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ON THE COMPUTATION OF THE RATLIFF-RUSH CLOSURE, ASSOCIATED GRADED RING AND INVARIANCE OF A LENGTH

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Dedicated to Professor Tony J. Puthenpurakal

ABSTRACT. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of positive dimension d and infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R, and let J be a minimal reduction of I. In this paper, we show that, if $\widetilde{I^k} = I^k$ and $J \cap I^n = JI^{n-1}$ for all $n \geq k+2$, then $\widetilde{I^n} = I^n$ for all $n \geq k$. As a consequence, we can deduce that, if $r_J(I) = 2$, then $\widetilde{I} = I$ if and only if $\widetilde{I^n} = I^n$ for all $n \geq 1$. Moreover, we recover some main results of **[5, 11]**. Finally, we give a counter example for **[21,** Question 3].

1. Introduction. Throughout this paper, we assume that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of positive dimension d, infinite residue field and I an $\mathfrak{m}$-primary ideal of R. An ideal $J \subseteq I$ is called a *reduction* of I if $I^{n+1} = JI^n$ for some $n \in \mathbb{N}$. A reduction J is called a *minimal reduction* of I if it does not properly contain a reduction of I. The least such n is called the *reduction number* of I with respect to J, and denoted by $r_J(I)$. These notions were introduced by Northcott and Rees in **[20]**, where they proved that minimal reductions of I always exist if the residue field of R is infinite. Recall that $x \in I$ is a superficial element of I if there exists a $k \in \mathbb{N}_0$ such that $I^{n+1} : x = I^n$ for all $n \geq k$. A set of elements $x_1, \ldots, x_d$ is a superficial sequence of I if x_i is a superficial element of $I/(x_1, \ldots, x_{i-1})$ for all $i = 1, \ldots, d$. A superficial sequence $x_1, \ldots, x_d$ of I is called *tame* if x_i is a superficial element of I, for all $i = 1, \ldots, d$. Elias **[8]** defined and proved the tame superficial sequence exists (see also **[6]**). Swanson **[27]** proved that,

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if $x_1, \dots, x_d$ is a superficial sequence of I, then $J = (x_1, \dots, x_d)$ is a minimal reduction of I. It is known that every minimal reduction can be generated by superficial sequence (see [**6, 26**]).

The Ratliff-Rush closure of I is defined as the ideal

$$\widetilde{I} = \cup_{n\geq 1}(I^{n+1} : I^n).$$

It is a refinement of the integral closure of I and $\widetilde{I} = I$ if I is integrally closed (see [**23**]). The Ratliff-Rush filtration $\widetilde{I^n}$, $n \in \mathbb{N}_0$, carries important information on the associated graded ring

$$G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}.$$

For example, Heinzer, Lantz and Shah [**13**] showed that the depth $G(I) \geq 1$ if and only if $\widetilde{I^n} = I^n$ for all $n \in \mathbb{N}_0$. The aim of this paper is to compute the Ratliff-Rush closure in some sense and, as an application, we shall reprove some of the main results of [**5, 11, 12**]. Finally, we reprove [**21,** Theorem 1] and [**2,** Theorem 1.6] with a much simpler proof, and we also give a counter example for [**21,** Question 3]. This example also states that [**2,** Theorem 1.8] does not hold, in general. For any unexplained notation or terminology, the reader is referred to [**3, 16**].

2. Ratliff-Rush closure, associated graded ring.

Proposition 2.1. *Let $d = 2$, x_1, x_2 be a superficial sequence of I and $J = (x_1, x_2)$. Let $k \in \mathbb{N}_0$ be such that*

$$J \cap I^n = JI^{n-1} \quad \textit{for all } n \geq k+1.$$

Then, $\widetilde{I^n} = I^n$ for all $n \geq 1$ if and only if $I^n : x_1 = I^{n-1}$ for $n = 1, \dots, k$.

Proof.

($\Rightarrow$) immediately follows by [**22,** Corollary 2.7].

