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UNIFORM L^1 BEHAVIOR IN CLASSES OF INTEGRODIFFERENTIAL EQUATIONS WITH CONVEX KERNELS

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Introduction. We consider families $\mathcal{R}$ of functions such that each a in $\mathcal{R}$ satisfies

$$\int_0^1 a(s)ds < \infty, \tag{1.1}$$

a is nonconstant, nonegative, nonincreasing, convex, and $-a'$ is convex.

We will show, for certain such families, that

$$\int_0^\infty \sup_{a\in\mathcal{R}} |u(t;a)|dt < \infty, \tag{1.2}$$

where $u(t) = u(t;a)$ is the solution of the scalar problem

$$u'(t) + \int_0^t a(t-\tau)u(\tau)d\tau = 0, \quad u(0) = 1, \; t \geq 0, \; a \in \mathcal{R}. \tag{1.3}$$

When $\mathcal{R} = \{\lambda a_0(t) : 0 < \lambda_0 \leq \lambda < \infty\}$, (1.2) is true. These and similar results were proved in [**1, 2, 4, 5, 10** and **11**]. The technique of proof relies on the methods of Shea and Wainger [**13**].

The estimate (1.2) was used in [**1, 4, 5** and **11**] to estimate the resolvent kernel

$$U(t) = \int_{\lambda_0}^\infty u(t;\lambda a_0)\; dE_\lambda,$$

of the problem

$$y'(t) + \int_0^t a_0(t-s)\; Ly(s)ds = f(t), \quad y(0) = y_0, \tag{1.4}$$

in a Hilbert space $\mathcal{H}$. The operator L is a densely defined self-adjoint linear operator with spectrum contained in $[\lambda_0, \infty)$ $(\lambda_0 > 0)$, y_0 and

$f(t)$ are prescribed elements of $\mathcal{H}$, and $\{E_\lambda\}$ is the spectral family corresponding to L.

Since (1.2) implies that

$$\int_0^\infty ||U(t)||dt < \infty, \tag{1.5}$$

the resolvent formula

$$y(t) = U(t)y_0 + \int_0^t U(t-s)f(s)ds, \tag{1.6}$$

for (1.4) gives information about the asymptotic behavior of $y(t)$ as $t \to \infty$.

For more general classes $\mathcal{R} = \{a(t;\lambda) : -\infty < \lambda < \infty\}$, (1.2) implies that (1.5) holds for the resolvent

$$U(t) = \int_{-\infty}^\infty u(t; a(\cdot,\lambda))\, dE_\lambda \tag{1.7}$$

for the problem

$$y'(t) + \int_0^t L(t-s)y(s)ds = f(t), \quad y(0) = y_0, \ t \geq 0, \tag{1.8}$$

with

$$L(t) = \int_{-\infty}^\infty a(t;\lambda)\, dE_\lambda, \tag{1.9}$$

where $\{E_\lambda\}$ is a fixed resolution of the indentity in $\mathcal{H}$.

Our results for (1.8) include some operators of the form

$$L(t) = \sum_{k=0}^n a_k(t)L_k, \tag{1.10}$$

and generalizes some of the results in [**7**]. The requirement that the L_k, $k = 0,\ldots,n$, have spectral decompositions with respect to a common resolution of the identity $\{E_\lambda\}$ greatly restricts the applicability of the result (1.5) with L as in (1.10), but see [**7**] for applications, including a linear model for heat flow in a rectangular, orthotropic material with memory in which the axes of orthotropy are parallel to the edges of the rectangle.

