates by $-\theta$, i.e.,

$$\mathcal{A}_{x,\theta}(y) = (y - x)e^{-i\theta}, \quad y \in \mathbf{C}.$$

Given $x \in \Gamma$ and $a > 0$, the triple (a, θ, ϕ) is said to be a *Lipschitz representation* of Γ at x if $\phi : [-a, a] \to [-a, a]$ is a Lipschitz function satisfying $\phi(0) = 0$ and

$$\begin{aligned}
\mathcal{A}_{x,\theta}(\Gamma) \cap (-a,a)^2 &= \{(y_1, y_2) : -a < y_1 < a \text{ and } y_2 = \phi(y_1)\} \\
\mathcal{A}_{x,\theta}(\Omega_+) \cap (-a,a)^2 &= \{(y_1, y_2) : -a < y_1 < a \text{ and } \phi(y_1) < y_2 < a\} \\
\mathcal{A}_{x,\theta}(\Omega_-) \cap (-a,a)^2 &= \{(y_1, y_2) : -a < y_1 < a \text{ and } -a < y_2 < \phi(y_1)\}.
\end{aligned}$$

In other words, $\mathcal{A}_{x,\theta}(\Gamma) \cap [-a, a]^2$ is the graph of the Lipschitz function ϕ. The contour Γ is said to be *Lipschitz* if a Lipschitz representation exists at every point of Γ. Note that a Lipschitz contour may have (infinitely many) corners, but may not possess cusps.

Henceforth, it is always assumed that Γ is a Lipschitz contour, and that Γ has a positive (i.e., counterclockwise) orientation. Thus, the winding number of Γ about any point of Ω_+ is $+1$, whereas the winding number of Γ about any point of Ω_- is 0.

The derivative of a Lipschitz function exists almost everywhere, and belongs to L_∞, therefore Γ is rectifiable. For points x and y lying on Γ, let $|x, y|$ denote the minimum of the lengths of the two sub-arcs of Γ having x and y as their end points. Thus, if $\gamma : \mathbf{R} \to \mathbf{C}$ is any periodic arc-length parametrization of Γ, i.e., if

$$\Gamma = \{\gamma(s) : 0 \le s \le L\}, \quad \gamma(s + L) = \gamma(s),$$

where L is the length of Γ, then

$$|x, y| = \min\{|s - t| : s, t \in \mathbf{R} \text{ with } x = \gamma(s) \text{ and } y = \gamma(t)\}.$$

It is not difficult to verify that the function $|\cdot, \cdot|$ is metric on Γ; this metric induces the usual topology, because there exists a constant c_0 such that

$$|x - y| \le |x, y| \le c_0 |x - y| \quad \text{for all } x, y \in \Gamma. \tag{1.1}$$

Indeed, the left hand inequality follows at once from the definition of arc-length as the supremum of the lengths of polygonal interpolants. The right hand inequality is called the *chord-arc condition*, and can be proved using the compactness of Γ, together with the fact that the derivative of a Lipschitz function is bounded. The inequalities (1.1) are crucial for the theory developed in the sequel.

For $x \in \Gamma$ and $0 < m < 1$, the *nontangential approach regions* $\mathcal{N}_+(x, m)$ and $\mathcal{N}_-(x, m)$ are defined by

$$\mathcal{N}_\pm(x, m) = \{z \in \Omega_\pm : \operatorname{dist}(z, \Gamma) > m|z - x|\},$$

where $\operatorname{dist}(z, \Gamma) = \inf\{|z - y| : y \in \Gamma\}$ is the distance between z and Γ; cf. Kenig [**6**, p. 177]. Figure 1 is a sketch, produced with the aid of computer graphics, depicting the nontangential approach regions to a point on the boundary of a polygon.

