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A DISCRETE GALERKIN METHOD FOR FIRST KIND INTEGRAL EQUATIONS WITH A LOGARITHMIC KERNEL

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ABSTRACT. Consider the first kind integral equation

$$\int_S g(Q)\log|P-Q|\,dS(Q) = h(P), \quad P \in S$$

with S a smooth simple closed curve in the plane. A special Galerkin method with trigonometric polynomial approximants has been shown by other authors to converge exponentially when solving the above integral equation. In this paper, Galerkin's method is further discretized by replacing the integrals with numerical integrals. The resulting discrete Galerkin method is shown to converge rapidly when the curve S and the data h are smooth. The method is also equivalent to a discrete collocation procedure with trigonometric polynomial approximants.

1. Introduction. Consider the numerical solution of

$$\int_S g(Q)\log|P-Q|\,dS(Q) = h(P), \quad P \in S, \tag{1.1}$$

with S the boundary of a simply-connected planar region D. This equation arises in solving the Dirichlet problem for Laplace's equation on D, using either a direct or indirect boundary integral equation reformulation of the Dirichlet problem. For this mathematical development of (1.1), see [**6**] or any of many other sources on boundary integral equation reformulations of Laplace's equation.

In this paper, we consider the restricted case that S is a smooth boundary curve. For simplicity, assume S has a C^∞ parameterization

$$r(s) = (\xi(s), \eta(s)), \quad 0 \le s \le 2\pi, \tag{1.2}$$

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with

$$|r'(s)| \neq 0, \quad 0 \leq s \leq 2\pi. \tag{1.3}$$

Following the development in [**8**] or [**11**], rewrite (1.1) as

$$-\frac{1}{\pi}\int_0^{2\pi} \rho(s)\log|r(t)-r(s)|\,ds = f(t), \quad 0 \leq t \leq 2\pi, \tag{1.4}$$

with

$$\rho(s) = g(r(s))|r'(s)|, \quad f(t) = -\frac{1}{\pi}h(r(t)).$$

Then decompose (1.4) as

$$A\rho + B\rho = f \tag{1.5}$$

with

$$A\rho(t) = -\frac{1}{\pi}\int_0^{2\pi} \rho(s)\log\left|2e^{-1/2}\sin\left[\frac{t-s}{2}\right]\right| ds \tag{1.6}$$

$$B\rho(t) = \int_0^{2\pi} b(t,s)\rho(s)\,ds \tag{1.7}$$

$$b(t,s) = \begin{cases} -\frac{1}{\pi}\log\left|\frac{e^{\frac{1}{2}[r(t)-r(s)]}}{2\sin[\frac{t-s}{2}]}\right|, & t-s \neq 2m\pi, \\ -\frac{1}{\pi}\log|e^{\frac{1}{2}}r'(t)|, & t-s = 2m\pi, \end{cases} \tag{1.8}$$

for $m = 0, \pm 1, \pm 2, \ldots$. The function b is 2π-periodic in both variables, and it is C^∞. The operator A arises from studying equation (1.1) on a circle.

To have unique solvability of (1.1) or (1.5), we assume that the *transfinite diameter* C_S of the boundary S is not equal to 1; see [**11**, §1] for the definition of transfinite diameter and a discussion of its properties. From $C_S \neq 1$, it follows that if $p > 1$ and if

$$\int_S q(Q)\log|P-Q|\,dS(Q) = 0, \quad P \in S, \tag{1.9}$$

for some $g \in L^p(S)$, then $g = 0$. From [**11**, §4.1], this ensures solvability and uniqueness for (1.1) for all function spaces on S of interest here.

The eigenfunctions of A are the trigonometric functions, and as a consequence, A^{-1} can be computed explicitly. This result has been used in the approximate solution of (1.5) by Galerkin's method with trigonometric polynomials as approximating solutions. Such a numerical method has been presented and analyzed in [**8**]. We will carry this approach further by analyzing the effect of the numerical integration errors that arise in the practical implementation of the Galerkin method.

In §2, some needed notation and function space results are presented, and the Galerkin method is given. §3 introduces a discrete Galerkin framework for solving (1.5), and the discrete Galerkin method is analyzed in §4. Numerical examples are given in §5.

A related paper is [**7**] in which the discretization of Galerkin's method is considered in a quite general setting, although differing from that presented here. Other numerical treatments of (1.1) for a smooth boundary S are given in [**2, 5, 8,** and **9**].