For families $\mathcal{R} = \{a_0(t) + c : \ 0 \le c \le 1\}$, where a_0 is a fixed function satisfying (1.1), Hannsgen and Wheeler show in [**6**] that $a_0 \notin L^1(1,\infty)$ is necessary for (1.2) to hold. They also show that (1.2) does not even hold for $a_0(t) = (1 - e^{-t})/t$ (which behaves like $1/t$ as $t \to \infty$). In [**12**], it is shown that the condition

$$\int_1^\infty \frac{\log u}{u\, A_{a_0}(u)} du < \infty, \tag{1.11}$$

where

$$A_{a_0}(u) \equiv \int_0^u a_0(s) ds, \ u > 0, \tag{1.12}$$

along with (1.1) implies (1.2). In [**6**], (1.2) is shown to follow when a_0 is completely nonotonic $((-1)^n a_0^{(n)}(t) \ge 0, \ n = 0,1,2,3,\ldots, \ t \ge 0)$ and satisfies a growth condition at ∞ that is similar to (1.11). Thus (1.11) and the condition used in [**6**] both allow functions a_0 that behave like $(\log^p t)/t$ as $t \to \infty$, for $p > 1$ and rule out functions a_0 that behave like $(\log^p t)/t$ as $t \to \infty$, for $0 < p \le 1$. As a corollary to our main result, Theorem 1, we show that (1.2) holds if a_0 satisfies (1.1) and

$$\int_1^\infty \frac{1}{u A_{a_0}(u)} du < \infty. \tag{1.13}$$

This improvement of the growth condition at ∞ allows functions a_0 that behave like $(\log^p t)/t$ as $t \to \infty$, even for $0 < p \le 1$.

The conditions on the family $\mathcal{R}$ that we will use are

$$\int_1^\infty \sup_{a \in \mathcal{R}} \frac{1}{u A_a(u)} du < \infty, \tag{1.14}$$

there exists a constant $L > 0$ such that

$$\inf_{a \in \mathcal{R}} \int_0^L t a(t) dt \ge 10, \tag{1.15}$$

and a condition stated in terms of the Fourier transform. Each a satisfying (1.1) has a Fourier transform

$$\hat{a}(\tau) = \frac{a(\infty)}{i\tau} + \int_0^\infty [a(t) - a(\infty)]e^{-i\tau t}dt, \ \tau \text{ real}, \tau \neq 0,$$

which we separate into real and imaginary parts as

$$\hat{a}(\tau) = \phi_a(\tau) - i\tau\theta_a(\tau). \tag{1.16}$$

By [**1**, Lemma 4.1], each $\theta_a(\tau)$ is nonnegative, continuous and strictly decreasing, with

$$\frac{1}{5}A_{1a}(\tau^{-1}) \leq \theta_a(\tau) \leq 12A_{1a}(\tau^{-1}), \ \tau > 0, \tag{1.17}$$

where

$$A_{1a}(u) = \int_0^u sa(s)ds, \ u > 0, \ a \in \mathcal{R}. \tag{1.18}$$

Note that (1.17) was originally proved for $a(t)$ with $a(\infty) = 0$. To see that (1.17) holds even with $a(\infty) > 0$, define $b(t) = a(t) - a(\infty)$. Then $\theta_a(\tau) = \theta_b(\tau) + a(\infty)\tau^{-2}$, so

$$\begin{aligned}\theta_a(\tau) &\leq 12A_{1b}(\tau^{-1}) + a(\infty)\tau^{-2} \leq 12A_{1b}(\tau^{-1}) + 6a(\infty)\tau^{-2} \\ &= 12A_{1a}(\tau^{-1}) \\ \theta_a(\tau) &\geq \frac{1}{5}A_{1b}(\tau^{-1}) + a(\infty)\tau^{-2} \geq \frac{1}{5}A_{1b}(\tau^{-1}) + \frac{1}{10}a(\infty)\tau^{-2} \\ &= \frac{1}{5}A_{1a}(\tau^{-1}).\end{aligned}$$

For each a in $\mathcal{R}$, we define $\tilde{\omega} = \tilde{\omega}(a)$ by $\theta_a(\tilde{\omega}) = 1$. Since (1.14) implies that for each a in $\mathcal{R}$, $\int_1^\infty a(t)dt = \infty$, it follows that $\int_0^\infty ta(t)dt = \infty$. Thus (1.17) shows that $\theta_a(0+) = \infty$ and $\theta_a(\infty) = 0$, so $\tilde{\omega}$ is well defined. Now define $\omega = \omega(a)$ by $\omega = \tilde{\omega}$ for $\tilde{\omega} \geq 2\epsilon$ and $\omega = 2\epsilon$ otherwise, where ϵ is the positive constant given in (2.9) below.

