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ON LINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF BARBASHIN TYPE IN SPACES OF CONTINUOUS AND MEASURABLE FUNCTIONS

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ABSTRACT. This paper surveys several important properties of linear integro-differential equations (1) of Barbashin type (2), especially those related to the geometric structure of the underlying function space. In contrast to Barbashin's classical results, discontinuous data (e.g. kernel functions) are also allowed. After discussing several classes of suitable kernels, the resolvent operator (Cauchy function) generated by the operator (2) is described. Moreover, stability results for equation (1) are proved. Finally, representation formulas for the corresponding Green's function are given; a perturbed version of such formulas applies to averaging procedures of Bogolyubov-Krylov type.

This paper is concerned with the linear differential equation

$$\frac{du}{dt} = A(t)u \tag{1}$$

in Banach spaces of continuous or measurable functions over some interval $[a,b]$, where the operator $A(t)$ is given for $t \in J$ (a bounded or unbounded interval in $\mathbf{R}$) by

$$A(t)x(s) = c(t,s)x(s) + \int_a^b k(t,s,\sigma)x(\sigma)d\sigma. \tag{2}$$

Here and in what follows, $c(t,s)$ and $k(t,s,\sigma)$ are measurable real functions on $J \times [a,b]$ and $J \times [a,b] \times [a,b]$, respectively. Appropriate function spaces for the operator (2) are, for instance, the space $X = C[a,b]$ of continuous functions or the space $X = L_p[a,b]$ of p -summable functions $(1 \leq p \leq \infty)$ over $[a,b]$; more generally, X may be an ideal space (see e.g. [50]) of measurable functions.

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Equations of the form (1) are natural "infinite-dimensional analogues" of finite systems of scalar differential equations. However, while the literature on general infinite linear systems is vast (see, e.g., the monograph [**47**] and the references therein), integro-differential equations like (1)/(2) have not been given much attention in the literature, although they occur in many fields of applied analysis (especially in control theory and in some parts of probability theory). Some early results in this direction may be found in the papers [**4-17**], [**22-24**], [**39**], [**45**], [**53**], for more recent results see, e.g., [**18-21**], [**25**], [**33**], [**34**], and [**36**]. The first essential achievements seem to be due to E.A. Barbashin and his pupils which refer to operators of type (2) with continuous functions $c(t,s)$ and $k(t,s,\sigma)$. After these papers appeared, equations of the above type were usually called *Barbashin equations.*

The purpose of this paper is two-fold. On the one hand, we shall apply the general theory of *differential equations in Banach spaces* (see, e.g., the book [**27**]) for studying the problem (1). On the other hand, we shall make extensive use of the very advanced theory of linear *integral operators in function spaces* (see, e.g., [**40**], [**52**]) taking into account the specific structure of the operator $A(t)$ in (1). This "combined approach" allows us not only to get the basic classical results on Barbashin equations as consequences of general theorems on both differential and integral operators in function spaces, but also to obtain essential generalizations and improvements of previous results; in particular, our approach allows us to consider *discontinuous data* $c(t,s)$ and $k(t,s,\sigma)$ as well, and to point out the influence of the geometric properties of the underlying function space X on the "analytical behaviour" of problem (1).

The plan of this paper is as follows. In the first section, we obtain minimal conditions for $c(t,s)$ and $k(t,s,\sigma)$ under which the equation (1) may be successfully studied in a specific given function space X. Afterwards, we describe the structure of the *resolvent operator* (Cauchy function) associated with equation (1). In the third section, we prove some results on the *stability* and *well-posedness* of problem (1) with respect to various scalar parameters. Finally, we give existence conditions and representation formulas for the *Green's function* with respect to bounded solutions on the whole real line; such conditions are particularly useful for applying the *Bogolyubov-Krylov averaging principle* to (both linear and nonlinear) integro-differential equations of Barbashin type.

1. Estimates for kernel functions

Let X be some Banach space of real functions on $[a,b]$, and denote by $L(X)$ the space of all bounded linear operators in X. By $L_0(X)$ we denote the subspace of all operators $A \in L(X)$ which are representable in the form

$$Ax(s) = c(s)x(s) + \int_a^b k(s,\sigma)x(\sigma)d\sigma. \tag{3}$$

The natural problem arises to find conditions on the functions $c(s)$ and $k(s,\sigma)$ under which (3) is in fact a bounded operator in X. It turns out that the solution to this problem heavily relies on the structure of the underlying space X.

In the case $X = C[a,b]$ it is useful to introduce the auxiliary functions

$$\alpha(s) = c(s) + \int_a^b k(s,\sigma)d\sigma, \tag{4}$$

$$\beta(z,s) = \int_a^z [c(s)\chi_s(\sigma) + (z-\sigma)k(s,\sigma)]d\sigma, \tag{5}$$

where $\chi_s(\sigma)$ is the characteristic function of the unbounded interval $[s,\infty)$, and

$$\gamma(s) = |c(s)| + \int_a^b |k(s,\sigma)|d\sigma. \tag{6}$$

By the classical results of F. Riesz and I. Radon (see, e.g., [29]), the following characterization holds true.