($\Leftarrow$). From [**22,** Corollary 2.7], it is sufficient for us to prove $I^n : x_1 = I^{n-1}$ for all $n \geq k$. By using induction on n, it is enough to prove the result for $n = k+1$. For this, firstly, we prove that

$JI^k : x_1 = I^k$. However, it is an elementary fact that

$$JI^k : x_1 = (x_1 I^k + x_2 I^k) : x_1 = I^k + (x_2 I^k : x_1),$$

as well as

$$x_2 I^k : x_1 = x_2 I^{k-1}.$$

Hence, $JI^k : x_1 = I^k$. Therefore, by our assumption, we have

$$(J \cap I^{k+1}) : x_1 = I^k,$$

and thus, we have $I^{k+1} : x_1 = I^k$, as desired. $\square$

The next result immediately follows by Proposition 2.1.

Corollary 2.2. *Let $d = 2$, x_1, x_2 be a superficial sequence of I and $J = (x_1, x_2)$. Let $k \in \mathbb{N}_0$ be such that $r_J(I) = k$. Then, $\widetilde{I^n} = I^n$ for all $n \geq 1$ if and only if $I^n : x_1 = I^{n-1}$ for $n = 1, \ldots, k$.*

Corollary 2.3. *Let $d = 2$, x_1, x_2 be a superficial sequence of I and $J = (x_1, x_2)$ such that $r_J(I) = 2$. Then, $\widetilde{I^n} = I^n$ for all $n \geq 1$ if and only if $I^2 : x_1 = I$.*

The Hilbert-Samuel function of I is the numerical function that measures the growth of the length of R/I^n for all $n \in \mathbb{N}$. For all n large, this function $\lambda(R/I^n)$ is a polynomial in n of degree d

$$\lambda(R/I^n) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{n+d-i-1}{d-i},$$

where $e_0(I), e_1(I), \ldots, e_d(I)$ are called the *Hilbert coefficients* of I. Let

$$A = \bigoplus_{m \geq 0} A_m$$

be a Noetherian graded ring where A_0 is an Artinian local ring, A is generated by A_1 over A_0, and

$$A_+ = \bigoplus_{m > 0} A_m.$$

Let $H^i_{A_+}(A)$ denote the ith local cohomology module of A with respect to the graded ideal A_+, and set

$$a_i(A) = \max\{m \mid [H^i_{A_+}(A)]_m \neq 0\},$$

with the convention $a_i(A) = -\infty$, if $H^i_{A_+}(A) = 0$. The Castelnuovo-Mumford regularity is defined by $\operatorname{reg}(A) := \max\{a_i(A) + i \mid i \geq 0\}$.

Proposition 2.4. *Let $d = 2$ and J be a minimal reduction of I such that $r_J(I) = 2$. If $\widetilde{I} = I$, then we have the following*:

(i) $\operatorname{reg} G(I) = 2$.
(ii) $e_2(I) = \lambda(I^2/JI)$.

Proof. Case (i) follows by Corollary 2.3 and [**19,** Theorem 2.1, Corollay 2.2], and case (ii) follows from Corollary 2.3 and [**4,** Theorem 3.1]. □

Remark 2.5. Let $d = 2$, $\widetilde{I} = I$ and J be a minimal reduction of I. If $\operatorname{reg} G(I) = 3$, then, by [**19,** Lemma 1.2, Corollary 2.2], [**28,** Proposition 3.2] and Proposition 2.4, we have $r_J(I) = 3$.

The next result is an improvement of [**15,** Theorem 2.11] and [**17,** Proposition 16].

Proposition 2.6. *Let $d = 2$, $\widetilde{I} = I$ and J be a minimal reduction of I. Then, $r_J(I) = 2$ if and only if $P_I(n) = H_I(n)$ for $n = 1, 2$, where $H_I(n)$ and $P_I(n)$ are the Hilbert-Samuel function and the Hilbert-Samuel polynomial, respectively.*

Proof.

($\Rightarrow$). Let $r_J(I) = 2$. Then, by Corollary 2.3, $\widetilde{I^n} = I^n$ for all $n \geq 1$, and so, by [**17,** Proposition 16], we have $H_I(n) = P_I(n)$ for all $n = 1, 2$.