Our last condition on the family $\mathcal{R}$ can now be given as

$$\sup_{a \in \mathcal{R}} \frac{1}{\phi_a(\omega(a))} < \infty. \tag{1.19}$$

A similar assumption is also used by Hannsgen and Wheeler [**7**, (2.6)]. Both (1.19) and [**7**, (2.6)] rule out the family $\mathcal{R} = \{a_0(t) + c: \ 1 \le c < \infty\}$, where a_0 satisfies (1.1). That (1.2) does not hold for such families is shown in [**7**].

THEOREM 1. *If $\mathcal{R}$ is a family of functions satisfying* (1.14), (1.15) *and* (1.19), *where each a in $\mathcal{R}$ satisfies* (1.1), *then* (1.2) *holds.*

COROLLARY. *If $\mathcal{R} = \{a_0(t) + c: \ 0 \le c \le 1\}$, where a_0 satisfies* (1.1) *and* (1.13), *then* (1.2) *holds.*

We give the proofs in §2.

In [**7**] it is shown that, for certain families $\mathcal{R}$ of completely monotonic functions,

$$\int_0^\infty \rho(t) \sup_{a\in\mathcal{R}} |u(t;a)| \, dt < \infty \tag{1.20}$$

where ρ is a weight function. Theorem 1 generalizes and improves their results for $\rho(t) \equiv 1$. In particular the growth condition that they use [**7**, (2.5)] rules out functions a in $\mathcal{R}$ that behave like $(\log^p t)/t$ as $t \to \infty$, $0 < p \le 1$.

The condition (1.15) is used to obtain (2.11) below. In its place, Hannsgen and Wheeler use a similar type of condition [**7**, (2.10)] ($\rho \equiv 1$). Although (1.15) allows for example the function $a(t) = 11/(t+1)^2$ and (1.14) rules it out, (1.15) does not in general follows from (1.14). For example, let $\mathcal{R} = \{a_T(t) : T \ge 1\}$, where

$$a_T(t) = \begin{cases} \frac{1}{(t+1)^2}, & 0 \le t \le T, \\ b_T(t), & T \le t \le T+3, \\ \frac{1}{(T+2)^2}, & T+3 \le t, \end{cases}$$

and b_T is chosen arbitrarily except that it is required that each a_T satisfies (1.1). Then (1.15) holds as long as L is chosen so large that $\log(L+1) + 1/(L+1) \ge 11$. Then we have

$$\begin{aligned} \inf_{a\in\mathcal{R}} \int_0^L ta(t)dt &= \inf_{T\ge 1} \int_0^L ta_T(t)dt = \int_0^L \frac{t}{(t+1)^2}dt \\ &= \log(L+1) + \frac{1}{L+1} - 1 \ge 10. \end{aligned}$$

But an easy calculation shows that, for each $u \geq 1$

$$\sup_{a \in \mathcal{R}} \frac{1}{uA_{a_T}(u)} \geq \frac{1}{uA_{a_u}(u)} = \frac{u+1}{u^2},$$

so (1.14) does not hold. I do not know if (1.2) holds for this family.

For families of the form

$$\mathcal{R} = \Big\{ \sum_{i=0}^{n} \lambda_i a_i(t) : \ \lambda_i \geq 1, \ i = 0, 1, 2, \ldots, n \Big\},$$

where all $a_i(t)$ satisfy (1.1), it is not clear that the assumptions (1.14) and (1.15) are needed to prove (1.2). Also when $n = 0$, as already mentioned, if a_0 satisfies (1.1), then (1.2) holds (note that (1.1) implies (1.19) in this case.). Thus we finish the introduction with a conjecture.