2. Background results. For our discussion of the operator A, we quote freely from [**11**]. Let H^t denote the Sobolev space of functions

$$\rho(s) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \hat{\rho}(m) e^{ims} \tag{2.1}$$

$$\hat{\rho}(m) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \rho(s) e^{-ims} ds$$

whose Fourier coefficients satisfy

$$||\rho||_t \equiv \left[|\hat{\rho}(0)|^2 + \sum_{|m|>0} |m|^{2t} |\hat{\rho}(m)|^2 \right]^{1/2} < \infty.$$

It is well-known that if $t > 1/2$, then $H^t \subset C_p[0, 2\pi]$, the 2π-periodic continuous functions.

The operator A of (1.6) can be shown to be equivalent to

$$A\rho(s) = \frac{1}{\sqrt{2\pi}} \left[\hat{\rho}(0) + \sum_{|m|>0} \frac{\hat{\rho}(m)}{|m|} e^{ims} \right] \tag{2.2}$$

where ρ is given by (2.1). Then

$$A : H^t \xrightarrow[\text{onto}]{1-1} H^{t+1} \tag{2.3}$$

for any real t, and

$$||A\rho||_{t+1} = ||\rho||_t. \tag{2.4}$$

For the inverse,

$$A^{-1}\rho(s) = \frac{1}{\sqrt{2\pi}}\Big[\hat{\rho}(0) + \sum_{|m|>0} |m|\hat{\rho}(m)e^{ims}\Big] \tag{2.5}$$

In addition,

$$A^{-1} = -\mathcal{D}\mathcal{H} + \mathcal{J} = -\mathcal{H}\mathcal{D} + \mathcal{J} \tag{2.6}$$

with $\mathcal{D}\rho(t) = \rho'(t)$,

$$\mathcal{J}\rho(s) = \frac{1}{\sqrt{2\pi}}\hat{\rho}(0)$$

and $\mathcal{H}$ the Hilbert transform:

$$\mathcal{H}\rho(s) = \frac{-1}{2\pi}\int_0^{2\pi} \cot\Big[\frac{s-\sigma}{2}\Big]\rho(\sigma)\,d\sigma$$

For the latter,

$$\mathcal{H} : H^t \to H^t \text{ is bounded}, \quad \text{all } t \geq 0. \tag{2.7}$$

The Galerkin method. We give a broad outline of the Galerkin method for solving (1.5). Let χ_n be the space of all trigonometric polynomials of degree $\leq n$. The dimension of χ_n is $2n+1$. For theoretical purposes, the most convenient basis for χ_n is the set

$$\{e^{-int}, \ldots, e^{-it}, 1, e^{it}, \ldots, e^{int}\}. \tag{2.8}$$

We will often write

$$\varphi_j(t) = e^{ijt}.$$

In computer programs, it may be faster to use the other standard basis, involving *sine* and *cosine* functions; but (2.8) is more convenient notationally.

Let P_n denote the orthogonal projection of $H^0 = L^2(0, 2\pi)$ onto χ_n:

$$(P_n\rho)(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-n}^{n} \hat{\rho}(m)e^{imt} \tag{2.9}$$

$$\hat{\rho}(m) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \rho(s)e^{-ims}ds$$

$P_n\rho$ is simply the truncation of the Fourier series expansion of $\rho(t)$ given in (2.1). It is straightforward to show that P_n is a bounded orthogonal projection on $L^2(0, 2\pi)$.

The Galerkin method for solving (1.5) consists of solving

$$P_n(A + B)\rho_n = P_n f, \quad \rho_n \in \chi_n. \tag{2.10}$$

Since $P_n A = AP_n$ (easily proven), we have the equivalent formulation

$$(A + P_n B)\rho_n = P_n f, \quad \rho_n \in L^2(0, 2\pi). \tag{2.11}$$

To analyze the convergence, consider instead the equivalent equation

$$(I + P_n A^{-1} B)\rho_n = P_n A^{-1} f, \tag{2.12}$$

using $A^{-1}P_n = P_n A^{-1}$.

Because of the smoothness of $b(t, s)$ in (1.8), the operator B is compact from H^0 into H^t, for all $t > 0$. Thus $A^{-1}B$ is a compact operator from $L^2(0, 2\pi)$ into $L^2(0, 2\pi)$. Since

$$P_n g \to g, \quad \text{all } g \in L^2(0, 2\pi),$$

it is straightforward that, on $L^2(0, 2\pi)$,

$$||(I - P_n)A^{-1}B|| \to 0 \quad \text{as } n \to 0.$$

Combined with the unique solvability of (1.5), we can obtain a convergence theory for (2.12).

