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INVERSE SCATTERING FOR SCATTERING DATA WITH POOR REGULARITY OR SLOW DECAY

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1. Introduction. Motivation to study the inverse scattering problem for scattering data with poor regularity or slow decay is an application which will be given in two subsequent papers [**7, 21**] for the Cauchy problem of the Korteweg-deVries equation (KdV) $u_t - 6uu_x + u_{xxx} = 0$ with irregular initial profile as, e.g., a smooth enough box shaped potential or a steplike a smoothed Heavyside function [**4,5**].

If we consider $u(x)$ as a potential for the Schrödinger equation $-y''(x) + u(x)y(x) = k^2y(x)$ we can associate to u, by a well known procedure [**8,9**], the scattering data of which a part is given by the so called scattering matrix (T_+, R_+, T_-, R_-). To find a solution $u(x,t)$ of the KdV $(t > 0)$ it is enough to study the evolution of the scattering in time and to construct $u(x,t)$ by the inverse problem [**3, 4, 5, 7, 10, 11, 12, 13**]. Often, however, the evolution of the scattering data, especially R_-, does not stay within the set where the inverse problem was known to be solvable [**4, 5**].

Let us briefly outline the organization of the paper. In §2 we discuss the Marchenko equation in $L_2(R_-)$. In §3 we study the inverse scattering problem under weaker decay and regularity properties of R_- and its Fourier transform than in [**8, 9**].

Let us introduce the following notation. Let f be a complex valued function defined on $\mathbf{R}$. By $\tau_x f$ we denote the translated function $\tau_x f(y) := f(x+y)$ (x and y in $\mathbf{R}$). If $h \neq 0$ we denote by $\Delta_h f$ the differential quotient $(\Delta_h f)(x) := \frac{(f(x+h)-f(x))}{h}$.

Let f be in $L_2(\mathbf{R})$. By $\hat{f}$ we denote the Fourier transform $\hat{f}(k) := \int_{-\infty}^{\infty} f(x)e^{2ikx}dx$. By $\tau_x f$ we define the operator on $L_2(\mathbf{R}_-)$ defined by

$$\tau_x f(g)(y) := \int_{-\infty}^{0} \tau_z f(x+z)g(z)dz \quad (g \text{ in } L_2(\mathbf{R}_-)).$$

By $'$ or ∂_x we denote the derivation with respect to x. For a complex number a we denote by a^* its complex conjugate. By H_2^+ we denote the Hardy space of all functions which are analytic in the upper half plane such that the supremum of the L_2-norms over lines of constant imaginary part is finite. By $L_2(-\infty)$ we denote the Fréchet space consisting of functions $f : \mathbf{R} \to \mathbf{R}$ with f in $L_2((-\infty, a))$ for all a in $\mathbf{R}$.

Finally let us remark that this paper is closely related to [**6**]. There, correcting results of [**2**], a characterization of a certain class of steplike potentials in terms of the corresponding scattering data is given. As in [**6**] we follow an approach to the inverse problem which is due to Faddeev [**1, 9**].

2. The Marchenko equation in $L_2(\mathbf{R}_-)$. Let us first restate a result due to Agranovich and Marchenko [**1**; Lemma 3.3.3, p. 73].

LEMMA 2.1. (AGRANOVICH and MARCHENKO). *Let $n \geq 0$ be given. Let f be a real valued function in $L_2(\mathbf{R})$ such that its Fourier transform $\hat{f}$ is continuous, $(2k)^n \hat{f}$ is in $L_2(\mathbf{R})$ and $\lim_{|k| \to \infty} (2k)^n \hat{f}(k) = 0$. Then*

1) $\underline{\tau_x(f^{(\ell)})}$ is a compact operator from $L_2(\mathbf{R}_-)$ to $L_2(\mathbf{R}_-)$ for $0 \leq \ell \leq n$.

2) $\lim_{h \to 0} \underline{\tau_x(\Delta_h f^{(\ell-1)})} = \underline{\tau_x f^{(\ell)}} (1 \leq \ell \leq n)$ in operator norm.

The following result can be proved by standard methods.

LEMMA 2.2. *Let $n \geq 1$ be given. Let f be a real-valued function in $L_2(\mathbf{R})$ such that its Fourier transform $\hat{f}(k)$ is continuous and $\lim_{|k| \to \infty} \hat{f}(k) = 0$. If $f^{(\ell)}(x)$ and $|x|^{1/2} f^{(\ell)}(x)$ are in $L_2(-\infty)$ for $1 \leq \ell \leq n$, then*

1) $\underline{\tau_x f^{(\ell)}}$ is a compact operator from $L_2(\mathbf{R}_-)$ to $L_2(\mathbf{R}_-)$ for $0 \leq \ell \leq n$, and

2) $\lim_{h \to 0} \underline{\tau_x(\Delta_h f^{(\ell-1)})} = \underline{\tau_x f^{(\ell)}} (1 \leq \ell \leq n)$ in operator norm.

Let f be in $L_2(\mathbf{R})$. f is said to have property P_N ($\mathbf{N}$ in $\mathbf{N}$) if the following conditions are satisfied:

There exists a decomposition $f = f_1 + f_2$ of f in $L_2(\mathbf{R})$ such that

1) for $i = 1, 2$, f_i is in $L_2(\mathbf{R})$, $\hat{f}_i$ is continuous and $\lim_{|k|\to\infty} \hat{f}_i(k) = 0$;
2) $k^N \hat{f}_1$ is in $L_2(\mathbf{R})$ and $\lim_{|k|\to\infty} k^N \hat{f}_1(k) = 0$; and
3) $f_2^{(n)}$ and $|x|^{1/2} f_2^{(n)}(x)$ are in $L_2(-\infty)(n = 1, \ldots, N)$.

Let F_- be an element in $L_2(\mathbf{R})$ with property P_N where $N \geq 1$ such that its Fourier transform $R_-(k) := \int_{-\infty}^{\infty} F_-(x) e^{2ikx} dx$ satisfies $|R_-(k)| \leq 1$ and $R_-(k) = 0(\frac{1}{k})$ as $|k| \to \infty$. Let $(c_{-j})_{j\in J}$ be real numbers where J is a finite set. Now let us introduce the function

$$\Omega_-(s) := F_-(s) + 2 \sum_{j\in J} c_{-j} e^{2\kappa_j s}.$$

Then $\underline{\tau_x \Omega_-}$ is a compact operator from $L_2(\mathbf{R}_-)$ to $L_2(\mathbf{R}_-)$. For any x in $\mathbf{R}$ and $y \leq 0$ let us consider the Marchenko equation

$$0 = B_-(x, y) + \Omega_-(x + y) + \int_{-\infty}^{0} B_-(x, z) \Omega_-(x + y + z) dz.$$

In operator form this equation can be written as an equation in $L_2(\mathbf{R}_-)$ in the following way: -

$$(\mathrm{Id} + \underline{\tau_x \Omega_-}) B_-(x, \cdot) = \tau_x \Omega_-,$$

where Id denotes the identity operator.

In the same way as Faddeev [**9**] did, one proves

PROPOSITION 2.3. *The homogeneous equation* $(\mathrm{Id} + \underline{\tau_x \Omega_-})h = 0$ *in* $L_2(\mathbf{R}_-)$ *has only the trivial solution* $h = 0$.

From Proposition 2.3 and from the compactness of $\underline{\tau_x \Omega_-}$, it follows that, for any x in $\mathbf{R}$, the Marchenko equation has a unique solution $B_-(x, \cdot)$ in $L_2(\mathbf{R}_-)$. To derive some properties for $B_-(x, \cdot)$ we need the following lemma which can be easily proved.

