JOURNAL OF COMMUTATIVE ALGEBRA
Volume 10, Number 4, Winter 2018

THE CLIQUE IDEAL PROPERTY

THOMAS G. LUCAS

ABSTRACT. For a commutative ring R, one can form a graph $\Gamma(R)^*$ whose vertices are the zero divisors of R (including 0) and whose edges are the pairs $\{a, b\}$ where $ab = 0$ with $a \neq b$. For this graph, a clique is a nonempty subset X such that $ab = 0$ for all $a \neq b$ in X. If R is a finite ring, there is always a maximum clique of $\Gamma(R)^*$–a clique X such that $|X| \geq |Y|$ for all cliques Y. We say that a finite ring R has the clique ideal property if each maximum clique of $\Gamma(R)^*$ is an ideal of R. If $R = S \oplus T$ where both S and T are finite rings with the clique ideal property and neither S nor T is a field, then R has the clique ideal property. The converse does not hold. For each positive integer $n > 1$, the ring $R = \mathbb{Z}_n[\mathrm{x}]/(\mathrm{x}^2)$ is a finite ring with the clique ideal property. In contrast, $\mathbb{Z}_n$ has the clique ideal property if and only if n is either a prime or a perfect square.

Throughout this article, each ring is assumed to be finite and commutative with a nonzero identity. In the event a ring R is (equal or) isomorphic to a sum $\bigoplus_{i=1}^n S_i$ for some family of rings $\{S_i\}_{i=1}^n$ where $n > 1$, then each S_i is presumed to have a nonzero identity (equivalently, no S_i is $\{0\}$).

For a ring R, we may form a graph $\Gamma(R)^*$ whose vertices are the zero divisors of R (including 0) where a pair of distinct vertices a and b form an edge if $ab = 0$. This is in contrast to the graph $\Gamma(R)$ where the vertices are restricted to the set of nonzero zero divisors of R (as in [**3, 8**]). Recall that a nonempty subset X of a graph is a *clique* if it forms a complete subgraph. In the case $X \subseteq \Gamma(R)^*$, X is a clique if and only if $xy = 0$ for all $x \neq y$ in X. A *maximum clique* of $\Gamma(R)^*$ is a clique X such that $|X| \geq |Z|$ for all cliques Z of $\Gamma(R)^*$. The size of a maximum clique is the *clique number* of $\Gamma(R)^*$. Note that a

---

2010 AMS *Mathematics subject classification.* Primary 13M99, Secondary 05C69.

Keywords and phrases. Zero divisor graph, clique, maximum clique.

Received by the editors on December 31, 2014, and in revised form on August 4, 2016.

DOI:10.1216/JCA-2018-10-4-499

maximal clique is simply a clique that is not a proper subset of some other clique; it need not be a maximum clique. For example, both $Y = \{0, 2, 16\}$ and $X = \{0, 4, 8, 16, 24\}$ are maximal cliques of $\Gamma(\mathbb{Z}_{32})^*$ (in fact, X is a maximum clique of this graph, see [**4,** Proposition 2.3]). Our concern is with the cliques of $\Gamma(R)^*$. In particular, we are interested in characterizing the finite rings for which each maximum clique is an ideal (thus, the inclusion of 0 as a vertex).

Consider the ring $R = \mathbb{Z}_{p^{2n}}$ where p is a prime. Then

$$Z(R) = \{s \mid p|s,\ 0 \leq s < p^{2n}\}$$

and the ideal $I = p^n R$ is the unique maximum clique of $\Gamma(R)^*$ (Theorem 3.1). For an alternate example, suppose $R = \mathbb{Z}_4 \oplus \mathbb{Z}_9$.Then $I = 2\mathbb{Z}_4 \oplus 3\mathbb{Z}_9$ is both a maximum clique of $\Gamma(R)^*$ and an ideal of R. On the other hand, $\{0, 2, 4\}$ is a maximum clique of $\Gamma(\mathbb{Z}_8)^*$ but it is not an ideal. Also, both

$$\{(0,0), (0,1), (1,0)\}$$

and

$$\{(0,0), (0,2), (1,0)\}$$

are maximum cliques of $\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_3)^*$, but neither is an ideal of $\mathbb{Z}_2 \oplus \mathbb{Z}_3$. We say that a finite ring R has the *clique ideal property* (or more simply, has *CIP*) if each maximum clique of $\Gamma(R)^*$ is an ideal of R. A consequence of Lemma 1.1 is that, if R is a finite reduced ring that is not a field, then it does not have the clique ideal property. An alternate proof of this can be derived from the fact that, if R is a reduced ring with finitely many minimal primes $P_1, P_2, \ldots, P_n$ such that $n > 1$, then a maximum clique of $\Gamma(R)^*$ is a set of the form

$$\{0, a_1, a_2, \ldots, a_n\}$$

where

$$a_i \in \bigcap_{j \neq i} P_j \setminus P_i$$

for each i (see [**4,** Theorem 3.3]). As we have restricted the vertices to only the zero divisors (including 0), each finite field trivially satisfies the clique ideal property.

For an ideal I of R, we say that I is a *clique ideal* of R if $I^2 = (0)$. In addition, I is a *maximal clique ideal* if I is a clique ideal that is not properly contained in some other clique ideal, and I is a *maximum clique ideal* if I is a (maximal) clique ideal such that $|I| \geq |J|$ for all clique ideals J. Note that a clique ideal is also a clique of $\Gamma(R)^*$; the converse holds for ideals when R is local (Lemma 1.1).

On the other hand, the nonzero idempotent ideal $J = \{(0,0),(1,0)\}$ is a clique of $\Gamma(\mathbb{Z}_2 \oplus \mathbb{Z}_2)^*$ but neither a clique ideal nor a maximal clique. If R is a (finite) local ring of odd characteristic, then Theorem 1.3 shows that R has CIP if and only if it contains at least one ideal that is a maximum clique of $\Gamma(R)^*$. In contrast, the ring in Example 3.11 is local of characteristic 2 with a maximum clique that is an ideal and a maximum clique that is not an ideal. It turns out that it is "almost true" that a finite local ring has CIP if and only if it contains at least one ideal that is a maximum clique. In Theorem 1.6, we show that, if R is a finite local ring that contains a set X that is a maximum clique but is not an ideal and there is an ideal I that is a maximum clique, then $|R| = 16$ and $|X| = 4 = |I|$. In addition to the ring in Example 3.11, examples exist of such rings with characteristic 4 (Examples 3.9 and 3.10) and with characteristic 8 (Example 3.8). Up to isomorphism, these are the only four finite rings with both a maximum clique that is an ideal and a maximum clique that is not an ideal.

Primarily, Section 1 is devoted to the local case. It begins with a few general results dealing with cliques, maximal and maximum cliques, and then looks at CIP in the local case. As mentioned above, a finite local ring R of odd characteristic has CIP if and only if it has at least one ideal that is a maximum clique of $\Gamma(R)^*$. More or less a restricted "contrapositive" statement holds for the even characteristic case. If R is finite and local with even characteristic and there is a maximum clique X of $\Gamma(R)^*$ such that $|X| \geq 5$ and X is not an ideal, then no ideal of R is a maximum clique of $\Gamma(R)^*$ (Theorem 1.3).

In Section 2, we look at the finite nonlocal case. Throughout the section, we make use of the fact that, if R is not local, then it is (naturally) isomorphic to

$$\bigoplus_{i=1}^{n} R_{M_i},$$

where $\{M_1, M_2, \ldots, M_n\}$ is the set of maximal ideals of R. As mentioned above, if R is a reduced ring that is not local, then it does not have CIP. If R is not local, then it also fails to have CIP if it is "partially reduced" in the sense that it is isomorphic to a sum $S \oplus T$ where S is reduced (Lemma 2.1). Note that such a sum exists if and only if R_{M_i} is a field for some maximal ideal M_i of R.

As a first positive result of Section 2, we show that, if S and T are finite rings such that neither is a field and each has an ideal that is a maximum clique of the respective zero divisor graphs $\Gamma(S)^*$ and $\Gamma(T)^*$, then $S \oplus T$ has CIP, and each maximum clique of $\Gamma^*(S \oplus T)$ has the form

$$I_S \oplus I_T,$$

where I_S is maximum clique of $\Gamma(S)^*$ and I_T is a maximum clique of $\Gamma(T)^*$, Theorem 2.3. Corollaries 2.4 and 2.5 extend the result to sums of finite families $\{S_i\}_{i=1}^n$ of finite rings for $n > 1$. Thus, the ring R in Example 3.11 is such that $W = R \oplus R$ has the clique ideal property even though R does not. The ring W has two maximal ideals:

$$M \oplus R \quad \text{and} \quad R \oplus M,$$

where M is the maximal ideal of R. Localizing W at either of these maximal ideals yields a ring that is naturally isomorphic to R. Thus, a finite direct sum of (finite) rings can have the clique ideal property even if one or more of the summands does not, and a ring may have the clique ideal property even though some localization at a maximal ideal does not.

In contrast, if V is a local ring that has CIP and F is a finite field (so a ring that trivially has CIP), then $V \oplus F$ does not have CIP. The maximal ideals of $V \oplus F$ are $V \oplus (0)$ and $N \oplus F$ where N is the maximal ideal of V. Localizing at $V \oplus (0)$ produces a ring isomorphic to F, and localizing at $N \oplus F$ produces a ring that is isomorphic to V, so both localizations have CIP. Hence, CIP may seem rather far from being a local property. However, if R has odd characteristic and is not a field, then it has CIP if and only if, for each maximal ideal M, R_M has CIP and is not a field, Corollary 2.10. In Theorem 2.7, we show that, if R is a finite ring that is not local and R_M has CIP for each maximal ideal M, then R has CIP if and only if there is no maximal ideal $N \in \mathrm{Max}(R)$ such that R_N is a field. A characterization of when a finite ring R that is not local has CIP is provided in Theorem 2.9. Specifically, the following are equivalent:

(1) R has CIP;

(2) some ideal of R is a maximum clique of $\Gamma(R)^*$;

(3) for each maximal ideal M of R, R_M is not a field, and some ideal of R_M is a maximum clique of $\Gamma(R_M)^*$;

(4) if

$$R \cong \bigoplus_{i=1}^{n} S_i$$

for some family of rings $\{S_i\}_{i=1}^n$ with $n > 1$, then no S_i is a field, and each S_i has an ideal that is a maximum clique of $\Gamma(S_i)^*$.

Section 3 is devoted to examples and a few special cases. Several of the results in this section deal with the idealization of a module. For a ring R and R-module B, the idealization of B (over R) is formed from $R \times B$ by defining addition and multiplication as follows: $(s, a) + (t, b) = (s + t, a + b)$ and $(s, a)(t, b) = (st, sb + ta)$. A standard notation for this ring is $R(+)B$ (see for example, [**6,** Section 25]). In Theorem 3.4, we give a complete characterization of when $R = \mathbb{Z}_{p^n}(+)B$ has CIP when p is a prime, n is a positive integer and B is a finite $\mathbb{Z}_{p^n}$-module. A consequence is that $\mathbb{Z}_{p^n}(+)\mathbb{Z}_{p^n}$ has CIP for each prime p and positive integer n, Corollary 3.6. Unlike $Z_{p^{2n}}$, $\mathbb{Z}_{p^n}(+)\mathbb{Z}_{p^n}$ has multiple maximum cliques, provided $n \geq 2$. In the case $n = 2k$ for some positive k, each ideal of the form

$$p^m Z_{p^n}(+)p^{n-m}\mathbb{Z}_{p^n}$$

is a maximum clique for $k \leq m \leq n$, Corollary 3.6. If, instead, $n = 2k + 1$, then $p^m\mathbb{Z}_{p^n}(+)p^{n-m}\mathbb{Z}_{p^n}$ is a maximum clique for each $k + 1 \leq m \leq n$.

The ring R in Example 3.14 is local and contains a pair of maximum cliques X and Y such that neither is an ideal and $\mathrm{Ann}(Y) \supsetneq \mathrm{Ann}(X)$. By Theorem 1.2 (and the fact that R is local), $q^3 = 0$ for all $q \in X \cup Y$, $\mathrm{Ann}(Y) = \{z \in Y \mid z^2 = 0\}$ and $\mathrm{Ann}(X) = \{t \in X \mid t^2 = 0\}$.

1. Finite local rings.

Lemma 1.1. *Let R be a finite ring that is not a field.*

(i) *If X is a clique of $\Gamma(R)^*$, then X is a maximal clique if and only if $X \supseteq \mathrm{Ann}(X)$.*
(ii) *If X is a maximal clique of $\Gamma(R)^*$, then X is an ideal if and only if $x^2 = 0$ for each $x \in X$.*
(iii) *If I is an ideal of R, then it is a maximal clique of $\Gamma(R)^*$ if and only if $I = \mathrm{Ann}(I)$.*
(iv) *If I is an ideal of R and R is local, then I is a clique of $\Gamma(R)^*$ if and only $I^2 = (0)$.*

Proof. For a clique Y, if $b \in \mathrm{Ann}(Y) \setminus Y$, then $Y \cup \{b\}$ is a strictly larger clique of $\Gamma(R)^*$. Hence, each maximal clique contains its annihilator. Conversely, if $Y \supseteq \mathrm{Ann}(Y)$, then certainly no element can be added to Y; thus, Y is a maximal clique if and only if $Y \supseteq \mathrm{Ann}(Y)$.

For (ii), let X be a maximal clique. If $x^2 = 0$ for each $x \in X$, then for all $a, b, c \in X$ (not necessarily distinct), we have $c(a+b) = ca + cb = 0$. Since X is a maximal clique, $a + b \in X$. Also, for $r \in R$, $(rc)a = r(ca) = 0$ so that we also have $rc \in X$. It follows that X is an ideal of R.

For the converse, suppose that X is an ideal (and a maximal clique). Since R is not a field, each maximal clique contains at least one nonzero element. We have two cases to consider. If $a \neq b$ are both nonzero elements of X, then $a + b \in X \setminus \{a, b\}$ and $ab = 0 = a(a + b)$. It follows that $a^2 = 0$. The other possibility is that $X = \{0, c\}$. Since $X \supseteq \mathrm{Ann}(X)$ (and $c \neq 0$ is a zero divisor), we have $c^2 = 0$.

For the statements in (iii) and (iv), let I be an ideal of R. If $I^2 = (0)$, then, clearly, I is a clique of $\Gamma(R)^*$ (even if R is not local). In particular, if $I = \mathrm{Ann}(I)$, then I is a clique. Moreover, in this case, no element of $R \setminus I$ can annihilate each element of I; thus, I is a maximal clique of $\Gamma(R)^*$.