It can be shown by standard arguments that $(I + P_n A^{-1} B)^{-1}$ exists and is uniformly bounded for all sufficiently large n, say $n \geq N$:

$$||(I + P_n A^{-1} B)^{-1}|| \leq M < \infty.$$

For example, see [**3**]. For convergence, use the identity

$$\rho - \rho_n = (I + P_n A^{-1} B)^{-1} (I - P_n)\rho. \tag{2.13}$$

Thus if $\rho \in L^2(0, 2\pi)$, we have convergence. If ρ is a smooth function, then $(I - P_n)\rho$ is very rapidly convergent to zero, and thus ρ_n is rapidly convergent to ρ. For a much more extensive analysis, see [**8**].

3. The discrete galerkin method. To understand the need for further discretization in the Galerkin method (2.11), consider the linear system that must be solved in order to calculate ρ_n. Let

$$\rho_n(t) = \sum_{j=-n}^{n} \alpha_j e^{ijt}. \tag{3.1}$$

The equation (2.10) says the Fourier coefficients of $(A + B)\rho_n$ and f must be the same for those of index $k = -n, \ldots, 0, \ldots, n$. This yields

$$\begin{aligned} \frac{2\pi\alpha_k}{\max\{1, |k|\}} + \sum_{j=-n}^{n} \alpha_j \int_0^{2\pi} \int_0^{2\pi} e^{i(js-kt)} b(t,s)\, ds\, dt \\ = \int_0^{2\pi} f(t) e^{-ikt} dt, \quad |k| \leq n \end{aligned} \tag{3.2}$$

The integral terms in (3.2) must be evaluated numerically. The numerical integration is to be chosen so that: (1) the resulting solution has the same speed of convergence as the original Galerkin solution ρ_n, if possible, and (2) the resulting method is efficient of computation time.

Let $C_p[0, 2\pi]$ denote the continuous complex-valued 2π-periodic functions on $-\infty < t < \infty$. To evaluate the integral

$$\int_0^{2\pi} g(t)\, dt$$

approximate it by the trapezoidal rule

$$T_m(f) = h\sum_{j=0}^{m}{}''f(t_j) = h\sum_{j=0}^{m-1} f(t_j), \quad f \in C_p[0, 2\pi],$$

with

$$h = \frac{2\pi}{m}, \quad t_j = jh, \quad j = 0, 1, \ldots, m.$$

The double prime notation on the summation means to halve the first and last terms before summing. The following is well-known and straightforward to prove.

LEMMA 1. *For any integer k,*

$$T_m(e^{ikt}) = \begin{cases} 2\pi, & k = 0(mod\ m) \\ 0, & k \neq 0(mod\ m). \end{cases} \tag{3.3}$$

To approximate the integral operator B, use

$$B_n\rho(t) = T_{2n+1}(b(t,\cdot)\rho(\cdot)) = h\sum_{j=0}^{2n} b(t, t_j)\rho(t_j) \tag{3.4}$$

with $h = 2\pi/(2n+1)$, $t_j = jh$.

Let

$$(f, g) = \int_0^{2\pi} f(t)\overline{g}(t)\,dt, \quad f, g \in C_p[0, 2\pi]$$

Introduce a discrete (semi-definite) inner product and associated norm:

$$(f, g)_h = T_{2n+1}(f(\cdot)\overline{g}(\cdot)) \tag{3.5}$$

$$||f||_h = \sqrt{(f, f)_h} \tag{3.6}$$

Using Lemma 1, $||\cdot||_h$ is a norm on χ_n. As in [**4**], introduce a *discrete orthogonal projection* $Q_n : C_p[0, 2\pi] \to \chi_n$ by

$$(Q_n f, \varphi)_h = (f, \varphi)_h, \quad \text{all } \varphi \in \chi_n. \tag{3.7}$$

The definitions (3.5)-(3.7) extend to the case where the number of integration nodes is greater than $2n+1$, but we need only the given case. See [**4**] for a more general presentation.

LEMMA 2. *The operator Q_n satisfies the following on $C_p[0,2\pi]$:*

(1) $Q_n^2 = Q_n$;

(2) $(Q_n f, g)_h = (f, Q_n g)_h, \quad f, g \in C_p[0,2\pi]$;

(3) $||Q_n f||_h \leq ||f||_h$;

(4) $Q_n f(t_j) = f(t_j), \quad j = 0, 1, \ldots, 2n, \quad f \in C_p[0,2\pi]$.

The proofs are straightforward and we omit them. The last property says that $Q_n f$ is the trigonometric polynomial in χ_n that interpolates f at the points in $\{t_0, \ldots, t_{2n}\}$ or equivalently, $\{t_{-n}, \ldots, 0, \ldots, t_n\}$. Thus Q_n is both an approximation to the orthogonal projection operator P_n and is the interpolating projection operator.