LEMMA 2.4. *$\underline{\tau_x \Omega_-}$ and thus $(\mathrm{Id} + \underline{\tau_x \Omega_-})^{-1}$ is a family of operators from $L_2(\mathbf{R}_-)$ to $L_2(\mathbf{R}_-)$ which depends continuously on the parameter x. Moreover there exists a nondecreasing constant $C(x)$ such that, in operator norm $\|(\mathrm{Id} + \underline{\tau_x \Omega_-})^{-1}\| \leq C(x)$ and $\lim_{x\to-\infty} \|\underline{\tau_x \Omega_-}\| = 0$.*

From Lemma 2.4 it follows that there exists a non decreasing function $C(x)$ such that

$$\|B_-(x,\cdot)\|_{L_2(\mathbf{R}_-)} \leq C(x).$$

PROPOSITION 2.5. *Let $N \geq 1$ be given. Let F_- have property P_N for some $N \geq 1$. Then $\partial_x^n B_-(x,\cdot)$ is in $L_2(\mathbf{R}_-)$ for $0 \leq n \leq N$. Further there exists a non decreasing function $C(x)$ such that*

$$\|\partial_x^n B_-(x,\cdot)\|_{L_2(\mathbf{R}_-)} \leq C(x) \quad 0 \leq n \leq N.$$

PROOF. Clearly $B_-(x,\cdot)$ is in $L_2(\mathbf{R}_-)$. We prove the statement only for $N = 1$. An inductive argument gives easily the conclusion for arbitrary N.

Let us start from the Marchenko equation

$$0 = B_-(x,y) + \Omega_-(x+y) + \int_{-\infty}^{0} B_-(x,z)\Omega_-(x+y+z)dz.$$

For $h \neq 0$ we have, in $L_2(\mathbf{R}_-)$,

$$(\mathrm{Id} + \underline{\tau_{x+h}\Omega_-})\,\frac{B_-(x+h,\cdot) - B(x,\cdot)}{h} = \Phi_h(x,\cdot),$$

where

$$\Phi_h(x,y) := \frac{\Omega_-(x+y+h) - \Omega_-(x+y)}{h} + \int_{-\infty}^{0} B_-(x,z)\frac{\Omega_-(x+y+z+h) - \Omega_-(x+y+z)}{h}dz.$$

Let us define

$$\Phi_0(x,y) := \Omega'_-(x+y) + \int_{-\infty}^{0} B_-(x,z)\Omega'_-(x+y+z)dz.$$

By Lemma 2.1 and Lemma 2.2 one concludes that

$$\lim_{h\to 0} \Phi_h(x,\cdot) = \Phi_0(x,\cdot) \quad \text{in} \quad L_2(\mathbf{R}_-).$$

Together with Lemma 2.4 we get

$$\begin{aligned}\partial_x B_-(x,\cdot) &= \lim_{h\to 0} \frac{B_-(x+h,\cdot) - B_-(x,\cdot)}{h} \\ &= \lim_{h\to 0} (\mathrm{Id} + \underline{\tau_{x+h}\Omega_-})^{-1}\Phi_h(x,\cdot) \\ &= (\mathrm{Id} + \underline{\tau_x\Omega_-})^{-1}\Phi_0(x,\cdot).\end{aligned}$$

Clearly there exists a non decreasing function $C(x)$ such that

$$||\partial_x B_-(x,\cdot)||_{L_2(\mathbf{R}_-)} \leq C(x).$$

In a similar way one can prove

PROPOSITION 2.6. *Let $N \geq 1$ be given. If F_- has property P_N for some $N \geq 1$, then $\partial_x^n B_-(x,\cdot)$ is in $H^{N-n}(\mathbf{R}_-)$ where $0 \leq n \leq N$ (i.e., $\partial_y^k\partial_x^n B_-(x,\cdot) \in L_2(\mathbf{R}_-)$ for $0 \leq k \leq N-n$).*

PROPOSITION 2.7. *Let F_- have property P_1. Let $\alpha \geq 1/2$ be given such that, for all a in $\mathbf{R}$*

$$\int_{-\infty}^{a} (a-x)^{2\alpha}|\Omega_-'(x)|^2 dx < \infty.$$

Then

$$\int_{-\infty}^{a} (a-x)^{2\alpha}|\partial_x B_-(x,0)|^2 dx < \infty.$$

PROOF. For the sake of convenience let us drop all subscripts "$-$". From the Marchenko equation it follows that, for $y \leq 0$ a.e.,

$$(*) \qquad \begin{aligned}\partial_x B(x,y) = &-\Omega'(x+y) - \int_{-\infty}^{0} \partial_x B(x,z)\Omega(x+y+z)dz \\ &- \int_{-\infty}^{0} B(x,z)\Omega'(x+y+z)dz.\end{aligned}$$

For x in $\mathbf{R}$ we have the estimate

$$|\Omega(x)|^2 \leq \int_{-\infty}^{x} |\Omega'(u)|^2 du.$$

There exists a non decreasing function $C(x)$ such that

$$||B(x,\cdot)||_{L_2(\mathbf{R}_-)} \leq C(x).$$

Using the Marchenko equation we estimate $|B(x,z)|^2$ by

$$\begin{aligned}|B(x,z)|^2 \leq & 2\int_{-\infty}^{x+z} |\Omega'(u)|^2 du \\ & + 2(\sup_{z\leq 0} |\Omega(x+z)|^2) \int_{-\infty}^{0} |B(x,z)|^2 dz \\ \leq & 2(1+C(x)) \int_{-\infty}^{x+z} |\Omega'(u)|^2 du.\end{aligned}$$

Now let us estimate $||\partial_x B(x,\cdot)||_{L_2(\mathbf{R}_-)}$. There exists a non decreasing function $C(x)$ such that

$$\begin{aligned}&||\partial_x B(x,\cdot)||^2_{L_2(\mathbf{R}_-)} \\ \leq & C(x)(||\tau_x \Omega'||^2_{L_2(\mathbf{R}_-)} + \sup_{z\leq 0} |B(x,z)|^2 \int_{-\infty}^{0} dy \int_{-\infty}^{0} |\Omega'(x+y+z)|^2 dz) \\ \leq & C(x) \int_{-\infty}^{x} |\Omega'(u)|^2 du (1 + C(x) \int_{-\infty}^{0} dy \int_{-\infty}^{0} |\Omega'(x+y+z)|^2 dz).\end{aligned}$$

This estimate can be used to get

$$\begin{aligned}&\left| \int_{-\infty}^{0} \partial_x B(x,z) \Omega(x+z) dz \right|^2 \\ & \leq \sup_{z\leq 0} |\Omega(x+z)|^2 \int_{-\infty}^{0} |\partial_x B(x,z)|^2 dz \\ & \leq C(x) \Big(\int_{-\infty}^{x} |\Omega'(u)|^2 du \Big)^2,\end{aligned}$$

where again $C(x)$ is a non decreasing function. Finally let us give the estimate

$$\Big|\int_{-\infty}^{0} B(x,z)\Omega'(x+z)dz\Big|^2$$
$$\leq \sup_{z\leq 0}|B(x,z)|^2\int_{-\infty}^{0}|\Omega'(x+z)|^2dz \leq C(x)\Big(\int_{-\infty}^{0}(\Omega'(x+z)|^2dz\Big)^2.$$

Using $(*)$ and the derived estimates, we conclude that there exists a non decreasing function $C(x)$ such that

$$|\partial_x B(x,0)|^2 \leq C(x)\Big(\int_{-\infty}^{x}|\Omega'(z)|^2dz\Big)^2.$$

So, for $x \leq -1$, we get

$$\int_{-\infty}^{x}|s|^{2\alpha}|\partial_s B(s,0)|^2ds$$
$$\leq C(x)\int_{-\infty}^{x}ds|\,s\,|^{2\alpha}\int_{-\infty}^{s}|\,\Omega'(z)\,|^2\,dz\int_{-\infty}^{s}|\Omega'(z)\,|^2\,dz$$
$$\leq C(x)\int_{-\infty}^{x}|\,z\,|^{2\alpha}|\,\Omega'(z)\,|^2dz\int_{-\infty}^{x}ds\int_{-\infty}^{s}|\,\Omega'(z)\,|^2dz$$

and the statement follows. □

Let us now introduce, for $\operatorname{Im} k > 0$, the functions

$$h_-(x,k) : 1 + \int_{-\infty}^{0} B_-(x,y)e^{-2iky}dy$$

and

$$f_-(x,k) := e^{-ikx}h_-(x,k).$$

One proves, in a similar way to what Faddeev **[9]** has done, the following two propositions.