LEMMA 1. *The linear operator* (3) *belongs to* $L_0(C)$ *if and only if the functions* (4) *and* (5) *are continuous, and the function* (6) *is bounded; the norm of* A *is then given by*

$$||A|| = \sup_{a\leq s\leq b}|\gamma(s)|. \tag{7}$$

Lemma 1 implies that $L_0(C)$ is a closed subspace of $L(C)$; moreover, the representation (3) for operators $A \in L_0(C)$ is unique, and the set $L_i(C)$ of integral operators in C (i.e., operators (3) with $c(s) \equiv 0$ is an *ideal* in $L_0(C)$.

It is interesting to note that the operator (3) may act in the space C even in the case when the single components of (3) do *not*. The set $L_d(C)$ of all operators (3) for which both terms belong to $L(C)$ is, by Lemma 1, a closed subspace of $L(C)$. The subspaces $L_i(C), L_{ic}(C)$ and $L_{iw}(C)$ of all *integral operators, compact integral operators*, and *weakly compact integral operators*, respectively, are also closed in $L(C)$.

Now let X be an *ideal space* of measurable functions on $[a, b]$ (for the terminology see, e.g., [**48-50**]). In this case, there do not exist simple conditions (both necessary and sufficient) for $c(s)$ and $k(s, \sigma)$ under which the operator (3) belongs to $L(X)$. Nevertheless, it turns out that the basic results for integral operators in ideal spaces (see, e.g., [**40**], [**49**], [**52**]) carry over to operators of the form (3).

Some definitions are in order. Recall (see, e.g., [**50**]) that the *associate space* X' of X consists of all measurable functions y on $[a, b]$ with $\langle x, y\rangle < \infty$ for all $x \in X$, where

$$\langle x, y\rangle = \int_a^b x(s)y(s)ds. \tag{8}$$

The space X' is, of course, a (possibly proper) subspace of the usual *adjoint space* X^*. The *associate operator* $A' \in L(X')$ of $A \in L(X)$ is defined by the relation $\langle Ax, y\rangle = \langle x, A'y\rangle$; A' is, of course, the restriction of the usual *adjoint operator* A^* to X'. We remark that the associate operator A' exists if and only if A preserves X'-weakly convergent sequences.

A linear operator A in an ideal space X is called *regular* if A preserves order-bounded subsets of X. Any regular operator A may be represented as a difference of two positive operators. Given a regular operator A, we may define its modulus $|A|$ as the minimal of all positive linear operators B in X such that

$$|Ax| \leq B|x| \quad (x \in X). \tag{9}$$

Finally, the operator A gives rise to the operators

$$A^{\#}x(s) = c(s)x(s) + \int_a^b k(\sigma, s)x(\sigma)d\sigma \tag{10}$$

and

$$[A]x(s) = |c(s)|x(s) + \int_a^b |k(s,\sigma)|x(\sigma)d\sigma. \tag{11}$$

By adapting the corresponding proof for general linear integral operators (see, e.g., [**40**], [**49**]), one proves the following

LEMMA 2. *Suppose that the operator* (3) *acts in an ideal space* X. *Then* A *is continuous and admits an associate operator* A', *and* $A'x = A^{\#}x$ *for each* x *for which* $A^{\#}x$ *is defined. Moreover, the operator* (3) *is regular if and only if* $[A] \in L(X)$; *in this case,* $|A| = [A]$ *and* $A' = A^{\#}$.

Lemma 2 implies that a *regular* operator A is bounded in X if and only if both of its components are bounded in X, hence $L_r(X) \subseteq L_d(X) \subseteq L(X)$. We remark that the subspace $L_r(X)$ is, in general, *not* closed in $L(X)$; however, equipped with the stronger norm

$$||A||_r = ||\,|A|\,||, \tag{12}$$

$L_r(X)$ becomes a Banach space. In this space, the sets $L_{ri}(X), L_{ric}(X)$, and $L_{riw}(X)$ of all *regular integral operators, compact regular integral operators*, and *weakly compact regular integral operators*, respectively, are closed ideals.

In order to formulate (sufficient) conditions, in terms of the functions $c(s)$ and $k(s,\sigma)$, under which the operator (3) acts in an ideal space X and has "nice" properties, we recall another important notion [49].