Using the basis (2.8), the projection Q_n is given by

$$\begin{aligned} Q_n f(t) &\sum_{j=-n}^{n} \alpha_j \varphi_j(t), \\ \alpha_j &= \frac{1}{2\pi}(f, \varphi_j)_h. \end{aligned} \tag{3.8}$$

The discrete Galerkin method. We approximate the Galerkin method (2.11) by

$$(A + Q_n B_n)\psi_n = Q_n f, \quad \psi_n \in C_p[0,2\pi], \tag{3.9}$$

replacing P_n and B by Q_n and B_n, respectively. To obtain a finite linear system from which ψ_n can be calculated, note first that if (3.9) is solvable, then

$$A\psi_n = Q_n(f - B_n\psi_n) \in \chi_n.$$

But $A\psi_n$ a trigonometric polynomial in χ_n implies ψ_n is a trigonometric polynomial of the same degree, from the definition of A. Thus (3.9) can be written as

$$Q_n(A + B_n)\psi_n = Q_n f, \quad \psi_n \in \chi_n. \tag{3.10}$$

Let

$$\psi_n(t) = \sum_{j=-n}^{n} \beta_j \varphi_j(t)$$

Calculate ψ_n from (3.10), Lemma 1, and the definitions of Q_n and A_n:

$$\frac{2\pi\beta_k}{\max\{1,|k|\}} + \sum_{j=-n}^{n} \beta_j (B_n\varphi_j, \varphi_k)_h = (f, \varphi_k)_h, \quad |k| \le n \tag{3.11}$$

No further numerical integrations are needed to set up this linear system.

An alternative to (3.11) is obtained by using the interpolating property of Q_n:

$$\sum_{j=-n}^{n} \beta_j \left[\frac{\varphi_j(t_k)}{\max\{1,|j|\}} + B_n\varphi_j(t_k)\right] = f(t_k), \quad -n \le k \le n. \tag{3.12}$$

This system is faster to set up, since the discrete inner products $(\cdot,\cdot)_h$ are no longer involved. In addition, the systems (3.11) and (3.12) have the same condition number if the latter is based on the operator matrix norm induced by the Euclidean vector norm on $\mathbf{C}^{2n+1}$.

4. Convergence analysis. Rewrite the boundary integral equation (1.5) as

$$(I + A^{-1}B)\rho = A^{-1}f \tag{4.1}$$

and rewrite the discrete Galerkin equation (3.9) as

$$(I + A^{-1}Q_nB_n)\psi_n = A^{-1}Q_nf. \tag{4.2}$$

We regard $A^{-1}B$ as a compact operator on $C_p[0,2\pi]$. The analysis of the solvability of (4.2) is carried out by using the framework of collectively compact operator approximations.

THEOREM 3. (i) *Assume the boundary curve S has the C^∞ parameterization of* (1.2) – (1.3). *Then the family $\{A^{-1}Q_nB_n|n \ge 1\}$ is collectively compact and pointwise convergent on $C_p[0,2\pi]$.*

(ii) *Assume the integral equation* (1.5) *is uniquely solvable for all right hand sides* $f \in H^1$. *Then, for all sufficiently large* n, *say* $n \geq N$, *the operators* $I + A^{-1}Q_nB_n$ *are invertible on* $C_p[0, 2\pi]$ *and satisfy*

$$\|(I + A^{-1}Q_nB_n)^{-1}\| \leq M < \infty, \quad n \geq N. \tag{4.3}$$

For the error in the discrete Galerkin solution ψ_n *of (3.9),*

$$\|\rho - \psi_n\|_\infty \leq M\{\|A^{-1}(f - Q_nf)\|_\infty + \|A^{-1}B\rho - A^{-1}Q_nB_n\rho\|_\infty\}. \tag{4.4}$$

The proof of this requires information on trigonometric interpolation. The results are not the best possible, but they are sufficient for our needs.

LEMMA 4. *For* $n \geq 1$,

$$(Q_nf)(t) = \frac{h}{\pi} \sum_{j=-n}^{n} D_n(t - t_j) f(t_j) \tag{4.5}$$

with $h = 2\pi/(2n + 1)$ *and*

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} \cos(ku) = \frac{\sin(n + \frac{1}{2})u}{2\sin(\frac{1}{2}u)} \tag{4.6}$$

which is called the Dirichlet kernel. Moreover,

$$\|f - Q_nf\|_\infty \leq I_n\rho_n(f), \tag{4.7}$$

with

$$I_n = \max_t \frac{h}{\pi} \sum_{-n}^{n} |D_n(t - t_j)| = O(\log n) \tag{4.8}$$

$$\rho_n(f) = \min_{\psi \in \chi_n} \|f - \psi\|_\infty$$

PROOF. See [**12**].