PROPOSITION 2.8. *Let F_- have property P_1. Then*

1) *$f_-(x,k)$ is analytic in* $\operatorname{Im} k > 0$.

2) *In* $\operatorname{Im} k > 0, f_-(x,k)$ *is continuously differentiable with respect to* x.

3) $\lim_{x\to-\infty} f_-(x,k)e^{ikx} = 1$ *and* $\lim_{x\to-\infty} \partial_x f_-(x,k)e^{ikx} = -ik$ *in* $\operatorname{Im} k > 0$.

4) $h_-(x,k) - 1 = O(\frac{1}{k})$ *for* x *in* $\mathbf{R}$ *and* $\operatorname{Im} k > 0, |k| \to \infty$. *The decay is uniform for* $\operatorname{Im} k > \varepsilon$, *where* $\varepsilon > 0$ *is arbitrarily small.*

PROPOSITION 2.9. *Let* F_- *have property* P_1. *Then* $f_-(x,k)$ *is, for* $\operatorname{Im} k > 0$, *a solution of the Shrödinger equation* $-y''(x) + q(x)y(x) = k^2 y(x)$, *where* $q(x) := \partial_x B_-(x,0)$.

3. Inverse Scattering in $L_2(\mathbf{R})$. Let F_- have a property P_1 and Ω_- be given as in the last section. Let T_- as well as R_- be functions which fulfill the conditions of the following theorem. With the notation of the last section let us introduce the following function defined a.e., in $\operatorname{Im} k = 0$:

$$g_-(x,k) := \{h_-^*(x,k) + R_-(k)e^{-2ikx}h_-(x,k)\}\frac{1}{T_-(k)}.$$

Then $T_-(k)g_-(x,k)$ can be represented by

$$\begin{aligned} T_-(k)g_-(x,k) = 1 + \int_0^\infty &\{F_-(x+y) \\ &+ \int_{-\infty}^0 B_-(x,z)F_-(x+y+z)dz\}e^{2iky}dy \\ &+ i\sum_{j\in J} c_{-j}e^{2\kappa_j x}h_-(x,i\kappa_j)\frac{1}{k - i\kappa_j}. \end{aligned}$$

As $T_-(k)$ has simple poles at $k = i\kappa_j$ and $T_-(k) \neq 0$ in $\operatorname{Im} k > 0$ one concludes that $g_-(x,k)$ is analytic in $\operatorname{Im} k > 0$ and that $\prod_{j\in J}((k - i\kappa_j)/(k + i\kappa_j))(T_-(k)g_-(x,k) - 1)$ is in H_2^+.

THEOREM 3.1. (Uniqueness). *Let the following conditions be met:*

X1) R_- *is a continuous complex valued function defined on the whole of* $\mathbf{R}$ *with the following properties:*

(i) $R_-^*(k) = R_-(-k)$,

(ii) $R_-(k) = O(\frac{1}{k})$ *as* $|k| \to \infty$,

(iii) $F_-(x) := \frac{1}{\pi}\int_{-\infty}^{\infty} R_-(k)e^{-2ikx}dx$ *has property* P_1.

X2) T_- *is meromorphic in* $\operatorname{Im} k > 0$, *has only simple poles, of the form all* $k = i\kappa_j$ *where* $\kappa_j > 0$ *and* j *is in the finite set* J. T_- *is continuous in* $\{\operatorname{Im} k \geq 0\}\backslash\{0\}$, *and, for* $k \neq 0, T_-(k) \neq 0$.

Then any pair of functions (h, g) *which satisfies* 1) *to* 5) *as stated below must be identical with* (h_-, g_-), *i.e., for* x *in* $\mathbf{R}$ *and* $\operatorname{Im} k > 0$,

$$h(x,k) = h_-(x,k) \text{ and } g(x,k) = g_-(x,k).$$

The conditions 1) *to* 5) *are:*

1) $h(x,\cdot) - 1$ *is in* $H_2^+(x$ *in* $\mathbf{R})$.

2) $g(x,k)$ *is analytic in* $\operatorname{Im} k > 0$ *and* $(T_-(k)_g(x,\kappa) - 1)\prod_{j\in J}\frac{k-i\kappa_j}{k+i\kappa_j}$ *is in* H_2^+.

3) $g(x,k) = 1 + 0(\frac{1}{k})$ *for* $|k| \to \infty$ with $\operatorname{Im} k > 0$ *and the decay is uniform in* $\operatorname{Im} k \geq \varepsilon$ *for any* $\varepsilon > 0$.

4) Res $(T_-(k)g(x,k); i\kappa_j) = ic_{-j}e^{2\kappa_j x}h(x, i\kappa_j)$ (j in J).

5) *For any fixed* x *in* $\mathbf{R}$, $T_-(k)g(x,k) - 1$ *and* $h(x,-k) - 1 + R_-(k)e^{-2ikx}h(x,k)$ *are in* $L_2(\mathbf{R})$ *as functions of the real variable* k *and are equal.*

PROOF. Let (h, g) be such a pair. By 1) there exists $A(x,\cdot)$ in $L_2(\mathbf{R}_-)$ such that $h(x,k) - 1 = \int_{-\infty}^{0} A(x,y)e^{-2iky}dy$ (for $\operatorname{Im} k > 0$ pointwise and for $\operatorname{Im} k = 0$ for $L_2(\mathbf{R})$). Then 5) can be written as an equation in $L_2(\mathbf{R})$ in the following way:

$$T_-(k)g(x,k) - 1 = \int_{-\infty}^{\infty}\{A(x,y) + F_-(x+y) + \int_{-\infty}^{0} A(x,z)F_-(x+y+z)dz\}e^{2iky}dy.$$

Take $w < 0$. By 2), $(T_-(k)g(x,k) - 1)e^{-2ikw}$ is in $L_2(\mathbf{R})$ as a function of k (in $\mathbf{R}$) and is meromorphic in $\operatorname{Im} k > 0$. By 2) we have in $L_2(\mathbf{R})$, with $k := k_1 + ik_2$ and $k_2 > 0$ sufficiently small, $\lim_{k_2 \downarrow 0}(T_-(k)g(x,k) - 1) = T_-(k_1)g(x,k_1) - 1$.