For the reverse implication in (iii), if I is a maximal clique, then $z^2 = 0$ for each $z \in I$ and $I \supseteq \mathrm{Ann}(I)$ by (i) and (ii). Since $zw = 0$ for all $z \neq w$ in I, we have $I = \mathrm{Ann}(I)$.

To complete the proof of (iv), suppose that R is local and I is a clique of $\Gamma(R)^*$. As above, if $a \neq b$ are both nonzero elements of I, then we have $a^2 = 0 = b^2$. The remaining case is when $I = \{0, c\}$ for some nonzero c. Since R is local and $c^2 \in I$, it must be that $c^2 = 0$ as the only idempotents of a local ring are 1 and 0. Thus, no matter the size of I, $I^2 = (0)$ whenever it is a clique. □

From Lemma 1.1, a necessary condition for an ideal to be a maximal clique is for it to be nilpotent. Thus, if R has the clique ideal property, each maximum clique is contained in every maximal ideal.

Theorem 1.2. *Let R be a finite local ring that is not a field, and let $S = \{b \in R \mid b^2 = 0\}$.*

(a) *If Z is a maximum clique of $\Gamma(R)^*$, then $z^3 = 0$ for each $z \in Z$.*
(b) *If X is a maximum clique that is not an ideal, then*
  (i) *there is an element $x \in X$ such that $x^2 \neq 0$ and $x^3 = 0$,*
  (ii) $\mathrm{Ann}(X) = \{y \in X \mid y^2 = 0\}$ *is the largest ideal contained in X, and*
  (iii) $XS \subseteq \mathrm{Ann}(X)$.

Proof. Since R is a finite local ring, each zero divisor is nilpotent. For a nilpotent $b \in R$, let

$$\rho(b) = \min\{i \in \mathbb{N} \mid b^i = 0\}.$$

Suppose that W is a clique with an element $w \in W$ such that $\rho(w) = k > 4$. Let $i = \lfloor k/2 \rfloor$ and $j = i + 1$. Since $i \geq (k-1)/2 > 1$, $i + j \geq k$. Hence, $w^i w^j = 0$ with $w^i \neq w^j$. Also, $w \cdot w^j \neq 0$, so neither w^i nor w^j is in W. It follows that the set $\{w^i, w^j\} \cup W \setminus \{w\}$ is a strictly larger clique of $\Gamma(R)^*$.

Next, suppose that Y is a clique with an element $y \in Y$ such that $\rho(y) = 4$. Clearly, y, y^2 and $y^2 + y^3$ are distinct elements of R and $y \cdot y^2 = y^3 = y(y^2 + y^3)$. Thus, neither y^2 nor $y^2 + y^3$ is in Y. On the other hand, $y^2(y^2 + y^3) = 0$, so $\{y^2, y^2 + y^3\} \cup Y \setminus \{y\}$ is a strictly larger clique of $\Gamma(R)^*$. Therefore, $z^3 = 0$ for each $z \in Z$ when Z is a maximum clique of $\Gamma(R)^*$.

Assume that X is a maximum clique of $\Gamma(R)^*$ which is not an ideal of R. Since R is a finite local ring that is not a field, $|R| = p^m$ for some prime p and positive integer $m > 1$. In this case, the maximal ideal is both nonzero and nilpotent. Hence, it has a nonzero annihilator. In addition, if I is an ideal of R, then $|I| = p^k$ for some integer $k \geq 0$. Thus, $|\operatorname{Ann}(X)| = p^a$ for some positive integer a. By Lemma 1.1, $X \supseteq \operatorname{Ann}(X)$, and there is an element $x \in X$ such that $x^2 \neq 0$. From (a), we have $x^3 = 0$.

Let $w \in X \setminus \{x\}$. Then, $xw = 0 = x \cdot x^2$. It follows that $x^2 \in \operatorname{Ann}(X)$. Hence, $|X| \geq 3$. Since $\operatorname{Ann}(X) \subseteq X$, each $r \in \operatorname{Ann}(X)$ is such that $r^2 = 0$. Thus,

$$\operatorname{Ann}(X) \subseteq \{y \in X \mid y^2 = 0\}.$$

For the reverse containment, simply note that, if $v \in \{y \in X \mid y^2 = 0\}$, then $v^2 = 0 = vw$ for each $w \in X \setminus \{v\}$. Hence, $v \in \operatorname{Ann}(X)$, and we have that $\operatorname{Ann}(X) = \{y \in X \mid y^2 = 0\}$. Since $x^2 \neq 0$, $\operatorname{Ann}(X) \subsetneq X$. Also note that, since x is nilpotent and $x^2 \neq 0$, $1 + x$ is a unit of R and $x \neq x(1 + x)$. Hence, $x(x(1 + x)) = x^2 + x^3 = x^2 \neq 0$, and thus, $x(1 + x)$ is not in X. It follows that $\operatorname{Ann}(X)$ contains each ideal of R that is contained in X.

For (iii), note that, if $s \in S$ and $x \in X$, then $(xs)^2 = 0$. Therefore, from above, we have $xs \in \operatorname{Ann}(X)$ if and only if $xs \in X$. Thus, to see that $XS \subseteq \operatorname{Ann}(X)$, suppose, by way of contradiction, that there are elements $t \in S$ and $q \in X$ such that $qt \notin X$. Since $\operatorname{Ann}(X) = \{y \in X \mid y^2 = 0\}$ is an ideal of R, $q \in X \setminus \operatorname{Ann}(X)$. Hence, we can replace q by qt to obtain a new maximum clique

$$V = \{qt\} \cup X \setminus \{q\}.$$

Since $(qt)^2 = 0$, $qt \in \operatorname{Ann}(V)$. As only q was replaced and $q \notin \operatorname{Ann}(X)$, $\operatorname{Ann}(V) = \operatorname{Ann}(X) \cup \{qt\}$, and we have

$$|\operatorname{Ann}(V)| = |\operatorname{Ann}(X)| + 1 = p^a + 1,$$

a contradiction of $\operatorname{Ann}(V)$ being an ideal. Hence, $XS \subseteq X$, and we have $XS \subseteq \operatorname{Ann}(X)$. □

The next three results show how rare it is for a finite local ring R to have an ideal that is a maximum clique of $\Gamma(R)^*$ and a subset X

that is a maximum clique of $\Gamma(R)^*$ but is not an ideal of R.

Theorem 1.3. *Let R be a finite local ring.*

(i) *If R has odd characteristic, then it has* CIP *if and only if at least one ideal is a maximum clique of $\Gamma(R)^*$.*
(ii) *If R has even characteristic and there is a maximum clique X of $\Gamma(R)^*$ such that X is not an ideal and $|X| \geq 5$, then no ideal of R is a maximum clique of $\Gamma(R)^*$.*

Proof. Since R is local, there is a prime p and positive integer r such that $|R| = p^r$. In addition, for each nonzero ideal H, $|H| = p^h$ for some positive integer h. In addition, if $b \in R \setminus H$, then

$$|bR + H| \geq p^{h+1} = p \cdot |H|.$$

A consequence is that, if B is a nonzero ideal containing p^n elements, then a minimal generating set for B requires at most n elements.

There is nothing to prove in the case R is a field, the only clique is (0). Also, $\Gamma(R)^*$ always has at least one maximum clique. Thus, if R has CIP, it has at least one ideal that is maximum clique, and each maximum clique is an ideal (no matter what the characteristic). For the reverse implication with regard to (i), we will prove the contrapositive. Thus, for both odd and even characteristic, we may assume there is a maximum clique X of $\Gamma(R)^*$ that is not an ideal. Then, $X \supsetneq \mathrm{Ann}(X)$ by Lemma 1.1. In the even characteristic case, we further assume that $|X| \geq 5$.

For both cases, we may assume that I is a clique ideal. Then, $I^2 = (0)$ (Lemma 1.1) and so,

$$I \subseteq S = \{f \in R \mid f^2 = 0\}.$$

Since $|X| > |\mathrm{Ann}(X)|$ and our goal is to show $|X| > |I|$, we may further assume $|X| \geq |I| = p^i > |\mathrm{Ann}(X)| = p^c$. Let $T = \{x_1, x_2, \ldots, x_t\} = X \setminus \mathrm{Ann}(X)$, and let J be the ideal generated by X. Then, $\mathrm{Ann}(X) = \mathrm{Ann}(J)$. Also, $t \geq p^i - p^c$ since $|X| = t + p^c \geq p^i$. By Theorem 1.2, $XI \subseteq \mathrm{Ann}(X)$ and $x_j^2 \neq 0$ for each $1 \leq j \leq t$.

Let $J_1 = x_1R + \mathrm{Ann}(X)$ and, for $1 < k \le t$, recursively define ideals $J_k = x_kR + J_{k-1}$ (so $J_t = J$). Since $x_1 \notin \mathrm{Ann}(X)$, we have

$$|J_1| \ge p \cdot |\mathrm{Ann}(X)|.$$

Also, since $x_nx_m = 0$ for all $n \ne m$, $x_jJ_k = (0)$ for $k < j \le t$. On the other hand, $x_j^2 \ne 0$, and thus, $x_j \notin J_k$. It follows that

$$|J_k| \ge p \cdot |J_{k-1}| \ge p^k \cdot |\mathrm{Ann}(X)|$$

for all $k > 1$. In particular, we have

$$|J| = |J_t| \ge p^t \cdot |\mathrm{Ann}(X)| = p^t \cdot p^c.$$

In addition $|(J+I)/I| = |J/J \cap I| \ge p^{t+c-i}$.

As noted above, $XI \subseteq \mathrm{Ann}(X)$. Hence, we have

$$XI = JI \subseteq I \cap \mathrm{Ann}(X)$$

since I is an ideal. Also, $XI \ne (0)$ as $|I| > |\mathrm{Ann}(X)|$. Thus, $|I \cap \mathrm{Ann}(X)| \ge p$, and we have

$$|I/(I \cap \mathrm{Ann}(X))| = p^h \le p^{i-1}$$

for some positive integer h with $i-c \le h \le i-1$. Since $|I \cap \mathrm{Ann}(X)| = p^{i-h}$ and $|I| = p^i$, a minimal generating set for $I \cap \mathrm{Ann}(X)$ requires at most $i-h$ elements, and one for I requires at most h more elements. Hence, we may select a set $E = \{e_1, e_2, \ldots, e_{i-h}\}$ that generates $I \cap \mathrm{Ann}(X)$ (not necessarily a minimal set) and an additional set

$$D = \{d_1, d_2, \ldots, d_h\} \subseteq I \setminus \mathrm{Ann}(X)$$

such that $D \cup E$ generates I.

For each $q \in J$, we may define an R-module homomorphism

$$\varphi_q : I \longrightarrow I \cap \mathrm{Ann}(X)$$

by $\varphi_q(b) = qb$. Note that $\varphi_q(d) = 0$ for each $d \in I \cap \mathrm{Ann}(X)$. Thus, $\varphi_q(e_j) = 0$ for all $e_j \in E$. It follows that each φ_q is determined by the values of the $\varphi_q(d_k)$s. For each d_k, there are at most $p^{i-h} = |I \cap \mathrm{Ann}(X)|$ potential values. Hence, $|N| \le p^{h(i-h)}$, where N is the set of distinct R-module homomorphisms of the form φ_q (for $q \in J$). In addition, φ_q is the zero map for each $q \in J \cap I$ since $I^2 = (0)$.

If $p^{t+c-i} > |N|$, then there are at least two elements $q, q' \in J$ such that $\varphi_q = \varphi_{q'}$ with $q - q' \in J \setminus I$. It follows that $q - q'$ is a nonzero annihilator of I that is not in I, and thus, I is not a maximum clique, nor even a maximal clique.

We will use induction to show that $p^{t+c-i} > p^{h(i-h)}$. When p is odd, this inequality holds for all positive integers t, c and i such that $t + p^c \geq p^i > p^c$. On the other hand, if $p = 2$, then the inequality fails for the case $t = 2 = i > c = 1$ as, in that case, $h = 1 = i - h$, and we have $t + c - i = 1 = 1^2$. However, the inequality

$$p^{t+c-i} > p^{h(i-h)}$$

does hold for all t, c and i when $p = 2$ and $|X| = t + 2^c \geq 5$. Also, note that, in the special case $t \geq 3 > i = 2 > c = 1$ and $p = 2$, we again have $h = 1 = i - h$ but now also have

$$2^{t+c-i} = 2^{t-1} \geq 2^2 > 2 = 2^1.$$

Thus, for the case $p = 2$, we may restrict to the case $i \geq 3$.

Since $t+p^c \geq p^i > p^c$, it suffices to show that $p^i - p^c + c - i > h(i-h)$ for all positive integers $i > c$ (with the above restriction to $i \geq 3$ when $p = 2$).

Consider the function $g(z) = z(i - z)$. The maximum value of this function is $i^2/4$ and occurs when $z = i/2$. Note that, in some cases, $i/2$ will not be an integer and it may be outside the range of h (which is restricted to $i - c \leq h \leq i - 1$). Also note that

$$p^i - p^c = p^c(p^{i-c} - 1) \geq 2^c(2^{i-c} - 1) = 2^i - 2^c$$

for all primes p. Thus, except for the special case that $c = 1$ and $i = 2$, it suffices to show $2^i - 2^c > i^2/4 - c + i$. From the argument above, if $c = 1$ and $i = 2$, then we only need to consider the case p is odd.

For the initial steps, we split the proof into odd and even characteristics. For p odd, we start with $c = 1$ and $i = 2$, and for $p = 2$, we establish the inequality $2^i - 2^c > i^2/4 - c + i$ for $i = 3$ and $c \in \{1, 2\}$.

If $c = 1$, $i = 2$ and p is odd, then $i/2 = 1$ and

$$\begin{aligned} p^i - p^c &= p^2 - p = p(p-1) \geq 2p > 2 \\ &= 2 - 1 + 1 = i - c + i^2/4 = (4(i-c) + i^2)/4. \end{aligned}$$

If $p = 2$ and $i = 3$, then $i/2 = 3/2$. For the case $c = 1$, we have

$$2^i - 2^c = 8 - 2 = 6 > 17/4 = 3 - 1 + 9/4 = i - c + i^2/4;$$

and, when $c = 2$, we have

$$2^i - 2^c = 8 - 4 = 4 > 13/4 = 3 - 2 + 9/4 = i - c + i^2/2.$$

Thus, in all three cases, $p^i - p^c > i - c + i^2/4$. In addition, we have

$$p^i - p^c = p^3 - p > 2^3 - 2 > i - c + i^2/4$$

for all odd primes p when $c = 1$ and $i = 3$; and

$$p^i - p^c = p^3 - p^2 > 2^3 - 2^2 > i - c + i^2/4$$

when p is odd, $c = 2$ and $i = 3$.