($\Leftarrow$) is clear by [**17,** Proposition 16]. □

Remark 2.7. Let J be a minimal reduction of I, $x_1 \in J$ and $\overline{I} = I/(x_1)$, $\overline{J} = J/(x_1)$. Then, by the definition of the reduction number, we have

(i) If $r_{\overline{J}}(\overline{I}) = k$ and $I^{k+1} : x_1 = I^k$, then $r_J(I) = k$.
(ii) If $d = 2$ and $I^2 : x_1 = I$, then $r_{\overline{J}}(\overline{I}) \leq 2$ if and only if $r_J(I) \leq 2$.

Lemma 2.8. *Let $d = 2$ and J be a minimal reduction of I such that $J \cap I^n = JI^{n-1}$ for $n = 1, \ldots, t$. If*

$$r_{\overline{J}}(\overline{I}) = k$$

and

$$\lambda(I^{n+1}/JI^n) = \lambda(\overline{I}^{n+1}/\overline{JI}^n)$$

for $n = t, \ldots, k-1$, then $I^{n+1} : x_1 = I^n$ for $n = 0, \ldots, k-1$.

Proof. By [**7,** Proposition 1.7(ii)], $(x_1) \cap I^n = x_1 I^{n-1}$ for $n = 1, \ldots, t$, and thus, $I^n : x_1 = I^{n-1}$ for $n = 1, \ldots, t$. Now, consider the exact sequence

$$(\dagger) \qquad 0 \longrightarrow I^{n+1} : x_1 / JI^n : x_1 \longrightarrow I^{n+1}/JI^n \longrightarrow \overline{I}^{n+1}/\overline{JI}^n \longrightarrow 0.$$

By our assumption, $I^{n+1} : x_1 = JI^n : x_1$ for $n = t, \ldots, k-1$. Assume that $yx_1 \in JI^t$. Then, we have $yx_1 = \alpha_1 x_1 + \alpha_2 x_2$ for some $\alpha_1, \alpha_2 \in I^t$. Hence, $(y - \alpha_1)x_1 = \alpha_2 x_2 \in x_2 I^t$, and, since x_1, x_2 is a regular sequence, we obtain $y - \alpha_1 = sx_2$ for some $s \in R$. Since $(y - \alpha_1)x_1 = sx_1x_2 \in x_2 I^t$, and x_2 is a non-zerodivisor, it follows that $sx_1 \in I^t$, and thus, $s \in I^t : x_1$. Therefore, $s \in I^{t-1}$, and it follows that $y \in I^t$. Thus, by repeating this argument, we obtain $I^{n+1} : x_1 = I^n$ for $n = 0, \ldots, k-1$, as desired. □

The following result was proven in [**5,** Theorem 3.10], [**14,** Theorem 2.4] and [**25,** Theorem 3.7]; here, we give a simplified proof.

Proposition 2.9. *Let J be a minimal reduction of I such that $J \cap I^n = JI^{n-1}$ for $n = 1, \ldots, t$ and $\lambda(I^{t+1}/JI^t) \leq 1$. Then,* $\operatorname{depth} G(I) \geq$ *$d - 1$.*

Proof. By using Sally's descent, we may reduce the problem to the case of $d = 2$. Set $r_{\overline{J}}(\overline{I}) = k$. Then, by using the exact sequence $(\dagger)$, we have

$$\lambda(\overline{I}^{n+1}/\overline{JI}^n) = \lambda(I^{n+1}/JI^n) \leq 1 \quad \text{for } n = t, \ldots, k-1.$$

From Lemma 2.8 , we have $I^{n+1} : x_1 = I^n$ for $n = 0, \ldots, k-1$. By [**14,** Proposition 1.1], we know that

$$\sum_{n\geq 0} \lambda(\widetilde{I^{n+1}}/J\widetilde{I^n}) = e_1(I) = e_1(\overline{I})$$
$$= \sum_{n=0}^{k-1} \lambda(I^{n+1}/JI^n) = \sum_{n=0}^{t-1} \lambda(I^{n+1}/JI^n) + k - t.$$

Therefore, from [**24,** Theorem 1.3], we have $r_J(I) \leq k$. Thus, by Lemma 2.8 and Corollary 2.2, we obtain $\widetilde{I^n} = I^n$ for all $n \geq 1$. Hence, $\operatorname{depth} G(I) \geq 1$, as required. □

Lemma 2.10. *Let $d = 2$ and $J = (x_1, x_2)$ be a minimal reduction of I such that $J \cap I^n = JI^{n-1}$ for all $n \geq 3$. If either $I^2 : x_1 = I$ or $I^2 : x_2 = I$, then $\widetilde{I^n} = I^n$ for all $n \geq 1$. In particular,* $\operatorname{depth} G(I) \geq 1$.