Put

$$m_\pm(x) = \sup\{m : 0 < m < 1 \text{ and } x \in \overline{\mathcal{N}_\pm(x,m)}\},$$

then it is not difficult to prove that there exists a number m_0 satisfying

$$m_\pm(x) \geq m_0 > 0 \quad \text{for all } x \in \Gamma.$$

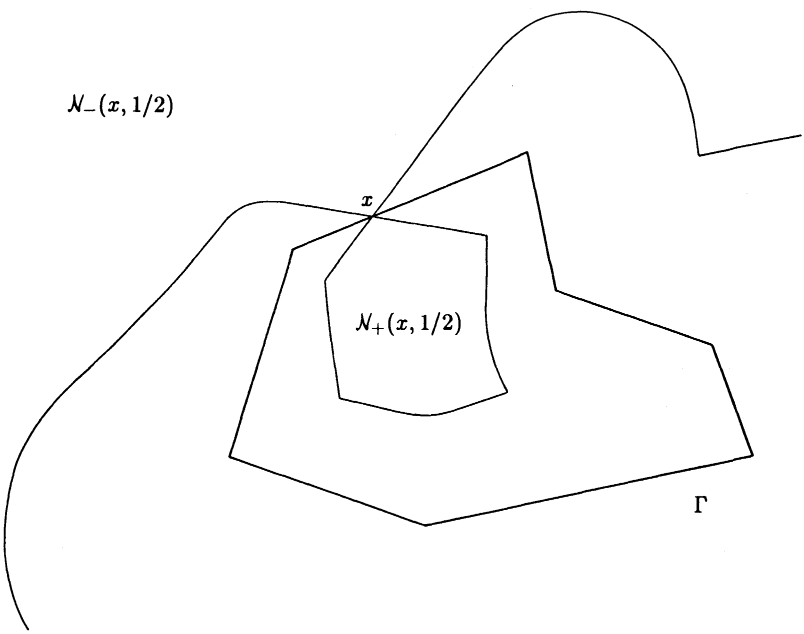

FIGURE 1. Nontangential approach regions with $m = 1/2$.

If $0 < m < m_\pm(x)$, then it makes sense to send $z \to x$ with $z \in \mathcal{N}_\pm(x,m)$. Thus, given a function $F : \mathbf{C}\backslash\Gamma \to \mathbf{C}$, we let

$$F_\pm(x;m) = \lim_{\substack{z\to x \\ z\in\mathcal{N}_\pm(x,m)}} F(z), \quad x \in \Gamma, 0 < m < m_\pm(x) \tag{1.2}$$

whenever these limits exist. Notice that

$$\mathcal{N}_\pm(x,m_2) \subseteq \mathcal{N}_\pm(x,m_1) \quad \text{for } 0 < m_1 < m_2 < 1,$$

so $F_\pm(x;m_1) = F_\pm(x,m_2)$ whenever $F_\pm(x;m_1)$ exists. Thus, if $F_\pm(x,m)$ exists for all m sufficiently small, then there exist unique *nontangential limits*

$$F_\pm(x) = F_\pm(x;m), \quad x \in \Gamma, 0 < m < m_\pm(x).$$

Of course, if the ordinary limits

$$F_\pm(x;0) = \lim_{\substack{z\to x \\ z\in\Omega_\pm}} F(z), \quad x \in \Gamma,$$

exist, then $F_\pm(x) = F_\pm(x;0)$, however it is necessary to allow for the possibility that in (1.2) the convergence is not uniform in m.

For $0 < \alpha \leq 1$, denote by $\Lambda^\alpha(\Gamma)$ the space of complex-valued functions defined on Γ which are Hölder continuous with exponent α, i.e., the function $f : \Gamma \to \mathbf{C}$ belongs to $\Lambda^\alpha(\Gamma)$ if and only if the value of the seminorm

$$[f]_\alpha = \sup_{x,y\in\Gamma, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is finite. Notice that

$$|f(x) - f(y)| \leq [f]_\alpha |x-y|^\alpha, \quad x, y \in \Gamma,$$

and that $\Lambda^1(\Gamma)$ consists of the Lipschitz continuous functions on Γ. In the usual way, we make $\Lambda^\alpha(\Gamma)$ into a (non-separable, non-reflexive) Banach space by defining the norm

$$||f||_{(\alpha)} = ||f||_\infty + [f]_\alpha,$$

where $||\cdot||_\infty$ is the norm in $L_\infty(\Gamma)$.

2. The operator S. For any $u \in L_1(\Gamma)$, the *Cauchy integral* of u is the function $\Phi u : \mathbf{C}\backslash\Gamma \to \mathbf{C}$ defined by

$$\Phi u(z) = \frac{1}{2\pi i}\int_\Gamma \frac{u(y)}{y-z}dy, \quad z \notin \Gamma; \tag{2.1}$$

obviously, Φu is holomorphic on Ω_+ and on Ω_-. In order to study the nontangential limits $\Phi_\pm u = (\Phi u)_\pm$, it is first necessary to make sense of the integral in (2.1) in the case when z is a point lying on the contour Γ. This is usually done by introducing the Cauchy principal value integral

$$\int_\Gamma \frac{u(y)}{y-x}dy = \lim_{h\to 0+} \int_{\Gamma\setminus\Gamma_h(x)} \frac{u(y)}{y-x}dy, \quad x \in \Gamma,$$

where $\Gamma_h(x) = \{y \in \Gamma : |y,x| < h\}$ is the sub-arc of Γ centered at x and having length $2h$.