Next, we use induction on c to establish the inequality in the special case $i = c+1$. Assume that the inequality $p^i - p^c > i - c + i^2/4$ holds for $c = k \geq 2$ and $i = k+1$, and consider the case $c = k+1$, $i = k+2$. We have

$$\begin{aligned} 2^i - 2^c &= 2^{k+2} - 2^{k+1} \\ &= 2(2^{k+1} - 2^k) > 2(4 + (k+1)^2)/4 \\ &= ((4 + (k+1)^2) + (4 + (k+1)^2))/4 \\ &> (4 + (k+1)^2 + 2k + 3)/4 \\ &= (4 + (k+2)^2)/4 = (4(i-c) + i^2)/4, \end{aligned}$$

as desired. It follows that $p^i - p^c > i - c + i^2/4$ for all primes p when $i = c + 1$ (and $c \geq 2$ when $p = 2$).

Finally, we do induction on i for fixed c. Assume that the inequality $2^i - 2^c > i - c + i^2/4$ holds for the case $i = k \geq c+1$ (with $k \geq 3$ when $c = 1$), and consider the case $i = k + 1$. We have

$$\begin{aligned} (4(k+1-c) + (k+1)^2)/4 = ((4(k-c) + k^2)/4) \\ + ((4 + 2k + 1)/4) < 2^k - 2^c \\ + ((2k+5)/4) < 2^k - 2^c + 2^k \\ = 2^{k+1} - 2^c = 2^i - 2^c. \end{aligned}$$

Therefore, by induction, $p^i - p^c \geq 2^i - 2^c > i - c + h(i-h)$ for all primes p and all integers $i > c \geq 1$ (with $i \geq 3$ when $c = 1$ and $p = 2$).

From the arguments above, we may now conclude that

$$|J| \geq p^{t+c-i} > p^{h(i-h)} \geq |N|,$$

and therefore, there is a pair of distinct elements $q, q' \in J$ such that $\varphi_q = \varphi_{q'}$ with $q - q' \in J \setminus I$. It follows that $q - q' \in \mathrm{Ann}(I) \setminus I$, and thus, I is not a maximal clique. Therefore, no ideal of R is a maximum clique when R has a maximum clique X that is not an ideal and either $|X| \geq 5$ or R has odd characteristic. $\square$

The following corollary provides a positive interpretation of statement (ii) from Theorem 1.3. It is independent of characteristic.

Corollary 1.4. *Let R be a finite local ring. If there is a clique Y of $\Gamma(R)^*$ such that $|Y| \geq 5$, then R has* CIP *if and only if there is at least one ideal that is maximum clique of $\Gamma(R)^*$.*

By way of the proof of Theorem 1.3, we can say a little more about the local case where there is a maximum clique that is not an ideal. The next result provides an upper bound for the size of an ideal of a local ring R that is a maximal clique $\Gamma(R)^*$ when no maximum clique of $\Gamma(R)^*$ is an ideal.

Corollary 1.5. *Let R be a finite local ring such that there is a maximum clique X of $\Gamma(R)^*$ that is not an ideal of R.*

(i) *If no ideal of R is a maximum clique of $\Gamma(R)^*$, then $|\mathrm{Ann}(X)| \geq |I|$ for each ideal I that is a maximal clique of R.*
(ii) *If $|X| \geq 5$ and I is an ideal of R that is a clique of $\Gamma(R)^*$ such that $|I| > |\mathrm{Ann}(X)|$, then $I \subsetneq \mathrm{Ann}(I)$, and thus, I is not a maximal clique.*

Proof. From Lemma 1.1, $X \supsetneq \mathrm{Ann}(X)$. Since R is finite and local, $\mathrm{Ann}(X) \neq (0)$ and $|R| = p^r$ for some prime p and positive integer r. Hence, $|H|$ is a positive power of p for each nonzero ideal H of R. In particular, $|\mathrm{Ann}(X)| = p^c$ for some positive integer c.

Let J be the ideal generated by X, and let I be an ideal that is a maximal clique of $\Gamma(R)^*$. Then, $I = \mathrm{Ann}(I)$ by Lemma 1.1. For both statements, it suffices to prove that $|\mathrm{Ann}(X)| \geq |I|$. By way of contradiction, we assume that $|I| = p^i > |\mathrm{Ann}(X)| \geq p$. Then, we have $|X| \geq |I| \geq p^2 \geq 4$. If no ideal of R is a maximum clique of $\Gamma(R)^*$, then $|X| \geq 5$. Thus, from this point, we can establish both (i) and (ii). From the proof of Theorem 1.3, there is an element $r \in J \setminus I$ such that $rI = (0)$, contradicting that I is a maximal clique. Hence, $|\mathrm{Ann}(X)| \geq |I|$ for each ideal I that is a maximal clique of $\Gamma(R)^*$. □

Our final result of this section characterizes those finite local rings that have both a maximum clique that is not an ideal and a maximum clique that is an ideal. In Section 3, we provide four distinct examples of such rings, one of characteristic 8, two of characteristic 4 and another of characteristic 2 (Examples 3.8, 3.9, 3.10 and 3.11, respectively).

Theorem 1.6. *Let R be a finite local ring with maximal ideal M. If there is a set X that is a maximum clique and not an ideal and an ideal I that is a maximum clique, then*:

(i) $|R| = 16$;
(ii) $|X| = 4 = |I|$;
(iii) $\mathrm{Ann}(X) = \mathrm{Ann}(M) = \{0, z\}$ *for some nonzero element* $z \in M$;
(iv) $X = \{0, z, x, y\}$ *for some* $x, y \in M$ *with* $x^2 = y^2 = z$;
(v) $I = (x + y)R = \{0, z, x + y, x + y + z\}$ *is the only ideal that is a maximum clique*; *and*
(vi) *there are exactly three other maximum cliques that are not ideals*: $\{0, z, x, y + z\}$, $\{0, z, x + z, y\}$ *and* $\{0, z, x + z, y + z\}$.

Proof. Let M be the maximal ideal of R. Since R is local and finite, $\mathrm{Ann}(M) \neq (0)$. Assume that X is a maximum clique that is not an ideal and I is a maximum clique that is an ideal. By Theorem 1.3, $|R| = 2^r$ for some positive integer r and $|I| = |X| \leq 4$. Also, $\mathrm{Ann}(M) \subseteq \mathrm{Ann}(X) \subsetneq X$ and $I = \mathrm{Ann}(I)$ by Lemma 1.1. It follows that $\mathrm{Ann}(M) = \mathrm{Ann}(X) = \{0, z\}$ for some nonzero element $z \in M$, $I = \{0, z, w, w + z\}$ for some $w \in M \setminus \{0, z\}$ such that $w^2 = 0$, and $X = \{0, z, x, y\}$ necessarily with $x^2 = z = y^2$, the latter by Theorem 1.2. Since $w^2 = 0 = (w+z)^2$, $|I \cup X| = 6$. It follows that R contains at least 12 elements. Thus, $|R| = 2^r$ for some integer $r \geq 4$.

Since $\mathrm{Ann}(X)$ is an ideal, $z + z = 0$, and therefore, $(x+y)^2 = 0$. In addition, $x(x+y) = x^2 = z = y^2 = y(x+y)$. Since $xz = 0 = yz$, $x + y \neq z$. Thus, the set $J = \{0, z, x+y, x+y+z\}$ is a clique such that the square of each member is 0. As no clique has more than four elements, J must be an ideal of R (Lemma 1.1).

Another application of Theorem 1.2 yields that

$$(0) \neq XI \subseteq \mathrm{Ann}(X),$$

and thus $XI = \mathrm{Ann}(X)$. As noted above, neither x nor y can be in I since $I^2 = (0)$ and $x^2 = z = y^2$. Also, $xz = 0 = yz$, so it must be that $xw = z = yw$. Therefore, $I = wR$. Since $z + z = 0$, we have $(x+y)w = 0$. It follows that $x + y \in \mathrm{Ann}(I) = I$. Recall from above that $x(x+y) = x^2 = z$ (and $xz = 0$), so either $x + y = w$ or $x + y = w + z$. If the latter, then $x + y + z = w$. Since $x(y+z) = 0 = z(y+z)$, $\{0, z, x, y+z\}$ is also a clique. Hence, we may assume $x + y = w$. Therefore, $I = J$ is the unique maximum clique that is an ideal.

Each of the sets $\{0, z, x, y+z\}$, $\{0, z, x+z, y\}$ and $\{0, z, x+z, y+z\}$ is a maximum clique of $\Gamma(R)^*$ that is not an ideal. Once we show $|R| = 16$, it will be sufficient to conclude that these are the only other maximum cliques that are not ideals.

Let $r \in R$, and consider the possible values of rw:

(i) $rw = 0$ implies $r \in \{0, z, w, w+z\}$;
(ii) $rw = w$ implies $r - 1 \in \mathrm{Ann}(w)$ so $r \in \{1, 1+z, 1+w, 1+w+z\}$;
(iii) $rw = z$ implies $r - x \in \mathrm{Ann}(w)$ since $xw = z$, and hence, $r \in \{x, x+z, x+w, x+w+z\}$;
(iv) $rw = w + z$ implies $(r-1)w = z$; thus, $r \in \{1+x, 1+x+z, 1+x+w, 1+x+w+z\}$.

Therefore, R has 16 elements,

$$M = \{0, x, y, z, x+y, x+z, y+z, x+y+z\}$$

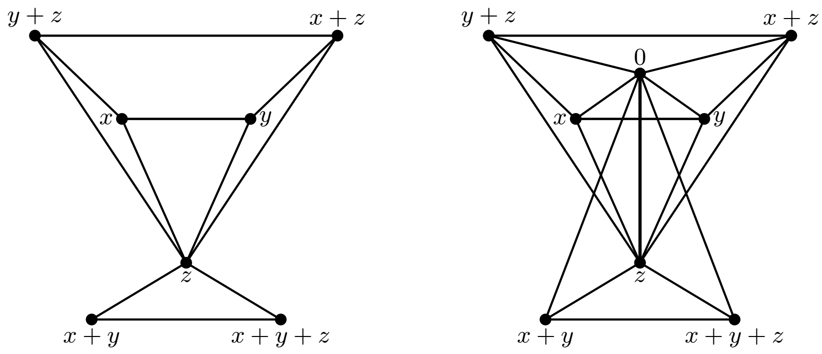

FIGURE 1. The graphs $\Gamma(R)$ and $\Gamma(R)^*$ for R satisfying Theorem 1.6.

(with $(x+z)+(y+z) = w = x+y$) and

$$R\backslash M = \{1, 1+x, 1+y, 1+z, 1+x+y, 1+x+z, 1+y+z, 1+x+y+z\}. \ \square$$

Remark 1.7. By Theorem 1.6, if R is a finite local ring such that there is a maximum clique of $\Gamma(R)^*$ that is an ideal I of R and another maximum clique X that is not an ideal, then $|R| = 16$ and $\Gamma(R)^*$ has eight vertices. Obviously, deleting 0 yields a graph $\Gamma(R)$ with seven vertices. Based on the more explicit description of what I and X look like in this case, it can easily be deduced that $\Gamma(R)$ is the graph labeled as "Figure 6" in **[8]** (also see **[9]** for a few unrelated corrections). We reproduce this graph in Figure 1 and include the corresponding graph $\Gamma(R)^*$.

Up to isomorphism, four finite local rings have these graphs (see **[8,** page 1160**]**), and each of these satisfies Theorem 1.6. One such ring is $\mathbb{Z}_4[\mathrm{w}]/(\mathrm{w}^2+2\mathrm{w})$ (see Example 3.9 below). The other three are given below as Examples 3.8, 3.10 and 3.11. Note that $2N \neq (0)$ for the maximal ideal N of the ring $\mathbb{Z}_4[\mathrm{w}]/(\mathrm{w}^2+2\mathrm{w})$. In contrast, $2M = (0)$ for the maximal ideal M of the ring in Example 3.10. It follows that these two rings are not isomorphic. For the ring $\mathbb{Z}_4[\mathrm{w}]/(\mathrm{w}^2+2\mathrm{w})$, if we wish to match the x, y, z notation in Theorem 1.6, we must set $z = w^2$ for x and y; a simple choice is $x = w$ and $y = 3w+2$, in which case, $x+y = 2$. An explicit description of values for x, y and z that is consistent with the notation in Theorem 1.6 is included in each of Examples 3.8, 3.10 and 3.11.

2. Finite rings that are not local. As noted earlier, a reduced finite ring has CIP if and only if it is a field. More generally, if $R = S \oplus T$ where S is a finite reduced ring and T is a finite ring, then not only does R not have CIP, but no maximal clique of $\Gamma(R)^*$ is an ideal of R.

Lemma 2.1. *If S and T are finite rings and S is reduced, then $R = S \oplus T$ is a finite ring which does not have the clique ideal property, and no maximal clique of $\Gamma(R)^*$ is an ideal of R.*

Proof. Each nonzero ideal of R has the form $I \oplus J$ where I is an ideal of S and J is an ideal of T. If $I = (0)$, then

$$(1, 0) \in \mathrm{Ann}(I \oplus J) \setminus I \oplus J.$$

Thus, in this special case, $I \oplus J$ is not a maximal clique of $\Gamma(R)^*$. Next, consider the case $I \neq (0)$. Since S is a reduced ring, $I + \mathrm{Ann}(I)$ has no nonzero annihilators. Thus, the only way to have $I \supseteq \mathrm{Ann}(I)$ is to have $\mathrm{Ann}(I) = (0)$. It follows that

$$I \oplus J \neq \mathrm{Ann}(I \oplus J).$$

That no ideal of R is a maximal clique now follows from Lemma 1.1. Therefore, R does not have CIP. □

Lemma 2.2. *Let $R = S \oplus T$, where S and T are finite rings. If no ideal of S is a maximal clique of $\Gamma(S)^*$, then no ideal of R is a maximal clique of $\Gamma(R)^*$, and R does not have the clique ideal property.*

Proof. Assume that no ideal of S is a maximal clique of $\Gamma(S)^*$, and let H be an ideal of R that is a clique of $\Gamma(R)^*$. Then, $H^2 = (0)$ by Lemma 1.1. Also, $H = I \oplus J$, where I is an ideal of S and J is an ideal of T. By assumption, I is not a maximal clique of $\Gamma(S)^*$, but we do have $I^2 = (0)$, so at least I is a clique of $\Gamma(S)^*$. From Lemma 1.1, there is an element $r \in \mathrm{Ann}(I) \setminus I$. Clearly, the element

$(r, 0) \in R \setminus H$ is in the annihilator of H. Hence, H is not a maximal clique of $\Gamma(R)^*$. $\square$

Theorem 2.3. *If S and T are finite rings that are not fields and each has an ideal that is a maximum clique of the respective zero divisor graphs $\Gamma(S)^*$ and $\Gamma(T)^*$, then $R = S \oplus T$ has the clique ideal property and each maximum clique of $\Gamma(R)^*$ has the form $I_S \oplus I_T$, where I_S is an ideal of S that is a maximum clique of $\Gamma(S)^*$, and I_T is an ideal of T that is a maximum clique of $\Gamma(T)^*$.*

Proof. Assume that both S and T are finite rings with nonzero ideals J_S of S and J_T of T such that J_S is a maximum clique of $\Gamma(S)^*$ and J_T is a maximum clique of $\Gamma(T)^*$. Then, it is clear that $J = J_S \oplus J_T$ is an ideal of $R = S \oplus T$ that is a clique of $\Gamma(R)^*$. Also,

$$|J_S \oplus J_T| = |J_S| \cdot |J_T| \geq 4.$$

Let Y be a clique of $\Gamma(R)^*$ such that $|Y| \geq 4$, and let

$$Y_S = \{s \in S \mid (s, t) \in Y \text{ for some } t \in T\}$$

and

$$Y_T = \{t \in T \mid (s, t) \in Y \text{ for some } s \in S\}.$$

Then,

$$|Y| \leq |Y_S \oplus Y_T| = |Y_S| \cdot |Y_T|.$$

If there is an element $(b, c) \in Y$ such that b is not a zero divisor of S, then all the other elements of Y have the form $(0, e)$, and there must be two such elements that are not $(0, 0)$. Thus, in this case, each element of Y_T is a zero divisor of T, and we have that Y_T is a clique of $\Gamma(T)^*$. It follows that $|Y| \leq 1 + |Y_T| \leq 1 + |J_T| < |J|$, and thus, Y is not a maximum clique.