Let X and Y be two ideal spaces. We denote by $R(X,Y)$ the set of all measurable functions $k(s,\sigma)$ on $[a,b] \times [a,b]$ for which the norm

$$||k(s,\sigma)|R(X,Y)|| = \sup_{\substack{||x|X||\leq 1 \\ ||y|Y'||\leq 1}} \int_a^b \int_a^b |k(s,\sigma)x(\sigma)y(s)|d\sigma ds \tag{13}$$

is finite; by $R^\circ(X,Y)$ we denote the subset of all functions from $R(X,Y)$ with *absolutely continuous norms.* Every kernel $k(s,\sigma)$ in $R(X,Y)$ gives rise to a *regular* linear integral operator from X into Y; in case $k(s,\sigma)$ belongs to $R^\circ(X,Y)$, this operator is even *compact.*

Lemma 2 implies that the operator A is regular if and only if $c(s)$ belongs to L_∞ and $k(s,\sigma)$ to $R(X,Y)$. This criterion, however, is not very suitable for applications, since calculating (or even estimating) the norm (13) for a given function $k(s,\sigma)$ is, in general, a hard problem. It is therefore useful to have a more explicit description of (at least some subspaces of) $R(X,Y)$ at hand.

Given again two ideal spaces X and Y, by $U(X,Y)$ and $V(X,Y)$ we denote the set of all measurable functions $k(s,\sigma)$ on $[a,b]\times[a,b]$ for which the "iterated norms"

$$||k(s,\sigma)|U(X,Y)|| = ||u(s)|Y|| \quad (u(s) = ||k(s,\cdot)|X||) \tag{14}$$

and

$$||k(s,\sigma)|V(X,Y)|| = ||v(\sigma)|Y|| \quad (v(\sigma) = ||k(\cdot,\sigma)|X||), \tag{15}$$

respectively, are finite; both sets $U(X,Y)$ and $V(X,Y)$ are ideal Banach spaces.

LEMMA 3. *Let X be an ideal space. Then $U(X',X)$ is continuously imbedded in $R(X,X)$ and*

$$||k(s,\sigma)|R(X,X)|| \le ||k(s,\sigma)|U(X',X)||. \tag{16}$$

LEMMA 4. *Let X be an (almost) perfect ideal space. Then $V(X,X')$ is continuously imbedded in $R(X,X)$ and*

$$||k(s,\sigma)|R(X,X)|| \le ||k(s,\sigma)|V(X,X')||. \tag{17}$$

Both lemmas are contained implicitly in [**48**], [**49**] for general ideal spaces; for special spaces (for example, Lebesgue and Orlicz spaces) they are of course well known.

In this connection, we mention that there exists a large literature on necessary and sufficient conditions for a linear operator A in L_p to be representable as a (Carleman) integral operator, i.e. $A \in L_i(L_p)$; such conditions are usually given in terms of one of the norms (14) and (15) (see, e.g., [**32**], [**37**], [**38**]).

We point out that the statements of Lemmas 3 and 4 are *sharp* in the sense that equality $U(X',X) = R(X,X)$ or $V(X,X') = R(X,X)$ may in fact occur; for instance, the first equality always holds in case $X = L_\infty$, the second one in case $X = L_1$. Another easily tractable case is that of *degenerate kernels*, i.e.,

$$k(s,\sigma) = \phi(s)\psi(\sigma) \quad (\phi \in Y, \psi \in X'). \tag{18}$$

Here we get immediately

$$u(s) = ||\psi|X'||\,|\phi(s)|, \quad v(\sigma) = ||\phi Y||\,|\psi(\sigma)|,$$

hence

$$||k(s,\sigma)|U(X,Y)|| = ||k(s,\sigma)|V(X,Y)|| = ||\phi|Y||\,||\psi|X'||,$$

and the estimate (16)/(17) is just Hölder's inequality. There is another feature in the general theory of integral operators which is worth mentioning. If the kernel function $k(s,\sigma)$ does not belong to the space $R^\circ(X,X)$, the corresponding operator (3) may be non-compact. However, it may happen that the operator (3) satisfies a *Darbo condition*

$$\alpha(AM) \leq k\alpha(M) \tag{19}$$

for some $k > 0$, where

$$\alpha(M) = \inf\{\eta : \eta > 0, M \text{ has a finite } \eta - \text{net in } X\}$$

denotes the *Hausdorff measure of noncompactness* (see, e.g., [**2**], [**44**]) in the space X. If the estimate (19) holds we write $A \in L_k^\alpha(X)$; in particular, $L_0^\alpha(X) \cap L_i(X) = L_{ic}(X)$.

Consider again, for example, the degenerate kernel function (18) in the case $X = Y = L_p (1 \leq p \leq \infty)$. Let

$$\alpha_{a,b}(t) = \Big[\int_t^b |\phi(s)|p_{ds}\Big]^{1/p}\Big[\int_a^t |\psi(\sigma)|^q d\sigma\Big]^{1/q},$$

where $1/p + 1/q = 1$, and let

$$\overline{\alpha} = \lim_{\varepsilon\to 0}[\sup_{a<t\leq a+\varepsilon}\alpha_{a,a+\varepsilon}(t) + \sup_{b-\varepsilon\leq t<b}\alpha_{b-\varepsilon,b}(t)].$$

Then the operator (3) belongs to $L_{ic}(L_p)$ if and only if $\overline{\alpha} = 0$; more generally, from estimates of C.A. Stuart and R.K. Juberg ([**35**], [**46**], see also [**1**]) it follows that $A \in L_k^\alpha(L_p) \backslash L_1^\alpha(L_p)$ for $k \geq p^{1/p}q^{1/q}\overline{\alpha}$ and $1 < (\frac{1}{2})^{1+1/p}\,\overline{\alpha}$.