Let $u \in L_p(\Gamma)$ where $1 < p < \infty$, and define

$$S_{pv}u(x) = \frac{1}{\pi i}\int_\Gamma \frac{u(y)}{y-x}dy, \quad x \in \Gamma.$$

It can be shown [**5**, pp. 55, 108] that $S_{pv}u(x)$ and $\Phi_\pm u(x)$ exist for almost every $x \in \Gamma$, and that the Plemelj-Sokhotski formulae hold, i.e.,

$$\Phi_\pm u = \frac{1}{2}(\pm I + S_{pv})u.$$

Furthermore, as mentioned in the Introduction,

$$S_{pv} : L_p(\Gamma) \to L_p(\Gamma), \quad 1 < p < \infty,$$

is a bounded linear operator.

The Plemelj-Privalov Theorem states that if the contour Γ is smooth, then $S_{pv} : \Lambda^\alpha(\Gamma) \to \Lambda^\alpha(\Gamma)$ for $0 < \alpha < 1$. This is no longer true, however, if Γ is permitted to have corners. Indeed, suppose for simplicity that Γ is piecewise smooth (e.g., a polygon) and let $\omega(x)$ denote the jump in the tangent angle at the point $x \in \Gamma$. The 'interior angle' at x is then $\pi - \omega(x)$, and the 'exterior angle' is $\pi + \omega(x)$; obviously, $\omega(x) = 0$ except when x is a corner point. Using the Cauchy Integral Theorem, it is not difficult to verify that

$$S_{pv}1(x) = 1 - \frac{\omega(x)}{\pi}, \quad x \in \Gamma,$$

which shows that $S_{pv}1$ is discontinuous at each corner point of Γ.

The way out of this difficulty is to introduce the 'singular integral'

$$\frac{1}{\pi i}\int_\Gamma^* \frac{u(y)}{y-x}dy = u(x) + \frac{1}{\pi i}\int_\Gamma \frac{u(y)-u(x)}{y-x}dy, \quad x \in \Gamma,$$

and to define a new operator

$$Su(x) = \frac{1}{\pi i} \int_{\Gamma}^{*} \frac{u(y)}{y-x} dy, \quad x \in \Gamma.$$

Observe that, in contrast to $S_{pv}1$, the function $S1 = 1$ is continuous. Also, $S = S_{pv}$ when Γ is smooth, since

$$Su(x) = \frac{\omega(x)}{\pi} u(x) + S_{pv} u(x), \quad x \in \Gamma. \tag{2.2}$$

Notice however that (2.2) cannot serve as a definition of S, because $\omega(x)$ does not make sense for a general Lipschitz contour.

We will now prove that

$$S : \Lambda^{\alpha}(\Gamma) \to \Lambda^{\alpha}(\Gamma), \quad 0 < \alpha < 1.$$

is a bounded linear operator. In all of the estimates in this paper, the generic constant c is a positive number depending only on the contour Γ; any dependence on Hölder exponents, functions, etc., is shown explicitly. The arc-length measure is denoted by $|dy|$, so that

$$\left| \int_{\Gamma} f(y) dy \right| \leq \int_{\Gamma} |f(y)| \, |dy|$$

for every function $f \in L_1(\Gamma)$.