Proof. By using the same argument that was used in the proof of Proposition 2.1, the result immediately follows. □

Lemma 2.11. *Let $d = 2$ and $J = (x_1, x_2)$ be a minimal reduction of I such that $\lambda(J \cap I^2/JI) \leq 1$. Then, either $I^2 : x_1 = I$ or $I^2 : x_2 = I$.*

Proof. If $\lambda(J \cap I^2/JI + I^2 \cap (x_1)) = 1$, then $I^2 \cap (x_1) \subseteq JI$, and thus, $I^2 \cap (x_1) \subseteq [x_1 I + x_2 I] \cap (x_1)$. Therefore, $I^2 \cap (x_1) = x_1 I$, and hence, $I^2 : x_1 = I$. If $\lambda(J \cap I^2/JI + I^2 \cap (x_1)) = 0$, then $I^2 \cap (x_1) + Ix_2 = J \cap I^2$. Hence,

$$I^2 \cap (x_1 x_2) + Ix_2 = I^2 \cap (x_2),$$

and thus,

$$Ix_2 = I^2 \cap (x_2).$$

Therefore, $I^2 : x_2 = I$. □

The following result was proven in [**11,** Theorem 3.2] and [**12,** Corollary 1.5]; we give a much easier proof.

Proposition 2.12. *Let J be a minimal reduction of I such that $J \cap I^n = JI^{n-1}$ for all $n \geq 3$. If $\lambda(J \cap I^2/IJ) \leq 1$, then* $\operatorname{depth} G(I) \geq d - 1$.

Proof. By Sally's descent, we may assume that $d = 2$. Now, using Lemmas 2.10 and 2.11, the result follows. □

Theorem 2.13. *Let $d \geq 3$ and $k \in \mathbb{N}_0$ be such that $\widetilde{I^k} = I^k$. If $x_1, \ldots, x_d$ is a tame superficial sequence of I and $J = (x_1, \ldots, x_d)$ such that $J \cap I^n = JI^{n-1}$ for all $n \geq k+2$, then*

$$\mathfrak{a}^m I^n : x_1 = \mathfrak{a}^m I^{n-1}$$

for all $n \geq k+1$ and all $m \in \mathbb{N}_0$, where $\mathfrak{a} = (x_2, \ldots, x_d)$. In particular, $\widetilde{I^n} = I^n$ for all $n \geq k$.

Proof. We will proceed by induction on n. Assume that $n = k+1$. Then, by [**18,** Lemma 2.7] and our assumption, we have

$$\mathfrak{a}^m I^{k+1} : x_1 \subseteq \mathfrak{a}^m \widetilde{I^{k+1}} : x_1 = \mathfrak{a}^m \widetilde{I^k} = \mathfrak{a}^m I^k.$$

Therefore, $\mathfrak{a}^m I^{k+1} : x_1 = \mathfrak{a}^m I^k$ for all $m \in \mathbb{N}_0$. Assume that $n \geq k+1$ and that, for all t with $k+1 \leq t \leq n$ and all $m \in \mathbb{N}_0$,

$$\mathfrak{a}^m I^t : x_1 = \mathfrak{a}^m I^{t-1}.$$

We show that, for all $m \in \mathbb{N}_0$,

$$\mathfrak{a}^m I^{n+1} : x_1 = \mathfrak{a}^m I^n.$$

Let yx_1 be an element of $\mathfrak{a}^m I^{n+1}$. Then, $yx_1 \in \mathfrak{a}^m$ and, by using [**18,** Lemma 2.1], we obtain $y \in \mathfrak{a}^m$. Therefore, we can write the expression