We point out that the minimal constant k in (19) (which is sometimes denoted by $\alpha(A)$) is closely related to the *radius of the essential spectrum* of A (see [**43**]) and may be essentially smaller than the norm of A. The (singular) *Hilbert kernel*

$$k(s,\sigma) = \frac{1}{\pi}\frac{1}{s-\sigma}, \tag{20}$$

for example, gives rise to an integral operator (3) (with $c(s) \equiv 0$) whose norm $||A||_p$ in the space $X = L_p[-1,1](1 < p < \infty)$ satisfies the two-sided estimate

$$\begin{aligned} -\cot\frac{\pi}{p} \leq ||A||_p \leq \tan\frac{\pi}{2p} \quad & \text{if } 1 < p \leq \frac{4}{3}, \\ 1 \leq ||A||_p \leq \tan\frac{\pi}{2p} \quad & \text{if } \frac{4}{3} \leq p \leq 2, \\ 1 \leq ||A||_p \leq \tan\frac{\pi}{2p} \quad & \text{if } 2 \leq p \leq 4, \\ \cot\frac{\pi}{p} \leq ||A||_p \leq \cot\frac{\pi}{2p} \quad & \text{if } 4 \leq p < \infty. \end{aligned}$$

In particular, this norm is minimal for $p = 2$, and tends to infinity if either $p \to 1$ or $p \to \infty$ (see [**28**], [**30**]). On the other hand, A fulfills the estimate (19) (with $k = 1$) in any of the spaces L_p (see [**3**]), since A has only eigenvalues on the circumference $|\lambda| = 1$.

Let us return to the space $R(X,Y)$ defined in (13). In order to formulate more precise conditions for $k(s,\sigma)$ to belong to some kernel class $R(X,Y)$, we shall need a special construction which is useful in the general *interpolation theory* of linear operators (see, e.g., [**26**], [**41**], [**42**]). Given an ideal space Z, we denote by $Z_p(0 < p < \infty)$ the ideal space of all functions Z for which the norm

$$||z|Z_p|| = ||\,|z|^p|Z||^{1/p} \tag{21}$$

is finite; for example, $Z_p = L_p$ if $Z = L_1$.

LEMMA 5. *Let X be an ideal space, and choose numbers $k_0, k_1, l_0, l_1 \in (0, \infty)$ and $\lambda, \mu \in [0,1]$ such that X_{k_0} and X_{k_1} are Banach spaces and*

$$\frac{1-\lambda}{k_0} + \frac{\lambda}{k_1} = \frac{1-\mu}{l_0} + \frac{\mu}{l_1} = 1.$$

Let $U = U(X'_{k_0}, X_{l_0})$ and $V = V(X_{l_1}, X'_{k_1})$. Then the space $W = U_{(1-\mu)/(1-\lambda)} \cap V_{\mu/\lambda}$ is continuously imbedded in $R(X,X)$ and

$$||k(s,\sigma)|R(X,X)|| \leq ||k(s,\sigma)|U_{(1-\mu)/(1-\lambda)}||^{1-\lambda}||k(s,\sigma)|V_{\mu/\lambda}||^{\lambda}. \tag{22}$$

We remark that, apart from Lemma 5, there exist other conditions for $k(s,\sigma)$ to belong to $R(X,X)$, mostly for the case $X = L - p$ [**40**]. For general ideal spaces, however, very little is presently known in this direction.

The Lemmas 3-5 given above allow us not only to find appropriate spaces for $k(s,\sigma)$, but also to study the continuous dependence of $k(s,\sigma)$ of various parameters; in this connection, the basic estimates (16), (17) and (22) are particularly useful.

2. The resolvent operator. Let us return to the Barbashin-type integro-differential equation (1). The notion of a solution to this equation essentially depends on whether we consider it as an *integro-differential equation* or a *differential equation in a Banach space X* (the latter is possible, of course, if (2) defines an operator function with values in $L(X)$). This problem is discussed in some aspects in the monograph [**31**]. As usual, a *classical solution* of (1) in X is a continuously differentiable function with values in X which satisfies the equation (1) for all $t \in J$. A sufficient condition for the existence of such solutions, as is well known, is the (strong) continuity of the operator function (2) in the space $L(X)$. In this case, one may define the *resolvent operator* (or *Cauchy function*) $U(t,\tau)$ with values in $L(X)$ which depends continuously on $(t,\tau) \in J \times J$. This operator may be defined as a solution of the linear integral equation

$$U(t,\tau) = I + \int_{\tau}^{t} A(s)U(s,\tau)ds. \tag{23}$$

As a consequence, the existence of the Cauchy function follows from the strong continuity of the operator function (2) in the space under consideration.