CONJECTURE. *Even for $n > 0$, if all a_i satisfy* (1.1), *then* (1.2) *holds.*

2. a. Proof of Theorem 1. Throughout this paper we will use M to denote a constant that is independent of the functions in $\mathcal{R}$, but whose value may change each time that it appears. To prove that (1.2) holds, we will find a constant $k > 0$ and a function $h(t)$ such that

(2.1) $$|u(t; a)| \leq h(t), \quad t \geq k, \ a \in \mathcal{R}$$

and

(2.2) $$\int_k^\infty h(t)\, dt < \infty.$$

To do this we will use the representation

(2.3) $$\pi u(t; a) = \int_0^\infty \operatorname{Re}\Big\{ \frac{e^{i\tau t}}{D(\tau; a)} \Big\}\, d\tau, \quad t > 0, \ a \in \mathcal{R},$$

(See (4.29) of [**1**]) where $D(\tau; a) \equiv \hat{a}(\tau) + i\tau$. Then (1.2) will follow by the estimate

(2.4) $$|u(t; a)| \leq 1, \quad t \geq 0, \ a \in \mathcal{R},$$

which is due to Levin [**9**]. (See [**3**, Theorem 2]. The number $\sqrt{2}$ appears in [**3**, Theorem 2], instead of the number 1 because of an error.) In our proof we will need the estimates

$$\frac{1}{2\sqrt{2}}A_a(\tau^{-1}) \le |\hat{a}(\tau)| \le 4A_a(\tau^{-1}), \quad \tau > 0, \tag{2.5}$$

($A_a(u)$ is defined in (1.12)) and

$$|\hat{a}'(\tau)| \le 40A_{1a}(\tau^{-1}), \quad \tau > 0, \tag{2.6}$$

from [**13**, Lemma 1], as well as the estimate

$$\frac{1}{5}B_a(\tau^{-1}) \le \phi_a(\tau) \le 12\ B_a(\tau^{-1}), \quad \tau > 0 \tag{2.7}$$

from [**8**, p. 236], where

$$B_a(u) = \int_0^u -sa'(s)\, ds, \quad u > 0,\ a \in \mathcal{R}. \tag{2.8}$$

Note that (2.5) and (2.6) originally were shown for $a(t)$ with $a(\infty) = 0$. An easy check shows that the proofs of (2.5) and (2.6) still are valid when $a(\infty) > 0$.

We define ϵ by

$$\epsilon = 1/L. \tag{2.9}$$

Then, for $0 < \tau \le \epsilon$, we use (1.17) and (1.15) to obtain

$$\theta_a(\tau) \ge \frac{1}{5}A_{1a}(\tau^{-1}) \ge \frac{1}{5}A_{1a}(\epsilon^{-1}) \ge 2. \tag{2.10}$$

This gives us the first inequality in the estimate

$$\begin{aligned} |D(\tau;a)|^2 &= |\hat{a}(\tau) + i\tau|^2 = \phi_a^2(\tau) + \tau^2(\theta_a(\tau) - 1)^2 \\ &\ge \phi_a^2(\tau) + \frac{1}{4}\tau^2\theta_a(\tau)^2 \ge \frac{1}{4}|\hat{a}(\tau)|^2 \\ &\ge \frac{1}{32}A_a^2(\tau^{-1}), \quad 0 < \tau \le \epsilon, \end{aligned} \tag{2.11}$$

where the last inequality follows from (2.5).