Since the Fourier transform is continuous on $L_2(\mathbf{R})$ we get (in $L_2(\mathbf{R})$)

$$\begin{aligned}&\lim_{k_2\downarrow 0}\int_{-\infty}^{\infty}(T_-(k)g(x,k)-1)e^{-2ik_1y}dk_1\\&=\int_{-\infty}^{\infty}(T_-(k_1)g(x,k_1)-1)e^{-2iky}dk_1.\end{aligned}$$

Thus, for $w < 0$, the residue formula furnishes the following equality in $L_2(\mathbf{R}_-)$:

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{\infty}(T_-(k)g(x,k)-1)e^{-2ikw}dk\\&=\sum_{j\in J}\operatorname{Res}\,(T_-(k)g(x,k);i\kappa_j)e^{2\kappa_j w}.\end{aligned}$$

By 5) one gets

$$\begin{aligned}&-2i\sum_{j\in J}\operatorname{Res}\,(T_-(k)g(x,k);i\kappa_j)e^{2\kappa_j w}\\&\quad=2\sum_{j\in J}c_{-j}e^{2\kappa_j(x+w)}h(x,i\kappa_j)\\&\quad=2\sum_{j\in J}c_{-j}e^{2\kappa_j(x+w)}+2\int_{-\infty}^{0}\sum_{j\in J}c_{-j}e^{2\kappa_j(x+w+z)}A(x,z)dz.\end{aligned}$$

Using the inverse formula of the Fourier transform in $L_2(\mathbf{R})$, we get, for $w < 0$ in $L_2(\mathbf{R}_-)$,

$$0 = A(x,w) + \Omega_-(x+w) + \int_{-\infty}^{0} A(x,z)\Omega_-(x+w+z)dz.$$

So $A(x,\cdot)$ is a solution of the Marchenko equation in $L_2(\mathbf{R}_-)$. But this solution is unique, so $A(x,\cdot) = B_-(x,\cdot)$ (in $L_2(\mathbf{R}_-)$) and $h(x,\cdot) = h_-(x,\cdot)$ in H_2^+. In $L_2(\mathbf{R})$ we get $T_-(\cdot)g(x,\cdot) - 1 = T_-(\cdot)g_-(x,\cdot) - 1$. By 2) and the same property for $g_-(x,k)$ we conclude that $g(x,k) = g_-(x,k)$ in $\operatorname{Im} k > 0$.□

Let $R_+(\ell)$ be a continuous function on $\mathbf{R}$ in $L_2(\mathbf{R})$ such that $|R_+(\ell)| \leq 1$ and $\lim_{|\ell|\to\infty} R_+(\ell) = 0$. Define $F_+(x) := \frac{1}{\pi}\int_{-\infty}^{\infty} R_+(\ell)$ $e^{2i\ell x}d\ell$ and

$$\begin{aligned}\Omega_+(x) :=&F_+(x) + 2\sum_{j\in J} c_{+j}\exp\{-2x\sqrt{\kappa_j^2+c^2}\}\\ &+\frac{1}{\pi}\int_0^c |T_-(t)|^2\exp(-2x\sqrt{c^2-t^2})dt,\end{aligned}$$

where $(c_{+j})_{j\in J}$ are numbers in $\mathbf{R}$, different from 0 and T_- is a complex valued function such that $|T_-(\sqrt{c^2-t^2})|^2/\sqrt{c^2-t^2}$ is in $L_1([0,c])$. Assume that, for all x in $\mathbf{R}$,

$$\int_x^\infty (s-x)|\Omega_+'(s)|ds < \infty.$$

Then, for any x in $\mathbf{R}$, there exists a unique solution $B_+(x,\cdot)$ in $L_1(\mathbf{R}_+)\cap L_2(\mathbf{R}_+)$ of the Marchenko equation $0 = B_+(x,y)+\Omega_+(x+y)$ $+\int_0^\infty B_+(x,z)\Omega_+(x+y+z)dz$. Set $B_+(x,y) = 0$ for $y<0$ and let us introduce $h_+(x,\ell) := 1+\int_0^\infty B_+(x,y)e^{2i\ell y}dy$, and, for ℓ in $\mathbf{R}\backslash\{0\}$, $g_+(x,\ell) := (h_+(x,-\ell)+R_+(\ell)e^{2i\ell x}h_+(x,\ell))\frac{1}{T_+(k(\ell))}$, where $T_+(k)$ is a complex valued function defined on $\{\operatorname{Im} k \geq 0\}\backslash\{0\}$ such that $T_+(k) \neq 0$ for $k \neq \pm c$ and $k(\ell) = \sqrt{\ell^2+c^2}$ is the inverse map of $\ell = \ell(k) = \sqrt{k^2-c^2}$, defined for $\{\operatorname{Im}\ell \geq 0\}\backslash\{i\lambda : 0\leq\lambda\leq c\}$. Using the definition of $h_+(x,\ell)$, $T_+(\ell)g_+(x,\ell)$ can be written as

$$\begin{aligned}&T_+(\ell)g_+(x,\ell)\\ &=1+\int_{-\infty}^{\infty}\Big\{B_+(x,y)+F_+(x+y)+\int_0^\infty B_+(x,z)F_+(x+y+z)dz\Big\}e^{-2i\ell y}dy\\ &=1+\int_{-\infty}^{0}\Big(F_+(x+y)+\int_0^\infty B_+(x,z)F_+(x+y+z)dz\Big)e^{-2i\ell y}dy\\ &\quad+i\sum_{j\in J} c_{+j}e^{-2\lambda_j x}h_+(x,i\lambda_j)\frac{1}{\ell-i\lambda_j}\\ &\quad+\frac{i}{2\pi}\int_0^c |T_-(t)|^2\frac{\exp\{-2x\sqrt{c^2-t^2}\}}{\ell-i\sqrt{c^2-t^2}}h_+(x,i\sqrt{c^2-t^2})dt,\end{aligned}$$

where by definition $\lambda_j := \sqrt{\kappa_j^2+c^2}$.

Assuming the conditions of either of the two next theorems, $T_+(k)$ $g_+(x,\ell(k))$ is analytic in $\{\operatorname{Im} k > 0\}\backslash\{i\kappa_j : j \text{ in } J\}$. The following result will be stated separately for the case where the step c^2 is strictly positive and where the step is 0 since, for the later case, we need much weaker assumptions.

THEOREM 3.2. (step $c^2 > 0$) *Let the following conditions be satisfied:*

Y1) *Let $R_+(\ell)$ be a complex valued function, defined and continuous on the whole of* $\mathbf{R}$ *with the following properties:*

(i) $R_+^*(\ell) = R_+(-\ell)$;

(ii) $|R_+(\ell)| \leq 1$ *and* $R_+(0) = -1$;

(iii) $R_+(\ell) = O(\frac{1}{\ell})$ *for* $|\ell| \to \infty$; and

(iv) $F_+(x) := \frac{1}{\pi}\int_{-\infty}^{\infty} R_+(\ell)e^{2i\ell x}d\ell$ *is absolutely continuous and* $\int_x^{\infty}|F_+'(s)|(1+|s|)ds < \infty$ *for all x in* $\mathbf{R}$.

Y2) *Let T_+ be a complex valued function defined on* $\{\operatorname{Im} k \geq 0\}\backslash\{0\}$ *and meromorphic in* $\operatorname{Im} k > 0$ *such that the points* $\{i\kappa_j : j \text{ in } J\}$ *are the only poles of T_+, all of them simple and purely imaginary* ($\kappa_j > 0$). *Moreover,*

(i) $T_+^*(k) = T_+(-k)$ *for k in* $\mathbf{R}\backslash\{0\}$,

(ii) $T_+(k) \neq 0$ *in* $\{\operatorname{Im} k \geq 0\}\backslash\{\pm c, 0\}$ *and* $T_+(\pm c) = 0$,

(iii) $T_+(k) = 1 + O(\frac{1}{k})$ *as* $|k| \to \infty$ *in* $\operatorname{Im} k \geq 0$ *uniformly, and*

(iv) T_+ *continuous in* $\{\operatorname{Im} k \geq 0\}\backslash\{0\}$.