To give actual rates of convergence for $Q_n f$, combine (4.7) with Jackson's theorem:

$$\rho_n(f) \leq \frac{c_k}{n^k} ||f^{(k)}||_\infty, \quad n \geq 1, \tag{4.9}$$

with $f \in C_p^k[0, 2\pi]$, the k-times differentiable 2π-periodic functions. A proof is given in [**10**]. □

LEMMA 5. *For a given $f \in C_p[0, 2\pi]$, let $F_n(t)$ denote the partial sum of terms of degree $\leq n$ for the Fourier series of f on $[0, 2\pi]$. Then*

$$F_n(t) = \int_0^{2\pi} D_n(t-s) f(s)\, ds \tag{4.10}$$

$$||f - F_n||_\infty \leq L_n \rho_n(f) \tag{4.11}$$

with

$$L_n = 1 + \int_0^{2\pi} |D_n(u)|\, du = O(\log n).$$

PROOF. See [**12**]. Fundamental to the proof is the identity

$$T(t) = \int_0^{2\pi} D_n(t-s) T(s)\, ds \tag{4.12}$$

for any trigonometric polynomial of degree $\leq n$. Then, for any such function $T(t)$,

$$f(t) - F_n(t) = [f(t) - T(t)] - \int_0^{2\pi} D_n(t-s)[f(s) - T(s)] ds$$

and

$$||f - F_n||_\infty \leq L_n ||f - T||_\infty,$$

leading to (4.11). □

We now consider the derivatives of $Q_n f$ for $f \in C_p[0, 2\pi]$: from (3.8) and (4.5),

$$(Q_n f)'(s) = \sum_{0<|j|\leq n} \alpha_j \varphi_j'(s) = \frac{h}{\pi} \sum_{j=-n}^{n} D_n'(t - t_j) f(t_j) \tag{4.13}$$

LEMMA 6. *Let* $f \in C_p^1[0, 2\pi]$. *Then*

$$||f' - (Q_n f)'||_\infty \le L_n \rho_n(f') + I_n^{(1)} ||f - F_n||_\infty \tag{4.14}$$

$$I_n^{(1)} = \max_t \frac{h}{\pi} \sum_{j=-n}^{n} |D_n'(t - t_j)| \le O(n^2). \tag{4.15}$$

PROOF. For $T(t)$, an arbitrary trigonometric polynomial of degree $\le n$,

$$T(t) = \frac{h}{\pi} \sum_{j=-n}^{n} D_n(t - t_j) T(t_j).$$

Thus

$$T'(t) = \frac{h}{\pi} \sum_{j=-n}^{n} D_n'(t - t_j) T(t_j) \tag{4.16}$$

and

$$f(t) - (Q_n f)'(t) = [f'(t) - T'(t)] - \frac{h}{\pi} \sum_{j=-n}^{n} D_n'(t - t_j)[f(t_j) - T(t_j)].$$

Let $T = F_n$. Then

$$\begin{aligned} |f'(t) - (Q_n f)'(t)| &\le |f'(t) - F_n'(t)| \\ &\quad + \frac{h}{\pi} \sum_{j=-n}^{n} |D_n'(t - t_j)|\, |f(t_j) - F_n(t_j)|. \end{aligned} \tag{4.17}$$

The last term is bounded by $I_n^{(1)} ||f - F_n||_\infty$. A crude bound for $I_n^{(1)}$ is obtained from

$$D_n'(u) = -\sum_{k=1}^{n} k \sin(ku),$$

$$||D_n'||_\infty \le \sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

$$I_n^{(1)} \leq n(n+1).$$

It is likely there is a smaller bound, and it would result in a slight improvement in our final convergence results.

For the term $|f'(t) - F_n'(t)|$ in (4.17), use (4.10) and the periodicity of f and D_n to obtain

$$F_n(t) = \int_0^{2\pi} D_n(s) f(t-s)\, ds$$
$$F_n'(t) = \int_0^{2\pi} D_n(s) f'(t-s)\, ds = \int_0^{2\pi} D_n(t-s) f(s)\, ds.$$

Thus the derivative of the Fourier approximation of degree n is just the Fourier approximant of degree n for f'. Thus

$$||f' - F_n'||_\infty \leq L_n \rho_n(f).$$

This proves (4.14). □

LEMMA 7. *Assume $f \in C_p^k[0, 2\pi]$ with $k \geq 3$. Then*

$$||f' - (Q_n f)'||_\infty \leq \frac{c_k \log(n+2)}{n^{k-2}} ||f^{(k)}||_\infty. \tag{4.18}$$

