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THE RADICAL OF THE DIFFERENTIAL IDEAL GENERATED BY XY IN THE RING OF TWO VARIABLE DIFFERENTIAL POLYNOMIALS IS NOT DIFFERENTIALLY FINITELY GENERATED

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ABSTRACT. Let (k, δ) be a differential field containing $\mathbf{Q}$. We prove that, in general, a radical differential ideal of the differential ring $k\{X, Y\}$ is not a differential ideal of finite type.

1. Introduction.

1.1. Let (k, δ) be a differential field containing $\mathbf{Q}$. Let $N \geq 1$ be an integer. The polynomial k-algebra $k[(X_{i,j})_{i\in\{1,\ldots,N\},j\in\mathbf{N}}]$ is denoted by $k\{X_1, \ldots, X_N\}$, when it is endowed with the unique derivation Δ of $k\{X_1, \ldots, X_N\}$ extending δ and satisfying the formula $\Delta(X_{i,j}) = X_{i,j+1}$ for every pair of integers $(i, j) \in \{1, \ldots, N\} \times \mathbf{N}$. The morphism of k-algebras $k[X_1, \ldots, X_N] \to k\{X_1, \ldots, X_N\}$, defined by $X_i \mapsto X_{i,0}$ for every integer $i \in \{1, \ldots, N\}$, gives rise to a structure of $k[X_1, \ldots, X_N]$-algebra on $k\{X_1, \ldots, X_N\}$. This process is called the *adjunction of differential indeterminates* and the differential k-algebra $k\{X_1, \ldots, X_N\}$, called the *ring of differential polynomials*, provides a suitable framework for the algebraic study of (polynomial) differential equations as proposed by J. Ritt and E. Kolchin. Let $S \subset k\{X_1, \ldots, X_N\}$. We denote by $[S]$ the differential ideal generated by S, and we set $\{S\} = \sqrt{[S]}$. The ideal $\{S\}$ is a radical differential ideal.

1.2. In the present article, we mainly prove the following statement (see also Question 1 and Corollary 2.6).

Theorem 1.3. *Let (k, δ) be a differential field which contains $\mathbf{Q}$. Let X, Y be differential indeterminates over the field k. Then, in the ring of*

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differential polynomials $k\{X,Y\}$, the radical differential ideal generated by the polynomial XY is not differentially finitely generated.

The basic idea of the proof is an explicit computation of an algebraic system of generators of the differential ideal $\{XY\}$, a descent argument, and a precise use of the weight grading of the differential k-algebra $k\{X_1,\dots,X_N\}$, which is recollected in Section 2.1. Let us mention that this result also has a natural interpretation in terms of the nilpotency at the level of arc schemes. We assume that the field k is endowed with the trivial derivation. With a k-algebra of finite type $A := k[X_1,\dots,X_N]/I$, we associate the so-called arc scheme $\mathscr{L}(\mathrm{Spec}(A))$, which can be defined, via differential algebra, to be $\mathrm{Spec}(k\{X_1,\dots,X_N\}/[I])$. So, Theorem 1.3 provides information on the reduced subscheme

$$\mathscr{L}(\mathrm{Spec}(k[X,Y]/\langle XY\rangle))_{\mathrm{red}} := \mathrm{Spec}(k\{X,Y\}/\{XY\}),$$

and, more generally, on the arc scheme associated with homogeneous reduced affine plane curves of degree 2 (see Corollary 2.6).

2. The proof of Theorem 1.3.

2.1. Recollection. The ring $k\{X_1,\dots,X_N\}$ is graded by the *weight* of differential polynomials. The weight of a given monomial

$$M := X_{i_1,j_1}^{\nu_1}\dots X_{i_N,j_N}^{\nu_N}$$

is defined by the formula $\mathrm{wt}(M)=\sum_{\ell=1}^{n}\nu_\ell j_\ell$. One extends this definition to every differential polynomial $P\in k\{X_1,\dots,X_N\}$ by taking the supremum of the weights of the monomials of P. As an illustration, let us note that, for every tuple $(i,j,n)\in\mathbf{N}\times\mathbf{N}\times\mathbf{N}^*$, the differential polynomial $\Delta^{(n)}(X_iY_j)$ is isobaric with weight $n+i+j$.

