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# A NOTE ON QUASI-MONIC POLYNOMIALS AND EFFICIENT GENERATION OF IDEALS

MD. ALI ZINNA

ABSTRACT. Let $A$ be a commutative Noetherian ring, and let $I$ be an ideal of $A[T]$ containing a quasi-monic polynomial. Assuming that $I/I^2$ is generated by $n$ elements, where $n \geq \dim(A[T]/I) + 2$, then, it is proven that any given set of $n$ generators of $I/I^2$ can be lifted to a set of $n$ generators of $I$. It is also shown that various types of Horrocks' type results (previously proven for monic polynomials) can be generalized to the setting of quasi-monic polynomials.

**1. Introduction.** It is well known that monic polynomials played a significant role in the development of the theory of projective modules and complete intersections. For instance, the affine Horrocks' theorem asserts that if a projective $A[T]$-module $P$ is such that $P_f$ is free for some monic polynomial $f \in A[T]$, then $P$ is free. This result was perhaps the most crucial step in proving Serre's conjecture.

Later, Mohan Kumar **[10]** and Mandal **[7]** used monic polynomials beautifully to obtain some significant results on complete intersections. In the more modern theory of Euler class groups, serious involvement of monic polynomials can be found in Das's work.

In this paper, we study "quasi-monic" polynomials (see Definition 2.7, which was introduced in **[14]**). Quasi-monic polynomials are a generalization of monic polynomials. In **[14]**, Zhu showed that the Suslin lemma and Horrocks' theorem for monic polynomials can be generalized to the setting of quasi-monic polynomials.

We now recall the theorem of Mandal, previously mentioned.

**Theorem 1.1** (**[7]**). *Let $A$ be a commutative Noetherian ring, and let $I$ be an ideal of $A[T]$ containing a monic polynomial. Suppose that $I/I^2$*

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*is generated by $n$ elements, where $n \geq \dim(A[T]/I) + 2$. Then, any given set of $n$ generators of $I/I^2$ can be lifted to a set of $n$ generators of $I$.*

In the context of quasi-monic polynomials, it is natural to ask the next question.

**Question 1.2.** *Let $A$ be a commutative Noetherian ring, and let $I$ be an ideal of $A[T]$ containing a quasi-monic polynomial. Let $I/I^2$ be generated by $n$ elements, where $n \geq \dim(A[T]/I)+2$. Suppose that $I = (f_1, \ldots, f_n) + I^2$. Do there exist $g_1, \ldots, g_n$ such that $I = (g_1, \ldots, g_n)$ with $f_i - g_i \in I^2$?*

In Section 3, we give an affirmative answer to the above question (Theorem 3.2).

Later, we prove the following result (see Theorem 3.6). This generalizes a result of Bhatwadekar and Raja Sridharan [**3,** Theorem 3.4].

**Theorem 1.3.** *Let $A$ be a commutative Noetherian ring of dimension $n$ containing an infinite field, and let $P$ be a projective $A[T]$-module of rank $n$. Suppose that the projective $A[T]_{f(T)}$-module $P_{f(T)}$ has a unimodular element for some quasi-monic polynomial $f(T) \in A[T]$. Then, $P$ has a unimodular element.*

Now we discuss another important application of monic polynomials.

Let $A = k[X_1, \ldots, X_n]$ be the polynomial ring in $n$ variables over an algebraically closed field $k$. Let $I$ be an ideal of $A$. Then, the famous *Hilbert Nullstellensatz* states that $V(I)$ is not empty. One of the proofs of this via *Noether normalisation* runs as follows: using a change of variables we may assume that $I$ contains a monic polynomial in one of the variables, say, $X_n$. Let $B = k[X_1, \ldots, X_{n-1}]$ be the polynomial ring in $n-1$ variables and $J = I \cap B$. Then, by the induction hypothesis, $V(J)$ is not empty.

In order to complete the proof, we would like to extend a zero of $J$ to a zero of $I$. We can do this using the following extension lemma due to Suslin ([**6,** page 79, Lemma 1.1]). *Let $A$ be a commutative ring and $I \subset A[X]$ be an ideal containing a monic polynomial. Let $\mathcal{M} \subset A$ be a maximal ideal containing $I \cap A$. Then, there exists a maximal ideal $\mathcal{N}$ of $A[X]$ containing $I + \mathcal{M}A[X]$.*

Let $(a_1, \ldots, a_{n-1}) \in V(J)$. Since $I$ contains a monic polynomial in $X_n$, we can apply the extension lemma to the maximal ideal $(X_1 - a_1, \ldots, X_{n-1} - a_{n-1})$ and find a maximal ideal $\mathcal{N} = (X_1 - a_1, \ldots, X_n - a_n)$ corresponding to a zero of $I$.

Note that we can also state the extension lemma in the following form.

**Lemma 1.4.** *Let $A$ be a commutative ring, and let $I \subset A[X]$ be an ideal containing a monic polynomial. Let $\mathcal{M}$ be a maximal ideal of $A$ such that $I + \mathcal{M}A[X] = A[X]$. Then, $(I \cap A) + \mathcal{M} = A$.*

Most proofs of Horrocks' theorem (other than Horrocks' own which uses cohomology) use the extension lemma in some form, in the same way that proofs of the Hilbert Nullstellensatz use Noether normalization. Motivated by the above discussions, we prove the following result which generalizes the extension lemma to the quasi-monic polynomials (3.7).

**Lemma 1.5.** *Let $A$ be a commutative ring, and let $I \subset A[T]$ be an ideal containing a quasi-monic polynomial. Let $\mathcal{M}$ be a maximal ideal of $A$ such that $I + \mathcal{M}A[T] = A[T]$. Then, $(I \cap A) + \mathcal{M} = A$.*

Another aspect of Horrocks' theorem is the extendability of vector bundles from the affine line to the projective line. Let $F \in \mathbb{C}[Y]$ be of degree $n$. Then, $F$ has $n$ roots. If the leading coefficient of $F$ tends to zero, then some of the roots of $F$ tend to $\infty$. Now, let $F(X, Y)$ be a polynomial of the form $F(X, Y) = a_n(X)Y^n$ with additional lower degree terms in $Y$. If we substitute any complex value of $X$, then the equation $F(X, Y) = 0$ has $n$ roots in $Y$ except for those values of $X$ such that $a_n(X) = 0$, in which case, some of those roots are $\infty$.

Let $(F_1(X, Y), \ldots, F_t(X, Y))$ be a unimodular row in $\mathbb{C}[X, Y]$, i.e., $F_1, \ldots, F_t$ generate the unit ideal. Hence, the polynomials have no common zero in $\mathbb{C}^2$. In particular, if $a \in \mathbb{C}$, then $F_1(a, Y), \ldots, F_t(a, Y)$ have no common zero. Let $F_i(X, Y) = a_{m_i}(X)Y^{m_i}$ with additional polynomials with lower degree terms in $Y$. If the leading coefficients of $F_i$ have no common complex zero in $X$, then, by what we stated earlier, the $F_i$ have no common zeros at $Y = \infty$. Any common zero $X = a$

yields a common zero $(a, \infty)$ of the polynomials. In particular, if the leading coefficients of the $F_i$ have no common zero, the polynomials $F_i$ have no common zero at $Y = \infty$, and therefore, the unimodular row $(F_1, \ldots, F_t)$ extends to a unimodular row at $Y = \infty$. Thus, we obtain a vector bundle on $\mathbb{P}^1$. For example, if $a_{m_1}(X) = 1$, this holds (in other words, $F_1(X, Y)$ is monic in $Y$). This is the reason monic polynomials are relevant to these questions.