Let X be a maximum clique of $\Gamma(R)^*$. From the argument in the previous paragraph,

$$X_S = \{s \in S \mid (s, t) \in X \text{ for some } t \in T\} \subseteq Z(S)$$

and

$$X_T = \{t \in T \mid (s, t) \in X \text{ for some } s \in S\} \subseteq Z(T).$$

It follows that X_S is a clique of $\Gamma(S)^*$ and X_T is a clique of $\Gamma(T)^*$. Also,

$$|X| \leq |X_S \oplus X_T| = |X_S| \cdot |X_T|$$

with $|X_S| \leq |J_S|$ and $|X_T| \leq |J_T|$. Hence, $|X| = |X_S| \cdot |X_T| = |J|$ and $J = J_S \oplus J_T$ is a maximum clique. In addition, $X = X_S \oplus X_T$ with $|X_S| = |J_S| \geq 2$ and $|X_T| = |J_T| \geq 2$.

Let $b \in X_S$ and $c \in X_T$ be nonzero. Then, it must be that all three of (b,c), $(b,0)$ and $(c,0)$ are in X with

$$(b^2, 0) = (b,0)(b,c) = (0,0) = (0,c)(b,c) = (0,c^2).$$

Thus, $X_S^2 = (0)$ and $X_T^2 = (0)$. Hence, X_S is an ideal of S and X_T is an ideal of T. Therefore, R has the clique ideal property, and each maximum clique of $\Gamma(R)^*$ is the direct sum of maximum cliques that are ideals. $\square$

If neither S nor T is a field and both have the clique ideal property, then $R = S \oplus T$ has the clique ideal property. Using recursion, we can extend this conclusion to arbitrary finite sums of finite rings that are not fields.

Corollary 2.4. *If $S_1, S_2, \ldots, S_n$ are finite rings that are not fields and each S_i has an ideal that is a maximum clique of $\Gamma(S_i)^*$, then $R = \bigoplus_{i=1}^n S_i$ has the clique ideal property. Moreover, an ideal J of R is a maximum clique of $\Gamma(R)^*$ if and only if*

$$J = \bigoplus_{i=1}^{n} J_i,$$

where

$$J_i = \{s_i \in S_i \mid (s_1, s_2, \ldots, s_n) \in J \text{ for some } s_j \in S_j,\ j \neq i\}$$

is an ideal of S_i that is a maximum clique of $\Gamma(S_i)^*$ for each i.

Corollary 2.5. *If $S_1, S_2, \ldots, S_n$ are finite rings that are not fields and each S_i has* CIP, *then $R = \bigoplus_{i=1}^n S_i$ has the clique ideal property.*

Remark 2.6. The converse of Corollary 2.5 does not hold. Let S be one of the rings in Examples 3.8, 3.9, 3.10 or 3.11. Then, both

$$R = S \oplus S \quad \text{and} \quad T = \mathbb{Z}_4 \oplus S$$

have the clique ideal property even though S does not. Moreover, R has two maximal ideals

$$N_1 = M \oplus S \quad \text{and} \quad N_2 = S \oplus M,$$

where M is the maximal ideal of S. While R has the CIP, the localizations

$$R_{N_1} \cong S \cong R_{N_2}$$

do not. The two maximal ideals of T are

$$P_1 = 2\mathbb{Z}_4 \oplus S$$

and

$$P_2 = \mathbb{Z}_4 \oplus M.$$

The localization $T_{P_1} \cong \mathbb{Z}_4$ has CIP while $T_{P_2} \cong S$ does not. On the other hand, if

$$W = \bigoplus_{i=1}^{n} V_i,$$

where W is a finite ring with odd characteristic and no V_i is a field, then W has CIP if and only if each V_i has CIP (see Corollary 2.11 below).

In addition, consider the ring $R = \mathbb{Z}_{12}$. It has two maximal ideals, $M = 3R$ and $N = 2R$. Since

$$R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4,$$

it does not have the clique ideal property (Lemma 2.1). For an alternate proof, we note that, since 12 is not a perfect square, R does not have the clique ideal property by Theorem 3.2 below. On the other hand, both

$$R_M \cong \mathbb{Z}_3 \quad \text{and} \quad R_N \cong \mathbb{Z}_4$$

have the clique ideal property (the "trivial" type for R_M as it is a field). The problem turns out to be that at least one of the localizations of R is a field.

If R is a finite ring that is not local, then there is a smallest positive integer m such that

$$\bigcap_{i=1}^{n} M_i^m = (0),$$

where $\mathrm{Max}(R) = \{M_1, M_2, \ldots, M_n\}$. Moreover, the natural map from R to

$$\bigoplus_{i=1}^{n} R/M_i^m$$

is an isomorphism and localizing yields $R_{M_i} \cong R/M_i^m$ for each i (see, for example, [**7,** 6.2 Theorem]).

Theorem 2.7. *Let R be a finite ring that is not local. If R_M has the clique ideal property for each maximal ideal M, then R has the clique ideal property if and only if there is no maximal ideal N such that R_N is a field.*

Proof. Assume that R_M has the clique ideal property for each maximal ideal M, and let

$$\mathrm{Max}(R) = \{M_1, M_2, \ldots, M_n\}.$$

Then, there is a natural isomorphism from R to

$$T = \bigoplus_{i=1}^{n} R_{M_i},$$

defined by $t \mapsto (t/1, t/1, \ldots, t/1)$. If there is no j such that R_{M_j} is a field, then R has the clique ideal property by Corollary 2.4. For the reverse implication, apply Lemma 2.1. □

Lemma 2.8. *Let $R = S \oplus T$ be a finite ring such that T is not a field. If I is a clique ideal of S and J is a clique ideal of T but not a* *maximal clique, then* $I \oplus J$ *is an ideal of* R *that is a clique of* $\Gamma(R)^*$, *but it is not a maximal clique of* $\Gamma(R)^*$.

Proof. Let I be a clique ideal of S, and let J be a clique ideal of T that is not a maximal clique of $\Gamma(T)^*$. Then, $I^2 = (0)$ and $J^2 = (0)$. Hence, $(I \oplus J)^2 = (0)$. By Lemma 1.1, there is an element $t \in \mathrm{Ann}(J) \setminus J$. It is clear that $(0, t)$ is an annihilator of $I \oplus J$ that is not contained in $I \oplus J$. Hence, $I \oplus J$ is a clique of $\Gamma(R)^*$, but it is not a maximal clique. $\square$

The next result provides both a refinement of Theorem 2.7 and a converse for both Theorem 2.3 and Corollary 2.4.

Theorem 2.9. *The following are equivalent for a finite ring* R *that is not local.*

(i) R *has* CIP.

(ii) *At least one ideal of* R *is a maximum clique of* $\Gamma(R)^*$.

(iii) *For each maximal ideal* M *of* R, R_M *is not a field, and at least one ideal of* R_M *is a maximum clique of* $\Gamma(R_M)^*$.

(iv) *If* $\{S_i\}_{i=1}^n$ *is a family of rings such that*

$$R \cong \bigoplus_{i=1}^{n} S_i,$$

then no S_i *is a field, and each* S_i *has an ideal that is a maximum clique of* $\Gamma(S_i)^*$.

(v) *There is a family of rings* $\{S_i\}_{i=1}^n$ *with* $n > 1$ *such that*

$$R \cong \bigoplus_{i=1}^{n} S_i,$$

no S_i *is a field, and each* S_i *has an ideal that is a maximum clique of* $\Gamma(S_i)^*$.

Proof. Let $\mathrm{Max}(R) = \{M_1, M_2, \ldots, M_m\}$ (with $m \geq 2$). Then,

$$R \cong \bigoplus_{i=1}^{m} R_{M_i}.$$

It follows that (iv) implies (iii).

It is clear that (i) implies (ii), and (iii) implies (v). Also each of (iii), (iv) and (v) implies (i) by Corollary 2.4. All that remains is to show (ii) implies (iv).

Assume that

$$R \cong \bigoplus_{i=1}^{n} S_i,$$

where $\{S_i\}_{i=1}^n$ is a family of rings. There is nothing to prove if $n = 1$ as R is not local. Thus, we may assume $n > 1$. For each S_i, let $\mathrm{Max}(S_i) = \{M_{i,1}, M_{i,2}, \ldots, M_{i,r_i}\}$. Then,

$$S_i \cong \bigoplus_{j=1}^{r_i} (S_i)_{M_{i,j}}$$

for each i and

$$R \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{r_i} (S_i)_{M_{i,j}}.$$

To simplify notation, we let $S_{i,j} = S_{M_{i,j}}$ for each (i, j).

Let J be an ideal of R that is a maximum clique of $\Gamma(R)^*$. Then, $J = \mathrm{Ann}(J)$ by Lemma 1.1, and $J \neq (0)$ since R is not local. Also, $J = \bigoplus_{i=1}^n J_i$, where each J_i is an ideal of the corresponding ring S_i. In addition,

$$J_i \cong \bigoplus_{j=1}^{r_i} J_{i,j},$$

where $J_{i,j}$ is an ideal of $S_{i,j}$ for each (i, j). Thus,

$$J \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{r_i} J_{i,j}.$$

By Lemma 2.1, no $S_{i,j}$ is a field. Since $J^2 = (0)$, we also have $J_{i,j}^2 = (0)$ for each (i, j). Hence, by Lemma 2.8, each $J_{i,j}$ is a maximal clique of the corresponding graph $\Gamma(S_{i,j})^*$ (and none are the zero ideal).

For a fixed i, if each $S_{i,j}$ has an ideal that is a maximum clique of $\Gamma(S_{i,j})^*$, then either S_i is local (in which case, S_i has an ideal that is a maximum clique of $\Gamma(S_i)^*$) or S_i has CIP, the latter by way of (iv) implies (i). Thus, by way of contradiction, we may assume some $S_{i,j}$ has no ideal that is a maximum clique of $\Gamma(S_{i,j})^*$. Without loss of generality, we may assume that no ideal of $S_{1,1}$ is a maximum clique of $\Gamma(S_{1,1})^*$. Let $X_{1,1}$ be a maximum clique of $\Gamma(S_{1,1})^*$. Then, by Corollary 1.5,

$$|\operatorname{Ann}(X_{1,1})| \geq |H_{1,1}|$$

for each ideal $H_{1,1}$ that is a maximal clique of $\Gamma(S_{1,1})^*$. In particular,

$$|\operatorname{Ann}(X_{1,1})| \geq |J_{1,1}|.$$

Let T be the direct sum of all $S_{i,j}$ except for $S_{1,1}$. In the special case $r_1 = 1$,

$$T = \bigoplus_{i=2}^{n} S_i;$$

otherwise, T is the direct sum of

$$\bigoplus_{j=2}^{r_1} S_{1,j} \quad \text{and} \quad \bigoplus_{i=2}^{n} S_i.$$

No matter which case holds, $R \cong S_{1,1} \oplus T$. Similarly, let J' be the direct sum of all $J_{i,j}$ except for $J_{1,1}$. Then, $J \cong J_{1,1} \oplus J'$. In addition, the ideal $H = \operatorname{Ann}(X_{1,1}) \oplus J'$ is a clique of $S_{1,1} \oplus T$ such that $|H| \geq |J_{1,1} \oplus J'|$. We have R isomorphic to $S_{1,1} \oplus T$ with J isomorphic to $J_{1,1} \oplus J'$. Since J is a maximum clique of $\Gamma(R)^*$, $J_{1,1} \oplus J'$ is a maximum clique of $\Gamma(S_{1,1} \oplus T)^*$. It follows that H is also a maximum clique of $\Gamma(S_{1,1} \oplus T)^*$. However, for each $x \in X_{1,1} \setminus \operatorname{Ann}(X_{1,1})$, the element $(x, 0)$ annihilates H but is not contained in H, providing a contradiction. Thus, each $S_{i,j}$ and each S_i has an ideal that is a maximum clique of the respective graphs $\Gamma(S_{i,j})^*$ and $\Gamma(S_i)^*$. $\square$

Since a finite local ring of odd characteristic has CIP if and only if it has at least one ideal that is a maximum clique, there are two variations on Theorem 2.9 when R is a finite ring with odd characteristic that is not a field.

Corollary 2.10. *Let R be a finite ring that is not a field. If R has odd characteristic, then the following are equivalent.*

(i) *R has* CIP*.*
(ii) *At least one ideal of R is a maximum clique of $\Gamma(R)^*$.*
(iii) *For each maximal ideal $M \in \mathrm{Max}(R)$, R_M has* CIP *and is not a field.*
(iv) *For each maximal ideal $M \in \mathrm{Max}(R)$, R_M is not a field and at least one ideal of R_M is a maximum clique of $\Gamma(R_M)^*$.*

Corollary 2.11. *Let*

$$T = \bigoplus_{i=1}^{n} S_i$$

be a ring with odd characteristic that is not a field. Then, T has CIP *if and only if each S_i has* CIP *and no S_i is a field.*

3. A few specific rings.

The first two results of this section are related to [**2,** Theorem 3.2] and [**4,** Proposition 2.3]. Essentially, both follow from the proof Beck gives for [**4,** Proposition 2.3].