Now, for $t \geq 1/\epsilon$, we use (2.11) to obtain

(2.12)

$$\Big|\int_0^{1/t} \mathrm{Re}\Big\{\frac{e^{i\tau t}}{D(\tau;a)}\Big\}\,d\tau\Big| \leq \int_0^{1/t} \frac{d\tau}{|D(\tau;a)|} \leq 4\sqrt{2}\int_0^{1/t} \frac{d\tau}{A_a(\tau^{-1})} = 4\sqrt{2}\int_t^{\infty} \frac{du}{u^2 A_a(u)} \leq \frac{4\sqrt{2}}{tA_a(t)}.$$

Now we integrate by parts to obtain

$$\mathrm{Re}\int_{1/t}^{\infty} \frac{e^{i\tau t}}{D(\tau;a)} = \mathrm{Im}\frac{1}{t}\Big\{\frac{-e^{i}}{D(t^{-1};a)} + \int_{t^{-1}}^{\infty} \frac{e^{i\tau t}D_\tau(\tau;a)}{D^2(\tau;a)}d\tau\Big\} \equiv \mathrm{Im}\frac{1}{t}\{B_1 + I_1\}. \tag{2.13}$$

By (2.11) we have

$$\Big|\frac{1}{t}B_1\Big| \leq \frac{4\sqrt{2}}{tA_a(t)}. \tag{2.14}$$

Combining (2.12)-(2.14) with (2.3), we obtain

$$\pi|u(t;a)| \leq \frac{8\sqrt{2}}{tA_a(t)} + \frac{|\mathrm{Im}I_1|}{t}. \tag{2.15}$$

To estimate $|\mathrm{Im}t^{-1}I_1|$, we first integrate by parts. This yields

(2.16)

$$\mathrm{Im}\ t^{-1}I_1 = \mathrm{Re}\ t^{-2}\Big(\frac{e^{i}D_\tau(t^{-1};a)}{D^2(t^{-1};a)} + \int_{t^{-1}}^{\infty} e^{i\tau t}\Big[\frac{\hat{a}''(\tau)}{D^2(\tau;a)} + \frac{2D_\tau^2(\tau;a)}{D^3(\tau;a)}\Big]d\tau\Big) \equiv \mathrm{Re}\ t^{-2}(B_2 + \int_{t^{-1}}^{\infty} J\ d\tau)$$

By (2.11), (2.6), (2.10) and the inequalities $t \geq 1/\epsilon$ and $A_{1a}(t) \leq tA_a(t)$, we have

$$|t^{-2}B_2| \leq \frac{32(40A_{1a}(t)+1)}{t^2A_a^2(t)} \leq \frac{1280}{tA_a(t)} + \frac{32}{tA_a(t)A_{1a}(t)} < \frac{1284}{tA_a(t)}. \tag{2.17}$$

To estimate the integral term in (2.16) we begin with

$$\begin{aligned}\left|t^{-2}\int_{t^{-1}}^{\epsilon} J d\tau\right| &\le \frac{M}{t^2}\int_{t^{-1}}^{\epsilon}\left[\frac{\tau^{-1}A_{1a}(\tau^{-1})}{A_a^2(\tau^{-1})}+\frac{(1+A_{1a}^2(\tau^{-1}))}{A_a^3(\tau^{-1})}\right]d\tau \\ &\le \frac{M}{t^2}\int_{t^{-1}}^{\epsilon}\frac{\tau^{-1}A_{1a}(\tau^{-1})}{A_a^2(\tau^{-1})}d\tau \\ &\le \frac{M}{t^2}\int_{t^{-1}}^{\epsilon}\frac{\tau^{-2}}{A_a(\tau^{-1})}\,d\tau = \frac{M}{t^2}\int_{1/\epsilon}^{t}\frac{du}{A_a(u)},\end{aligned} \tag{2.18}$$

where the first inequality follows from (2.11), (2.6) and the inequality

$$|\hat{a}''(\tau)| \le 6000\int_0^{1/\tau} r^2 a(r) d\tau \le 6000\,\tau^{-1}A_{1a}(\tau^{-1}),\ \ \tau > 0, \tag{2.19}$$