Y3) *Let R_- be a complex valued function defined and continuous on the whole of* $\mathbf{R}$ *such that*

(i) $R_-^*(k) = R_-(-k)$,

(ii) $|R_-(k)| \leq 1$, *and* $R_-(0) = -1$,

(iii) $R_-(k) = O(\frac{1}{k})$ as $|k| \to \infty$, *and*

(iv) $F_-(x) := \frac{1}{k}\int_{-\infty}^{\infty} R_-(k)e^{-2ikx}dx$ *has property* P_1.

Y4) *$T_-(k)$ is given by* $(kT_+(k))/\ell(k)$ *in* $\{\operatorname{Im} k \geq 0\}\backslash\{\pm c, 0\}$.

Y5) (i) *For all k in* $\mathbf{R}\backslash[-c,c]$:

$$1 = \frac{k}{\ell}|T_+|^2(k) + |R_+|^2(\ell(k))$$
$$1 = \frac{\ell}{k}|T_-|^2(k) + |R_-|^2(\ell(k))$$

and

$$0 = \ell T_-(k)R_-^*(k) + kT_+^*(k)R_+(\ell(k)).$$

(ii) $R_-(0) = -1$ *and, for real k with* $0 < |k| \leq c$,

$$R_-(k) = \frac{T_-(k)}{T_-^*(k)}.$$

Further, let $(c_{-j})_{j\in J}$ be given positive numbers and $(\mu_j)_{j\in J}$ numbers different from 0 *such that* $\operatorname{Res}(T_-(k); i\kappa_j) = \mu_j ic_{-j}$. *Moreover let us introduce $c_{+j} := \mu_j^2$ and define Ω_+ as above. If, in addition, $e^{-i\ell x}e^{ikx}g_+(x,\ell(k))$ is in H_2^+, then, for any x in* $\mathbf{R}$ *and* $\operatorname{Im} k \geq 0$,

$$e^{-i\ell x}g_+(x,\ell(k)) = e^{-ikx}h_-(x,k)$$

and

$$e^{ikx}g_-(x,k) = e^{i\ell x}h_+(x,\ell(k)).$$

Moreover, $e^{-i\ell x}e^{ikx}g_+(x,\ell(k))$ is an element in H_2^+ for any fixed x in $\mathbf{R}$ *if the following conditions are satisfied:*

I) *There exists $\alpha > 0$ such that, for all x in* $\mathbf{R}$,

$$\int_x^\infty |F_+'(s)|(1+|s|^{1+\alpha})ds < \infty$$

and $R_+(\ell)$ is α-Hölder continuous in a neighborhood of $\ell = 0$.

II) a) *$k/T_-(k)$ is bounded in $U\backslash\{0\}$, where U is a neighborhood of* 0 *in* $\operatorname{Im} k \geq 0$ *which is sufficiently small.*

b) *$|T_-(\sqrt{c^2-t^2}|^2 t/\sqrt{c^2-t^2}$ is γ Hölder continuous in $[0,c]$ for some* $\gamma > 0$.

REMARK. It is shown in Lemma 3.5 that a) and b) hold if there exist $1/2 < \beta \leq 1$ and $M, M' > 0$ such that T_- is continuously differentiable in the open interval $(0, c)$ and

$$M't \leq |T_-(t)| \leq Mt^\beta, \quad 0 \leq t \leq c,$$

$$|t^{7/2-\beta}T'_-(\sqrt{c^2-t^2})| \leq M, \quad 0 < t \leq c/2,$$

$$|(\sqrt{c^2-t^2})^{1-\beta}T'_-(\sqrt{c^2-t^2})| \leq M, \quad c/2 \leq t < c.$$

PROOF. Let us introduce the functions

$$\begin{aligned} h(x,k) &:= e^{-i\ell k}e^{ikx}g_+(k,\ell) \ \text{ in } \operatorname{Im} k \geq 0 \\ g(x,k) &:= e^{i\ell x}e^{-ikx}h_+(x,\ell) \ \text{ in } \operatorname{Im} k \geq 0. \end{aligned}$$

It suffices to show that (h, g) satisfies the conditions 1) to 5) of Theorem 3.1. Clearly g satisfies 2) and 3). 1) follows by assumption. So it remains to show 4) and 5). To consider 5) let k be in $\mathbf{R}$ with $|k| > c$. Then $R_-(k) = -R^*_+(k)T_+(k)/T^*_+(k)$, and we get

$$\begin{aligned} &h(x,-k) + R_-(k)h(x,k)e^{-2ikx} \\ &\quad = e^{i\ell x}e^{-ikx}\frac{1+|R_+(k)|^2}{T^*_+(k)}h_+(x,\ell(k)) = T_-(k)g(x,k), \end{aligned}$$

where we used Y5 (i). For $0 < |k| < c$, we have $R_-(k) = T_-(k)/T^*_-(k) = -T_+(k)/T^*_+(k)$, and with $k := k_1 + ik_2 (k_2 \geq 0)$ obtain

$$\begin{aligned} &h^*(x,k_1) + R_-(k_1)h(x,k_1)e^{-2ik_1x} \\ =&e^{-i\ell(k_1)x}e^{-ik_1x}\frac{1}{T^*_+(k_1)}((T_+(k_1)g_+(x,\ell(k_1)))^* - T_+(k_1)g_+(x,\ell(k_1))) \\ =&e^{-i\ell(k_1)x}e^{ik_1x}\frac{(-2i)}{T^*_+(k_1)}\operatorname{Im}(T_+(k_1)G_+(x,\ell(k_1))) \\ =&e^{-i\ell(k_1)x}e^{ik_1x} \\ &\frac{1}{T^*_+(k_1)}i\operatorname{Im}\Big(\frac{1}{\pi}\int_0^c |T-(t)|^2\exp(-2\sqrt{c^2-t^2}x)\frac{h_+(x,i\sqrt{c^2-t^2})}{\sqrt{c^2-t^2}+i\ell}dt\Big). \end{aligned}$$

It suffices to show that

$$\begin{aligned}
&\operatorname{Im}\left(\frac{1}{2\pi}\int_0^c |T_-(t)|^2 \exp(-2\sqrt{c^2-t^2}x)\frac{h_+(x,i\sqrt{c^2-t^2})}{\sqrt{c^2-t^2}+il}dt\right)\\
&=\frac{-i}{2}T_+^*(k_1)T_-(k_1)e^{2i\ell(k_1)x}h_+(x,\ell(k_1))\\
&=-\frac{1}{2}\frac{|\ell(k_1)|}{k_1}|T_-(k_1)|^2e^{2i\ell(k_1)x}h_+(x,\ell(k_1)).
\end{aligned}$$

This follows from Lemma 3.6, and 5) is proved. Now let us come to 4). By the definition of g, $\operatorname{Res}(T_-(k)g(x,k);i\kappa_j) = \operatorname{Res}(T_-(k);i\kappa_j)e^{-\lambda_j}$ $e^{\kappa_j x}h_+(x,i\lambda_j)$ where as usual $\lambda_j := \sqrt{c^2+\kappa_j^2}$. On the other hand

$$\begin{aligned}
&ic_{-j}e^{2\kappa_j x}h(x,i\kappa_j)\\
&=-c_{-j}e^{\kappa_j x}e^{-\lambda_j x}c_{+j}h_+(x,i\lambda_j)\frac{\lambda_j}{\kappa_j}\frac{1}{\operatorname{Res}(T_+(k);i\kappa_j)}\\
&=\operatorname{Res}(T_-(k);i\kappa_j)h_+(x,i\lambda_j)e^{\kappa_j x}e^{-\lambda_j x},
\end{aligned}$$

where we used that $(\operatorname{Res}(T_-(k);i\kappa_j))^2 = -c_{+j}c_{-j}$, and so 4) follows. The last part of the theorem will be proved in the following lemmas.