Motivated by the above considerations, we have the following definition: If $A$ is a commutative ring and $(F_1(Y), \ldots, F_t(Y))$ in $A[Y]$ is unimodular, we say that this row is unimodular if $Y = \infty$ if $(L(F_1), \ldots, L(F_t))$ is unimodular in $A$ where $L(F_i)$ is the leading coefficient of $F_i$.

In this context, Rao proved the next result ([**12**]).

**Theorem 1.6.** *Let $A$ be a commutative ring and $(f_1, \ldots, f_n) \in Um_n(A[X])$, i.e., a unimodular row of length $n$, with $n \geq 3$ such that $(L(f_1), \ldots, L(f_n)) \in Um_n(A)$. Then, $(f_1, \ldots, f_n)$ can be transformed to $(L(f_1), \ldots, L(f_n))$ via elementary transformations. In particular, when one of the $a_i$ is 1, i.e., when $F_i$ is monic, then the row can be completed to an elementary matrix.*

In the context of quasi-monic polynomials we prove the following result (3.8).

**Theorem 1.7.** *Let $A$ be a commutative ring, $(f_1, \ldots, f_n) \in Um_n(A[T])$ with $n \geq 3$, and let $f_1$ be a quasi-monic polynomial. Then, $(f_1, \ldots, f_n)$ can be completed to an elementary matrix.*

Careful investigation of our methods adapted to prove the above results reveal that we are essentially taking advantage of decomposing the given ring $A = A_0 \oplus \cdots \oplus A_{k+1}$. The decomposition is induced by a quasi-monic polynomial. With this observation, we consider the following set up.

Let $A$ be a commutative Noetherian ring and assume that $\mathrm{Spec}(A)$ is disconnected. Let $A = A_1 \oplus \cdots \oplus A_k$ be a decomposition such that

$$\mathrm{Spec}(A) = \mathrm{Spec}(A_1) \cup \cdots \cup \mathrm{Spec}(A_k)$$

with $\mathrm{Spec}(A_i)$ is connected. In Section 4, we investigate the relationship between the top Euler class group of $A$ and the top Euler class groups of $A_i$.

In fact, we prove the following result.

**Theorem 1.8.** *Let $A$ be a commutative Noetherian ring of dimension $n \geq 2$ and $A = A_1 \oplus \cdots \oplus A_k$ such a decomposition of $A$ so that* $\mathrm{Spec}(A_1), \ldots, \mathrm{Spec}(A_k)$ *are the connected components of* $\mathrm{Spec}(A)$. *Then, $E(A) \simeq E(A_1) \oplus \cdots \oplus E(A_k)$.*

In order to prove Theorem 1.8, we consider the notion of the generalized Euler class group of a Noetherian ring $R$ (denoted by $\widetilde{E}(R)$) from **[5]**. It is easy to see that the natural map from $\widetilde{E}(A)$ to

$$\widetilde{E}(A_1) \oplus \cdots \oplus \widetilde{E}(A_k)$$

is well defined. Finally, we show that the natural map is an isomorphism. In **[5]**, it was proven that $E(R) \simeq \widetilde{E}(R)$. Therefore, the theorem follows.

## 2. Preliminaries.

**Notation 2.1.** All of the rings considered in this paper are assumed to be commutative. By dimension of a ring $A$ we mean its Krull dimension, denoted by $\dim(A)$. Modules are assumed to be finitely generated. Projective modules are assumed to have constant rank.

We begin with the following definition.

**Definition 2.2.** Let $A$ be a ring. An element $a \in A$ is called an *$m$-idempotent* in case $a^m = a$ for some integers $m \geq 2$.

**Definition 2.3.** Let $e_1, e_2, \ldots, e_n$ be idempotents of a ring $A$. Then, $e_1, e_2, \ldots, e_n$ are called *orthogonal idempotents* if $e_i e_j = 0$ whenever $i \neq j$.

The next lemma is proven in **[14]**.

**Lemma 2.4.** *Let $A$ be a ring and $d_0, d_1, \ldots, d_k \in A$. If each $d_i$ is an $m_i$-idempotent, then there exist orthogonal idempotents*

$$s_0, s_1, \ldots, s_k, s_{k+1} \in A$$

*with*

$$s_0 + s_1 + \cdots + s_{k+1} = 1,$$

*and there is a ring decomposition of $A$:*

$$A = As_0 \oplus \cdots \oplus As_{k+1}.$$

**Remark 2.5.** In Lemma 2.4 above, the orthogonal idempotents $s_0, s_1, \cdots, s_k, s_{k+1} \in A$ are precisely of the form:

$$\begin{aligned}
s_0 &= d_0^{m_0-1}, \\
s_1 &= (1 - d_0^{m_0-1})d_1^{m_1-1}, \\
\cdots&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
s_k &= (1 - d_0^{m_0-1}) \cdots (1 - d_{k-1}^{m_{k-1}-1})d_k^{m_k-1}, \\
s_{k+1} &= (1 - d_0^{m_0-1}) \cdots (1 - d_{k-1}^{m_{k-1}-1})(1 - d_k^{m_k-1}).
\end{aligned}$$

**Remark 2.6.** Let $A$ be a ring and $e \in A$ an idempotent element. Then, $A$ has a decomposition, namely, $A = Ae \oplus A(1-e)$. Now, choose finitely many idempotent elements in $A$, say, $e_1, \ldots, e_n$. Take $e'_1 = e_1$, $e'_2 = (1-e_1)e_2, \ldots,$ $e'_n = (1-e_1)\cdots(1-e_{n-1})e_n$ and $e'_{n+1} = (1-e_1)\cdots(1-e_{n-1})(1-e_n)$. Then, we have $n+1$ pairwise orthogonal idempotents $e'_1, \ldots, e'_n, e'_{n+1}$ such that $e'_1 + \cdots + e'_{n+1} = 1$. Thus, the choice of $s_i$ in Remark 2.5 is the natural generalization of the $e'_i$.

**Definition 2.7.** A polynomial $f(X) = d_0X^n + d_1X^{n-1} + \cdots + d_{n-1}X + d_n$ with $d_0 \neq 0$ over a commutative ring $A$ is called a *quasi-monic polynomial* if it is monic or there exists an integer $k$, with $0 \leq k < n$, such that the coefficients of $f(X)$ satisfy the following conditions:

(i) each $d_i$ $(0 \leq i \leq k)$ is an $m_i$-idempotent;
(ii) $s_{k+1}d_{k+1} = s_{k+1}$ ($s_{k+1} \in A$ as in Lemma 2.4), or there exist $t_0, \ldots, t_{k+1} \in A$ such that $t_0d_0 + \cdots + t_{k+1}d_{k+1} = 1$.

**Remark 2.8.** The definition of a quasi-monic polynomial over an arbitrary ring (not necessarily commutative) may be found in [**14**].