(See [**1**, Lemma 5.1]; (2.19) holds even when $a(\infty) < 0$), the next two inequalities use $A_{1a}(\tau^{-1}) \le \tau^{-1}A_a(\tau^{-1})$ and (2.10), and the equality follows by a change of variables. Note that

$$\sup_{a\in R}\frac{1}{t^2}\int_{1/\in}^{t}\frac{du}{A_a(u)} \le \frac{1}{t^2}\int_{1/\in}^{t}\sup_{a\in R}\frac{1}{A_a(u)}du,$$

therefore, by the Fubini theorem (1.14), and (2.18) we have

$$\begin{aligned}\int_{1/\epsilon}^{\infty}\sup_{a\in\mathcal{R}}\left|t^{-2}\int_{t^{-1}}^{\epsilon}J d\tau\right|dt &\le M\int_{1/\epsilon}^{\infty}\left[t^{-2}\int_{1/\in}^{t}\sup_{a\in\mathcal{R}}\left[\frac{1}{A_a(u)}du\right]dt\right. \\ &= \int_{1/\epsilon}^{\infty}\left[\sup_{a\in\mathcal{R}}\frac{1}{A_a(u)}\int_u^{\infty}t^{-2}dt\right]du \\ &\le \int_1^{\infty}\sup_{a\in\mathcal{R}}\frac{1}{uA_a(u)}du < \infty.\end{aligned} \tag{2.20}$$

We will need the inequalities

$$24,000|D(\tau;a)| \ge \tau A_{1a}(\tau^{-1}),\ \ \epsilon \le \tau \le \frac{\omega}{2},\ a \in \mathcal{R}, \tag{2.21}$$

and

$$144,000|D(\tau;a)| \ge |\tau - \omega|,\ \ \frac{\omega}{2} \le \tau,\ a \in \mathcal{R}. \tag{2.22}$$

These are essentially the inequalities in [**1**; Lemma 5.2]. The proof, except for very minor changes is identical with the one given in [**1**] (see also §8 on [**2**] for the correction of an error in part of the proof of [**1**; Lemma 5.2]), and we will omit it. The main point is that we were able to choose the constants (24,000 and 144,000) in (2.21) and (2.22) independently of a in $\mathcal{R}$.

We use (2.19), (2.6), (2.21), (1.17) and the definition of ω to obtain

(2.23)
$$\begin{aligned} t^{-2}\left|\int_\varepsilon^{\omega/2} J d\tau\right| &\leq M t^{-2}\int_\epsilon^{\omega/2}\left[\frac{\tau^{-1}A_{1a}(\tau^{-1})}{(\tau A_{1a}(\tau^{-1}))^2}+\frac{A_{1a}^2(\tau^{-1})+1}{(\tau A_{1a}(\tau^{-1}))^3}\right]d\tau \\ &\leq M t^{-2}\int_\epsilon^{\omega/2}\tau^{-3}d\tau\left[\frac{1}{A_{1a}(\omega^{-1})}+\frac{1}{A_{1a}^3(\omega^{-1})}\right] \\ &\leq \frac{M}{t^2}(a\in\mathcal{R}). \end{aligned}$$

(Note that if a in $\mathcal{R}$ is such that $\epsilon=\omega/2$, then clearly $t^{-2}|\int_\epsilon^{\omega/2} J d\tau|$

$\leq M/t^2$).

Next we use (2.22), (2.19), (2.6), (1.17) and the definition of ω to obtain

$$\begin{aligned} & t^{-2}\left|\left[\int_{\omega/2}^{\omega-\epsilon}+\int_{\omega+\epsilon}^{\infty}\right]J\, d\tau\right| \\ &\leq \frac{M}{t^2}\left[\int_{\omega/2}^{\omega-\epsilon}+\int_{\omega+\varepsilon}^{\infty}\right]\left[\frac{\tau^{-1}A_{1a}(\tau^{-1})}{|\tau-\omega|^2}+\frac{A_{1a}^2(\tau^{-1})+1}{|\tau-\omega|^3}\right]d\tau \\ (2.24)\quad &\leq \frac{M}{\epsilon t^2}A_{1a}\left(\frac{2}{\omega}\right)\left[\int_{\omega/2}^{\omega-\epsilon}+\int_{\omega+\epsilon}^{\infty}\right]\frac{1}{|\tau-\omega|^2}d\tau \\ &\quad + Mt^{-2}\left[A_{1a}^2\left[\frac{2}{\omega}\right]+1\right]\left[\int_{\omega/2}^{\omega-\epsilon}+\int_{\omega+\epsilon}^{\infty}\right]\frac{1}{|\tau-\omega|^3}d\tau \\ &\leq \frac{M}{t^2}\int_\epsilon^\infty \tau^{-2}+\tau^{-3}d\tau\leq\frac{M}{t^2},\quad a \text{ in } \mathcal{R}. \end{aligned}$$