Theorem 3.1. *Let $R = \mathbb{Z}_{p^n}$ for some prime p and positive integer n. Then, R has* CIP *if and only if n is even. Moreover, if $n = 2k$, then the only maximum clique is the ideal $p^k R$.*

Proof. Let $k = \lceil n/2 \rceil$. Then, $2k \geq n$. Each nonzero ideal of R is principal and generated by p^m for some positive integer $m < n$. Thus, throughout the proof, we let $I_m = p^m R$ for $1 \leq m < n$. Note that the ideals

$$I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_{n-1}$$

for $m \geq k$, $I_m^2 = (0)$, and for $m < k$, $I_m^2 \neq (0)$. In addition, regardless how m compares with k, $|I_m| = p^{n-m}$ and $\mathrm{Ann}(I_m) = I_{n-m}$.

If n is odd, then $n = 2k - 1$, and $p^{k-1} \in R \setminus I_k$ is a nonzero annihilator of I_k. Thus, no maximum clique of $\Gamma(R)^*$ is an ideal of R when n is odd. On the other hand, if $n = 2k$, then I_k is an ideal whose square is the zero ideal, and no element outside of I_k annihilates I_k. It is clear that I_k is the unique largest subset of $\Gamma(R)^*$ that forms a clique. Hence, in this case, R has CIP, and I_k is the unique maximum clique of $\Gamma(R)^*$. □

Theorem 3.2. *Let $R = \mathbb{Z}_n$, where $n > 1$ is a positive integer that is not prime. Then, R has the clique ideal property if and only if n is a perfect square. Moreover, if $n = m^2$ for some m, then mR is the unique maximum clique of $\Gamma(R)^*$.*

Proof. If n is square-free, then R is reduced, and thus, $\Gamma(R)^*$ does not have the clique ideal property. Hence, we may assume $n = km^2$ for some positive integers k and $m > 1$ with k square-free.

Each nonzero ideal of R is principal and generated by a positive integer j that divides n. For such an ideal jR, $\mathrm{Ann}(jR) = (n/j)R$. In addition, $|jR| = n/j$, and $j^2R = (0)$ if and only if mk divides j. Thus, no matter whether $k = 1$ or is strictly greater than 1, the ideal mkR is the unique largest ideal of R whose square is (0). Moreover, mkR contains each ideal I whose square is zero. If $k > 1$, then $m \in R \setminus mkR$ annihilates mkR. Hence, mkR is not a maximal clique in this case. It follows that no maximum clique of $\Gamma(R)^*$ is an ideal when $k > 1$.

If $k = 1$, then $n = m^2$ and $mR = \mathrm{Ann}(mR)$. Hence, mR is a maximal clique of $\Gamma(R)^*$. There are primes $p_1 < p_2 < \cdots < p_s$ and corresponding positive integers $r_1, r_2, \ldots, r_s$ such that

$$n = \prod_{i=1}^{s} p_i^{2r_i}.$$

We also have

$$R \cong \bigoplus_{i=1}^{s} \mathbb{Z}_{p_i^{2r_i}}.$$

From Theorem 3.1, each $\mathbb{Z}_{p_i^{2r_i}}$ has CIP with unique maximum clique $p_i^{r_i}\mathbb{Z}_{p_i^{2r_i}}$. Since none of these rings are fields, R has CIP with unique maximum clique mR by Corollary 2.4. $\square$

For each positive integer m, there is a finite ring with the clique ideal property where the clique number of $\Gamma(R)^*$ is m. A similar statement holds for the clique number of $\Gamma(R)$.

Corollary 3.3. *For each positive integer n, there is a finite ring R with the clique ideal property such that the clique number of $\Gamma(R)^*$ (of $\Gamma(R)$) is n.*

Proof. For $n = 1$, each finite field has the clique ideal property since 0 is the only zero divisor. The clique number for $\Gamma(R)^*$ is 1 in this case. Also, the ring $R = \mathbb{Z}_4$ has a single nonzero zero divisor; the set $\{0, 2\}$ is both an ideal of R and the complete set of zero divisors of R with $2 \cdot 2 = 0$. Thus, R has the clique ideal property, the clique number of $\Gamma(R)^*$ is 2 and the clique number of $\Gamma(R)$ is 1.

Let $n \geq 2$, and factor it into a product of powers of distinct primes: $n = p_i^{r_1} p_2^{r_2} \cdots p_m^{r_m}$. By Theorem 3.1, $R_i = \mathbb{Z}_{p_i^{2r_i}}$ has the clique ideal property with a unique clique ideal $p_i^{r_i} R_i$, an ideal of cardinality $p_i^{r_i}$. By Corollary 2.4,

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_m$$

has the clique ideal property with (unique) clique ideal

$$I = p_1^{r_1} R_1 \oplus p_2^{r_2} R_2 \oplus \cdots \oplus p_m^{r_m} R_m$$

such that $|I| = n$.

In order to obtain a finite ring with the clique ideal property such that the clique number of $\Gamma(R)$ is n, simply use the above scheme to get a finite ring R with the clique ideal property such that the clique number of $\Gamma(R)^*$ is $n + 1$. $\square$

For an R-module B, we may form a ring $R(+)B$ (via the "idealization of B") from $R \times B$ by setting

$$(r, a) + (s, b) = (r + s, a + b) \quad \text{and} \quad (r, a)(s, b) = (rs, rb + sa).$$

Theorem 3.4. *Let $T = R(+)B$, where $R = \mathbb{Z}_{p^n}$ for some prime p and positive integer n, and B is a finite nonzero R-module. Also, let*

$k = \lceil n/2 \rceil$, and let $q \le n$ be such that $\mathrm{Ann}_R(B) = p^q R$ *(*$= (0)$ *if* $q = n$*).*

(i) *The following are equivalent.*

(a) *T has* CIP.
(b) *T has an ideal that is a maximum clique of $\Gamma(T)^*$.*
(c) *Either n is even or $q \ge k$.*

(ii) *If n is odd and $q < k$, then $p^k R(+)B$ is a maximum clique ideal but not a maximal clique, $\{(g,b)\} \cup p^k R(+)B$ is a maximum clique for each $g \in I_{k-1} \setminus I_k$ and each $b \in B$, and each maximum clique has this form.*

(iii) *If n is even and $q < k$, then $p^k R(+)B$ is the unique maximum clique of $\Gamma(T)^*$.*

(iv) *If $q \ge k$, then $p^q R(+)B$ is a maximum clique of $\Gamma(T)^*$. Moreover, there is an integer $k \le s \le q$ such that $p^t R(+)B_t$ is a maximum clique for each $s \le t \le q$, where $B_t = \{b \in B \mid p^t b = 0\}$, and there are no other maximum cliques.*

Proof. First note that, if C is a nonzero finite R-module, then $|C|$ is a positive power of p. Moreover, $\mathrm{Ann}_R(C) = p^i R$ for some $1 \le i \le n$. In particular, $\mathrm{Ann}_R(B) = p^q R$ for some $1 \le q \le n$. Since $(0,b)^2 = (0,0) = (0,b)(0,c)$ for all $b, c \in B$, the ideal $(0)(+)B$ is a clique of $\Gamma(T)^*$.

For each integer $0 \le i \le n$, let $I_i = p^i R$ with $I_n = (0)$ and $I_0 = R$. Also, let $B_i = \{b \in B \mid p^i b = 0\}$. Then, $B_i = \mathrm{Ann}_B(I_i)$ and, for $1 \le r < s < q \le t \le n$, we have

$$B_r \subsetneq B_s \subsetneq B = B_q = B_t.$$

Each ideal of R is principal, and they are linearly ordered as

$$(0) = I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1 \subsetneq I_0 = R.$$

In addition, $|I_i| = p^{n-i}$, and, for $a \in I_i \setminus I_{i+1}$, $\mathrm{Ann}_R(a) = I_{n-i} = \mathrm{Ann}_R(I_i)$ with $|\mathrm{Ann}_R(a)| = |\mathrm{Ann}_R(I_i)| = p^i$.

We consider cases based on the comparison of q and k.

Relevant to all three cases is the fact that, if $g \in R \setminus I_{k-1}$ is a nonunit, then $g^2 \neq 0$, and there is an integer $1 \leq j < k-1$ such that $g \in I_j \setminus I_{k-1}$ with $|\operatorname{Ann}_R(g)| = p^j < p^{k-1}$. Thus, each clique of $\Gamma(R)^*$ which contains g has cardinality at most $1+p^j$ which is strictly smaller than p^{k-1}. For $f \in I_{k-1} \setminus I_k$, $f^2 \neq 0$ and $\operatorname{Ann}_R(f) = I_{n-k+1}$ with $|\operatorname{Ann}_R(f)| = p^{k-1}$. It follows that each clique of $\Gamma(R)^*$ that contains f has cardinality at most $1 + p^{k-1}$. In addition, the set

$$\operatorname{Ann}_R(f) \cup \{f\}$$

is a maximal clique of $\Gamma(R)^*$. If n is odd, then $\operatorname{Ann}_R(f) \cup \{f\}$ is a maximum clique of $\Gamma(R)^*$. On the other hand, if n is even, then $n = 2k$ and I_k is a maximum clique of cardinality p^k.

Case 1. $q < k$ and n is odd. In this case, $I_q \neq (0)$, and thus, $I_k(+)B$ is a clique of $\Gamma(T)^*$. However,

$$(p^{k-1}, 0) \in T \setminus I_k(+)B$$

annihilates $I_k(+)B$. Thus, $I_k(+)B$ is not a maximal clique of $\Gamma(T)^*$. Since no element outside of $I_k(+)B$ annihilates $(p^{k-1}, 0)$, the set

$$I_k(+)B \cup \{(p^{k-1}, 0)\}$$

is a maximal clique of $\Gamma(T)^*$. As above, if $g \in R \setminus I_{k-1}$, then $\operatorname{Ann}_R(g) \subsetneq I_k$. It follows that each clique of $\Gamma(T)^*$ which contains (g, b) for some $b \in B$ has cardinality strictly smaller than $|I_k| \cdot |B|$. On the other hand, if $g \in I_{k-1} \setminus I_k$, then $\operatorname{Ann}_R(g) = I_k$. In this case,

$$\{(g, b)\} \cup I_k(+)B$$

is a maximal clique. Therefore, each maximum clique of $\Gamma(T)^*$ has this form, and we have that T does not have CIP when $q < k$ and n is odd.

Case 2. $q < k$ and n is even. We again have $I_q^2 \neq (0)$, and thus, $I_q(+)B$ is not a clique of $\Gamma(T)^*$. Since $q < k$, I_q properly contains I_k and $I_k B = (0)$. In addition, $\operatorname{Ann}_R(I_k) = I_k$ since n is even. Thus, $I_k(+)B$ is a maximal clique of $\Gamma(T)^*$. In this case, $\operatorname{Ann}(g) = I_m$ for some $m > k$ when $g \in R \setminus I_k$. Thus, no other clique is as large as $I_k(+)B$. Therefore, $I_k(+)B$ is the unique maximum clique of $\Gamma(T)^*$ when $q < k$ and n is even. Also, T has CIP in this case.

Case 3. $q \geq k$. In this case, the ideal $I_q(+)B$ is a maximal clique of $\Gamma(T)^*$. We will show that there are no larger cliques and all others of the same size are also ideals of T. For this, we need to know more about the structure of B. Since B is an Abelian group and $|B| = p^m$ for some $m \geq 1$, B is isomorphic (as a group) to a direct sum

$$C_1 \oplus C_2 \oplus \cdots \oplus C_f,$$

where each $C_i \cong Z_{p^{s_i}}$ for some s_i (as a group). Since B is also a $\mathbb{Z}_{p^n}$-module, each $s_i \leq n$, and we may further assume that $C_i = p^{r_i}\mathbb{Z}_{p^n}$, where $r_i = n - s_i$ with $0 \leq r_1 \leq r_2 \leq \cdots \leq r_f < n$ and $r_1 = n - q$. We have

$$|I_q(+)B| = p^{n-q}p^m = p^{n-q}\prod_{i=1}^{f} p^{n-r_i} = p^n \prod_{i=2}^{f} p^{n-r_i}$$

since $n - r_1 = q$.

For $s < q$,

$$B_s = \bigoplus_{i=1}^{f} D_{i,s},$$

where $D_{i,s} = p^{g_i}C_i$, with $g_i = 0$ when $s + r_i \geq n$ and $g_i = n - (r_i + s)$ when $s + r_i < n$. We have $|D_{i,s}| = p^{n-r_i}$ when $s + r_i \geq n$ and $|D_{i,s}| = p^{n-(r_i+g_i)} = p^s$ when $s + r_i < n$. In particular, $|D_{1,s}| = p^s$. Thus,

$$\begin{aligned} |I_s(+)B_s| &= |I_s| \cdot \prod_{i=1}^{f} |D_{i,s}| = p^{n-s} \prod_{i=1}^{f} p^{n-(r_i+g_i)} \\ &= p^n \prod_{i=2}^{f} p^{n-(r_i+g_i)} \leq |I_q(+)B| \end{aligned}$$

since $g_i \geq 0$ for each $i \geq 2$. Moreover, for $k \leq s < t \leq q$,

$$|I_s| \cdot |D_{1,s}| = p^n = |I_t| \cdot |D_{1,t}|,$$

$$D_{i,s} \subseteq D_{i,t} \subseteq C_i$$

and

$$|D_{i,s}| \leq |D_{i,t}| \quad \text{for all } i.$$

Hence,

$$|I_k(+)B_k| \le |I_s(+)B_s| \le |I_t(+)B_t| \le |I_q(+)B|.$$

Further note that, if there is an i such that $t + r_i < n$, then $|D_{i,s}| < |D_{i,t}| < |C_i|$; thus, in this case, $|I_s(+)B_s| < |I_t(+)B_t| < |I_q(+)B|$.

For a nonunit $a \in R \setminus I_k$, $\mathrm{Ann}_R(a) = p^s R$ for some $s \ge k$. Thus, for each $c \in B$, any clique that contains (a, c) is properly contained in $I_t(+)B_t$ for some $t \ge s$. Each such graph is strictly smaller than $|I_q(+)B|$. Hence, $I_q(+)B$ is a maximum clique of $\Gamma(T)^*$. Moreover, there is an integer s such that $k \le s \le q$, where $I_t(+)B_t$ is a maximum clique of $\Gamma(T)^*$ for each $s \le t \le q$ and there are no other maximum cliques. $\square$

Note that $\mathbb{Z}_p(+)B$ has CIP for each finite nonzero $\mathbb{Z}_p$-vector space B, and $(0)(+)B$ is the unique maximum clique. For the case $R = \mathbb{Z}_{p^n}$ with $n \ge 2$, the values of "q" and "s" in statement (iii) of Theorem 3.4 can range anywhere from $k = \lceil n/2 \rceil$ up to n.