If there is a least integer $k \geq 0$ such that the above two conditions hold, then we say that $f(X)$ is a quasi-monic polynomial of length $k+1$, and a monic polynomial is called a quasi-monic polynomial of length 0.

The following facts are well known. Therefore, we state them without proofs.

**Lemma 2.9.** *Let $A = A_1 \oplus \cdots \oplus A_n$ be a finite direct product of rings and $e_i := (0, \dots, 0, 1, 0, \dots, 0)$, where the ith component is* 1. *Then*:

(i) ${e_i}^2 = e_i$;

(ii) $e_i e_j = 0$ *for* $i \neq j$; *and*

(iii) $\Sigma e_i = 1$.

**Theorem 2.10.** *Let $A_1, \dots, A_n$ be rings and $A = A_1 \oplus \cdots \oplus A_n$. Let $e_1, \dots, e_n$ be the idempotents defined by this splitting, as above. Then, any $A$-module $M$ admits the splitting $M = M_1 \oplus \cdots \oplus M_n$ where $M_i := Me_i$. Note that $M_i$ is an $A_i$-module in an obvious manner.*

**Lemma 2.11.** *Let $A = A_1 \oplus \cdots \oplus A_n$ be as above. Let $I$ be an ideal of $A$. Then, $I = I_1 \oplus \cdots \oplus I_n$, where $I_j$ is a suitable ideal of $A_j$ for $j = 1, \dots, n$. In this situation, there is a canonical isomorphism*

$$A/I \simeq A_1/I_1 \oplus \cdots \oplus A_n/I_n.$$

**Remark 2.12.** Let $A_1, \dots, A_n$ be rings and $A = A_1 \oplus \cdots \oplus A_n$. Then, any projective $A$-module $P$ admits the splitting $P = P_1 \oplus \cdots \oplus P_n$, where $P_i$ is a projective $A_i$-module. In addition, we know that, if

$$\phi : R \longrightarrow S$$

is a ring homomorphism and $P$ is a projective $R$-module of rank $n$, then $P \otimes_R S$ is also a projective $S$-module of rank $n$. Therefore, each $P_i$ is a projective $A_i$-module of rank $n$ (where $n$ is rank$(P)$).

**Definition 2.13.** Let $R$ be a ring and $P$ a projective $R$-module. An element $p \in P$ is called *unimodular* if there is a surjective $R$-linear map

$$\phi : P \twoheadrightarrow R$$

such that $\phi(p) = 1$.

**Definition 2.14.** An ideal $I$ of a ring $A$ is said to be *efficiently generated* if $\mu(I) = \mu(I/I^2)$, where $\mu(-)$ stands for the minimal number of generators as an $A$-module.

Let $A$ be a Noetherian ring of dimension $n$. We recall the definition of the $n$th Euler class group $E^n(A)$ of $A$ with respect to $A$ (from [**2**]). For brevity, we shall denote $E^n(A)$ by $E(A)$. However, we can define the $n$th Euler class group $E^n(A, L)$ of $A$ with respect to any rank 1 projective $A$-module $L$.

**Definition 2.15** (The Euler class group $E(A)$)**.** Write $F = A^n$. Let $J \subset A$ be an ideal of height $n$ such that $J/J^2$ is generated by $n$ elements. Two surjections $\alpha$ and $\beta$ from $F/JF$ to $J/J^2$ are said to be *related* if there exists a $\sigma \in SL(F/JF)$ such that $\alpha\sigma = \beta$. Clearly, this is an equivalence relation on the set of surjections from $F/JF$ to $J/J^2$. Let $[\alpha]$ denote the equivalence class of $\alpha$. Such an equivalence class $[\alpha]$ is called a *local orientation* of $J$. By abuse of notation, we shall identify an equivalence class $[\alpha]$ with $\alpha$. A local orientation $\alpha$ is called a *global orientation* if $\alpha : F/JF \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : F \twoheadrightarrow J$.

Let $G$ be the free abelian group on the set of pairs $(\mathfrak{n}, \omega_{\mathfrak{n}})$, where $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ of height $n$ such that $\mathfrak{n}/\mathfrak{n}^2$ is generated by $n$ elements and $\omega_{\mathfrak{n}}$ is a local orientation of $\mathfrak{n}$. Let $J \subset R$ be an ideal of height $n$ such that $J/J^2$ is generated by $n$ elements and $\omega_J$ is a local orientation of $J$. Let $J = \cap_i \mathfrak{n}_i$ be the (irredundant) primary decomposition of $J$. We associate to the pair $(J, \omega_J)$ the element

$$\sum_i (\mathfrak{n}_i, \omega_{\mathfrak{n}_i})$$

of $G$, where $\omega_{\mathfrak{n}_i}$ is the local orientation of $\mathfrak{n}_i$ induced by $\omega_J$. By abuse of notation, we denote $\sum_i (\mathfrak{n}_i, \omega_{\mathfrak{n}_i})$ by $(J, \omega_J)$. Let $H$ be the subgroup of $G$ generated by set of pairs $(J, \omega_J)$, where $J$ is an ideal of height $n$ and $\omega_J$ is a global orientation of $J$. The Euler class group of $A$ with respect to $A$ is $E(A) \stackrel{\text{def}}{=} G/H$.

Now we recall the definition of the generalized Euler class group of a Noetherian ring [**5**]. The definition of generalized Euler class group is very similar to the Euler class group.

**Definition 2.16** (The generalized Euler class group $\widetilde{E}(A)$)**.** Let $A$ be a Noetherian ring of dimension $n \geq 2$. Let $\mathcal{G}$ be the free abelian group on the pairs $(J, \omega_J)$, where:

(1) $J$ is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ of $A$, *not necessarily of height* $n$;
(2) $\omega_J$ is an $SL_n(A/J)$-equivalence class of surjections from $(A/J)^n \twoheadrightarrow J/J^2$, i.e., a local orientation of $J$.

Given any zero-dimensional ideal $I$ of $A$, i.e., $\dim(A/I) = 0$, and a surjection $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$, an element of $\mathcal{G}$ may be associated in an obvious manner; we call it $(I, \omega_I)$. Let $\mathcal{H}$ be the subgroup of $\mathcal{G}$ generated by all elements of the type $(I, \omega_I)$, where $\dim(A/I) = 0$, and $\omega_I$ can be lifted to a surjection from $A^n$ to $I$. We define $\widetilde{E}(A) = \mathcal{G}/\mathcal{H}$.

The theory of Euler class groups as developed in **[2]** and adapt similar methods to prove the following assertion.

**Theorem 2.17** ([**5**])**.** *Let $A$ be a Noetherian ring of dimension $n \geq 2$. Let $I \subset A$ be an ideal with* $\dim(A/I) = 0$ *such that $I/I^2$ is generated by $n$ elements, and let is isomorphic to the Euler class group $E(A)$.*

$$\omega_I : (A/I)^n \twoheadrightarrow I/I^2$$

*be a local orientation of $I$. Suppose that the image of $(I, \omega_I)$ is zero in the Euler class group $\widetilde{E}(A)$ of $A$. Then, $I$ is generated by $n$ elements and $\omega_I$ is a global orientation of $I$.*

It was proven in [**5**] that the generalized Euler class group $\widetilde{E}(A)$ is isomorphic to the Euler class group $E(A)$.