2.2. Preliminary results. Let us state a technical preliminary result about the irreducible decomposition of the differential ideal $\{XY\}$ of the differential ring $k\{X,Y\}$.

Lemma 2.3. *Let (k,δ) be a differential field containing $\mathbf{Q}$. Then we have*

$$\{XY\} = [X]\cap[Y] = [X]\cdot[Y]$$

in the differential ring $k\{X,Y\}$. In particular, the differential ideal $\{XY\}$ is algebraically generated by the polynomials X_iY_j for i,j running over $\mathbf{N}$.

Proof. By the Kolchin irreducibility theorem [**2,** IV.17, Proposition 10], one knows that $\{XY\}=\{X\}\cap\{Y\}$. We remark that the differential ideals $[X]$ and $[Y]$ are prime; hence, $\{XY\}=[X]\cap[Y]$. Let us prove that $[X]\cap[Y]\subset[X]\cdot[Y]$. Let $Z\in[X]\cap[Y]$. There exist integers $m,n\in\mathbf{N}$ and polynomials $\alpha_1,\dots,\alpha_m\in k\{X,Y\}$, $\beta_1,\dots,\beta_n\in k\{X,Y\}$, such that

$$Z=\sum_{i=0}^{m}\alpha_iX_i=\sum_{j=0}^{n}\beta_jY_j. \tag{2-1}$$

For every integer $i\in\{0,\dots,m\}$, we write $\alpha_i=r_i+\sum_{j\geq 0}Y_j\alpha_{ij}$, with $r_i\in k\{X\}$; hence, equation (2-1) reads

$$Z=\left(\sum_{i,j}\alpha_{ij}Y_jX_i\right)+\left(\sum_{i=0}^{m}r_iX_i\right)=\sum_{j=0}^{n}\beta_jY_j.$$

By specializing all the Y_j to zero, we conclude that the polynomial $\sum_{i=0}^{m}r_iX_i$ vanishes. This proves the required property. □

2.4. The proof of Theorem 1.3. Let us assume that there exist polynomials $b_1,\dots,b_m\in k\{X,Y\}$, with $\mathrm{wt}(b_1)\leq\cdots\leq\mathrm{wt}(b_m)$, such that

$$\{XY\}=[b_1,\dots,b_m]. \tag{2-2}$$

There exists an integer $n\in\mathbf{N}^*$ such that $2n-2\geq\mathrm{wt}(b_m)$; hence, (2-2) and Lemma 2.3 imply, for $\Gamma:=\{(i,j)\in\{0,\dots,2n-2\}^2 : i+j\leq 2n-2\}$, that

$$\{XY\}=[(X_iY_j)_{(i,j)\in\Gamma}]=[(X_0Y_j)_{j\in\{0,\dots,2n-2\}}]. \tag{2-3}$$

To prove the last equality in (2-3), let us note that it is sufficient to show that, for every pair of integers $(i,j)\in\Gamma$, we have

$$X_iY_j\in[(X_0Y_j)_{j\in\{0,\dots,2n-2\}}].$$

This last claim comes, by induction, from the formula

$$X_iY_j=\Delta(X_{i-1}Y_j)-X_{i-1}Y_{j+1}.$$

We are going to prove that the polynomial $X_nY_n \in \{XY\}$ does not belong to $[(X_0Y_j)_{j\in\{0,\dots,2n-2\}}]$. Let us assume that the converse assertion holds. There exist polynomials $P_{i,j} \in k\{X,Y\}$ such that

$$X_nY_n = \sum_{i,j} P_{i,j}\Delta^{(n_{i,j})}(X_0Y_j). \tag{2-4}$$

By Section 2.1,we observe that the differential polynomial $\Delta^{(n_{i,j})}(X_0Y_j)$ is isobaric with weight $j+n_{i,j}$, and homogeneous of degree 2 for every pair of integers (i,j). Let us set $F_j := \Delta^{(2n-j)}(X_0Y_j)$ for every integer $j \in \{0,\dots,2n-2\}$. Since the differential polynomial X_nY_n is homogeneous of degree 2 and isobaric of weight $2n$, we may assume that (2-4) reads as follows: there exist $\lambda_0,\dots,\lambda_{2n-2} \in k$ such that

$$X_nY_n = \sum_{j=0}^{2n-2} \lambda_j F_j. \tag{2-5}$$

Let us note that, for every integer $j \in \{0,\dots,2n-2\}$, there exist non-zero positive integers $\delta_{j,i}$ such that

$$F_j = X_{2n-j}Y_j + \delta_{j,1}X_{2n-j-1}Y_{j+1} + \dots + \delta_{j,2n-j-1}X_1Y_{2n-1} + X_0Y_{2n}. \tag{2-6}$$