PROOF. Combine (4.14)-(4.15) with the Jackson result (4.9) and the Fourier series result (4.11). It is likely that this result can be improved by finding a sharper bound for $I_n^{(1)}$. □

LEMMA 8. *Let $l \geq 1$. Assume $f \in C_p^k[0, 2\pi]$, with $k \geq l+2$. Then*

$$||f^{(l)} - (Q_n f)^{(l)}||_\infty \leq \frac{c_{k,l} \log(n+2)}{n^{k-l-1}} ||f^{(k)}||_\infty. \tag{4.19}$$

PROOF. Just repeat the arguments used in Lemmas 6 and 7. In particular,

$$f^{(l)}(t) - (Q_n f)^{(l)}(t) = [f^{(l)}(t) - F_n^{(l)}(t)] - \frac{h}{\pi} \sum_{j=-n}^{n} D_n^{(l)}(t-t_j)[f(t_j) - F_n(t_j)]$$

$$||f^{(l)} - (Q_n f)^{(l)}||_\infty \le L_n \rho_n(f^{(l)}) + I_n^{(l)} ||f - F_n||_\infty,$$

$$I_n^{(l)} = \max_t \frac{h}{\pi} \sum_{j=-n}^{n} |D_n^{(l)}(t - t_j)| \le O(n^{l+1}).$$

□

PROOF OF THEOREM 3. (a). We find an explicit formula for $C_n \equiv A^{-1} Q_n B_n$, analyze its properties, and compare it to $C \equiv A^{-1} B$. Recall from (3.4) that

$$B_n \rho(t) = h \sum_{j=0}^{2n} b(t, t_j) \rho(t_j).$$

Write

$$Q_n B_n f(t) = h \sum_{j=0}^{2n} b_n(t, t_j) f(t_j) \tag{4.20}$$

$$b_{n,s}(t) \equiv b_n(t, s) = (Q_n b_s)(t), \quad b_s(t) \equiv b(t, s). \tag{4.21}$$

For each $s, b_{ns} \in \chi_n$ and interpolates $b_s(t)$ at $t = t_0, t_1, \ldots, t_{2n}$ define

$$C_n f(t) = h \sum_{j=0}^{2n} c_n(t, t_j) f(t_j) \tag{4.22}$$

$$c_{n,s} = A^{-1} b_{n,s}, \quad c_n(t, s) = c_{n,s}(t) \tag{4.23}$$

The function $c_{n,s} \in \chi_n$, for each s. Also define

$$C f(t) = \int_0^{2\pi} c(t, s) f(s)\, ds \tag{4.24}$$

$$c_s = A^{-1} b_s, \quad c(t, s) = c_s(t) \tag{4.25}$$

From the smoothness of $b(t,s)$, it is straightforward that c_s is C^∞ and an element of H^l for all $l \geq 0$.

(b). From Lemmas 4, 7, and 8, we can show that, for all $l \geq 0$,

$$\sup_{t,s} \left| \frac{\partial^l b_n(t,s)}{\partial t^l} - \frac{\partial^l b(t,s)}{\partial t^l} \right| \to 0 \text{ as } n \to \infty \tag{4.26}$$

Recall that $C_p^l[0,2\pi] \subset H^l$ and that convergence in C_p^l implies convergence in $H^l, l \geq 0$. Thus (4.26) implies

$$||b_{n,s} - b_s||_l \to 0 \text{ as } n \to \infty, \quad l \geq 0. \tag{4.27}$$

To examine the convergence of $\{C_n\}$, use (2.6) to write

$$c_{n,s} = (-\mathcal{H}\mathcal{D} + \mathcal{H})b_{n,s} = -\mathcal{H}(\mathcal{D}b_{n,s}) + \mathcal{J}b_{n,s} \tag{4.28}$$

Then

$$(\mathcal{J}b_{n,s})(t) = \frac{\hat{b}_n(0,s)}{\sqrt{2\pi}} \to \frac{\hat{b}(0,s)}{\sqrt{2\pi}} \tag{4.29}$$

uniformly in s. Next, the result (4.26) implies

$$\sup_s ||\mathcal{D}b_{n,s} - \mathcal{D}b_s||_l \to 0 \text{ as } n \to \infty \tag{4.30}$$

for all $l \geq 0$. Combined with (2.7) and (4.27)-(4.30), we have

$$\sup_s ||c_{n,s} - c_s||_l \to 0 \text{ as } n \to \infty, \ l \geq 0. \tag{4.31}$$

(c). To show $\{C_n\}$ is collectively compact in $C_p[0,2\pi]$, we must show that the set

$$\mathcal{B} = \{C_n f \,|\, ||f||_\infty \leq 1\}$$

is uniformly bounded and equicontinuous. The convergence in (4.31) implies the functions in $\{c_n(t,s) | n \geq 1\}$ are uniformly bounded in (t,s), and thus $\mathcal{B}$ is bounded. The equicontinuity of $\mathcal{B}$ comes from the uniform continuity of $c(t,s)$ and the uniform convergence of c_n to c.