**Proposition 2.18.** *Let $A$ be a Noetherian ring of dimension $n \geq 2$. Then, the canonical map from $E(A)$ to $\widetilde{E}(A)$ is an isomorphism.*

**Definition 2.19** (The weak Euler class group $E_0(R)$)**.** Let $R$ be a Noetherian ring of dimension $n \geq 2$. Let $G_0$ be the free abelian group on the set of all ideals $\mathfrak{n}$, where $\mathfrak{n}$ is $\mathfrak{m}$-primary for some maximal ideal $\mathfrak{m}$ of height $n$ such that there is a surjection

$$F \twoheadrightarrow \mathfrak{n}/\mathfrak{n}^2.$$

Given any ideal $J$ of height $n$, we take the (irredundant) primary decomposition $J = \cap_i \mathfrak{n}_i$ and associate to $J$ the element

$$\sum_i \mathfrak{n}_i$$

of $G_0$. We denote this element by $(J)$. Let $H_0$ be the subgroup of $G_0$ generated by all $(J)$ such that $J$ is a surjective image of $F$. The weak Euler class group of $R$ with respect to $R$ is defined as $E_0(R) = G_0/H_0$.

**Remark 2.20.** It is clear from the above definitions that there is an obvious canonical surjective group homomorphism

$$\Theta : E(R) \twoheadrightarrow E_0(R)$$

which sends an element $(J, \omega_J)$ of $E(R)$ to $(J)$ in $E_0(R)$.

**3. Generalizations of Mandal's theorem.** Some results regarding monic polynomials can be generalized to the setting of quasi-monic polynomials. For monic polynomials, Mandal proved the following.

**Theorem 3.1** (**[7]**). *Let $A$ be a Noetherian ring, and let $I$ be an ideal of $A[T]$ containing a monic polynomial. Suppose that $I/I^2$ is generated by $n$ elements, where $n \geq \dim(A[T]/I) + 2$. Then, any given set of $n$ generators of $I/I^2$ can be lifted to a set of $n$ generators of $I$.*

We generalize Mandal's theorem in the following form.

**Theorem 3.2.** *Let $A$ be a Noetherian ring, and let $I$ be an ideal of $A[T]$ containing a quasi-monic polynomial. Suppose that $I/I^2$ is generated by $n$ elements, where $n \geq \dim(A[T]/I) + 2$. Then, any given set of $n$ generators of $I/I^2$ can be lifted to a set of $n$ generators of $I$.*

*Proof.* Let $g_1, g_2, \ldots, g_n \in I$ be such that $I = (g_1, g_2, \ldots, g_n) + I^2$. By hypothesis, $I$ contains a quasi-monic polynomial, say, $f$. Let

$$f = d_0T^m + \cdots + d_kT^{m-k} + \cdots + d_m.$$

We may assume that $f$ is not monic (if $f$ is monic, then it is exactly Mandal's theorem). Therefore, let

$$f = d_0T^m + \cdots + d_kT^{m-k} + \cdots + d_m$$

be a quasi-monic polynomial of length $k+1$ for some $k \geq 0$. Then, by Lemma 2.4, there exists a decomposition of $A$:

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{k+1},$$

where $A_j = As_j$ and each $s_j$ is as in Lemma 2.4. This yields a decomposition of $f$:

$$f = f_0 + f_1 + \cdots + f_{k+1},$$

where

$$f_j = fs_j = d_j s_j T^{m-j} + d_{j+1} s_j T^{m-j-1} + \cdots + d_m s_j \in A_j[T]$$

for $0 \leq j \leq k+1$. Suppose that each $d_j$ is an $m_j$-idempotent for $0 \leq j \leq k$. Now, from Remark 2.5, it follows that $d_j^{m_j-2}(d_j s_j) = d_j^{m_j-1} s_j = s_j$ for $0 \leq j \leq k$. In addition, when $j = k+1$, it follows from Definition 2.7 (i) that $d_{k+1} s_{k+1}$ is a unit. Therefore, the leading co-efficient of $f_j$ $(= d_j s_j)$ is invertible for all $0 \leq j \leq k+1$, and hence, $f_j$ $(0 \leq j \leq k+1)$ is a monic polynomial in $A_j[T]$.

Furthermore, by Lemma 2.11, we have $I = I_0 + I_1 + \cdots + I_{k+1}$, where $I_j = Is_j$ and

$$A[T]/I \simeq A_0[T]/I_0 \oplus A_1[T]/I_1 \oplus \cdots \oplus A_{k+1}[T]/I_{k+1}.$$

Clearly, $f_j \in I_j$ for each $j = 0, 1, \ldots, k+1$. Now, $\dim(A[T]/I) = \max\{\dim(A_0[T]/I_0), \dim(A_1[T]/I_1), \ldots, \dim(A_{k+1}[T]/I_{k+1})\}$. Hence, $n \geq \dim(A[T]/I) + 2 \geq \dim(A_j[T]/I_j) + 2$ for $j = 0, 1, \ldots, k+1$.

Let $g_i = g_{i0} + g_{i1} + \cdots + g_{ik+1}$, for $i = 1, 2, \ldots, n$, where $g_{ij} \in I_j$. Then, it is easy to verify that

$$I_j = (g_{1j}, g_{2j}, \ldots, g_{nj}) + I_j^2 \quad \text{for } j = 0, 1, \ldots, k+1.$$

However, we have already observed that $f_j \in I_j$ is a monic polynomial. Therefore, by Theorem 3.1, there exist $h_{1j}, h_{2j}, \ldots, h_{nj}$ such that $I_j = (h_{1j}, h_{2j}, \ldots, h_{nj})$ with $g_{ij} - h_{ij} \in I_j^2$ for $1 \leq i \leq n$, $0 \leq j \leq k+1$.

Now our claim is $I = (h_1, h_2, \ldots, h_n)$, where $h_i = h_{i0} + h_{i1} + \cdots + h_{ik+1}$ for $1 \leq i \leq n$. Clearly, $(h_1, h_2, \ldots, h_n) \subset I$. Therefore, we will show that $I \subset (h_1, h_2, \ldots, h_n)$. Let $\alpha \in I$. Then, $\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_{k+1}$, where $\alpha_j = \alpha s_j \in I_j$ for $0 \leq j \leq k+1$. Now, since $I_j = (h_{1j}, h_{1j}, \ldots, h_{nj})$, we therefore have

$$\alpha_j = a_{1j} h_{1j} + a_{2j} h_{2j} + \cdots + a_{nj} h_{nj},$$

where $a_{ij} \in A_j[T]$ for $1 \leq i \leq n$, $0 \leq j \leq k+1$. Since $s_i s_j = 0$ for $i \neq j$, it follows that $a_{ri} h_{rj} = 0$ for $i \neq j$. Therefore, it is easy to verify that

$$\alpha = a_1 h_1 + a_2 h_2 + \cdots + a_n h_n,$$

where $a_i = a_{i0} + a_{i1} + \cdots + a_{ik+1} \in A[T]$. Hence, the claim is proven.