LEMMA 2.1. *For all $x \in \Gamma$ and $h > 0$,*

$$\int_{\Gamma_h(x)} |y-x|^{\alpha-1} |dy| \leq \frac{ch^{\alpha}}{\alpha}, \quad 0 < \alpha \leq 1,$$

$$\int_{\Gamma \backslash \Gamma_h(x)} |y-x|^{\alpha-2} |dy| \leq \frac{ch^{\alpha-1}}{1-\alpha}, \quad 0 \leq \alpha < 1,$$

$$\left| \int_{\Gamma \backslash \Gamma_h(x)} \frac{dy}{y-x} \right| \leq c.$$

PROOF. The chord-arc condition (1.1) implies

$$\int_{\Gamma_h(x)} |y-x|^{\alpha-1} |dy| \leq c \int_{\Gamma_h(x)} |y,x|^{\alpha-1} |dy|$$

$$\leq c \int_0^h s^{\alpha-1} ds = c\alpha^{-1} h^{\alpha}$$

for $0 < \alpha \le 1$, and

$$\int_{\Gamma\backslash\Gamma_h(x)} |y - x|^{\alpha-2}|dy| \le c\int_{\Gamma\backslash\Gamma_h(x)} |y, x|^{\alpha-2}|dy|$$
$$\le c\int_h^\infty s^{\alpha-2}ds = c(1-\alpha)^{-1}h^{\alpha-1}$$

for $0 \le \alpha < 1$.

If $0 < h \le L/2$, where L is the length of Γ, then

$$\left|\int_{\Gamma\backslash\Gamma_h(x)} \frac{dy}{y - x}\right| \le c\int_h^{L/2} s^{-1}ds = c\log\left(\frac{L}{2h}\right),$$

so, for the third inequality, it suffices to consider h sufficiently small. (If $h > L/2$, then $\Gamma\backslash\Gamma_h(x) = \emptyset$ and there is nothing to prove.) Let $x_\pm^h$ be the two points on Γ satisfying $|x_\pm^h, x| = h$, with x_-^h preceding x, and x_+^h following x, as Γ is traversed in the counterclockwise sense near x.

The compactness of Γ implies there is a fixed $a > 0$ such that, for every $x \in \Gamma$, there exists a Lipschitz representation (a, θ, ϕ) of Γ at x. Fix x, then since

$$\int_{\Gamma\backslash\Gamma_h(x)} \frac{dy}{y - x} = \int_{\mathcal{A}_{x,\theta}[\Gamma\backslash\Gamma_h(x)]} \frac{dy}{y - 0},$$

we may assume $x = 0$ and $\theta = 0$, and thereby simplify the notation.

For $0 < h < a$, there is a rectifiable arc Π_h beginning at x_-^h, finishing at x_+^h, and lying wholly within $\Omega_+ \cap [-a, a]^2$. The point $x = 0$ lies outside the (closed) Jordan curve $\Pi_h \cup \Gamma\backslash\Gamma_h(x)$, therefore

$$\int_{\Gamma\backslash\Gamma_h(x)} \frac{dy}{y - x} + \int_{\Pi_h} \frac{dy}{y - x} = 0.$$

Choose the branch of the logarithm so that

$$-\pi/2 < \arg(y - x) < 3\pi/2 \quad \text{for all } y \in \Pi_h,$$

then

$$\left|\int_{\Pi_h} \frac{dy}{y - x}\right| = \left|\left[\log(y - x)\right]_{y=x_-^h}^{x_+^h}\right|$$
$$\le \Big|\log|x_+^h - x| - \log|x_-^h - x|\Big|$$
$$+ |\arg(x_+^h - x)| + |\arg(x_-^h - x)|$$
$$\le \left|\log\frac{|x_+^h - x|}{|x_-^h - x|}\right| + 3\pi.$$

This completes the proof because the chord-arc condition implies that the ratio $|x_+^h - x|/|x_-^h - x|$ is bounded away from 0 and ∞, uniformly in x. □

THEOREM 2.2. *If* $0 < \alpha < 1$, *then*

$$||Su||_{(\alpha)} \leq c\Big(\frac{1}{\alpha} + \frac{1}{1-\alpha}\Big)||u||_{(\alpha)}$$

for all $u \in \Lambda^\alpha(\Gamma)$.

PROOF. Define the function

$$\psi(x) = \frac{1}{\pi i}\int_\Gamma \frac{u(y)-u(x)}{y-x}dy, \quad x \in \Gamma,$$

then $Su = u + \psi$, so it suffices to estimate $||\psi||_{(\alpha)}$. Firstly,

$$|\psi(x)| \leq \frac{1}{\pi}\int_\Gamma [u]_\alpha |y-x|^{\alpha-1}|dy| \leq \frac{c[u]_\alpha}{\alpha},$$

therefore $||\psi||_\infty \leq c\alpha^{-1}[u]_\alpha$. To estimate $[\psi]_\alpha$, let $x, z \in \Gamma$ and put $h = |x, z|$. Then