Example 3.5. Let $R = \mathbb{Z}_{p^n}$ for some prime p and positive integer $n \ge 2$. Also, let $k = \lceil n/2 \rceil$. For integers $k \le s \le q \le n$, let

$$B = p^{n-q}\mathbb{Z}_{p^n} \oplus p^{n-s}\mathbb{Z}_{p^n}.$$

Then, the maximum cliques of $\Gamma(T)^*$, where $T = R(+)B$ are the ideals $p^t R(+)(p^{n-t}\mathbb{Z}_{p^n} \oplus p^{n-s}\mathbb{Z}_{p^n})$ for $s \le t \le q$.

Proof. For each integer t such that $s \le t \le q$, the ideal

$$J_t = p^t\mathbb{Z}_{p^n}(+)(p^{n-t}\mathbb{Z}_{p^n} \oplus p^{n-s}\mathbb{Z}_{p^n})$$

has cardinality $p^n \cdot p^s$. In addition, $J_t^2 = \{(0,0)\}$. From Theorem 3.4, each J_t is a maximum clique (since J_q is a maximum clique) and, from the proof of Case 3, there are no other cliques. $\square$

The next result follows easily from Theorem 3.4. It can also be derived from [**1,** Theorems 1 and 2] by viewing $\mathbb{Z}_{p^n}[\mathrm{x}]/(\mathrm{x}^2)$ via idealization as $\mathbb{Z}_{p^n}(+)\mathbb{Z}_{p^n}$.

Corollary 3.6. *Let $T = \mathbb{Z}_{p^n}(+)\mathbb{Z}_{p^n}$ for some prime p and positive integer n. Then, T has* CIP, *the ideal $p^s\mathbb{Z}_{p^n}(+)p^{n-s}\mathbb{Z}_{p^n}$ is a maximum clique of $\Gamma(T)^*$ for each integer $\lceil n/2 \rceil \le s \le n$ and there are no other maximum cliques.*

Let $n = p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r}$ be a positive integer with distinct prime factors

$$p_1 < p_2 < \cdots < p_r.$$

Then, the ring $R = \mathbb{Z}_n(+)\mathbb{Z}_n$ is naturally isomorphic to the sum

$$\bigoplus_{i=1}^{r} R_i,$$

where

$$R_i = \mathbb{Z}_{p_i^{c_i}}(+)\mathbb{Z}_{p_i^{c_i}}.$$

Combining Corollaries 2.4 and 3.6, we see that R has the clique ideal property.

Corollary 3.7. *For each positive integer $n > 1$, the ring $R_n = \mathbb{Z}_n(+)\mathbb{Z}_n$ has the clique ideal property. If $n = km^2$ where k is square-free, then, for each positive integer s that divides m, the ideal*

$$I_s = (kms)\mathbb{Z}_n(+)(m/s)\mathbb{Z}_n$$

is a maximum clique, and there are no other maximum cliques. In particular, $(0)(+)\mathbb{Z}_n$ is the only maximum clique in the case where n is square-free.

Proof. As noted above, R_n has CIP by Corollaries 2.4 and 3.6. Suppose that $n = km^2$ with k square-free. Each ideal of $\mathbb{Z}_n$ has the form $J_r = r\mathbb{Z}_n$ for some positive integer $r \le n$ that divides n (with $J_1 = \mathbb{Z}_n$ and $J_n = (0)$). The annihilator of J_r (in $\mathbb{Z}_n$) is the ideal $J_{n/r}$. Since k is square-free, the following are equivalent:

(i) $J_r^2 = (0)$;
(ii) $J_r \subseteq J_{n/r}$;
(iii) n/r divides r; and
(iv) km divides r.

Let s be a positive integer that divides m. Then, the ideal $I_s = J_{kms}(+)J_{m/s}$ is such that $I_s^2 = (0)$. Moreover, since $J_{kms} = \mathrm{Ann}(J_{m/s})$ and $J_{m/s} = \mathrm{Ann}(J_{kms})$, $I_s = \mathrm{Ann}_R(I_s)$. Thus, I_s is a maximal clique of $\Gamma(R)^*$. Also, note that

$$|I_s| = (m/s)(kms) = n.$$

Factor n as

$$n = p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r},$$

where $p_1 < p_2 < \cdots < p_r$ are distinct primes and each $c_i \geq 1$. Then,

$$m = \prod_{i=1}^{r} \lfloor c_i/2 \rfloor,$$

and a given p_i divides k if and only if c_i is odd. For s as above,

$$kms = \prod_{i=1}^{r} p_i^{s_i},$$

where $\lceil c_i/2 \rceil \leq s_i \leq c_i$. Under the natural isomorphism

$$R \cong \bigoplus_{i=1}^{r} (\mathbb{Z}_{p^{c_i}}(+)\mathbb{Z}_{p^{c_i}}),$$

we have

$$I_s \cong \bigoplus_{i=1}^{r} (p_i^{s_i}\mathbb{Z}_{p_i^{c_i}}(+)p_i^{c_i - s_i}\mathbb{Z}_{p_i^{c_i}}).$$

That each I_s is a maximum clique now follows from Corollaries 2.4 and 3.6. $\square$

Each of the finite local rings in the next four examples has an ideal that is a maximum clique and a maximum clique that is not an ideal. The first is a ring of characteristic 8, the second and third have characteristic 4 and the fourth has characteristic 2. By way of Theorems 1.3, 1.6 and 2.9 and the tables in **[8, 9]** (also see **[5]**), these are the only four finite rings (up to isomorphism) with this property.

Example 3.8. Let $R = \mathbb{Z}_8[\mathrm{w}]/(\mathrm{w}^2, 2\mathrm{w}+4)$, and let w denote the image of w in R. Then,

$$R = \{0,1,2,3,4,5,6,7,w,w+1,w+2,w+3,w+4,w+5,w+6,w+7\}.$$

The set $X = \{0,2,4,w+2\}$ is a maximum clique that is not an ideal and $I = wR = \{0,4,w,w+4\}$ is a maximum clique that is an ideal. To match with the notation "x, y, z" in Theorem 1.6, we have $z = 4$ and can use $x = 2$ and $y = w+6$ (so that $x + y = w$).

Proof. Since $2w = 4$ and $w^2 = 0$, $R = \{0,1,2,3,4,5,6,7,w,w+1,w+2,w+3,w+4,w+5,w+6,w+7\}$ and

$$M = \{0,2,4,6,w,w+2,w+4,w+6\}$$

is the maximal ideal. We have $4w = 0$; thus, $\mathrm{Ann}(M) \supseteq \{0,4\}$. For w, $\mathrm{Ann}(w) = \{0,4,w,w+4\}$ since $2w = 4 \neq 0$. Hence, $\mathrm{Ann}(M) = \{0,4\}$. Since $2^2 = 4 \neq 0$ but

$$2 \cdot 4 = 0 = 2w + 4 = 2(w+2),$$

$\mathrm{Ann}(2) = \{0,4,w+2,w+6\} = \mathrm{Ann}(6)$. Also, $\mathrm{Ann}(w+2) = \{0,2,4,6\} = \mathrm{Ann}(w+6)$. It follows that $I = wR$ is a maximum clique of R that is an ideal and $X = \{0,2,4,w+6\}$ is a maximum clique that is not an ideal. $\square$

Example 3.9. Let $R = \mathbb{Z}_4[\mathrm{w}]/(\mathrm{w}^2 + 2\mathrm{w})$, and let w denote the image of w in R. Then,

$$\begin{aligned} R = \{&0,1,2,3,w,w+1,w+2,w+3,\\ &2w,2w+1,2w+2,2w+3,3w,3w+1,3w+2,3w+3\}. \end{aligned}$$

The set $X = \{0,2w,w,3w+2\}$ is a maximum clique that is not an ideal, and $I = 2R = \{0,2w,2,2w+2\}$ is a maximum clique that is an ideal. To match the notation "x, y, z" in Theorem 1.6, we have $z = 2w (= w^2)$ and can use $x = w$ and $y = 3w+2$ so that $x + y = 2$.

Proof. In R, $w^2 = 2w \neq 0$. Also,

$$M = \{0,2,w,w+2,2w,2w+2,3w,3w+2\}$$

is the maximal ideal of R with $0 = 2w \cdot 2 = 2w \cdot w$. In addition,

$$\begin{aligned} w \cdot 3w = w^2 = 2w &= (w+2)^2 = (w+2)(3w+2), \\ 2(w+2) &= 2w = 2(3w+2), \\ w(3w+2) &= 0 = w(w+2), \end{aligned}$$

and

$$2^2 = 0 = (2w+2)^2.$$

Thus, $\mathrm{Ann}(w) = \{0, 2w, w+2, 3w+2\} = \mathrm{Ann}(3w)$ and $\mathrm{Ann}(3w+2) = \{0, 2w, w, 3w\} = \mathrm{Ann}(w+2)$. Thus, $X = \{0, 2w, w, 3w+2\}$ is a maximum clique of $\Gamma(R)^*$ that is not an ideal. On the other hand, $I = 2R = \{0, 2w, 2, 2w+2\}$ is a maximum clique of $\Gamma(R)^*$ that is an ideal of R. $\square$

Example 3.10. Let

$$R = \mathbb{Z}_4[\mathrm{X}, \mathrm{Y}]/(\mathrm{X}^2+2, \mathrm{Y}^2+2, \mathrm{XY}, 2\mathrm{X}, 2\mathrm{Y}),$$

and let x and y denote the respective images of X and Y in R. Then,

$$\begin{aligned} R = \{&0, 1, 2, 3, x, 1+x, 2+x, 3+x, y, 1+y, \\ &2+y, 3+y, x+y, 1+x+y, 2+x+y, 3+x+y\}. \end{aligned}$$

The set $X = \{0, 2, x, y\}$ is a maximum clique that is not an ideal, and $I = (x+y)R = \{0, 2, x+y, 2+x+y\}$ is a maximum clique that is an ideal. To match the notation "x, y, z" in Theorem 1.6, we have $z = 2$, and x and y can stay as they are.

Proof. The maximal ideal of R is

$$M = \{0, 2, x, 2+x, y, 2+y, x+y, 2+x+y\}$$

with $0 = 2 \cdot 2 = 2x = 2y$. In addition,

$$x(x+y) = x^2 = 2 = y^2 = y(x+y)$$

and

$$(x+y)^2 = 0 = (2+x+y)^2.$$

Thus, $\operatorname{Ann}(M) = \{0, 2\}$. For x and y, we have $\operatorname{Ann}(x) = \{0, 2, y, 2 + y\} = \operatorname{Ann}(2 + x)$ and $\operatorname{Ann}(y) = \{0, 2, x, 2 + x\} = \operatorname{Ann}(2 + y)$. Hence, $X = \{0, 2, x, y\}$ is a maximum clique that is not an ideal. In contrast, the ideal

$$I = (x + y)R = \{0, 2, x + y, 2 + x + y\}$$

is a maximum clique that is an ideal. □

As noted in Remark 1.7, a simple way to see that the rings of Examples 3.9 and 3.10 are not isomorphic is to note that 2 annihilates the maximal ideal in Example 3.10 but does not annihilate the maximal ideal in Example 3.9.

The ring in the next example can also be constructed as a factor ring of $\mathbb{Z}_2[\mathrm{X}, \mathrm{Y}]$. We have begun with $\mathbb{Z}_2[\mathrm{X}, \mathrm{Y}, \mathrm{Z}]$ to more closely match the notation in Theorem 1.6. In particular, x, y and z of Theorem 1.6 can exactly be matched with the respective images of X, Y and Z in the factor ring of this example.

Example 3.11. Let

$$R = \mathbb{Z}_2[\mathrm{X}, \mathrm{Y}, \mathrm{Z}]/(\mathrm{XY}, \mathrm{XZ}, \mathrm{YZ}, \mathrm{X}^2 + \mathrm{Z}, \mathrm{Y}^2 + \mathrm{Z}).$$

Let x, y and z denote the respective images of X, Y and Z. Note that $z = x^2 = y^2 \neq 0$ and $z^2 = x^3 = y^3 = 0$.

(a) The ring R has 16 elements:

$$\{0, x, y, z, x + y, x + z, y + z, x + y + z, 1, 1 + x, 1 + y, 1 + z, \\ 1 + x + y, 1 + x + z, 1 + y + z, 1 + x + y + z\}.$$

(b) $M = xR + yR$ is the maximal ideal of R and $\operatorname{Ann}(M) = zR = \{0, z\}$.

(c) Both $X = \{0, x, y, z\}$ and $I = \{0, x+y, z, x+y+z\}$ are maximum cliques of $\Gamma(R)^*$ with $XI = \{0, z\} = \operatorname{Ann}(X)$.

(d) Since I is an ideal and X is not, R does not have the clique ideal property, but $T = R \oplus R$ has the clique ideal property, as does R^n for each $n \geq 3$.

(e) The maximal ideals of $T(= R \oplus R)$ are $M_1 = M \oplus R$ and $M_2 = R \oplus M$. While T has the clique ideal property, the localizations $T_{M_1} \cong R_M \cong T_{M_2}$ do not.

Proof. Since $z = x^2 = y^2$ and $xy = 0 = xz = yz$,

$$\begin{aligned} R = \{&0, x, y, z, x+y, x+z, y+z, x+y+z, 1, 1+x, \\ &1+y, 1+z, 1+x+y, 1+x+z, 1+y+z, 1+x+y+z\} \end{aligned}$$

and

$$M = xR + yR = \{0, x, y, z, x+y, x+z, y+z, x+y+z\}.$$

The element z can be obtained as a product of two elements of M in several ways. Each of the following products equals z:

$$\begin{gathered} x^2, \quad y^2, \quad (x+z)^2, \\ (y+z)^2, \quad x(x+y), \quad y(x+y), \\ x(x+z), \quad y(y+z), \quad x(x+y+z), \\ y(x+y+z), \quad (x+z)(x+y), \quad (y+z)(x+y), \\ (x+z)(x+y+z), \quad (y+z)(x+y+z). \end{gathered}$$

It may easily be verified that the following hold:

$$\begin{aligned} M^2 &= \{0, z\} = \mathrm{Ann}(M), \\ xR &= \{0, x, z, x+z\} = (x+z)R \end{aligned}$$

and

$$yR = \{0, y, z, y+z\} = (y+z)R.$$

The ideal

$$I = \{0, x+y, z, x+y+z\} = (x+y)R = (x+y+z)R$$

is such that $I^2 = (0)$, and no element of $M \setminus I$ annihilates I. Hence, I is a maximum clique ideal of R that is also a maximal clique. In addition, I is the only ideal with four elements whose square is the zero ideal. Thus, it is the unique maximum clique ideal.

The principal ideals xR, yR and $(x+y)R$ are the only ideals with four elements, and zR is the unique ideal with only two elements. We also have $yR = \mathrm{Ann}(xR)$ and $xR = \mathrm{Ann}(yR)$.