To attack the problem of characterizing the strong continuity of (2) in the space C, for instance, we consider the functions

$$\alpha(t,s) = c(t,s) + \int_a^b k(t,s,\sigma)d\sigma, \tag{24}$$

$$\beta(t,z,s) = \int_a^z [c(t,s)\chi_s(\sigma) + (z-\sigma)k(t,s,\sigma)]d\sigma, \tag{25}$$

and

$$\gamma(t,s) = |c(t,s)| + \int_a^b |k(t,s,\sigma)|d\sigma \tag{26}$$

which are of course parallel to (4), (5), and (6). Lemma 1 and general results on operator valued functions imply the following

THEOREM 1. *The operator function* (2) *is strongly continuous in* $L(C)$ *if and only if the following two conditions are satisfied:*

(a) *the functions* $(t,s) \to \alpha(t,s)$ *and* $(t,z,s) \to \beta(t,z,s)$ *are continuous on* $J \times [a,b]$ *and* $J \times [a,b] \times [a,b]$, *respectively;*

(b) *for each* $T > 0$, *the function* $(t,s) \to \gamma(t,s)$ *is bounded on* $([-T,T] \cap J) \times [a,b]$.

Similarly, the operator function (2) *is norm continuous in* $L(C)$ *if and only if the following four conditions are satisfied:*

(a) *for each* $t \in J$ *and* $z \in [a,b]$, *the functions* $s \to \alpha(t,s)$ *and* $s \to \beta(t,z,s)$ *are continuous on* $[a,b]$;

(b) *for each* $t \in J$, *the function* $s \to \gamma(t,s)$ *is bounded on* $[a,b]$;

(c) *the function* $t \to c(t,s)$ *is continuous on* J, *uniformly with respect to* $s \in [a,b]$;

(d) *the function* $t \to k(t,s,\cdot)$ *is continuous from* J *into* L_1, *uniformly with respect to* $s \in [a,b]$.

Theorem 1 gives conditions for the existence of the Cauchy function in C which are both necessary and sufficient. In case of an ideal space X, the situation is much more complicated. We confine ourselves to indicating just one sufficient condition.

THEOREM 2. *Let X be an ideal space and suppose that, for each $t \in J$, the function $s \to c(t,s)$ belongs to L_∞ and the function $(s,\sigma) \to k(t,s,\sigma)$ to $R(X,X)$. Then the operator function* (2) *is strongly continuous in $L(X)$ if the following two conditions are satisfied:*

(a) *the function $t \to c(t,\cdot)$ is continuous from J into L_∞ (resp. the function $t \to c(t,\cdot)$ is locally bounded from J into L_∞, and continuous from J into L_1);*

(b) *the function $t \to k(t,\cdot,\cdot)$ is continuous from J into $R(L_1,X)$ (resp. the function $t \to k(t,\cdot,\cdot)$ is locally bounded from J into $R(X,X)$, and continuous from J into $R(L_\infty,X)$).*

Similarly, the operator function (2) *is norm continuous in $L(X)$ if the following two conditions are satisfied:*

(a) *the function $t \to c(t,\cdot)$ is continuous from J into L_∞;*

(b) *the function $t \to k(t,\cdot,\cdot)$ is continuous from J into $R(X,X)$.*

In contrast to Theorem 1, the conditions given in Theorem 2 are *only sufficient* in general. However, they become also necessary if $L(X)$ is replaced by $L_r(X)$. Consequently, in case $L(X) = L_r(X)$ Theorem 2 gives necessary and sufficient conditions; this holds, as already observed, for $X = L_1$ or $X = L_\infty$.

The verification of the hypotheses of Theorem 2 is, generally speaking, not difficult, except for the assertions concerning the kernel space $R(X,X)$. Here the results contained in Lemmas 2 - 5 are of course useful to guarantee, say, the continuity of the map $t \to k(t,\cdot,\cdot)$ from J into $R(X,X)$.

Let us return to the resolvent operator $U(t,\tau)$. By definition (2), the operator $A(t)$ is for each $t \in J$ the sum of a *multiplication operator*

$$C(t)x(s) = c(t,s)x(s) \tag{27}$$

and an *integral operator*

$$K(t)x(s) = \int_a^b k(t,s,\sigma)x(\sigma)d\sigma. \tag{28}$$

It seems therefore reasonable to expect that also the Cauchy function $U(t,\tau)$ is the sum of a "multiplicative part" and an "integral part". It turns out that, under the hypotheses of Theorems 1 and 2, one can prove even more; in fact, the integral part of $U(t,\tau)$ shares many properties with $K(t)$. We call a Banach algebra M of integral operators an X-*ideal* if the following holds: in case $X = C$, M is a Banach ideal in $L_0(C)$ and continuously imbedded into $L_i(C)$; in case X is an ideal space, M is a Banach ideal in $L_r(X)$ and continuously imbedded into $L_{ri}(X)$.