$$\psi(x) - \psi(z) = \frac{1}{\pi i}(I_1 + I_2 + I_3),$$

where

$$\begin{aligned}
I_1 &= \int_{\Gamma_{2h}(x)} \Big\{\frac{u(y)-u(x)}{y-x} - \frac{u(y)-u(z)}{y-z}\Big\}dy \\
I_2 &= \int_{\Gamma\setminus\Gamma_{2h}(x)} \frac{u(z)-u(x)}{y-x}dy \\
I_3 &= \int_{\Gamma\setminus\Gamma_{2h}(x)} [u(y)-u(z)]\Big\{\frac{1}{y-x} - \frac{1}{y-z}\Big\}dy.
\end{aligned}$$

If $y \in \Gamma_{2h}(x)$, then $|y, z| \leq |y, x| + |x, z| \leq 3h$, so $y \in \Gamma_{3h}(z)$ and hence

$$\begin{aligned}
|I_1| &\leq [u]_\alpha\Big\{\int_{\Gamma_{2h}(x)} |y-x|^{\alpha-1}|dy| + \int_{\Gamma_{3h}(z)} |y-z|^{\alpha-1}|dy|\Big\} \\
&\leq \frac{c[u]_\alpha}{\alpha}h^\alpha.
\end{aligned}$$

Next,

$$|I_2| = |u(z) - u(x)|\Big|\int_{\Gamma\backslash\Gamma_{2h}(x)} \frac{dy}{y-x}\Big| \le c[u]_\alpha|x-z|^\alpha.$$

If $y \in \Gamma\backslash\Gamma_{2h}(x)$, then $|y,z| \ge |y,x| - |x,z| \ge 2h - h = h$ and so $|y-x| \le |y,x| \le |y,z| + h \le 2|y,z| \le c|y-z|$. Hence, the integrand of I_3 can be estimated as follows:

$$\begin{aligned}\Big|[u(y)-u(z)]\Big\{\frac{1}{y-x} - \frac{1}{y-z}\Big\}\Big| &\le [u]_\alpha \frac{|y-z|^\alpha|z-x|}{|y-x||y-z|} \\ &\le \frac{c[u]_\alpha h}{|y-x||y-z|^{1-\alpha}} \le \frac{c[u]_\alpha h}{|y-x|^{2-\alpha}}.\end{aligned}$$

Therefore,

$$|I_3| \le c[u]_\alpha h \int_{\Gamma\backslash\Gamma_{2h}(x)} |y-x|^{\alpha-2}|dy| \le \frac{c[u]_\alpha}{1-\alpha}h^\alpha,$$

and so, combining the estimates for I_1, I_2 and I_3, we find

$$|\psi(x) - \psi(z)| \le c\Big(\frac{1}{\alpha} + \frac{1}{1-\alpha}\Big)[u]_\alpha|x-z|^\alpha,$$

noting that the chord-arc condition implies $h \le c|x-z|$. □

3. The Plemelj-Sokhotski formulae. The Cauchy Integral Theorem implies

$$\Phi 1(z) = \begin{cases} 1, & \text{if } z \in \Omega_+ \\ 0, & \text{if } z \in \Omega_-, \end{cases}$$

therefore, if $u \in \Lambda^\alpha(\Gamma)$ and $x \in \Gamma$, then

$$\Phi u(z) = \begin{cases} u(x) + \Psi u(z,x) & z \in \Omega_+ \\ \Psi u(z,x), & z \in \Omega_-, \end{cases} \tag{3.1}$$

where

$$\Psi u(z,x) = \frac{1}{2\pi i}\int_\Gamma \frac{u(y)-u(x)}{y-z}dy, \quad z \in \Omega_+ \cup \{x\} \cup \Omega_-.$$

Hence, to find the nontangential limits $\Phi_{\pm}u(x)$, it suffices to determine the limiting values of $\Psi u(z,x)$ as $z \to x$ with $z \in \mathcal{N}_{\pm}(x,m)$. In fact,

$$\lim_{\substack{z\to x\\ z\in\mathcal{N}_{\pm}(x,m)}} \Psi u(z,x) = \Psi u(x,x) = \frac{1}{2}[Su(x) - u(x)], \quad x \in \Gamma, \tag{3.2}$$

as the second part of the following lemma shows.