More precisely, by the Leibniz formula, we deduce that, for every pair of integers $(j,s) \in \{0,\dots,2n-2\}\times\{1,\dots,2n-j-1\}$, we have

$$\delta_{j,s} = \binom{2n-j}{s} \tag{2-7}$$

◦ By specializing, for every integer $j \geq n+1$, the variable Y_j to zero, we conclude that

$$X_nY_n = \sum_{j=0}^{n} \lambda_j\Big(X_{2n-j}Y_j + \sum_{\ell=1}^{n-j} \delta_{j,\ell}X_{2n-j-\ell}Y_{j+\ell}\Big). \tag{2-8}$$

By identifying in (2-8) the coefficients of $X_{2n}Y_0, X_{2n-1}Y_1, \dots, X_{n+1}Y_{n-1}$

respectively, we obtain

$$\begin{aligned} \lambda_0 &= 0, \\ \lambda_1 + \delta_{0,1}\lambda_0 &= 0, \\ \vdots \qquad & \quad \vdots \\ \lambda_{n-1} + \delta_{n-2,1}\lambda_{n-2} + \cdots + \delta_{0,n-1}\lambda_0 &= 0. \end{aligned} \tag{2-9}$$

Then, from the study of system (2-9), it follows that, for every integer $j \in \{0, \dots, n-1\}$, we have $\lambda_j = 0$; hence, by (2-8), we deduce

$$X_n Y_n = \lambda_n X_n Y_n. \tag{2-10}$$

The relation (2-10) clearly implies that $\lambda_n = 1$. Let us note that, if $n \in \{1, 2\}$, these last assertions imply that $0 = 1$, which is a contradiction.

- Let us assume that $n \geq 3$. Then, equation (2-5) now reads

$$X_n Y_n = F_n + \sum_{j=n+1}^{2n-2} \lambda_j F_j. \tag{2-11}$$

From now on, we analyze the above equation (2-11). For every integer $m \in \{n+1, \dots, 2n-2\}$, we claim that

$$\lambda_m = (-1)^{m-n} \binom{n}{m-n}. \tag{2-12}$$

Let us prove this claim. By identifying the coefficients of $X_{n-1}Y_{n+1}$ in (2-11), we obtain the formula $\lambda_{n+1} = -\delta_{n,1}$; hence, by (2-7), we have

$$\lambda_{n+1} = -\binom{n}{1}.$$

Let us fix $s \in \{n+2, \dots, 2n-2\}$ and assume that, for every integer $t \in \{n+1, \dots, s-1\}$, (2-12) holds true for λ_t. Let us set $s = n + \ell$ with $\ell \in \{2, \dots, n-2\}$; then $2n - s = n - \ell \geq 0$. By identifying the coefficients of $X_{2n-s}Y_s$ in (2-11), we obtain the formula

$$\begin{aligned} \lambda_s + \delta_{n+\ell-1,1}\lambda_{n+\ell-1} + \delta_{n+\ell-2,2}\lambda_{n+\ell-2} \\ + \cdots + \delta_{n+1,\ell-1}\lambda_{n+1} + \delta_{n,\ell} = 0. \end{aligned} \tag{2-13}$$

Let us note that, for every integer $r \in \{1, \dots, \ell-1\}$, we have by the induction hypothesis and (2-7)

$$\delta_{n+\ell-r,r}\lambda_{n+\ell-r} = (-1)^{\ell-r}\binom{n+r-\ell}{r}\binom{n}{\ell-r} = (-1)^{\ell-r}\binom{\ell}{r}\binom{n}{\ell}. \tag{2-14}$$

From (2-14), we deduce that

$$\begin{aligned} \lambda_s &= \binom{n}{\ell}\cdot\left((-1)+\ell+\cdots+(-1)^{\ell-1}\binom{\ell}{\ell-2}+(-1)^{\ell}\ell\right) \\ &= -\binom{n}{\ell}\cdot\left(1-\ell+\cdots+(-1)^{\ell-2}\binom{\ell}{\ell-2}+(-1)^{\ell-1}\ell\right) \\ &= -\binom{n}{\ell}\cdot\left(\sum_{i=0}^{\ell-1}\binom{\ell}{i}(-1)^i\right) \\ &= (-1)^{\ell}\binom{n}{\ell}. \end{aligned} \tag{2-15}$$

This concludes the proof of the claim.