For instance, the sets $L_i(C), L_{ic}(C)$ and $L_{iw}(C)$ introduced above are C-ideals. Similarly, in case X is an ideal space, the sets $L_{ri}(X), L_{ric}(X)$ and $L_{riw}(X)$ are X-ideals; other examples in this case are operators with kernels from $U(X',X), V(X,X')$, or even $U(X'_{k_0}, X_{l_0})_{(1-\mu)/(1-\lambda)}$ $\cap V(X_{l_1}, X'_{l_0})_{\mu/\lambda}$ (see Lemmas 3-5). On the other hand, the linear subspace $L_k^\alpha(X)$ of $L_0(X)$ (see (19)) is not an X-ideal in general.

THEOREM 3. *Suppose that the hypotheses of either Theorem* 1 *or Theorem* 2 *hold. Then the resolvent operator* $U(t,\tau)$ *may be represented as a sum*

$$U(t,\tau) = V(t,\tau) + W(t,\tau) \tag{29}$$

of a multiplication operator

$$V(t,\tau)x(s) = v(t,\tau,s)x(s) \tag{30}$$

and an integral operator

$$W(t,\tau)x(s) = \int_a^b w(t,\tau,s,\sigma)x(\sigma)d\sigma. \tag{31}$$

Moreover, if the integral part $K(t)$ *of* $A(t)(-\infty < t < \infty)$ *belongs to some* X-*ideal* M, *then the integral part* $W(t,\tau)$ *of* $U(t,\tau)(-\infty < t,\tau < \infty)$ *also belongs to* M.

We remark that the explicit form of the multiplication part (30) is always known; simply

$$v(t,\tau,s) = \exp \int_\tau^t c(\xi,s)d\xi. \tag{32}$$

Moreover, it follows from the integral representation [**27**]

$$U(t,\tau) = I + \sum_{n=1}^{\infty} \int_{\tau}^{t} \int_{\tau}^{t_1} \cdots \int_{\tau}^{t_{n-1}} A(t_1)A(t_2)\ldots A(t_n)dt_1 \cdots dt_n$$

that one may also obtain an analytic expression for the kernel $w(t,\tau,s,\sigma)$ of the integral part (31) in terms of a series which converges in the norm of M (in particular, in the norm of $U(L_1, L_\infty)$ if $X = C$, and in the norm of $R(X,X)$ if X is an ideal space).

As an application of Theorem 3, we consider the Floquet - Lyapunov theory for equation (1) in the case when J is the whole real line and the operator (2) is ω-periodic in t. As a matter of fact [27], this theory applies if and only if the logarithm of the operator $U(\omega,0)$ may be defined; this is the case, in particular, if the spectrum $\sigma(U(\omega,0))$ of $U(\omega,0)$ is bounded away from 0 and ∞. Now, if we suppose in addition that $K(t) \in L_{ic}(X)$ for all $t \in \mathbf{R}$, all limit points of $\sigma(U(\omega,0))$ coincide with the closure of the essential range of the function $v(\omega,0,s)$ (see (32)); since this set is contained in the positive real half-axis, the entire spectrum of $U(\omega,0)$ is in fact bounded away from 0 and ∞, and thus the Floquet - Lyapunov theory applies. Similar results on the stability of $A(t)$ will be given in the following section.

3. Some stability results. Suppose now that J is the whole axis $(-\infty,\infty)$, or the half-axis $(0,\infty)$. In this section we shall be concerned with stability results for the trivial (and hence any) solution of the integro-differential equation (1), considered as a linear differential equation in some Banach space X. As is well known [**27**], a basic notion is here the so called *Lyapunov - Bohl exponent* $\gamma[A(t)]$ of the operator (2) defined by

$$\gamma[A(t)] = \overline{\lim_{t,\tau,t-\tau\to\infty}} \frac{\log||U(t,\tau)||}{|t-\tau|} \tag{33}$$

More precisely, the inequality $\gamma[A(t)] \leq 0$ is necessary, and the strict inequality $\gamma[A(t)] < 0$ is sufficient for the stability of the trivial solution of (1).

The problem of calculating the Lyapunov - Bohl exponent (33) is very difficult, even in the case of a finite dimensional system. If the operator

(2) does not depend on t, however, $\gamma[A]$ is just the least upper bound of the real part of the spectrum $\sigma(A)$, i.e.,

$$\gamma[A] = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}. \tag{34}$$

In our case, this applies if A is of the *stationary* form (3), i.e., the functions $c(s)$ and $k(s, \sigma)$ are independent of t. If we suppose again that $K \in L_{ic}(X)$, the representation formula (29) yields the important estimate

$$\gamma[A] \geq \operatorname*{ess\,sup}_{a \leq s \leq b} c(s) \tag{35}$$

From (35) it follows, in turn, that the inequality $c(s) \leq 0$ is necessary for the stability of the trivial solution of (1). The strict inequality $c(s) < 0$ alone is, of course, not sufficient for stability; one must require in addition that all eigenvalues of A have strictly negative real part.