$$y = \sum_{i_2+\cdots+i_d=m} r_{i_2\cdots d} x_2^{i_2} \cdots x_d^{i_d}.$$

Since the element yx_1 belongs to $\mathfrak{a}^m I^{n+1}$, as well, we obtain the following equalities

$$\sum_{i_2+\cdots+i_d=m} r_{i_2\cdots i_d} x_1 x_2^{i_2} \cdots x_d^{i_d} = yx_1 = \sum_{i_2+\cdots+i_d=m} s_{i_2\cdots i_d} x_2^{i_2} \cdots x_d^{i_d},$$

where $s_{i_2\cdots i_d} \in I^{n+1}$ for all $i_2, \ldots, i_d$ such that $i_2 + \cdots + i_d = m$. As $x_1, \ldots, x_d$ is a regular sequence in R, by equating coefficients in the previous expressions, we obtain

$$r_{i_2\cdots i_d} x_1 - s_{i_2\cdots i_d} \in (x_2, \ldots, x_d)$$

for all $i_2, \ldots, i_d$ such that $i_2 + \cdots + i_d = m$. Hence, $s_{i_2\cdots i_d} \in J \cap I^{n+1}$ and, by our assumption, we obtain $s_{i_2\cdots i_d} \in JI^n$ for all $i_2, \ldots, i_d$ such that $i_2 + \cdots + i_d = m$. Hence, returning to the equalities we wrote for

yx_1, we obtain $yx_1 \in \mathfrak{a}^m JI^n = \mathfrak{a}^{m+1}I^n + x_1\mathfrak{a}^m I^n$. Therefore, we have $\mathfrak{a}^m I^{n+1} \cap (x_1) \subseteq \mathfrak{a}^{m+1}I^n \cap (x_1) + x_1\mathfrak{a}^m I^n = x_1(\mathfrak{a}^{m+1}I^n : x_1) + x_1\mathfrak{a}^m I^n$. By applying the induction hypothesis, we get

$$\mathfrak{a}^m I^{n+1} \cap (x_1) \subseteq x_1\mathfrak{a}^{m+1}I^{n-1} + x_1\mathfrak{a}^m I^n = x_1\mathfrak{a}^m I^n.$$

This proves that $\mathfrak{a}^m I^{n+1} : x_1 \subseteq \mathfrak{a}^m I^n$, and thus, $\mathfrak{a}^m I^{n+1} : x_1 = \mathfrak{a}^m I^n$ for all $m \in \mathbb{N}_0$. In particular, if we set $m = 0$, then $I^{n+1} : x_1 = I^n$ for all $n > k$; hence, by **[22,** Corollary 2.7], $\widetilde{I^n} = I^n$ for all $n \geq k$, as desired. $\square$

The next result easily follows by Theorem 2.13.

Corollary 2.14. *Let $x_1, \ldots, x_d$ be a tame superficial sequence of I and $J = (x_1, \ldots, x_d)$.*

(i) *If $\widetilde{I} = I$ and $J \cap I^n = JI^{n-1}$ for all $n \geq 3$, then $\widetilde{I^n} = I^n$ for all $n \geq 1$. In particular,* $\operatorname{depth} G(I) \geq 1$.
(ii) *If $r_J(I) = 2$, then $\widetilde{I} = I$ if and only if* $\operatorname{depth} G(I) \geq 1$.
(iii) *Let $k \in \mathbb{N}_0$ be such that $r_J(I) = k+1$ and $\widetilde{I^k} = I^k$. Then, $\widetilde{I^n} = I^n$ for all $n \geq k$.*

The following example shows that the equality of Corollary 2.14 (ii) may occur.

Example 2.15. Let K be a field, $R = K[\![x, y]\!]$, $I = (x^6,\ x^4y^2,\ x^3y^3,\ x^2y^4,\ xy^5, y^6)$ and $J = (x^6, y^6 + x^4y^2)$. Then, $r_J(I) = 2$, $\operatorname{depth} G(I) = 1$, and thus, $G(I)$ is not Cohen-Macaulay.