Therefore, we have $I = (h_1, h_2, \ldots, h_n)$. Furthermore, it is easy to verify that $g_i - h_i \in I^2$ for $1 \leq i \leq n$. $\square$

As a consequence, the following result is derived in terms of the Euler class group of $A[T]$ (denoted by $E(A[T])$). For the definition of the Euler class group of $A[T]$, the reader is referred to [**4**].

**Corollary 3.3.** *Let $A$ be a Noetherian ring of dimension $n \geq 3$, containing $\mathbb{Q}$. Let $(I, \omega_I) \in E(A[T])$. Assume that $I$ contains some quasi-monic polynomial. Then, $(I, \omega_I) = 0$ in $E(A[T])$.*

*Proof.* By hypothesis, $(I, \omega_I) \in E(A[T])$, where $I \subset A[T]$ is an ideal of height $n$ and $\omega_I$ is a local orientation of $I$ induced by, say, $I = (f_1, \ldots, f_n) + I^2$.

Now, $\dim(A[T]/I) \leq \dim(A[T]) - \mathrm{ht}(I) = 1$ and $n \geq 3$; therefore, the hypothesis of Theorem 3.2 is satisfied. Furthermore, applying Theorem 3.2, there exist $g_1, \ldots, g_n$ such that $I = (g_1, \ldots, g_n)$ with $f_i - g_i \in I^2$, in other words, $\omega_I$ is a global orientation and $(I, \omega_I) = 0$ in $E(A[T])$. Hence, the corollary is proven. $\square$

Now, using the same method of Theorem 3.2, we can also generalize a variant of Mandal's theorem to the setting of the quasi-monic polynomial.

**Theorem 3.4.** *Let $A$ be a Noetherian ring. Let $I \subset A[T]$ be an ideal containing a quasi-monic polynomial. Let $I = (f_1, \ldots, f_n) + I^2$, where $n \geq \dim(A[T]/I) + 2$. Suppose that $I(0) = (a_1, \ldots, a_n)$ is such that $f_i(0) - a_i \in I(0)^2$. Then, there exists a set of generators $h_1, \ldots, h_n$ of $I$ such that $h_i(0) = a_i$.*

The next theorem was proven by Bhatwadekar and Raja Sridharan [**3**].

**Theorem 3.5.** *Let $A$ be a Noetherian ring of dimension $n$ containing an infinite field and $P$ a projective $A[T]$-module of rank $n$. Suppose that the projective $A[T]_{f(T)}$-module $P_{f(T)}$ has a unimodular element for some monic polynomial $f(T) \in A[T]$. Then, $P$ has a unimodular element.*

Now, we generalize the previous theorem to the setting of the quasi-monic polynomial.

**Theorem 3.6.** *Let $A$ be a Noetherian ring of dimension $n$ containing an infinite field and $P$ a projective $A[T]$-module of rank $n$. Suppose that the projective $A[T]_f$-module $P_f$ has a unimodular element for some quasi-monic polynomial $f \in A[T]$. Then, $P$ has a unimodular element.*

*Proof.* If $f$ is monic, then it is exactly the above theorem. Thus, let

$$f = d_0T^m + \cdots + d_kT^{m-k} + \cdots + d_m$$

be a quasi-monic polynomial of length $k+1$ for some $k \geq 0$. Then, by Lemma 2.4, there exists a decomposition of $A$:

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{k+1},$$

where $A_j = As_j$ and each $s_j$ is as in Lemma 2.4. We have a decomposition of $f$:

$$f = f_0 + f_1 + \cdots + f_{k+1},$$

where $f_j = fs_j = d_js_jT^{m-j} + d_{j+1}s_jT^{m-j-1} + \cdots + d_ms_j \in A_j[T]$ is a monic polynomial for $0 \leq j \leq k+1$.

Furthermore, by Remark 2.12,

$$P = P_0 \oplus \cdots \oplus P_{k+1},$$

where $P_j$ is a projective $A_j[T]$-module of rank $n$. Now, it is easy to verify that

$$A[T]_f \simeq (A_0[T])_{f_0} \oplus \cdots \oplus (A_{k+1}[T])_{f_{k+1}}$$

and

$$P_f \simeq (P_0)_{f_0} \oplus \cdots \oplus (P_{k+1})_{f_{k+1}}.$$

Since $P_f$ has a unimodular element, it follows that $(P_j)_{f_j}$, $0 \leq j \leq k+1$, also has a unimodular element. Now, by Theorem 3.5, each $P_j$ has a unimodular element. Therefore, $P$ has a unimodular element. Hence, the theorem follows. $\square$

Next is the proof of the quasi-monic version of the extension lemma (1.4).

**Lemma 3.7.** *Let $A$ be a ring and $I \subset A[T]$ an ideal containing a quasi-monic polynomial. Let $\mathcal{M}$ be a maximal ideal of $A$ such that $I + \mathcal{M}A[T] = A[T]$. Then, $(I \cap A) + \mathcal{M} = A$.*

*Proof.* By hypothesis, $I$ contains a quasi-monic polynomial, say, $f$. Let

$$f = d_0T^m + \cdots + d_kT^{m-k} + \cdots + d_m.$$

We may assume that $f$ is not monic (if $f$ is monic, then it is exactly the extension lemma). Therefore, let $f = d_0T^m + \cdots + d_kT^{m-k} + \cdots + d_m$ be a quasi-monic polynomial of length $k+1$ for some $k \geq 0$. Then, by Lemma 2.4, there exists a decomposition of $A$:

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{k+1},$$

where $A_j = As_j$ and each $s_j$ is as in Lemma 2.4, and we have a decomposition of $f$:

$$f = f_0 + f_1 + \cdots + f_{k+1},$$

where

$$f_j = fs_j = d_js_jT^{m-j} + d_{j+1}s_jT^{m-j-1} + \cdots + d_ms_j \in A_j[T]$$

is a monic polynomial for $0 \leq j \leq k+1$. In addition, by Lemma 2.11, we have

$$I = I_0 + I_1 + \cdots + I_{k+1}$$

and

$$\mathcal{M} = A_0 + A_1 + \cdots + \mathcal{M}_j + \cdots + A_{k+1} \quad \text{for some j},$$

where $\mathcal{M}_j$ is a maximal ideal of $A_j$. We have already observed that $f_j \in I_j$ is a monic polynomial. By the hypothesis, we have $I_j + \mathcal{M}_jA_j[T] = A_j[T]$. Therefore, by the extension lemma (1.4), $(I_j \cap A_j) + \mathcal{M}_j = A_j$. It is now easy to conclude that $(I \cap A) + \mathcal{M} = A$. $\square$

We now prove (1.7), mentioned in the introduction.