A fairly convenient sufficient stability condition which generalizes corresponding conditions from [**5**], [**8**], [**10**], [**11**], [**15**] may be obtained by means of *perturbation techniques* for the Lyapunov - Bohl exponent [**27**] which are based on the following observation: if the operator (2) does not contain an integral part (28), but just reduces to the multiplication operator (27), its Lyapunov - Bohl exponent is simply

$$\gamma[C(t)] = \overline{\lim_{t,\tau,t-\tau\to\infty}} \frac{1}{|t-\tau|} \operatorname*{ess\,sup}_{a \leq s \leq b} \int_\tau^t c(\xi, s)d\xi. \tag{36}$$

This leads to

THEOREM 4. *Suppose that the hypotheses of either Theorem* 1 *or Theorem* 2 *hold. Assume, moreover, that*

$$\begin{aligned} &\overline{\lim_{t,\tau,t-\tau\to\infty}} \frac{1}{|t-\tau|} \operatorname*{ess\,sup}_{a \leq s \leq b} \int_\tau^t c(\xi, s)d\xi \\ &+ \overline{\lim_{t,\tau,t-\tau\to\infty}} \frac{1}{|t-\tau|} \int_\tau^t ||K(\xi)|X||d\xi < 0. \end{aligned} \tag{37}$$

Then the trivial solution of the integro - differential equation (1) *is stable in the space* X.

The above approach to the investigation of stability properties of (1) is closely related to the so called *first Lyapunov method.* In the case of a finite dimensional system, the *second Lyapunov method* (which builds on certain properties of the Lyapunov function) is equally important. However, in case of integro - differential equations like (1), the second Lyapunov method seems to be far less suitable, since equation (1) exhibits various typical "infinite dimensional features". On the one hand, this leads to serious difficulties in the study of Lyapunov functions which are hard to overcome, at least by presently available techniques. On the other hand, the set of Lyapunov functions themselves is essentially reduced in this case. The only exception is the *Hilbert space case*, where almost all results carry over from the finite dimensional case (see, e.g., [**27**]).

We point out that specific properties of Lyapunov functions related with integro - differential equations like (1) are practically unknown and should merit a more detailed investigation in future.

In order to illustrate this situation for equation (1) we restrict ourselves to the *stationary case*, i.e., when A has the form (3). Recall that the *Lyapunov function* (resp. the *strong Lyapunov function*) for equation (1) in a space X is a real functional ϕ on X such that $\phi(x) > 0$ for $x \neq 0$, $\phi(x) \to 0$ implies $||x|X|| \to 0$, and $V(x, Ax) \leq 0$ (resp. $V(x, Ax) \leq 0$ and $V(x, Ax) \to 0$ implies that $||x|X|| \to 0$), where

$$V(x,h) = \overline{\lim_{t,z\to 0}} \frac{1}{t}[\phi(x + t(h + z)) - \phi(x)].$$

The basic Lyapunov theorem states that the existence of a Lyapunov function (resp. strong Lyapunov function) for equation (1) implies the stability (resp. asymptotic stability) of the trivial solution of (1).

From the definition of a Lyapunov function it is clear that its existence depends essentially on geometric properties of the function space X (in particular, the existence of smooth functionals on X). If one can find a functional ϕ on X satisfying the first two conditions given above, the problem arises to describe all linear operators A in X (if there are any) for which the third condition is also satisfied. Such a description, however, is easy to carry out [**27**] only in case X is the Hilbert space L_2 and $\phi(x) = (Tx, x)$, where T is some uniformly positive selfadjoint operator on X. The condition $V(x, Ax) \leq 0$ means in this case that the operator $TA^* + A^*T$ is negative definite or, equivalently, that

Re $(Tx, Ax) \leq 0$ for all $x \in X$. In the special case when $T = I$ and A is defined by (3), this condition may be written in the form

$$\int_a^b c(s)x^2(s)ds + \int_a^b \int_a^b k_{sym}(s,\sigma)x(\sigma)x(s)d\sigma ds \leq 0 \tag{38}$$

($x \in L_2$), where $k_{sym}(s,\sigma) = \frac{1}{2}[k(s,\sigma) + k(\sigma, s)]$ is the *symmetrization* of the kernel $k(s,\sigma)$. Condition (38) holds, in turn, if $c(s) \leq -\lambda_0 < 0$, and the spectrum $\sigma(K_{sym})$ of the integral operator $K_{sym} = \frac{1}{2}(K+K^*)$ defined by the kernel $k_{sym}(s,\sigma)$ lies entirely on the left side of λ_0. Similar criteria may be obtained in the case when T is a uniformly positive definite operator in $L_0(C)$ or $L_r(X)$ with X being an ideal space.

4. The Green's Function The Theorems 1 - 3 allow us to extend the basic results on linear differential equations (such as the Lagrange formula for the solution of inhomogeneous equations, several results on the continuous dependence of the Cauchy function on parameters, existence and representation theorems for the Green's function for boundary value problems, or existence results for ω-periodic and other bounded solutions) to integro-differential equations of Barbashin type. In what follows, we give two results which are particularly useful for applying the *Bogolyubov - Krylov averaging principle* to (linear and nonlinear) Barbashin - type integro - differential equations.