Let us give immediately the corresponding result for the case where there is no step ($c^2 = 0$). Then $\ell = k$ and $T(k) := T_-(k) = T_+(k)$ in $\operatorname{Im} k \geq 0$.

THEOREM 3.3. (step $c^2 = 0$) *Let the following conditions be satisfied:*

Z1) *Let R_+ be a complex valued function, defined and continuous on the whole of* $\mathbf{R}$ *such that:*

(i) $R_+^*(k) = R_+(-k)$;

(ii) $|R_+(k)| \leq 1$;

(iii) $R_+(k) = 0(\frac{1}{k})$ as $|k| \to \infty$;

(iv) $F_+(k) := \frac{1}{\pi}\int_{-\infty}^{\infty} R_+(k)e^{2ikx}dk$ *is absolutely continuous and* $\int_x^\infty |F_+'(s)|(1+|s|)ds < \infty$ *for all x in* $\mathbf{R}$.

Z2) *Let T be a complex valued function, defined on* $\{\operatorname{Im} k \geq 0\}\backslash\{0\}$ *and meromorphic in* $\operatorname{Im} k > 0$ *such that the points* $\{i\kappa_j : j \text{ in } J\}$ *are* *the only poles of* T, *all of them simple and purely imaginary* $(\kappa_j > 0)$ *such that*:

(i) $T^*(k) = T(-k)$ *for* k *in* $\mathbf{R}\backslash\{0\}$;

(ii) $T(k) \neq 0$ *in* $\{\operatorname{Im} k \geq 0\}\backslash\{0\}$;

(iii) $T(k) = 1 + O(\frac{1}{k})$ *as* $|k| \to \infty$ *uniformly in* $\operatorname{Im} k \geq 0$; *and*

(iv) T *is continuous in* $\{\operatorname{Im} k \geq 0\}\backslash\{0\}$.

Z3) *Let* R_- *be a complex valued function, defined and continuous on the whole of* $\mathbf{R}$ *such that*:

(i) $R_-^*(k) = R_-(-k)$,

(ii) $|R_-(k)| \leq 1$,

(iii) $R_-(k) = O(\frac{1}{k})$ *as* $|k| \to \infty$, *and*

(iv) $F_-(x) := \frac{1}{\pi}\int_\infty^\infty R_-(k)e^{-2ikx}dx$ *has property* P_1.

Z4) *For all* k *in* $\mathbf{R}\backslash\{0\}$,

$$1 = |T|^2(k) + |R_+|^2(k) = |T|^2(k) + |R_-|^2(k)$$
$$0 = T(k)R_-^*(k) + T^*(k)R_+(k).$$

Further let $(c_{-j})_{j\in J}$ *be given positive numbers and* $(\mu_j)_{j\in J}$ *be different from* 0 *such that* $\operatorname{Res}(T; i\kappa_j) = \mu_j i c_{-j}$. *Moreover let us introduce* $c_{+j} := \mu_j^2 c_{-j}$ *and define* Ω_+ *as above. If in addition* $g_+(x,k) := (h_+^*(x,k) + R_+(k)h_+(x,k))\frac{1}{T(k)}$ *is in* H_2^+ *for all* x *in* $\mathbf{R}$, *where* h_+ *is given as above, then, for all* x *in* $\mathbf{R}$ *and* $\operatorname{Im} k > 0$,

$$g_+(x,k) = h_-(x,k)$$

and

$$g_-(x,k) = h_+(x,k).$$

In either of the following cases $g_+(x,k)$ *is in* H_2^+:

A) *There exists* $0 \leq \alpha < 1/2$ *and* $M > 0$ *such that* $|k|^\alpha/|T(k)| \leq M$ *in a neighborhood of* $k = 0$ *in* $\operatorname{Im} k \geq 0$.

B) (i) *There exist* $1 \geq \beta \geq 1/2$ *and* $M > 0$ *such that* $|k|^\beta|T(k)| \leq M$ *in a neighborhood of* $k = 0$ *in* $\operatorname{Im} k = 0$.

(ii) $R_+(0) = -1$ *and there exists* $\beta > \alpha > \beta - 1/2$ *such that* R_+ *is* α*-Hölder continuous in a neighborhood of* $k = 0$ *and* $\int_x^\infty |F_+'(s)|$

$(1+|s|^{1+\alpha})ds < \infty$ *for all* x *in* $\mathbf{R}$.

PROOF. Let us introduce the functions

$$\begin{aligned} h(x,k) &:= g_+(x,k) \quad \text{in } \{\operatorname{Im} k \geq 0\}\backslash\{0\} \\ g(x,k) &:= h_+(x,k) \quad \text{in } \operatorname{Im} k \geq 0. \end{aligned}$$

It suffices to show that the pair (h,g) satisfies condition 1) to 5) of Theorem 3.1. Clearly g satisfies 2) and 3). 4) and 5) are shown in a similar way as in the proof of Theorem 3.2. So let us prove 1). Let us start with the representation

$$\begin{aligned} T(k)g_+(x,k) = 1 + \int_{-\infty}^{0} \{F_+(x+y) \\ + \int_0^\infty B_+(x,z)F_+(x+y+z)dz\}e^{-2iky}dy \\ + i\sum_{j\in J} c_{+j}e^{-2\kappa_j x}h_+(x,i\kappa_j)\frac{1}{k-i\kappa_j}. \end{aligned}$$

We conclude that $(T(k)g_+(x,k)-1)\prod_{j\in J}((k-i\kappa_j)/(k+i\kappa_j))$ is in H_2^+ (x in $\mathbf{R}$). In case A it follows right away that $g_+(x,k)-1$ is in H_2^+. In case $B, \lim_{k\to 0, \operatorname{Im} k=0} T(k)g(x,k) = 0$ and so it is enough to show that $T(k)g_+(x,k)$ is α-Hölder continuous in a neighborhood of $k=0$ in $\operatorname{Im} k = 0$. Consider $T(k)g_+(x,k) = h_+^*(x,k) + R_+(k)h_+(x,k)$. By Corollary 3.8, $h_+(x,k)$ and $h_+(x,-k)$ are α-Hölder continuous in $\operatorname{Im} k \geq 0$ (x in $\mathbf{R}$). By assumption $R_+(k)$ is α-Hölder continuous in a neighborhood of $k=0$. So there exists $C>0$ such that, in a neighborhood of $k=0$ in $\{\operatorname{Im} k \geq 0\}\backslash\{0\}$, we have

$$|g_+(x,k)| \leq c/|k|^{\beta-\alpha}.$$

Now we conclude that $g_+(x,k)-1$ is in H_2^+. □

LEMMA 3.4. *Let all the hypothesis of Theorem* 3.2 *together with* I *and* II *be fulfilled. Then* $g_+(x,\ell(k))-1$ *is in* H_2^+ *as a function of* k(x *in* $\mathbf{R}$).