**Theorem 3.8.** *Let $A$ be a ring and $(f_1, \ldots, f_n) \in Um_n(A[T])$ with $n \geq 3$. Assume that $f_1$ is a quasi-monic polynomial. Then, $(f_1, \ldots, f_n)$ can be completed to an elementary matrix.*

*Proof.* Let $f_1 = d_0T^m + \cdots + d_kT^{m-k} + \cdots + d_m$. If $f_1$ is monic, then we are done. Therefore, let $f_1 = d_0T^m + \cdots + d_kT^{m-k} + \cdots + d_m$ be a quasi-monic polynomial of length $k+1$ for some $k \geq 0$. Then, by Lemma 2.4, there exists a decomposition of $A$:

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{k+1},$$

where $A_j = As_j$, and each $s_j$ is as in Lemma 2.4, and we have a decomposition of $f_1$:

$$f_1 = f_{01} + f_{11} + \cdots + f_{(k+1)1},$$

where

$$f_{j1} = f_1s_j = d_js_jT^{m-j} + d_{j+1}s_jT^{m-j-1} + \cdots + d_ms_j \in A_j[T]$$

is a monic polynomial for $0 \leq j \leq k+1$.

It is easy to see that $GL_n(A[T]) = GL_n(A_0[T]) \oplus \cdots \oplus GL_n(A_{k+1}[T])$, i.e., any $\sigma \in GL_n(A[T])$ is of the form $(\sigma_0, \ldots, \sigma_{k+1}) \in GL_n(A_0[T]) \oplus \cdots \oplus GL_n(A_{k+1}[T])$. It may also be observed that, if $\sigma \in E_n(A[T])$, then $\sigma_j \in E_n(A_j[T])$ for $0 \leq j \leq k{+}1$. On the other hand, if $\sigma_j \in E_n(A_j[T])$, then $\sigma_j = \prod \sigma_{jk_j}$, where $\sigma_{jk_j} = \varepsilon_{il}(\lambda)$ for some $i$, $l$, i.e., the $il$th entry is $\lambda$ for the matrix whose diagonal entries are 1, and other entries are 0. Then, clearly, we have

$$\sigma = \prod \tau_{jk},$$

where $\tau_{jk} = (I_n, \ldots, \sigma_{jk}, \cdots, I_n)$ and $\sigma_{jk}$ is at the $j$th position. Therefore, $\sigma \in E_n(A[T])$.

Now, $(f_1, \ldots, f_n) \in Um_n(A[T])$ is of the form $((f_{01}, \ldots, f_{0n}), \ldots, (f_{(k+1)1}, \ldots, f_{(k+1)n})$, where $(f_{j1}, \ldots, f_{jn}) \in Um_n(A_j[T]))$ with $f_{j1}$ is monic for $0 \leq j \leq k+1$. By Rao's result **[12]**, there exist $\sigma_j \in E_n(A_j[T])$ such that $(f_{j1}, \ldots, f_{jn})\sigma_j = (1, 0, \ldots, 0)$ for $0 \leq j \leq k+1$. If we consider $\sigma = (\sigma_0, \ldots, \sigma_{k+1})$, then $\sigma \in E_n(A[T])$. In addition, note that $(f_1, \ldots, f_n)\sigma = (1, 0, \ldots, 0)$. We are done. $\square$

We conclude this section by generalizing a result on set-theoretic generation of ideals to the setting of quasi-monic polynomial. The following is a result due to Mandal **[8,** Theorem 1.1].

**Theorem 3.9.** *Let $R = A[X]$ be a polynomial ring over a Noetherian ring $A$. Let $I \subset R$ be a locally complete intersection ideal of height $r$*

*with* $\dim(R/I) \le 1$*. If it contains a monic polynomial, then* $I$ *is set theoretically generated by* $r$ *elements.*

The next result is the quasi-monic version of the affine Horrocks' theorem [**11, 13**]. Since the proof is along the same lines, we do not repeat it here.

**Theorem 3.10.** *Let* $A$ *be a Noetherian ring. Let* $P$ *be a projective* $A[X]$*-module such that the* $A[X]_f$*-module* $P_f$ *is free for some quasi-monic polynomial* $f \in A[X]$*. Then,* $P$ *is free.*

The next theorem generalizes Theorem 3.9 to the setting of the quasi-monic polynomial. The proof is along the same lines as [**8,** Theorem 1.1]; therefore, we do not give the details.

**Theorem 3.11.** *Let* $R = A[X]$ *be a polynomial ring over a Noetherian ring* $A$*. Let* $I \subset R$ *be a locally complete intersection ideal of height* $r$ *with* $\dim(R/I) \le 1$*. If* $I$ *contains a quasi-monic polynomial, then* $I$ *is set theoretically generated by* $r$ *elements.*

*Proof.* The case $r = 1$ follows from Theorem 3.10. For details, see the proof of [**8,** Theorem 1.1].

We assume that $r \ge 2$. Then, from Ferrand and Szpiro [**9,** Theorem 6.1.3], there is a locally complete intersection ideal $J$ of height $r$ such that

(i) $\sqrt{J} = \sqrt{I}$, and
(ii) $J/J^2$ is a free $R/J$-module of rank $r$.

Let $r \ge 3$ and $f \in I$ be a quasi-monic polynomial. Furthermore, we can assume that $f$ is a quasi-monic polynomial of length $\ge 1$. Then, by Lemma 2.4, there exists a decomposition of $A$:

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{k+1},$$

where $A_j = As_j$, and each $s_j$ is as in Lemma 2.4.

Note that

$$\sqrt{J} = \sqrt{Js_0} \oplus \cdots \oplus \sqrt{Js_{k+1}}.$$

Since $\sqrt{J}$ contains $f$, therefore, each $Js_j$ contains a monic polynomial (some power of $fs_j$). As $Js_j/(Js_j)^2$ is generated by $r$ elements and $\dim(As_j/Js_j) \leq 1$, applying Theorem 3.1, it follows that $Js_j$, $0 \leq j \leq k+1$, is generated by $r$ elements. Hence, $J$ is generated by $r$ elements. Therefore, $I$ is set-theoretically generated by $r$ elements.

If $r = 2$, we have that $J/J^2$ is generated by two elements. From the standard patching argument, it may be shown that $J$ is a surjective image of a projective module $P$ of a trivial determinant. Since $J$ contains a quasi-monic polynomial, the result follows from Theorem 3.10. $\square$

**4. Ring decomposition and the Euler class groups.** Let $A$ be a Noetherian ring, and let $X = \operatorname{Spec}(A)$. Now, consider $X$ together with the Zariski topology. Assume that $\operatorname{Spec}(A)$ is not connected. Let

$$\operatorname{Spec}(A) = X_1 \cup \cdots \cup X_k$$

be the disjoint union of the connected components. Since connected components are closed, therefore, for $1 \leq i \leq k$, $X_i = \mathbb{V}(J_i)$ for some ideal $J_i \subset A$.

Since $\operatorname{Spec}(A) = \operatorname{Spec}(A_{\text{red}})$; thus, we may assume that $A$ is reduced. This yields

$$\operatorname{Spec}(A) = \mathbb{V}(J_1) \cup \cdots \cup \mathbb{V}(J_k) = \mathbb{V}(J_1 \cap \cdots \cap J_k).$$

Hence, $J_1 \cap \cdots \cap J_k \subset \mathbf{N}(A)$, where $\mathbf{N}(A)$ denotes the nilradical of $A$. However, since $A$ is reduced, $\mathbf{N}(A) = 0$ and $J_1 \cap \cdots \cap J_k = 0$. Also note that $J_i + J_j = A$ for $i \neq j$. By the Chinese remainder theorem, we have

$$A \simeq A/J_1 \oplus \cdots \oplus A/J_k.$$

Therefore, we have a decomposition of $A$, say, $A = A_1 \oplus \cdots \oplus A_n$, such that $\operatorname{Spec}(A) = \operatorname{Spec}(A_1) \cup \cdots \cup \operatorname{Spec}(A_k)$. Let $\dim(A) = n$. Then, $\dim(A_i) \leq n$ for $1 \leq i \leq k$.