∘ Let us identify the coefficient of X_0Y_{2n} in (2-11). By the specific form (2-6) of the F_j, we obtain

$$\begin{aligned} 0 = 1+\sum_{\ell=1}^{n-2}(-1)^{\ell}\binom{n}{\ell} &= \sum_{\ell=0}^{n-2}(-1)^{\ell}\binom{n}{\ell} \\ &= -((-1)^{n-1}n+(-1)^n) = (-1)^n(n-1). \end{aligned}$$

This is a contradiction, which concludes the proof of the theorem.

2.5. The nodal plane curve singularity. We deduce from Theorem 1.3 the following statement:

Corollary 2.6. *Let (k,δ) be a differential field containing $\mathbf{Q}$. Let $f\in k[X,Y]$ be a non-degenerate reduced homogeneous polynomial of degree 2. Then, in the ring of differential polynomials $k\{X,Y\}$, the radical differential ideal generated by the polynomial f is not differentially finitely generated.*

Proof. Let $f \in k[X,Y]$ be an irreducible homogeneous polynomial of degree 2, which we may assume to be unitary. Let k' be an algebraic closure of k. The derivation δ may be extended in a unique way to k'. Let us assume that there exist polynomials $b_1, \ldots, b_m$ in the ring $k\{X,Y\}$ such that $\{f\} = [b_1, \ldots, b_m]$. In particular, the radical differential ideal $\{f\} \otimes_k k'$ is finitely generated. Furthermore, there exists a differential k'-automorphism Φ of $k'\{X,Y\}$ such that $\Phi(\{f\} \otimes_k k') = \{XY\}$. Our assumption implies that the radical differential ideal $\{XY\}$ is finitely generated, which contradicts Theorem 2.4. □

2.7. Further comments. The main result of the present article provides a negative answer to the following general question:

Question 1. Let (k, δ) be a differential field containing $\mathbf{Q}$. Let $N \geq 2$ be an integer. Let I be a radical differential ideal of $k\{X_1, \ldots, X_N\}$. Does there exist an integer $m \geq 1$ and $a_1, \ldots, a_m \in I$ such that $I = [a_1, \ldots, a_m]$?

By the Ritt-Raudenbusch basis theorem, one knows that, for every radical differential ideal I of $k\{X_1, \ldots, X_N\}$, there exist an integer $m \geq 1$ and $a_1, \ldots, a_m \in I$ such that $I = \{a_1, \ldots, a_m\}$ (see [**1,** Theorem 7.4]). On the other hand, by Ritt's works, we know that there exist differential ideals *not* of finite type in arbitrary finitely generated differential k-algebras.[1] Let us emphasize the fact that Question 1 is stronger than these results, insofar as it asks for the existence of this finiteness property among the radical differential ideals.

To conclude, let us note that our negative answer to Question 1 crucially relies on the homogeneity of the polynomial XY. So, it seems interesting to us to study the following question:

Question 2 (D. Bourqui, A. Reguera and J. Sebag)**.** Let (k, δ) be a differential field containing $\mathbf{Q}$. Let $f \in k[X,Y]$ be an analytically irreducible polynomial. Is the radical differential ideal generated by the polynomial f differentially finitely generated in the ring of differential polynomials $k\{X,Y\}$?

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[1]As an illustration of this remark, we may consider the ideals $J_1 := [(X_i^2)_{i \in \mathbf{N}}]$ or $J_2 := [(X_i X_{i+1})_{i \in \mathbf{N}}]$ of the finitely generated differential $\mathbf{Q}$-algebra $\mathbf{Q}\{X\}$. In particular, we observe that $\{J_1\} = [X_0]$ and $\{J_2\} = [X_1]$; hence, these examples do not provide a negative answer to Question 1.

To the best of our knowledge, Question 2 is open, even for the polynomial $f = X^3 - Y^2$.

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