PROOF. We start with the representation

$$\begin{aligned}\prod_{j\in J}\frac{\ell-i\lambda_j}{\ell+i\lambda_j}\Big(&T_+(k)g_+(x,\ell)-1\\ &+\frac{1}{2\pi}\int_0^c |T_-(t)|^2\exp(-2\sqrt{c^2-t^2}x)\frac{h_+(x,i\sqrt{c^2-t^2}}{\sqrt{c^2-t^2}+i\ell}dt\Big)\\ =\prod_{j\in J}&\frac{\ell-i\lambda_j}{\ell+i\lambda_j}\\ &\cdot\int_{-\infty}^0\Big\{F_+(x+y)+\int_0^\infty B_+(x,z)F_+(x+y+z)dz\Big\}e^{-2i\ell y}dy\\ &+i\sum_{n\in J}c_{+n}e^{-2\lambda_n x}h_+(x,i\lambda_n)\prod_{j\in J}\frac{\ell-i\lambda_j}{\ell+i\lambda_j}\frac{1}{\ell-i\lambda_n}.\end{aligned}$$

Clearly the function on the right hand side is in H_2^+ as a function of ℓ (x in $\mathbf{R}$). For sake of convenience only we may assume that $R_+(\ell)$ is α-Hölder continuous on the whole of $\mathbf{R}$ and that $\alpha=\gamma$. The next step is to show that the function on the right hand side is α-Hölder continuous in $\operatorname{Im}\ell\geq 0$. It suffices to prove that the function on the left hand side is α-Hölder continuous on $\operatorname{Im}\ell=0$.

By Lemma 3.6 $\frac{1}{2\pi}\int_0^c|T_-(t)|^2\exp(-2\sqrt{c^2-t^2}x)((h_+(x,i\sqrt{c^2-t^2}))/(\sqrt{c^2-t^2}+i\ell))dt$ is α-Hölder continuous in ℓ for $\operatorname{Im}\ell=0$. So let us turn to the term $T_+(k)g_+(x,\ell)$. For $\operatorname{Im}\ell=0, T_+(k)g_+(x,\ell)=h_+^*(x,\ell)+R_+(\ell)e^{2i\ell x}h_+(x,\ell)$. By Corollary 3.8, $h_+(x,\ell)$ is α-Hölder continuous. By assumption, $R_+(\ell)$ is α-Hölder continuous and thus $T_+(k)g_+(x,\ell)$ is α-Hölder continuous for $\operatorname{Im}\ell=0$. Moreover

$$\lim_{\substack{\ell\to 0\\ \operatorname{Im}\ell=0}} T_+(k)g_+(x,\ell)=0$$

and thus $\prod_{j\in J}((\ell-i\lambda_j)/(\ell+i\lambda_j))T_+(k)g_+(x,\ell)$ is $\alpha/2$-Hölder continuous as a function of k in $\operatorname{Im}k\geq 0$. So there exists $C>0$ such that, for $|k|\leq 2c$,

$$|g_+(x,\ell(k))|\leq\frac{C}{|k-c|^{1/2-\alpha/2}|k+c|^{1/2-\alpha/2}},$$

where we used $kT_+(k)=\ell(k)T_-(k)$ and $M'|k|<|T_-(k)|$ for some $M'>0$. These two facts together imply that $T_+(k)$ is bounded away from 0 in $U\backslash\{0\}$ where U is a neighborhood of $k=0$ in $\operatorname{Im}k\geq 0$.

With this estimate in hand and $1/T_+(k) = 1 + O(\frac{1}{k})$ for $|k| \to \infty$ in $\operatorname{Im} k \geq 0$, one can see that $g_+(x, \ell(k)) - 1$ is an element in H_2^+. □

Let us introduce the function

$$\phi(t) := |T_-(\sqrt{c^2 - t^2})|^2 \frac{t}{\sqrt{c^2 - t^2}} 1_{(0,c)}(t).$$

Here $1_{(0,c)}$ denotes the characteristic function of the open interval $(0, c)$.

LEMMA 3.5. *Let the following conditions be met.*

1) T_- is continuous on the closed interval $[0, c]$ and there exists $1/2 < \beta \leq 1$ and $M > 0$ such that $|T_-(t)| \leq M|t|^\beta$.

2) T_- is continuously differentiable in the open interval $(0, c)$ such that

$$|T'_-(\sqrt{c^2 - t^2})| \leq \frac{M}{t^{-\beta+7/2}} \textit{ for } t \textit{ in } (0, c/2)$$

and

$$|T'_-(\sqrt{c^2 - t^2})| \leq \frac{M}{(\sqrt{c^2 - t^2})^{1-\beta}} \textit{ for } t \textit{ in } (c/2, c).$$

Then ϕ is $(\beta - 1/2)$ Hölder continuous.

PROOF. Straight-forward verification shows that, for t in $(0, c)$ and $\gamma := \beta - 1/2$, there exists $A > 0$ such that $|\phi'(t)| < A/t^{1-\gamma}$. Moreover $(\phi(t) - \phi(0))/t^\gamma = \phi(t)/t^\gamma$ and $(\phi(t) - \phi(0))/(c - t)^\gamma = \phi(t)(t + c)^\gamma/(c^2 - t^2)^\gamma$ are bounded in $(0, c)$ and the Hölder continuity of ϕ follows.□

Let us introduce the function

$$v(x, t) := \frac{1}{2}\left|T_-(\sqrt{c^2 - t^2})\right|^2 \frac{t}{\sqrt{c^2 - t^2}} e^{-2tx} 1_{(0,c)}(t).$$

By $\mathcal{H}$ we denote the Hilbert transformation on $L_2(\mathbf{R})$.

LEMMA 3.6. *Let the condition* II *of Theorem* 3.2 *be satisfied. If in addition* $\int_x^\infty |F'_+(s)|(1+|s|^{1+\gamma})ds < \infty$ *(x in* $\mathbf{R}$*), where* $\gamma := \beta - 1/2$, *then:*

a) $v(x,t)h_+(x, i\sqrt{c^2-t^2})$ *is as a function of* t *in* $L_2(\mathbf{R})$ *and is* γ-*Hölder continuous (x in* $\mathbf{R}$*).*

b) $\frac{1}{2\pi}\int_0^c |T_-(t)|^2 e^{-2\sqrt{c^2-t^2}x} \frac{h_+(x,i\sqrt{c^2-t^2})}{\sqrt{c^2-t^2}+i\ell}dt$ *defines an analytic function in* $\operatorname{Re}\ell \neq 0$ *which is* γ-*Hölder continuous in* $\operatorname{Re}\ell \leq 0$ *as well as in* $\operatorname{Re}\ell \geq 0$.

c) *With* $\ell := \ell_1 + i\ell_2$ *one has*

$$\mathcal{H}(v(x,t)h_+(x,i\sqrt{c^2-t^2}))(\ell_2)\overset{+}{(\;-)}iv(x,\ell_2)h_+(x,i\ell_2)$$
$$= \lim_{\substack{\ell_1\uparrow 0\\ (\ell_1\downarrow 0)}} \frac{1}{2\pi}\int_0^c |T_-(t)|^2 \exp(-2\sqrt{c^2-t^2}x)\frac{h_+(x,i\sqrt{c^2-t^2})}{\sqrt{c^2-t^2}+i\ell}dt.$$

(d) $$\frac{1}{2\pi}\int_0^c |T_-(t)|^2 \exp(-2\sqrt{c^2-t^2}x)\frac{h_+(x,i\sqrt{c^2-t^2})}{\sqrt{c^2-t^2}+i\ell}dt$$

is in H_2^+ *and* $\gamma/2$ *Hölder continuous in* $\operatorname{Im} k \geq 0$.

(e) $$\frac{1}{2\pi}\int_0^c |T_-(t)|^2 \exp(-2\sqrt{c^2-t^2}x)\frac{h_+(x,i\sqrt{c^2-t^2})}{\sqrt{c^2-t^2}+i\ell}dt$$

is γ-Hölder continuous in ℓ with $\operatorname{Im}\ell = 0$.