In this section, we establish a natural relation between the Euler class group of $A = A_1 \oplus \cdots \oplus A_k$ and the Euler class groups of $A_1, \ldots, A_k$. Note that, if $\dim(A_i) < n$, then $E(A_i) = 0$. The main theorem of this section is the following.

**Theorem 4.1.** *Let $A$ be a Noetherian ring and $A = A_1 \oplus \cdots \oplus A_k$ such a decomposition of $A$ so that* $\mathrm{Spec}(A_1), \ldots, \mathrm{Spec}(A_k)$ *are the connected components of* $\mathrm{Spec}(A)$. *Then,* $E(A) \simeq E(A_1) \oplus \cdots \oplus E(A_k)$.

If $\dim(A_i) < n$ for some $i$, then $E(A_i) = 0$, and we may drop that summand. Therefore, without loss of generality, we may assume that each $A_i$ has dimension $n$.

In order to prove Theorem 4.1 (using Theorem 2.18), it is sufficient to show that

$$\widetilde{E}(A) \simeq \widetilde{E}(A_1) \oplus \cdots \oplus \widetilde{E}(A_k),$$

for which we must ensure that there is a group homomorphism from $\widetilde{E}(A)$ to $\widetilde{E}(A_1) \oplus \cdots \oplus \widetilde{E}(A_k)$.

**Remark 4.2.** Let $A_1, \ldots, A_k$ be Noetherian rings of dimension $n$, and let $A = A_1 \oplus \cdots \oplus A_k$. It follows that $\dim(A) = n$. Now, we can give a natural map from $\widetilde{E}(A)$ to $\widetilde{E}(A_1) \oplus \cdots \oplus \widetilde{E}(A_k)$ in the following manner.

Let $(I, \omega_I) \in \widetilde{E}(A)$, where $I$ is an ideal of $A$ such that $\dim(A/I) = 0$ and $\omega_I$ is a local orientation of $I$. Then, by Lemma 2.11, $I$ is of the form $I = I_1 \oplus \cdots \oplus I_k$, where $I_j$ is an ideal of $A_j$. Furthermore, $A/I \simeq A_1/I_1 \oplus \cdots \oplus A_k/I_k$.

Since $\dim(A/I) = \max\{\dim(A_1/I_1), \ldots, \dim(A_k/I_k)\}$ and $\dim(A/I) = 0$, therefore, $\dim(A_j/I_j) = 0$, for $j = 1, \ldots, k$.

Let $\omega_I$ be a local orientation of $I$ induced by, say, $I = (f_1, \ldots, f_n) + I^2$. Then, by Lemma 2.4, each $f_i$ has a decomposition, say $f_i = f_{i1} + \cdots + f_{ik}$, where $f_{ij} \in I_j$. It is easy to verify that

$$I_j = (f_{1j}, \ldots, f_{nj}) + I_j^2 \quad \text{for } j = 1, \ldots, k.$$

We define

$$\varphi : \widetilde{E}(A) \longrightarrow \widetilde{E}(A_1) \oplus \cdots \oplus \widetilde{E}(A_k)$$

by sending $(I, \omega_I)$ to $((I_1, \omega_{I_1}), \ldots, (I_n, \omega_{I_k}))$, where $\omega_{I_j}$ is the local orientation of $I_j$ induced by $I_j = (f_{1j}, \ldots, f_{nj}) + I_j^2$.

*Proof of Theorem* 4.1. First, we show that $\varphi$ is one-to-one. Let $(I, \omega_I) \in \widetilde{E}(A)$ be such that $(I_j, \omega_{I_j}) = 0$ in $E(A_j)$ for $i = 1, \ldots, n$.

Then, by [**2,** Theorem 4.2], $\omega_{I_j}$ is a global orientation, for $i = 1, \dots, k$. This tells us that $I_j = (g_{1j}, \dots, g_{nj})$ such that $f_{ij} - g_{ij} \in I_j^2$ for all $j$, and for $1 \le i \le n$.

Now, following the proof of Theorem 3.2, it can easily be verified that there exist $g_1, \dots, g_n \in I$ such that $I = (g_1, \dots, g_n)$ with $f_i - g_i \in I^2$, in other words, $\omega_I$ is a global orientation and $(I, \omega_I) = 0$ in $E(A)$. Hence, $\varphi$ is one-to-one.

Next, we show that $\varphi$ is surjective. Let $((I_1, \omega_{I_1}), \dots, (I_k, \omega_{I_k})) \in \widetilde{E}(A_1) \oplus \cdots \oplus \widetilde{E}(A_k)$. Consider $I = I_1 \oplus \cdots \oplus I_k$. Then, $I$ is an ideal of $A$ and

$$A/I \simeq A_1/I_1 \oplus \cdots \oplus A_k/I_k.$$

Since $\dim(A_i/I_i) = 0$, for $i \le i \le k$, we have $\dim(A/I) = 0$ as well as the local orientations $\omega_{I_j}$, for $1 \le j \le n$, which will naturally induce a local orientation of $I$, say $\omega_I$. Therefore, $(I, \omega_I) \in E(A)$. It follows that $\varphi(I, \omega_I) = ((I_1, \omega_{I_1}), \dots, (I_k, \omega_{I_k}))$.

Finally, we have

$$E(A) \simeq \widetilde{E}(A) \simeq \widetilde{E}(A_1) \oplus \cdots \oplus \widetilde{E}(A_k) \simeq E(A_1) \oplus \cdots \oplus E(A_k).$$

This completes the proof of the theorem. $\square$

**5. Ring decomposition and the weak Euler class groups.** Let $A$ be a Noetherian ring of dimension $n \ge 2$. Let $A = A_1 \oplus \cdots \oplus A_k$ be a decomposition such that $\operatorname{Spec}(A) = \operatorname{Spec}(A_1) \cup \cdots \cup \operatorname{Spec}(A_k)$ as in Section 4. In this section, we study the relation between the weak Euler class group of $A$ and the weak Euler class groups of $A_1, \dots, A_k$.

We now define a group $\widetilde{E}_0(A)$ which may be regarded as the *generalized* weak Euler class group of $A$. The definition of $\widetilde{E}_0(A)$ is similar to that of $E_0(A)$, defined in [**2**].

**Definition 5.1** (The generalized weak Euler class group $\widetilde{E}_0(A)$)**.** Let $A$ be a Noetherian ring of dimension $n \ge 2$. Let $\mathcal{G}_0$ be the free abelian group on the set of all ideals $\mathfrak{n}$, where $\mathfrak{n}$ is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ of $A$ (*not necessarily of height* $n$). Given any zero-dimensional ideal $I$ of $A$, an element of $\mathcal{G}_0$ may be associated in an obvious manner; we call it $(I)$. Let $\mathcal{H}_0$ be the subgroup of $\mathcal{G}_0$ generated by all elements of type $(I)$, where $\dim(A/I) = 0$ and $I$ is generated by $n$ elements. We define $\widetilde{E}_0(A) = \mathcal{G}_0/\mathcal{H}_0$.