To show pointwise convergence of the family $\{C_n\}$, write

$$Cf(s) - C_n f(s) = \int_0^{2\pi} c(t,s)f(s)\,ds - h\sum_{j=0}^{2n} c(t,t_j)f(t_j) + h\sum_{j=0}^{2n}[c(t,t_j) - c_n(t,t_j)]f(t_j)$$

$$\begin{aligned} ||Cf - C_n f||_\infty \le \max_t \left| \int_0^{2\pi} c(t,s)f(s)\,ds - h\sum_{j=0}^{2n} c(t,t_j)f(t_j)\right| \\ + 2\pi\Big[\max_{t,s}|c(t,s) - c_n(t,s)|\Big]||f||_\infty \end{aligned} \tag{4.32}$$

The first term on the right side goes to zero by standard arguments on the convergence of numerical integration operators. The second term goes to zero from (4.31) with $l = 1$.

(d). The theory of the solvability of (1.1), given in [**11**], leads to the existence and uniform boundedness of $(I-C)^{-1}$ on $C_p[0,2\pi]$ into itself. [Recall the assumption involving (1.9)]. The existence and uniform boundedness of the inverses $(I + C_n)^{-1}$,

$$||(I + C_n)^{-1}|| \le M < \infty, \tag{4.33}$$

follows from part (i) and the standard theory of collectively compact operator approximations, given in [**1**].

For convergence of the discrete Galerkin method (3.9), rewrite (1.6) and (3.9), respectively, as

$$\begin{aligned} (I + C)\rho &= A^{-1}f \\ (I + C_n)\psi_n &= A^{-1}Q_n f. \end{aligned} \tag{4.34}$$

Subtract and manipulate these equations to get

$$(I + C_n)(\rho - \psi_n) = A^{-1}[f - Q_n f] - [C - C_n]\rho \tag{4.35}$$

$$||\rho - \psi_n||_\infty \leq M\Big[||A^{-1}[f - Q_n f]||_\infty + ||[C - C_n]\rho||_\infty\Big]. \tag{4.36}$$

□

THEOREM 9. *Assume the hypotheses of Theorem* 3. *Assume* $f \in C_p^k[0, 2\pi]$ *with* $k \geq 4$. *Then*

$$||\rho - \psi_n||_\infty \leq \frac{c \log n}{n^{k-3}} \tag{4.37}$$

for all sufficiently large n.

PROOF. The space $C_p^k[0, 2\pi] \subset H^k$; and it then follows from [**11**] that $\rho \in H^{k-1}$. From (4.31), (4.32), and the Euler-MacLaurin formula for the error in the trapezoidal numerical integration rule,

$$||[C - C_n]\rho||_\infty \leq \frac{c}{n^{k-1}} \tag{4.38}$$

for some c.

From Lemma 8, $Q_n f \to f$ in H^2, with

$$||f - Q_n f||_2 \leq \frac{c \log n}{n^{k-3}}$$

for some c. Since $A^{-1} : H^2 \to H^1$, we have that

$$||A^{-1}[f - Q_n f]||_1 \leq \frac{c \log n}{n^{k-3}} ||A^{-1}||. \tag{4.39}$$

The space H^1 is compactly embedded in $C_p[0, 2\pi]$, and consequently (4.39) implies the same bound for $||A^{-1}[f - Q_n f]||_\infty$.

5. Numerical example. We give only a simple illustration of the preceding material. Consider the interior Dirichlet problem for Laplace's equation:

$$\begin{aligned} \Delta u(P) &= 0, \quad P \in D, \\ u(P) &= h(P), \quad P \in S, \end{aligned} \tag{5.1}$$

We represent the solution u as the single layer potential

$$u(P) = \int_S g(Q) \log |P - Q| \, dS(P), \quad P \in D \tag{5.2}$$

The unknown density function g is obtained by solving (1.1).