From the list of factorizations for z given above, it is clear that a maximum clique contains at most one of x and $x+z$ and at most one of y and $y+z$. Since

$$(y+z)R = yR = \mathrm{Ann}(xR) \quad \text{and} \quad (x+z)R = xR = \mathrm{Ann}(yR),$$

the following sets of four elements are all maximal cliques of $\Gamma(R)^*$: $X = \{0, x, y, z\}$, $W = \{0, x, y+z, z\}$, $V = \{0, x+z, y, z\}$ and $U = \{0, x+z, y+z, z\}$. These sets, along with I, are the maximum cliques of $\Gamma(R)^*$. Hence, R does not have the clique ideal property, even though it has an ideal that is a maximum clique. By Theorem 2.3, $T = R \oplus R$ does have the clique ideal property, and so does R^n for each $n \geq 3$. □

The ring in Example 3.11 has characteristic 2. Thus, in R, we have $z = x^2 = y^2 = -z$. If, instead, the base ring is $\mathbb{Z}_p$ for some odd prime p, then there are essentially two variations we may examine: both x^2 and y^2 equal to z, or $x^2 = z = -y^2$. For both, we still have $xy = 0 = xz = yz$ and $x^3 = y^3 = z^2 = 0$. It remains the case that the annihilator of the maximal ideal is the principal ideal generated by z, but, now, this ideal contains p elements which we may view as $z\mathbb{Z}_p$. The set

$$X_p = z\mathbb{Z}_p \cup \{x, y\}$$

is a maximal clique; however, it need not be a maximum clique in the case $x^2 = z = y^2$. Moreover, it is not a maximum clique in the case $x^2 = z = -y^2$ since the principal ideal generated by $x+y$ contains p^2 elements and is its own annihilator.

Example 3.12. Let

$$R = \mathbb{Z}_p[\mathrm{X}, \mathrm{Y}, \mathrm{Z}]/(\mathrm{XY}, \mathrm{XZ}, \mathrm{YZ}, \mathrm{Z} - \mathrm{X}^2, \mathrm{Z} + y^2)$$

with p an odd prime. Let x, y and z denote the respective images of X, Y and Z. Note that $x^2 = z = -y^2 \neq 0$ and $z^2 = x^3 = y^3 = 0$.

(a) For positive integers $b_1, b_2, b_3, c_1, c_2, c_3 \in \{1, 2, \dots, p\}$,

$$(b_1x + b_2y + b_3z)(c_1x + c_2y + c_3z) = b_1c_1x^2 + b_2c_2y^2 = (b_1c_1 - b_2c_2)z.$$

The product is 0 if and only if $b_1c_1 \equiv b_2c_2 \pmod{p}$.

(b) The ring $R = \{a_0 + a_1x + a_2y + a_3z \mid a_i \in \mathbb{Z}_p\}$ has cardinality p^4, and the maximal ideal $M = xR + yR$ has cardinality p^3. Also $M^2 = zR$ and $\mathrm{Ann}(M) = zR = \{0, z, 2z, \dots, (p-1)z\}$.

(c) There are $p+1$ ideals of cardinality p^2. Specifically, yR, xR, $(x+y)R$, $(x+2y)R$, $\dots$, $(x+(p-1)y)R$. Of these,

$$(x+y)R = \mathrm{Ann}((x+y)R) \quad \text{and} \quad (x-y)R = \mathrm{Ann}((x-y)R),$$

while $xR = \mathrm{Ann}(yR)$, $yR = \mathrm{Ann}(xR)$ and, when $p > 3$ and $2 \le b \le p-2$, $(x+ay)R = \mathrm{Ann}((x+by)R)$ for a such that $ab \equiv 1 \pmod{p}$.

(d) The ideals $I = (x+y)R$ and $J = (x-y)R$ are maximum cliques of $\Gamma(R)^*$. The other maximal cliques are the sets of the form

$$X_{f,g} = zR \cup \{f, g\},$$

where

$$f, g \in M \setminus (I \cup J)$$

with $fg = 0$ (and necessarily with $fR \cap gR = zR$). Each of the sets $X_{f,g}$ has cardinality $p+2$. Hence, R has the clique ideal property.

Proof. Since $x^2 = z = -y^2$ and $xy = xz = yz = 0$, each element of R has the (unique) form $a = a_0 + a_1x + a_2y + a_3z$ for some $a_0, a_1, a_2, a_3 \in \mathbb{Z}_p$. Obviously, a is a unit if and only if $a_0 \neq 0$. Thus, $M = xR + yR$, $|R| = p^4$ and $|M| = p^3$. For integers $b_1, b_2, b_3, c_1, c_2, c_3 \in \{0, 1, \dots, p-1\}$,

$$(b_1x + b_2y + b_3z)(c_1x + c_2y + c_3z) = b_1c_1x^2 + b_2c_2y^2 = (b_1c_1 - b_2c_2)z.$$

Thus, $M^2 = zR$ and $\mathrm{Ann}(M) = zR$.

Let $f = f_1x + f_2y + f_3z$ and $g = g_1x + g_2y + g_3z$ be elements of $M \setminus zR$ with $f_i, g_i \in \{0, 1, 2, \dots, p-1\}$ for each i. Then, the product $fg = (f_1g_1 - f_2g_2)z$ is the same as the products

$$(f_1x + f_2y)(g_1x + g_2y), \qquad (f_1x + f_2y)(g_1x + g_2y + g_3z)$$

and

$$(f_1x + f_2y + f_3z)(g_1f + g_2y).$$

Note that the product is 0 if and only if $f_1g_1 \equiv f_2g_2 \pmod{p}$. Also, $xf = f_1z$ and $yf = -f_2z$. It follows that zR is contained in each nonzero principal ideal of R. Since f is not in zR, at least one of f_1 and f_2 is not 0, and we have $fR = (f_1x + f_2y)R$. In the case $f_1 \neq 0$, we also have $fR = (x + hy)R$, where h is the product of f_1^{-1} and f_2 in $\mathbb{Z}_p$. If $f_1 = 0$, then $f_2 \neq 0$ and $fR = yR$.

Each of the following ideals has cardinality

$$p^2 : yR,\ xR,\ (x+y)R,\ (x+2y)R, \dots,\ (x-y)R.$$

These are the only ideals of cardinality p^2. In addition, $I = (x+y)R$ and $J = (x-y)R$ are their own annihilators. For the others, the annihilator is a different ideal in the list. Specifically, $xR = \mathrm{Ann}(yR)$ and $yR = \mathrm{Ann}(xR)$. In the case $p = 3$, these four are the only ideals of cardinality 9.

For $a \in \{2, 3, \dots, p-2\}$, when $p > 3$,

$$\mathrm{Ann}(x + ay)R) = (x + by)R,$$

where $b \in \{2, 3, \dots, p-2\}$ is such that $ab \equiv 1 \pmod{p}$. In this case, $(x + ay)R \neq (x + by)R$, and thus, neither of these is a clique ideal. It follows that, for each odd prime p, $(x+y)R$ and $(x-y)R$ are the only maximum clique ideals. Moreover, for the ideals xR, yR and $(x + ay)R$, when $a \in \{2, 3, \dots, p-2\}$ and $p > 3$, the product of any pair of elements in the ideal is 0 if and only if at least one of them is in zR. Otherwise, the product is a nonzero element of zR.

Let $f, g \in M \setminus (I \cup J)$ be such that $fg = 0$. As above, write $f = f_1x + f_2y + f_3z$ and $g = g_1x + g_2y + g_3z$. Then, we have $f_1g_1 \equiv f_2g_2 \pmod{p}$. Since neither f nor g is in $I \cup J$, $f^2 \neq 0 \neq g^2$. Moreover, $g \notin fR$ and $f \notin gR$; thus, $fR \cap gR = zR$. The set

$$X_{f,g} = zR \cup \{f, g\}$$

is a clique of $\Gamma(R)^*$. Since $gR = \mathrm{Ann}(f)$ and $fR = \mathrm{Ann}(g)$, $X_{f,g}$ is a maximal clique.

The ideals I and J are the only maximum cliques of $\Gamma(R)^*$. Thus, R has the clique ideal property. $\square$

The case $x^2 = z = y^2$ is more complicated (for an odd prime p). For each odd prime p, the corresponding zero divisor graph of

$$R_p = \mathbb{Z}_p[\mathrm{X}, \mathrm{Y}, \mathrm{Z}]/(\mathrm{XY}, \mathrm{XZ}, \mathrm{YZ}, \mathrm{Z} - \mathrm{X}^2, \mathrm{Z} - \mathrm{Y}^2)$$

has maximal cliques of the form

$$X_{f,g} = zR_p \cup \{f, g\}$$

for certain choices of $f, g \in M \setminus zR$. In some cases, this set is a maximum clique, while in others it is not. For $p = 5$, the corresponding ring

$$R_5 = \mathbb{Z}_5[\mathrm{X}, \mathrm{Y}, \mathrm{Z}]/(\mathrm{XY}, \mathrm{XZ}, \mathrm{YZ}, \mathrm{Z} - \mathrm{X}^2, \mathrm{Z} - y^2)$$

has the clique ideal property. Specifically, the ideals $I = (x+2y)R$ and $J = (x+3y)R$ are maximum cliques and, as in the previous example, the other maximal cliques have cardinality $7 = 5 + 2$. On the other hand, if $p = 3$, then R_3 does not have the clique ideal property, and the various $X_{f,g}$ sets are the maximum cliques, each with $5 = 3 + 2$ elements. It turns out that R_p has the clique ideal property if and only if $\mathbb{Z}_p$ contains a square root of -1. Alternately, R_p has CIP if and only $p = 4k + 1$ for some integer k. Thus, for example, R_5, R_{13} and R_{17} have CIP, while R_3, R_7 and R_{11} do not.

Example 3.13. Let $R = \mathbb{Z}_p[\mathrm{X}, \mathrm{Y}, \mathrm{Z}]/(\mathrm{XY}, \mathrm{XZ}, \mathrm{YZ}, \mathrm{Z} - \mathrm{X}^2, \mathrm{Z} - y^2)$ with p an odd prime. Let x, y and z denote the respective images of X, Y and Z. Note that $x^2 = z = y^2 \neq 0$ and $z^2 = x^3 = y^3 = 0$.

(a) The ring

$$R = \{a_0 + a_1 x + a_2 y + a_3 z \mid a_i \in \mathbb{Z}_p\}$$

has cardinality p^4, and the maximal ideal $M = xR + yR$ has cardinality p^3. Also, $\mathrm{Ann}(M) = zR = \{0, z, 2z, \ldots, (p-1)z\}$.

(b) For elements $f = f_1 x + f_2 y + f_3 z$ and $g = g_1 x + g_2 y + g_3 z$, the product $fg = f_1 g_1 x^2 + f_2 g_2 y = (f_1 g_1 + f_2 g_2) z$. Moreover, $fg = 0$ if and only if p divides $f_1 g_1 + f_2 g_2$. In particular, $f^2 = 0$ if and only if p divides $f_1^2 + f_2^2$.

(c) There are $p + 1$ ideals of cardinality p^2. Specifically, yR, xR, $(x + y)R$, $(x + 2y)R$, $\ldots$, $(x + (p-1)y)R$. Of these, $xR = \mathrm{Ann}(yR)$

and $yR = \mathrm{Ann}(xR)$. For $1 \le a \le p-1$, the annihilator of $(x+ay)R$ is the ideal $(x+by)R$, where b is such that $ab \equiv -1 \pmod{p}$.

(d) Let $f = f_1x + f_2y + f_3z$ and $g = g_1x + g_2y + g_3z$ be elements of $M \setminus zR$ such that $f_1g_1 \equiv -f_2g_2 \pmod{p}$. The set

$$X_{f,g} = zR \cup \{f, g\}$$

is a maximal clique of $\Gamma(R)^*$ if and only if $fR \neq gR$.

(e) If $\mathbb{Z}_p$ does not contain a square root of -1, then each set $X_{f,g}$ is a maximum clique of $\Gamma(R)^*$ and R does not have the clique ideal property.

(f) If $h \in \mathbb{Z}_p$ is a square root of -1, then the ideals $I = (x+hy)R$ and $J = (x-hy)R$ are maximum cliques of $\Gamma(R)^*$. Moreover, these two ideals are the only maximum cliques of $\Gamma(R)^*$ and R has the clique ideal property.

(g) R has CIP if and only if $p = 4k+1$ for some integer k.

Proof. The proof is similar to that given for Example 3.12. First, note that $\mathrm{Ann}(xR) = yR$ and $\mathrm{Ann}(yR) = xR$. Thus, $\{x, y\} \cup zR$ is a maximal clique of $\Gamma(R)^*$.

If p is such that $\mathbb{Z}_p$ contains an element h that is a square root of -1, then

$$(x+hy)^2 = z - z = 0 = z - z = (x-hy)^2$$

(since $xy = 0$ and $x^2 = z = -y^2$). Thus, $I = (x+hy)R$ and $J = (x-hy)R$ are such that $I = \mathrm{Ann}(I)$ and $J = \mathrm{Ann}(J)$. There are no larger cliques in this case, so R_p has CIP when $\mathbb{Z}_p$ contains a square root of -1.

If $\mathbb{Z}_p$ does not contain a square root of -1, then there is no nonzero $b \in \mathbb{Z}_p$ such that $(x+by)^2 = 0$. Thus, in this case, the only maximal clique ideal is $\mathrm{Ann}(M) = zR$, an ideal of cardinality p. Hence, R has CIP if and only $p = 4k+1$ for some integer k.

For the case that $\mathbb{Z}_p$ does not contain a square root of -1, let $u, v \in Z(R) \setminus zR$ be such that $uv = 0$. Then, there are integers a, b, c, d, e and f between 0 and $p-1$ such that $u = ax + by + cz$, $v = dx + ey + fz$, $ad + be = 0$, and at least one of a and b is positive and at least one of e and f is positive. Since $\mathbb{Z}_p$ does not contain a square root of -1, we cannot have both $a = d$ and $b = e$. It follows that $\{u, v\} \cup zR$ is a maximum clique of $\Gamma(R)^*$. $\square$

The local ring in the next example has maximum cliques X and Y where neither is an ideal and $\mathrm{Ann}(X) \subsetneq \mathrm{Ann}(Y)$. For another example of such behavior, see [**2,** Theorem 2.1].

Example 3.14. Let

$$R = \mathbb{Z}_2[\mathrm{W}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}]/(\mathrm{WX}, \mathrm{WY}, \mathrm{WZ}, \mathrm{XY}, \mathrm{XZ}, \mathrm{YZ}, \mathrm{Z} + \mathrm{W}^2, \mathrm{Z} + \mathrm{X}^2, \mathrm{Z} + \mathrm{Y}^2),$$

and let w, x, y and z denote the respective images of $\mathrm{W}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}$ in R. We have $z = w^2 = x^2 = y^2$ and $z^2 = w^3 = x^3 = y^3 = 0$.