LEMMA 3.1. *Suppose* $0 < \alpha < 1$ *and* $0 < m < 1$. *If* $u \in \Lambda^{\alpha}(\Gamma)$ *and* $x \in \Gamma$, *then*

$$|\Psi u(z,x) - \Psi u(w,x)| \le c\frac{1}{m^2}\Big(\frac{1}{\alpha} + \frac{1}{1-\alpha}\Big)[u]_{\alpha}|z-w|^{\alpha} \tag{3.3}$$

and

$$|\Psi u(z,x) - \Psi u(x,x)| \le c\frac{1}{m}\Big(\frac{1}{\alpha} + \frac{1}{1-\alpha}\Big)[u]_{\alpha}|z-x|^{\alpha}, \tag{3.4}$$

for all $z, w \in \mathcal{N}_{+}(x,m) \cup \mathcal{N}_{-}(x,m)$.

PROOF. Put $h = |z-w|$, and write

$$\begin{aligned}\Psi u(z,x) - \Psi u(w,x) &= \frac{1}{2\pi i}\int_{\Gamma} \frac{[u(y)-u(x)](z-w)}{(y-z)(y-w)}dy \\ &= \frac{z-w}{2\pi i}(I_1 + I_2),\end{aligned} \tag{3.5}$$

where

$$I_1 = \int_{\Gamma_h(x)} \frac{u(y)-u(x)}{(y-z)(y-w)}dy, \quad I_2 = \int_{\Gamma\setminus\Gamma_h(x)} \frac{u(y)-u(x)}{(y-z)(y-w)}dy.$$

Assuming $z \in \mathcal{N}_{+}(x,m) \cup \mathcal{N}_{-}(x,m)$ and $y \in \Gamma$, we have

$$|y-z| \ge \operatorname{dist}(z,\Gamma) > m|x-z|,$$

and so

$$\frac{1}{|y-z|} = \left|\frac{1}{y-x}\Big\{1 + \frac{z-x}{y-z}\Big\}\right| \le \frac{1+m^{-1}}{|y-x|}.$$

Similarly, the inequality

$$\frac{1}{|y-w|} \le \frac{1+m^{-1}}{|y-x|}$$

holds for $w \in \mathcal{N}_+(x,m) \cup \mathcal{N}_-(x,m)$ and $y \in \Gamma$. Without loss of generality, we may assume $|x-z| \ge |x-w|$, then

$$|z-w| \le |z-x| + |x-w| \le 2|x-z|.$$

Hence, $1/|x-z| \le 2/|z-w|$, from which it follows that

$$\begin{aligned}|I_1| &\le \int_{\Gamma_h(x)} \frac{[u]_\alpha |y-x|^\alpha}{m|x-z|} \frac{1+m^{-1}}{|y-x|}|dy| \\ &\le \frac{2(1+m^{-1})[u]_\alpha}{m|z-w|} \int_{\Gamma_h(x)} |y-x|^{\alpha-1}|dy| \\ &\le \frac{c[u]_\alpha}{m^2\alpha} \frac{h^\alpha}{|z-w|}\end{aligned}$$

and

$$\begin{aligned}|I_2| &\le \int_{\Gamma\setminus\Gamma_h(x)} [u]_\alpha |y-x|^\alpha \Big(\frac{1+m^{-1}}{|y-x|}\Big)^2 |dy| \\ &\le (1+m^{-1})^2 [u]_\alpha \int_{\Gamma\setminus\Gamma_h(x)} |y-x|^{\alpha-2}|dy| \\ &\le \frac{c[u]_\alpha}{m^2(1-\alpha)} h^{\alpha-1}.\end{aligned}$$

Inserting these bounds in (3.5), we arrive at (3.3).

To prove (3.4), put $h = |z-x|$ and write

$$\Psi u(z,x) - \Psi u(x,x) = \frac{z-x}{2\pi i}(I_1 + I_2),$$

where

$$I_1 = \int_{\Gamma_h(x)} \frac{u(y)-u(x)}{(y-z)(y-x)}dy, \quad I_2 = \int_{\Gamma\setminus\Gamma_h(x)} \frac{u(y)-u(x)}{(y-z)(y-x)}dy.$$