PROOF. a) follows from Corollary 3.8. That implies that the Hilbert transform $\mathcal{H}(v(x,\cdot)h_+(x,i\sqrt{c^2-\cdot}))(t)$ is γ-Hölder continuous also. Now b) can be deduced from c). d) follows from b) and e) follows from c). So it remains to prove c). For $\ell_1 \neq 0$ we get

$$\frac{1}{2\pi}\int_0^c |T_-(t)|^2 \exp(-2\sqrt{c^2-t^2}x)h_+(x,i\sqrt{c^2-t^2})\frac{1}{\sqrt{c^2-t^2}+i\ell}dt$$
$$= \frac{1}{2\pi}\int_0^c |T_-(t)|^2 \exp(-2\sqrt{c^2-t^2}x)\frac{1}{\sqrt{c^2-t^2}+i\ell}dt$$
$$= \frac{1}{\pi}\int_0^c v(x,t)h_+(x,i\sqrt{c^2-t^2})\frac{(t-\ell_2)-i\ell_1}{(t-\ell_2)^2+\ell_1^2}dt.$$

For the imaginary part the convergence follows by a well known property of the Poisson kernel. The convergence of the real part one gets directly by the definition of the Hilbert transformation. □

LEMMA 3.7. *Let $\alpha > 0$. If $\int_x^\infty |F'_+(s)|(1+|s|^{1+\alpha})ds < \infty$ for all x in $\mathbf{R}$, then $\int_0^\infty |B_+(x,y)|y^\alpha dy < \infty$ (x in $\mathbf{R}$).*

PROOF. There exists a non increasing function $C(x)$ such that $|B_+(x,y)| \leq C(x)\int_{x+y}^\infty |\Omega'_+(s)|ds$.

By interchanging the order of integration,

$$\int_0^\infty |B_+(x,y)|y^\alpha dy \leq C(x)\int_x^\infty |\Omega'_+(s)|\Big(\int_0^{s-x} y^\alpha dy\Big)ds$$
$$\leq C(x)\int_x^\infty |\Omega'_+(s)|(s-x)^{1+\alpha}ds$$

and the lemma follows.

Immediately we get

COROLLARY 3.8. *If $\int_x^\infty |\Omega'_+(s)|(1+|s|^{1+\alpha})ds < \infty$ for all x in $\mathbf{R}$, then $h_+(x,\ell) := 1 + \int_0^\infty B_+(x,y)e^{2i\ell y}dy$ is α-Hölder continuous in $\operatorname{Im}\ell \geq 0$ (x in $\mathbf{R}$).*

Let us summarize the main result of this paper in the following theorem, recalling that by definition $q_+(x) := c^2 - \partial_x B_+(x,0)$ and $q_-(x) := \partial_x B_-(x,0)$ (in $L_1^{\ell oc}(\mathbf{R})$).

THEOREM 3.9. *Under the hypothesis of Theorem 3.2 or Theorem 3.3, q_- is in $L_2(-\infty)$, q_+ in $L_1(+\infty)$ and $q_+ = q_-$ in $L_1^{\ell oc}(\mathbf{R})$. If moreover there exists $\alpha \geq 1/2$ such that for all x in $\mathbf{R}$ $\int_{-\infty}^x (x-s)^{2\alpha}|\Omega_{-'}(s)|^2dx < \infty$, then $\int_{-\infty}^x |s|^{2\alpha}|q_-(s)|^2 ds < \infty$ for all x in $\mathbf{R}$.*

PROOF. The last statement follows from Proposition 2.7. As concerns the equality of the two potentials we remark that $e^{i\ell x}h_+(x,\ell)$ is a solution of the Schrödinger equation for $\operatorname{Im}\ell \geq 0$:

$$1) \qquad -y''(x) + q_+(x)y(x) = (\ell^2 + c^2)y(x) = k^2 y(x).$$

By Theorem 3.2 or Theorem 3.3, we have for $\operatorname{Im} k > 0$,

$$h_-(x,k)e^{-ikx} = g_+(x,\ell(k))e^{-i\ell x}.$$

The right hand side is continuous in $\{\operatorname{Im} k \geq 0\}\backslash\{|k| \leq c : k \text{ in } \mathbf{R}\}$ and is twice continuously differentiable with respect to x. So $h_-(x,k)$ is continuously defined in $\{\operatorname{Im} k \geq 0\}\backslash\{|k| \leq c : k \text{ in } \mathbf{R}\}$ and is twice continuously differentiable with respect to x there. Because q_+ is real, $e^{-i\ell x}g_+(x,\ell(k))$ is a solution of 1) for k in $\mathbf{R}\backslash[-c,c]$. But for $\operatorname{Im} k > 0, e^{-ikx}h_-(x,k)$ is a solution of

$$2) \qquad -y''(x) + q_-(x)y(x) = k^2 y(x).$$

Due to the smoothness properties of $h_-(x,k)$, 2) must hold even for k in $\mathbf{R}\backslash[-c,c]$. We conclude that $e^{-ikx}h_-(x,k)$ satisfies for k in $\mathbf{R}\backslash[-c,c]$ both equations 1) and 2). Subtracting them one gets

$$(q_+(x) - q_-(x))e^{-ikx}h_-(x,k) = 0.$$

It follows that $q_+(x) = q_-(x)$ in $L_1^{\ell oc}(\mathbf{R})$.

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REFERENCES

1. Z.S. Agranovich and V.A. Marchenko, *The Inverse Problem of Scattering Theory* (English Translation), Gordon and Breach, New York, 1963.

2. V.S. Buslaev and V.N. Fomin, *An inverse scattering problem for the one-dimensional Schrödinger equation on the axis,* Vestnik Leningrad Univ. **17** (1962), 56-64 (in Russian).

3. A. Cohen, *Existence and regularity for solutions of the Korteweg-de Vries equation,* Archive Rat. Mech. Anal. **71** (1979), 143-175.

4. ———, *Solutions of the Korteweg-de Vries equation from irregular data,* Duke Math. J. **45** (1978), 149-181.

5. ———, *Solutions of the Korteweg-de Vries equation with steplike initial profile,* Part. Diff. Eq. **9** (1984), 751-807.

6. ———, and T. Kappeler, *Scattering and inverse scattering for steplike potentials in the Schrödinger equation*, Indiana Univ. Math. J. **43** (1985), 127-180.

7. ——— and ———, *Solutions of the Korteweg-de Vries equation with intial profile in* $L^1_1(\mathbf{R}) \cap L^1_N(\mathbf{R}^+)$, SIAM J. Math. Anal. **18** (1987), 991-1025.

8. P. Deift and E. Trubowitz, *Inverse scattering on the line,* Comm. Pure Appl. Math. **32** (1979), 121-252.

9. L. Faddeev, *Properties of the s-matrix of the one dimensional Schrödinger equation,* A.M.S. Translations, Series 2, **65** (1967), 129-166.

10. C. Gardner, J. Greene, M. Kruskal and R. Miura, *Method for solving the Kroteweg-de Vries equation,* Phys. Rev. Lett. **19** (1967), 1095-1097.

11. E.J. Hruslov, *Asymptotics of the solution of the Cauchy problem for the Korteweg-de Vries equation with initial data of step type,* Math. USSR Sbornik **28** (1976), 229-248.

12. T. Kappeler, *Solutions of the Korteweg-de Vries equation with irregular steplike initial data,* J. Diff. Eq. **63** (1986), 306-331.

13. S. Tanka, *Korteweg-de Vries equation: Construction of solutions in terms of scattering data,* Osaka Math. J. **11** (1974), 49-59.

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