(a) $R = \mathbb{Z}_2 + w\mathbb{Z}_2 + x\mathbb{Z}_2 + y\mathbb{Z}_2 + z\mathbb{Z}_2$ with maximal ideal $M = w\mathbb{Z}_2 + x\mathbb{Z}_2 + y\mathbb{Z}_2 + z\mathbb{Z}_2$.
(b) $M^2 = \{0, z\} = \mathrm{Ann}(M)$.
(c) Both $V = \{0, w, x, y, z\}$ and $W = \{0, w, z, x + y, x + y + z\}$ are maximum cliques of $\Gamma(R)^*$.
(d) $\mathrm{Ann}(V) = \{0, z\} \subsetneq \mathrm{Ann}(W) = (x + y)R$.

Proof. It is clear that each element a of R has a unique representation as a sum

$$a = a_0 + a_1 w + a_2 x + a_3 y + a_4 z$$

with each a_i in $\mathbb{Z}_2$ with $a \in M$ if and only if $a_0 = 0$. Hence, $|R| = 32$ and $|M| = 16$. Since $wz = xz = yz = z^2 = 0$, $z \in \mathrm{Ann}(M)$. Also, for a pair of elements b and c in M, the product

$$b \cdot c = b_1c_1w^2 + b_2c_2x^2 + b_3c_3y^2 = (b_1c_1 + b_2c_2 + b_3c_3)z \in \{0, z\}$$

since $w^2 = x^2 = y^2 = z$. For the case $b = c$, we have $b^2 = (b_1{+}b_2{+}b_3)z$, which is z if an even number of b_is (for $i = 1, 2, 3$) are 0, and is 0 otherwise. Hence,

$$S = \{s \in R \mid s^2 = 0\} = \{0, z, w{+}x, w{+}y, x{+}y, w{+}x{+}z, w{+}y{+}z, x{+}y{+}z\}$$

and

$$M \backslash S = \{t \in R \mid t^2 = z\} = \{w, x, y, w{+}x{+}y, w{+}z, x{+}z, y{+}z, w{+}x{+}y{+}z\}.$$

Except for zR, each of the nonzero principal ideals of R contains four elements, and each has an annihilator that contains eight elements. There are seven such ideals. Specifically:

(i) $wR = (w+z)R$ with $\mathrm{Ann}(wR) = \{0, x, y, z, x+y, x+z, y+z, x+y+z\} = xR + yR$;
(ii) $xR = (x+z)R$ with $\mathrm{Ann}(xR) = \{0, w, y, z, w+y, w+z, w+y+z\} = wR + yR$;
(iii) $yR = (y+z)R$ with $\mathrm{Ann}(yR) = \{0, w, x, z, w+x, w+z, x+z, w+x+z\} = wR + xR$;
(iv) $(w+x)R = (w+x+z)R$ with $\mathrm{Ann}((w+x)R) = \{0, y, z, w+x, w+x+y, w+x+z, w+x+y+z\} = yR + (w+x)R$;
(v) $(w+y)R = (w+y+z)R$ with $\mathrm{Ann}((w+y)R) = \{0, x, z, w+y, w+x+y, w+y+z, w+x+y+z\} = xR + (w+y)R$;
(vi) $(x+y)R = (x+y+z)R$ with $\mathrm{Ann}((x+y)R)) = \{0, w, z, x+y, w+x+y, x+y+z, w+x+y+z\} = wR + (x+y)R$; and
(vii) $(w+x+y)R = (w+x+y+z)R$ with $\mathrm{Ann}((w+x+y)R) = \{0, z, w+x, w+y, x+y, w+x+z, w+y+z, x+y+z\} = (w+x)R + (w+y)R$.

Each of the principal ideals $(w+x)R = (w+x+z)R$, $(w+y)R = (w+y+z)R$ and $(x+y)R = (x+y+z)R$ is such that its square is (0). On the other hand, each of the principal ideals wR, xR, yR and $(w+x+y)R$ is such that the square of the ideal is zR. Since $zM = (0)$, each maximal clique contains $\{0, z\} = zR$.

Suppose that T is a maximal clique which contains w. We have $0, z \in T$, and the other elements come from the set $\{x, y, x+y, x+z, y+z, x+y+z\}$. The following are the only possible pairs of distinct elements from $T \setminus \{0, w, z\}$ whose corresponding product is 0: $\{x, y\}$, $\{x, y+z\}$, $\{y, x+z\}$, $\{x+z, y+z\}$ and $\{x+y, x+y+z\}$. Thus, $|T| = 5$. The same analysis applies to the case $w+z \in T$ instead of w. Similarly, $|U| = 5$ if U is a maximal clique that contains at least one of x, $x+z$, y or $y+z$.

Next, suppose that Q is a maximal clique which contains $w+x+y$ (or $w+x+y+z$). In this case, the elements of $Q \setminus \{0, z, w+x+y\}$ come from the set

$$\{w+x, w+y, x+y, w+x+z, w+y+z, x+y+z\}.$$

The following are the only pairs of distinct elements from this set whose corresponding product is 0: $\{w+x, w+x+z\}$, $\{w+y, w+y+z\}$, $\{x+y, x+y+z\}$. It follows that $|Q|=5$.

Each of the principal ideals $(w+x)R=(w+x+z)R$, $(w+y)R=(w+y+z)R$ and $(x+y)R=(x+y+z)R$ is properly contained in its annihilator. Thus, none are maximal cliques. From the analysis above, a maximal clique that contains one of these three ideals contains exactly one more element. It follows that each of $W=\{w\}\cup(x+y)R$, $X=\{x\}\cup(w+y)R$, $Y=\{y\}\cup(w+x)R$ and $V=\{0,w,x,y,z\}$ is a maximum clique of $\Gamma(R)^*$. We have $\operatorname{Ann}(V)=\{0,z\}\subsetneq\operatorname{Ann}(W)=(x+y)R$. □

The ring R in the next example has a maximum clique ideal I that is a maximal clique of $\Gamma(R)^*$ but not a maximum clique and a maximum clique X of $\Gamma(R)^*$ that is not an ideal. From the proof of Theorem 1.3, it must be that $|I|=|\operatorname{Ann}(X)|$. In addition, $|X|=|\operatorname{Ann}(X)|+1$. Moreover, each maximum clique Y is such that $|Y|=|\operatorname{Ann}(Y)|+1$ with $\operatorname{Ann}(Y)=\operatorname{Ann}(X)$.

Example 3.15. Let $R=\mathbb{Z}_p[\mathrm{X},\mathrm{Y},\mathrm{Z}]/(\mathrm{X}^3,\mathrm{Y}^2,\mathrm{Z}^2,\mathrm{XZ},\mathrm{X}^2\mathrm{Y})$ with p a prime, and let M be the maximal ideal of R. Also, let x, y and z denote the respective images of X, Y and Z in R, and let

$$S=\{f\in R\mid f^2=0\}.$$

(a) Each element of R can be expressed as a unique sum of the form

$$a=a_0+a_1x+a_2y+a_3z+a_4x^2+a_5xy+a_6yz,$$

where each $0\le a_i\le p-1$, with $a\in M$ if and only if $a_0=0$. Thus, $|R|=p^7$ and $|M|=p^6$.

(b) $\operatorname{Ann}(M)=x^2\mathbb{Z}_p+xy\mathbb{Z}_p+yz\mathbb{Z}_p$.

(c) The ideals $I=yR+\operatorname{Ann}(M)$ and $J=zR+\operatorname{Ann}(M)$ are maximum clique ideals of R. While I is a maximal clique, J is properly contained in the maximum clique $X=\{x\}\cup J$. In addition, each maximum clique Y is such that $Y=\{f\}\cup J$.

(d) If $p = 2$, then

$$H = (y + z)R + \mathrm{Ann}(M)$$

is another maximum clique ideal of R that is a maximal clique.

(e) If p is odd, I is the only ideal of R that is both a maximum clique ideal and a maximal clique.

(f) R does not have the clique ideal property.

Proof. The statements in (a) are clear. Consider the product $a \cdot c$ where $a, c \in M$. Then,

$$a = a_1x + a_2y + a_3z + a_4x^2 + a_5xy + a_6yz$$

and

$$c = c_1x + c_2y + c_3z + c_4x^2 + c_5xy + c_6yz.$$

Since x^3, y^2, z^2, xz and x^2y are all 0, all three of x, y and z annihilate the ideal

$$x^2R + xyR + yzR = x^2\mathbb{Z}_p + xy\mathbb{Z}_p + yz\mathbb{Z}_p.$$

Thus,

$$a \cdot c = a_1c_1x^2 + (a_1c_2 + a_2c_1)xy + (a_2c_3 + a_3c_2)yz.$$

Suppose that $a \cdot c = 0$. If $a_1 \neq 0$, then $c_1 = 0 = c_2$ and at least one of a_2 and c_3 is 0. Thus (for example) $\mathrm{Ann}(x+y) = x^2\mathbb{Z}_p + xy\mathbb{Z}_p + yz\mathbb{Z}_p$. It follows that

$$\mathrm{Ann}(M) = x^2\mathbb{Z}_p + xy\mathbb{Z}_p + yz\mathbb{Z}_p$$

with $|\mathrm{Ann}(M)| = p^3$.

Let $f = f_1x + f_2y + f_3z + h \notin \mathrm{Ann}(M)$, where $h \in \mathrm{Ann}(M)$ and $0 \leq f_i \leq p - 1$ for $i = 1, 2, 3$. We consider several cases.

Case 1. $f_1 \neq 0$ and $f_2 \neq 0$. In this case, $\mathrm{Ann}(f) = \mathrm{Ann}(M)$ no matter the values of f_3 and h.

Case 2. $f_1 \neq 0$ and $f_2 = 0$. In this case, $zf = 0$ and $f^2 \neq 0$. From the analysis in the first paragraph, we have $\mathrm{Ann}(f) = z\mathbb{Z}_p + \mathrm{Ann}(M)$, again no matter the values of f_3 and h.

For the next four cases we let $g = g_1x+g_2y+g_3z$ with $0 \le g_i \le p-1$ for $i = 1, 2, 3$, and consider the product $f \cdot g$.

Case 3. $f_2 \neq 0$ and $f_1 = 0 = f_3$. We have

$$f \cdot g = f_2g_1xy + f_2g_2y^2 + f_2g_3yz.$$

Such a product is 0 if and only if $g_1 = 0 = g_3$. Hence, $\mathrm{Ann}(f) = y\mathbb{Z}_p + \mathrm{Ann}(M)$. It follows that the ideal

$$I = y\mathbb{Z}_p + x^2\mathbb{Z}_p + xy\mathbb{Z}_p + yz\mathbb{Z}$$

contains p^4 elements and not only is $I^2 = (0)$, but $I = \mathrm{Ann}(I)$. Hence, I is a maximal clique.

Case 4. $f_3 \neq 0$ and $f_1 = 0 = f_2$. We have

$$f \cdot g = g_2f_3yz.$$

Thus,

$$\mathrm{Ann}(f) = x\mathbb{Z}_p + z\mathbb{Z}_p + \mathrm{Ann}(M).$$

We have $|\mathrm{Ann}(f)| = p^5$, but $x^2 \neq 0$, so $\mathrm{Ann}(f)$ is not a clique ideal. However, the subideal $J = z\mathbb{Z}_p + \mathrm{Ann}(M)$ is such that $|J| = p^4$ and $J^2 = (0)$. Thus, J is a clique. It is not a maximal clique as $xJ = (0)$. By Case 2, $\{x\} \cup J$ is a maximal clique of $\Gamma(R)^*$. In addition, $\{h\} \cup J$ is a maximal clique for each $h \in \mathrm{Ann}(f) \setminus J$.

Case 5. $f_2 \neq 0$, $f_3 \neq 0$ and $f_1 = 0$ with p odd. In this case,

$$f \cdot g = f_2g_1xy + (f_2g_3 + f_3g_2)yz.$$

For this product to be 0, we must have $g_1 = 0$ and $f_2g_3+f_3g_2$ a multiple of p. Since both f_2 and f_3 are units in $\mathbb{Z}_p$, given any $0 \le k_2 \le p-1$, there is a unique integer k_3 between 0 and $p-1$ such that p divides $f_2k_3 + f_3k_2$. In addition, either both k_2 and k_3 are 0 or neither is. Let m be such that p divides $f_2m + f_3$. Then,

$$\mathrm{Ann}(f) = (y + mz)\mathbb{Z}_p + \mathrm{Ann}(M).$$

Note that $f^2 = 2f_2f_3 \neq 0$ since p is odd. Thus, $f\mathbb{Z}_p + \mathrm{Ann}(M)$ is not a clique ideal when p is odd.

Case 6. $f_2 \neq 0$, $f_3 \neq 0$ and $f_1 = 0$ with $p = 2$. As in Case 5, $\mathrm{Ann}(f) = (y+mz)\mathbb{Z}_p + \mathrm{Ann}(M)$ for some positive integer m, but the only choice for each of f_2, f_3 and m is 1. Thus, in this special case,

$$H = f\mathbb{Z}_p + \mathrm{Ann}(M)$$

is a clique ideal. As with the ideal I (when $p = 2$), $|H| = 2^4$. It follows that H is a maximum clique ideal that is also a maximal clique.

Based on the ideals that are annihilators, the set

$$X = \{x\} \cup J$$

is a maximum clique of $\Gamma(R)^*$, and I is a maximum clique ideal that is a maximal clique but not a maximum clique. □

REFERENCES

1. O.A. AbuGhneim, E.E. AbdAlJawad and H. Al-Ezeh, *The clique number of* $\Gamma(Z_{p^n}(\alpha))$, Rocky Mountain J. Math. **42** (2012), 1–14.

2. D.D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra **159** (1993), 500–514.

3. D.F. Anderson and P. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434–447.

4. I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), 208–226.

5. G.A. Cannon, K.A. Neuerburg and S.P. Redmond, *Zero-divisor graphs of nearrings and semigroups*, in *Nearrings and nearfields*, Springer, Dordrecht, 2005.

6. J. Huckaba, *Commutative rings with zero divisors*, Dekker, New York, 1988.

7. B. McDonald, *Finite rings with identity*, Pure Appl. Math. **28** (1974).

8. S.P. Redmond, *On zero-divisor graphs of small finite commutative rings*, Discr. Math. **307** (2007), 1155–1166.

9. ______, *Corrigendum to On zero-divisor graphs of small finite commutative rings*, Discr. Math. **307** (2007), 2449–2452.

University of North Carolina Charlotte, Department of Mathematics and Statistics, Charlotte, NC 28223
Email address: **tglucas@uncc.edu**