(Note that $A_{1a}(2x)\leq 4A_{1a}(x)$, $x>0$, a in $\mathcal{R}$, since $\int_x^{2x} sa(s)\,ds$ $\leq a(x)\int_x^{2x} sds = 3a(x)\int_0^x sds\leq 3A_1(x)$.)

Next we use (2.6), (2.19), $\operatorname{Re} D(\tau; a) = \phi_a(\tau)$, (1.17), the definition of ω and (2.7) to obtain
(2.25)

$$\begin{aligned}\left|t^{-2}\int_{\omega-\epsilon}^{\omega+\epsilon} J d\tau\right| &\le \frac{M}{t^2}\int_{\omega-\epsilon}^{\omega+\epsilon}\left[\frac{\tau^{-1}A_{1a}(\tau^{-1})}{\phi_a^2(\tau)} + \frac{A_{1a}^2(\tau^{-1})+1}{\phi_a^3(\tau)}\right] d\tau \\ &\le \frac{M}{t^2}\left[\frac{1}{B_a(\omega^{-1})^2} + \frac{1}{B_a(\omega^{-1})^3}\right] \\ &\le \frac{M}{t^2}\left[\frac{1}{\phi_a(\omega)^2} + \frac{1}{\phi_a(\omega)^3}\right] \le \frac{M}{t^2}.\end{aligned}$$

Thus (2.25), (2.24), (2.23), (2.18), (see (2.20)), (2.17), (2.16), (2.15) and (2.4) prove that (1.2) holds.

Finally, to see that the corollary follows from Theorem 1, let $\mathcal{R} = \{a_0(t) + c : 0 \le c \le 1\}$ where a_0 satisfies (1.1) and (1.13). Then clearly

$$\sup_{a\in\mathcal{R}} \frac{1}{uA_a(u)} = \frac{1}{uA_{a_0}(u)},$$

so (1.13) implies (1.14). The assumption (1.15) is used in the proof of Theorem 1 to obtain inequality (2.11). However (2.11) holds for this family $\mathcal{R}$, without the additional assumption (1.15) as is proved in [**12**]. Finally, to show (1.19), we note that

$$\begin{aligned}\hat{a}(\tau) &= \phi_a(\tau) - i\tau\theta_a(\tau) = \hat{a}_0(\tau) + \frac{c}{i\tau} \\ &= \phi_{a_0}(\tau) - i\tau\theta_{a_0}(\tau) - i\tau^{-1}c.\end{aligned}$$

Thus it follows that

$$1 = \theta_a(\tilde{\omega}) = \theta_{a_0}(\tilde{\omega}) + c\tilde{\omega}^{-2},$$

or

$$c = \tilde{\omega}^2(1 - \theta_{a_0}(\tilde{\omega})).$$

Since $\theta_{a_0}(\infty) = 0$, by (1.17), and $0 \le c \le 1$, clearly $\tilde{\omega}$ is bounded from above, thus so is ω (say $\omega = \omega(a) \le M_1$). Then, by (2.7) and the fact that $a'(t) = a_0'(t)$, we have

$$\frac{1}{\phi_a(\omega(a))} \le \frac{5}{B_a(\omega^{-1})} \le \frac{5}{B_a(M_1^{-1})} = \frac{5}{B_{a_0}(M_1^{-1})} \equiv M < \infty.$$

The corollary now follows by applying Theorem 1.

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