THEOREM 5. *Suppose that the operator* (3) *belongs to* $L_0(X)$ *for* $X = C$ *(resp. to* $L_r(X)$ *for* X *being an ideal space). Assume that the essential range of the function* $c(s)$ *does not contain* 0, *and that the operator* A *has neither* 0 *as eigenvalue, nor purely imaginary eigenvalues. Then the differential equation* $\frac{du}{dt} = Au$ *admits a Green's function* $G(t,\tau) \in L_0(C)$ *(resp.* $G(t,\tau) \in L_r(X)$*) for bounded solutions, and* $G(t,\tau)$ *may be represented as a sum*

$$G(t,\tau) = G_0(t,\tau) + H(t,\tau) \quad (-\infty < t, \tau < \infty) \tag{39}$$

of a multiplication operator

$$G_0(t,\tau)x(s) = g_0(t,\tau,s)x(s) \tag{40}$$

and an integral operator

$$H(t,\tau)x(s) = \int_a^b h(t,\tau,s,\sigma)x(\sigma)d\sigma. \tag{41}$$

Moreover, the function $g_0(t,\tau,s)$ *in* (40) *is given explicitly by*

$$g_0(t,\tau,s) = \begin{cases} \exp c(s)(t-\tau) & \text{if } c(s)(t-\tau) < 0, \\ 0 & \text{if } c(s)(t-\tau) > 0. \end{cases} \tag{42}$$

Finally, if the integral part K *of* A *(see* (28)*) belongs to some* X*-ideal* M*, then the integral part* $H(t,\tau)$ *of* $G(t,\tau)$ $(-\infty < t,\tau < \infty)$ *belongs also to* M.

We point out that, if the integral part K of A is compact, the conditions given in Theorem 5 are also *necessary* for the existence of the Green's function.

THEOREM 6. *Suppose that the hypotheses of either Theorem* 1 *or Theorem* 2 *hold for the operator* (2), *and those of Theorem* 5 *hold for the operator* (3). *Moreover, assume that*

$$\lim_{T\to\infty}\text{sum}_{-\infty<t<\infty}\left|\left|\frac{1}{T}\int_t^{t+T} c(\xi,x)d\xi - c(s)|L_\infty\right|\right| = 0 \tag{43}$$

and

$$\lim_{T\to\infty}\sup_{-\infty<t<\infty}\left|\left|\frac{1}{T}\int_t^{t+T} k(\xi,s,\sigma) - k(s,\sigma)|N\right|\right| = 0, \tag{44}$$

where $N = U(L_1, L_\infty)$ *in case* $X = C$*, and* $N = R(X,X)$ *in case* X *is an ideal space. Then for sufficiently small* ε *the differential equation* $\frac{du}{dt} = \varepsilon A(t)u$ *admits a Green's function* $G(\varepsilon,t,\tau) \in L_0(C)$ *(resp.* $G(\varepsilon,t,\tau) \in L_r(X)$*) for bounded solutions, and* $G(\varepsilon,t,\tau)$ *may be represented as a sum*

$$G(\varepsilon,t,\tau) = G_0(\varepsilon,t,\tau) + H(\varepsilon,t,\tau) \tag{45}$$

of a multiplication operator

$$G_0(\varepsilon,t,\tau)x(s) = g_0(\varepsilon,t,\tau,s)x(s) \tag{46}$$

and an integral operator

$$H(\varepsilon,t,\tau)x(s) = \int_a^b h(\varepsilon,t,\tau,s,\sigma)x(\sigma)d\sigma. \tag{47}$$

Moreover, the function $g_0(\varepsilon,t,\tau,s)$ *in* (46) *is given explicitly by*

$$g_0(\varepsilon,t,\tau,s) = \begin{cases} \exp\, \varepsilon \int_\tau^t c(\xi,s)d\xi & \text{if } c(s)(t-\tau) < 0, \\ 0 & \text{if } c(s)(t-\tau) > 0. \end{cases} \tag{48}$$

Finally, if the integral part $K(t)$ *of* $A(t)(-\infty < t < \infty)$ *belongs to some* X*-ideal* M*, then the integral part* $H(\varepsilon,t,\tau)$ *of* $G(\varepsilon,t,\tau)$ $(-\infty < t,\tau < \infty)$ *belongs also to* M*.*

We remark that the proof of both Theorems 5 and 6 follows the idea of [51]. Under the hypotheses of Theorem 6, the Green's function $G(\varepsilon,t,\tau)$ for the equation

$$\frac{du}{dt} = A(\frac{t}{\varepsilon})u$$

tends uniformly to the Green's function $G(t,\tau)$ for the equation

$$\frac{du}{dt} = Au$$

as $\varepsilon \to 0$.

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