3. Invariance of a length. Let $J = (x_1, \ldots, x_d)$ be a minimal reduction of I. In **[29]**, Wang defined the following exact sequence for all n, k as

$$(*) \quad 0 \longrightarrow T_{n,k} \longrightarrow \oplus^{\binom{k+d-1}{d-1}} I^n/JI^{n-1} \xrightarrow{\phi_k} J^kI^n/J^{k+1}I^{n-1} \longrightarrow 0,$$

where $\phi_k = (x_1^k, x_1^{k-1}x_2, \ldots, x_1^{k-1}x_d, \ldots, x_d^k)$ and $T_{n,k} = \ker(\phi_k)$. He also showed that $T_{1,k} = 0$ for all k and, if $d = 1$, then $T_{n,k} = 0$ for all n, k. Using the exact sequence $(*)$, we derive the following, simple lemma, and the proof is left to the reader.

Lemma 3.1. *Let $t \in \mathbb{N}_0$ and $J = (x_1, \ldots, x_d)$ be a minimal reduction of I. Then, we have the following*:

(i) *if $J \cap I^n = JI^{n-1}$ for $n = 1, \ldots, t$, then $T_{n,k} = 0$ for $n = 1, \ldots, t$ and all k.*
(ii) *If I is integrally closed, then $T_{2,k} = 0$ for all k. In particular, if $I = m$, then $T_{2,k} = 0$ for all k.*

The next lemma is well known; see the proof of [**5,** Proposition 2.1].

Lemma 3.2. *Let $J = (x_1, \ldots, x_d)$ be a minimal reduction of I. Then, $\lambda(I/J) = e_0(I) - \lambda(R/I)$ and*

$$\lambda(I^{n+1}/J^n I) = e_0(I)\binom{n+d-1}{d} + \lambda(R/I)\binom{n+d-1}{d-1} - \lambda(R/I^{n+1})$$

for $n \geq 1$, which are independent of J.

In [**21**], Puthenpurakal proved that $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2)$ is independent of the minimal reduction J of $\mathfrak{m}$ and, subsequently, Ananthnarayan and Huneke [**2**] extended it for n-standard admissible I-filtrations.

The following result was proven in [**2,** Theorem 3.5] and [**21,** Theorem 1]. We reprove it here with a much simpler proof.

Theorem 3.3. *Let $t \in \mathbb{N}_0$ and $J = (x_1, \ldots, x_d)$ be a minimal reduction of I. If $J \cap I^n = JI^{n-1}$ for $n = 1, \ldots, t$, then $\lambda(I^{n+1}/JI^n)$ is independent of J for $n = 1, \ldots, t$.*

Proof. We have

$$\lambda(I^{n+1}/JI^n) = \lambda(I^{n+1}/J^n I) - \sum_{k=1}^{n-1} \lambda(J^k I^{n+1-k}/J^{k+1} I^{n-k}).$$

Therefore, by Lemma 3.1 and the exact sequence $(*)$, we obtain

$$\lambda(I^{n+1}/JI^n) = \lambda(I^{n+1}/J^n I) - \sum_{k=1}^{n-1} \binom{k+d-1}{d-1} \lambda(I^{n+1-k}/JI^{n-k}).$$

Now by using Lemma 3.2 and induction on n, the result follows. $\square$

The next example is a counterexample for [**21,** Question 3], and it also states that [**2,** Theorem 1.8] does not hold, in general. The computations were performed by using `Macaulay2` [**9**], `CoCoA` [**1**] and `Singular` [**10**].

Example 3.4. Let K be a field and $S = K[\![x, y, z, u, v]\!]$, where $I = (x^2 + y^5, xy + u^4, xz + v^3)$. Then, $R = S/I$ is a Cohen-Macaulay local ring of dimension two, ideals $J_1 = (y, z)R$ and $J_2 = (z, u)R$ are minimal reduction of $\mathfrak{m} = (x, y, z, u, v)R$ and $\lambda(\mathfrak{m}^4/J_1\mathfrak{m}^3) = 17$, $\lambda(\mathfrak{m}^4/J_2\mathfrak{m}^3) = 20$.

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