This time,

$$|I_1| \le \int_{\Gamma_h(x)} \frac{[u]_\alpha |y-x|^{\alpha-1}}{m|x-z|}|dy| \le \frac{c[u]_\alpha}{m\alpha} \frac{h^\alpha}{|x-z|}$$

and

$$|I_2| \le \int_{\Gamma \setminus \Gamma_h(x)} [u]_\alpha (1+m^{-1})|y-x|^{\alpha-2}|dy| \le \frac{c[u]_\alpha}{m(1-\alpha)} h^{\alpha-1},$$

from which the result follows immediately. □

THEOREM 3.2. *Let* $u \in \Lambda^\alpha(\Gamma)$, *where* $0 < \alpha < 1$. *The Plemelj-Sokhotski formulae*

$$\Phi_\pm u = \frac{1}{2}(S \pm I)u$$

hold, and the inequality

$$|\Phi u(z) - \Phi_\pm u(x)| \le c\Big(\frac{1}{\alpha} + \frac{1}{1-\alpha}\Big)||u||_{(\alpha)}|z-x|^\alpha \tag{3.6}$$

is valid for $z \in \Omega_\pm$ *and* $x \in \Gamma$.

PROOF. The Plemelj-Sokhotski formulae follow at once from (3.1) and (3.2).

To prove (3.6), let $z \in \Omega_\pm$ and choose $y \in \Gamma$ such that $|z-y| = \operatorname{dist}(z, \Gamma)$, then

$$\begin{aligned}\Phi u(z) - \Phi_\pm u(y) &= \begin{cases} u(y) + \Psi u(z,y) - [u(y) + \Psi u(y,y)], & z \in \Omega_+ \\ \Psi u(z,y) - \Psi u(y,y), & z \in \Omega_- \end{cases} \\ &= \Psi u(z,y) - \Psi u(y,y).\end{aligned}$$

Since $z \in \mathcal{N}_\pm(y,m)$ for every $m < 1$, it follows from (3.4) that

$$|\Phi u(z) - \Phi_\pm u(y)| \le c\Big(\frac{1}{\alpha} + \frac{1}{1-\alpha}\Big)[u]_\alpha |z-y|^\alpha, \tag{3.7}$$

while Theorem 2.2 implies

$$\begin{aligned}|\Phi_\pm u(y) - \Phi_\pm u(x)| &\le \frac{1}{2}(|Su(y) - Su(x)| + |u(y) - u(x)|) \\ &\le c\Big(\frac{1}{\alpha} + \frac{1}{1-\alpha}\Big)||u||_{(\alpha)}|y-x|^\alpha.\end{aligned} \tag{3.8}$$

Notice $|z-y| = \operatorname{dist}(z,\Gamma) \le |z-x|$ and $|y-x| \le |y-z|+|z-x| \le 2|z-x|$, so (3.7) and (3.8) together imply the result. □

At this point, it is convenient to extend $\Phi_\pm u$ to $\overline{\Omega}_\pm$ in the obvious way: by setting $\Phi_\pm u = \Phi u$ on $\Omega_\pm$. The final theorem for this section asserts that the function $\Phi_\pm u$ is Hölder continuous on $\overline{\Omega}_\pm$, whenever u is Hölder continuous on Γ. This fact is, of course, well known if Γ is smooth, and the reader may like to compare our proof with that in Muskhelishvili [**10**, pp. 53-55]; the latter relies on the maximum modulus principle.

THEOREM 3.3. *If $0 < \alpha < 1$ and $u \in \Lambda^\alpha(\Gamma)$, then*

$$|\Phi_\pm u(z) - \Phi_\pm u(w)| \le c\Big(\frac{1}{\alpha} + \frac{1}{1-\alpha}\Big)\|u\|_{(\alpha)}|z-w|^\alpha \tag{3.9}$$

for all $z, w \in \overline{\Omega}_\pm$.

PROOF. If at least one of the points z and w lies on Γ, then (3.9) follows at once from Theorems 2.2 and 3.2. Thus, we may assume that both z and w belong to the open set $\Omega_\pm$.

Choose $x, y \in \Gamma$ such that $|z-x| = \operatorname{dist}(z,\Gamma)$ and $|w-y| = \operatorname{dist}(w,\Gamma)$, then the formula (3.1) implies

$$\Phi_\pm u(z) - \Phi_\pm u(w) = \Psi u(z,x) - \Psi u(w,x) = \Psi u(z,y) - \Psi u(w,y).$$

Therefore, by Lemma 3.1,

$$|\Phi_\pm u(z) - \Phi_\pm u(w)| \le c\frac{1}{m^2}\Big(\frac{1}{\alpha} + \frac{1}{1-\alpha}\Big)[u]_\alpha|z-w|^\alpha$$

if $w \in \mathcal{N}_\pm(x,m)$ or if $z \in \mathcal{N}_\pm(y,m)$.