**Remark 5.2.** We note that there is a canonical surjective group homomorphism from $\widetilde{E}(A)$ to $\widetilde{E}_0(A)$, sending $(I, \omega_I) \in \widetilde{E}_0(A)$ to $(I) \in \widetilde{E}_0(A)$.

The proof of the following lemma is the same as that of [**1,** Lemma 3.3].

**Lemma 5.3.** *Let $A$ be a Noetherian ring of dimension $n \geq 2$. Let $\widetilde{H}$ be a subgroup of $\widetilde{E}(A)$, generated by all elements of the type $(I, \omega_I)$, where $I$ is generated by $n$ elements. Let*

$$\Phi : \widetilde{E}(A) \longrightarrow \widetilde{E}_0(A)$$

*be the canonical surjection. Then,* $\mathrm{Ker}(\Phi) = \widetilde{H}$.

The proof of the next proposition is along the same lines as [**1,** Theorem 3.9].

**Proposition 5.4.** *Let $A$ be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n$, where $n$ is even. Let $I \subset A$ be a zero-dimensional ideal such that $I/I^2$ is generated by $n$ elements. Then, $(I) = 0$ in $\widetilde{E}_0(A)$ if and only if $I$ is the surjective image of a stably free projective $A$-module of rank $n$.*

**Remark 5.5.** The natural map

$$\psi_0 : E_0(A) \longrightarrow \widetilde{E}_0(A)$$

which sends $(I) \in E_0(A)$ to $(I) \in \widetilde{E}(A)$, is a group homomorphism.

**Lemma 5.6.** *Let $A$ be a Noetherian ring (containing $\mathbb{Q}$) of dimension $n$, where $n$ is even. Then, the map $\psi_0 : E_0(A) \to \widetilde{E}_0(A)$, as described above, is an isomorphism of groups.*

*Proof.* $\psi_0$ is clearly surjective due to the following commutative diagram:

$$\begin{tikzcd} E(A) \arrow[r, "\simeq"] \arrow[d] & \widetilde{E}(A) \arrow[d] \\ E_0(A) \arrow[r] & \widetilde{E}_0(A) \end{tikzcd}$$

where the top horizontal map is an isomorphism, and the vertical maps are surjective.

Now, we show that $\psi_0$ is injective. Let $(I) \in E_0(A)$ be such that $(I) = 0$ in $\widetilde{E}_0(A)$. By Proposition 5.4, there exists a stably free projective $A$-module $P$ such that $I$ is surjective image of $P$. Therefore, by [**2,** Proposition 6.2], $(I) = 0$ in $E_0(A)$. □

**Remark 5.7.** Let $A = A_1 \oplus \cdots \oplus A_k$ be a ring decomposition, as above. Let $(I) \in \widetilde{E}_0(A)$. Then, $I = I_1 \oplus \cdots \oplus I_k$ with each $I_j$ zero-dimensional. Then, we have a natural map $\psi$ from $\widetilde{E}_0(A)$ to

$$\widetilde{E}_0(A_1) \oplus \cdots \oplus \widetilde{E}_0(A_k),$$

sending $(I)$ to $((I_1), \ldots, (I_k))$.

**Proposition 5.8.** *Let $A$ be a Noetherian ring of dimension $n$. Let $A = A_1 \oplus \cdots \oplus A_k$ be a ring decomposition, as above. Then, the induced homomorphism*

$$\psi : \widetilde{E}_0(A) \longrightarrow \widetilde{E}_0(A_1) \oplus \cdots \oplus \widetilde{E}_0(A_k)$$

*is an isomorphism.*

*Proof.* It is easy to prove that the canonical homomorphism $\psi$ is surjective since we have already proved that the canonical homomorphism

$$\widetilde{E}(A) \longrightarrow \widetilde{E}(A_1) \oplus \cdots \oplus \widetilde{E}(A_k)$$

is an isomorphism. Now, we have a natural surjection

$$\widetilde{E}(A) \twoheadrightarrow \widetilde{E}_0(A),$$

the assignment sending $(J, \omega_J) \in \widetilde{E}(A)$ to $(J) \in \widetilde{E}_0(A)$. Hence, $\psi$ is surjective.

Now we will show the injectivity of $\psi$. Let $J$ be a zero-dimensional ideal of $A$ such that $J/J^2$ is generated by $n$ elements. Suppose that

$(J) \in \widetilde{E}_0(A)$ such that $(J_i) = 0$ in $\widetilde{E}_0(A_i)$ for $1 \le i \le k$. Now, using Lemma 5.3, we have

$$(J_i, \omega_{J_i}) + \sum_{j=1}^{r} (J_{ij}, \omega_{J_{ij}}) = \sum_{j=r+1}^{m} (J_{ij}, \omega_{J_{ij}})$$

for $1 \le i \le k$ in $\widetilde{E}(A_i)$, where the $J_{ij}$s are zero-dimensional ideals of $A_i$ for $1 \le j \le m$ such that they are generated by $n$ elements.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\phi"] & \widetilde{E}(A) \arrow[r] \arrow[d, "\varphi"] & \widetilde{E}_0(A) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \widetilde{K} \arrow[r] & \bigoplus_{i=1}^{k} \widetilde{E}(A_i) \arrow[r] & \bigoplus_{i=1}^{k} \widetilde{E}_0(A_i) \arrow[r] & 0.
\end{tikzcd}$$

Now, we have already proved that $\widetilde{E}(A) \simeq \oplus_{i=1}^{k} \widetilde{E}(A_i)$. For each $((J_{i1}, \omega_{J_{i1}}), \ldots, (J_{ik}, \omega_{J_{ik}})) \in \oplus_{i=1}^{k} \widetilde{E}(A_i)$, there exists a unique preimage in $\widetilde{E}(A)$, say $(I_i, \omega_{I_i})$.

Therefore, we have the following

$$(J, \omega_J) + \sum_{i=1}^{r} (I_i, \omega_{I_i}) = \sum_{i=r+1}^{m} (I_i, \omega_{I_i})$$

in $\widetilde{E}(A)$ where the $I_i$s are zero-dimensional ideals of $A$ such that they are generated by $n$ elements. The proof follows from Lemma 5.3. $\square$

**Corollary 5.9.** *Let $A$ be a Noetherian ring, containing $\mathbb{Q}$, of even dimension $n \ge 2$. Let $A = A_1 \oplus \cdots \oplus A_k$ be a ring decomposition, as above. Then, the induced homomorphism*

$$\psi : E_0(A) \longrightarrow E_0(A_1) \oplus \cdots \oplus E_0(A_k)$$

*is an isomorphism.*

*Proof.* Since $R$ containing $\mathbb{Q}$ and $n$ is even, by Lemma 5.6, $E_0(R) \simeq \widetilde{E}_0(R)$. Therefore, the proof follows from the above proposition. $\square$

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National Institute of Science Education and Research Bhubaneswar (HBNI), School of Mathematical Sciences, 752050 India
**Email address**: **zinna2012@gmail.com**