We give numerical results for the case of the elliptical region

$$(x, y) = (ar\cos(t), br\sin(t)), \quad 0 \le t \le 2\pi, \quad 0 \le r \le 1, \tag{5.3}$$

with $a, b > 0$. Then (1.8) yields

$$b(t, s) = -\frac{1}{2\pi}\left\{1 + \log\left[a^2 \sin^2\left[\frac{s+t}{2}\right] + b^2 \cos\left[\frac{s+t}{2}\right]\right]\right\}$$

After solving the equation (1.5) for the approximate solution ψ_n, the approximate density function g_n is given by

$$g_n(t) = \psi_n(t)/|r'(t)|, \quad 0 \le t \le 2\pi,$$

as in (1.4). We obtain an approximation u_n to u by substituting g_n into (5.2) and numerically integrating. The integral is evaluated with the trapezoidal rule T_{2n+1}, using, as the quadrature nodes, the same points $\{t_j\}$ as were used in solving for ψ_n. An error analysis for the resulting solution u_n can be given to show that the effect of this integration preserves the rate of convergence associated with ψ_n.

We give the results of this integration at a selected set of points (x_j, y_j) inside D. In particular, define

$$(x_j, y_j) = r_j(a\cos(\pi/4),\ b\sin(\pi/4)), \quad j = 1, 2, 3, 4$$

with $r_j = 0, .4, .8, .99$. The point (x_4, y_4) is very close to the boundary S, making the integrand in (5.2) very peaked. The problem being solved here has the true solution

$$u = e^x \cos(y)$$

and we let $(a, b) = (1, .4)$. Table 1 contains the errors in u_n for selected values of the degree n.

TABLE 1. Errors in $u_n(x_j, y_j)$.

n	j=1	j=2	j=3	j=4
1	1.99E-1	5.21E-1	6.83E-1	7.78E-1
2	6.84E-4	-1.32E-1	2.40E-1	5.96E-1
3	1.21E-3	2.68E-2	-2.71E-1	-1.22E-1
4	-8.02E-5	-5.87E-4	5.40E-2	-1.49E-1
5	1.95E-4	-4.58E-3	6.39E-2	2.06E-1
6	-6.38E-5	3.91E-3	-5.18E-2	1.88E-1
7	2.78E-5	-2.04E-3	-5.49E-3	-1.07E-1
8	-1.06E-5	7.00E-4	2.48E-2	-5.70E-2
9	4.23E-6	-7.26E-5	-6.72E-3	1.21E-1
10	-1.66E-6	-1.15E-4	-9.79E-3	1.06E-1
11	6.63E-7	1.14E-4	7.85E-3	-9.11E-1
12	-2.64E-7	-6.51E-5	1.67E-3	-2.96E-2
13	1.06E-7	2.45E-5	-4.95E-3	8.39E-2
14	-4.25E-8	-3.32E-6	1.39E-3	7.02E-2
15	1.71E-8	-3.79E-6	1.99E-3	-8.07E-2
16	-6.94E-9	4.12E-6	-1.64E-3	-1.69E-2
17	2.81E-9	-2.48E-6	-3.37E-4	6.34E-2
18	-1.15E-9	9.82E-7	1.04E-3	5.02E-2
19	4.67E-10	-1.56E-7	2.81E-4	-7.29E-2
20	-1.91E-10	-1.39E-7	-4.51E-4	-9.70E-3

As can be seen from the table entries, the convergence of u_n to u is quite rapid, in fact, exponential. But as the point of evaluation (x, y) becomes closer to the boundary S, the convergence becomes significantly worse. In addition, the convergence is also somewhat

erratic, making it difficult to predict errors for individual values of n and (x, y).

To improve the accuracy in our approximate potential, we use a more accurate numerical integration to evaluate the integral formula defining $u_n(P)$ [formula (5.2) with $\rho(P)$ replaced by $\psi_n(P)$]. We again integrate (5.2) with the trapezoidal rule, but now we use $2q + 1$ evenly spaced node points on the boundary, with $q \geq n$. The results are shown in Table 2 for the case of $n = 10$, with varying values of q. Note that the new values of $u_{10}(P)$ are much improved over those in Table 1, even for only moderately sized values of q.

TABLE 2. Errors in $u_{10}(x_j, y_j)$: varying q.

q	j=1	j=2	j=3	j=4
10	-1.66E-6	-1.15E-4	-9.79E-3	1.06E-1
20	1.23E-9	-1.38E-7	-4.51E-4	-9.70E-3
40	1.42E-9	1.10E-9	-1.85E-6	2.04E-3
80	1.42E-9	1.10E-9	1.11E-9	3.86E-3
160	1.42E-9	1.10E-9	1.17E-9	1.35E-3
320	1.42E-9	1.10E-9	1.17E-9	-5.29E-5
640	1.42E-9	1.10E-9	1.17E-9	3.88E-7

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Added in proof. A related paper, D. Arnold and R. Cheng, "The delta-trigonometric method using the single-layer potential representation" will appear later. It contains a similar, but different, method, with a different method of analysis.

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