This leaves the case when $w \in \Omega_\pm \backslash \mathcal{N}_\pm(x,m)$ and $z \in \Omega_\pm \backslash \mathcal{N}_\pm(y,m)$, i.e., when

$$\begin{aligned} |w-y| &= \operatorname{dist}(w,\Gamma) \le m|x-w| \\ |z-x| &= \operatorname{dist}(z,\Gamma) \le m|y-z|. \end{aligned} \tag{3.10}$$

By Theorems 2.2 and 3.2,

$$\begin{aligned}|\Phi_{\pm}u(z)-\Phi_{\pm}u(w)| &\leq m|\Phi_{\pm}u(z)-\Phi_{\pm}u(x)|+|\Phi_{\pm}u(x)-\Phi_{\pm}u(y)| \\ &\quad + |\Phi_{\pm}u(y)-\Phi_{\pm}u(w)| \\ &\leq c\Big(\frac{1}{\alpha}+\frac{1}{1-\alpha}\Big)||u||_{(\alpha)}E\end{aligned}$$

where $E=|z-x|^{\alpha}+|x-y|^{\alpha}+|y-w|^{\alpha}$. Thus, to complete the proof, it suffices to show $E\leq c|z-w|^{\alpha}$ for $m\leq 1/4$ (with c independent of m).

The inequalities (3.10) imply

$$|w-y|\leq\ m|x-y|+m|y-w|,\quad |z-x|\leq m|y-x|+m|x-z|,$$

therefore if $m<1$, then

$$|w-y|\leq\frac{m}{1-m}|x-y|,\quad |z-x|\leq\frac{m}{1-m}|y-x|.$$

Next,

$$|x-y|\leq|x-z|+|z-w|+|w-y|\leq|z-w|+\frac{2m}{1-m}|x-y|,$$

so, for $m<1/3$,

$$|x-y|\leq\frac{1-m}{1-3m}|z-w|,$$

and finally

$$\begin{aligned}E &\leq\Big\{1+2\Big(\frac{m}{1-m}\Big)^{\alpha}\Big\}|x-y|^{\alpha} \\ &\leq\Big\{1+2\Big(\frac{m}{1-m}\Big)^{\alpha}\Big\}\Big(\frac{1-m}{1-3m}\Big)^{\alpha}|z-w|^{\alpha}\leq c|z-w|^{\alpha}\end{aligned}$$

for $m\leq 1/4$. □

To conclude, we prove an important corollary of the above theorem, namely, the fact that $S^{-1}=S:\Lambda^{\alpha}(\Gamma)\to\Lambda^{\alpha}(\Gamma)$ for $0<\alpha<1$.

THEOREM 3.4. *If $u\in\Lambda^{\alpha}(\Gamma)$ for some $\alpha>0$, then $S^2u=u$.*

PROOF. By Theorem 3.3, the function $\Phi_+ u$ is holomorphic on Ω_+ and (Hölder) continuous on $\overline{\Omega}_+$. Since $\overline{\Omega}_+$ is compact and $\Omega_- = \mathbf{C} \backslash \overline{\Omega}_+$ is connected, Mergelyan's Theorem [**2**, p. 97] implies that there exists a sequence of polynomials P_n with coefficients in $\mathbf{C}$, such that $P_n \to \Phi_+ u$ uniformly on $\overline{\Omega}_+$. Obviously, each polynomial P_n is holomorphic on an open neighborhood of $\overline{\Omega}_+$, so

$$P_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{P_n(y)}{y - z} dy, \quad z \in \Omega_+.$$

Sending $n \to \infty$, we see $\Phi u(z) = \Phi(\Phi_+ u)(z)$ for all $z \in \Omega_+$, and so the Plemelj-Sokhotski formulae imply

$$\frac{1}{2}(I + S)u = \frac{1}{2}(I + S)\frac{1}{2}(I + S)u.$$

After some simple algebra, this gives the result. □

In [**8**], there is an alternative (and longer) proof of Theorem 3.4, which does not use Mergylan's Theorem, but instead requires one to prove the Poincaré-Bertrand formula for a Lipschitz contour.

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