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DUAL F-SIGNATURE OF SPECIAL COHEN-MACAULAY MODULES OVER CYCLIC QUOTIENT SURFACE SINGULARITIES

YUSUKE NAKAJIMA

ABSTRACT. The notion of F-signature was defined by Huneke and Leuschke and this numerical invariant characterizes some singularities. This notion is extended to finitely generated modules by Sannai and is called dual F-signature. In this paper, we determine the dual F-signature of a certain class of Cohen-Macaulay modules (so-called "special") over cyclic quotient surface singularities. Also, we compare the dual F-signature of a special Cohen-Macaulay module with that of its Auslander-Reiten translation. This gives a new characterization of the Gorensteinness.

1. Introduction. Throughout this paper, we suppose that k is an algebraically closed field of prime characteristic $p > 0$. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with $\operatorname{char} R = p > 0$. Since $\operatorname{char} R = p > 0$, we can define the Frobenius map

$$F : R \longrightarrow R, \quad r \mapsto r^p.$$

For $e \in \mathbb{N}$, we also define the e-times iterated Frobenius map

$$F^e : R \longrightarrow R, \quad r \mapsto r^{p^e}.$$

For any R-module M, we denote the module M with its R-module structure pulled back via the e-times iterated Frobenius map F^e by ${}^eM$, namely, ${}^eM$ is merely M as an abelian group, and its R-module structure is defined by $r \cdot m := F^e(r)m = r^{p^e} m$ for all $r \in R$, $m \in M$. We say that R is F-finite if ${}^1R$ is a finitely generated R-module.

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In order to investigate the properties of R, Huneke and Leuschke introduced the notion of F-signature.

Definition 1.1 ([**6**])**.** Let $(R,\mathfrak{m},k)$ be a reduced F-finite local ring of prime characteristic $p>0$. For each $e\in\mathbb{N}$, decompose ${}^eR$ as follows

$$ {}^eR\cong R^{\oplus a_e}\oplus M_e, $$

where M_e has no free direct summands. We call a_e the eth F-splitting number of R. Then, the F-signature of R is

$$ s(R):=\lim_{e\to\infty}\frac{a_e}{p^{ed}}, $$

if it exists, where $d:=\dim R$.

Note that Tucker showed its existence in a general situation [**16**]. As Kunz's theorem [**10**] shows, this invariant measures the deviation from regularity (see also Theorem 1.4 (1)).

For a finitely generated R-module, Sannai extended the notion of F-signature as follows.

Definition 1.2 ([**14**])**.** Let $(R,\mathfrak{m},k)$ be a reduced F-finite local ring of prime characteristic $p>0$. For a finitely generated R-module M and $e\in\mathbb{N}$, we set

$$ b_e(M):=\max\{n\mid \text{there exists a } \varphi: {}^eM\twoheadrightarrow M^{\oplus n}\}, $$

and call it the eth F-surjective number of M. Then, we call the limit

$$ s(M):=\lim_{e\to\infty}\frac{b_e(M)}{p^{ed}} $$

the dual F-signature of M if it exists, where $d=\dim R$.

Remark 1.3. The dual F-signature of R coincides with the F-signature of R since the morphism ${}^eR\twoheadrightarrow R^{\oplus b_e(R)}$ is split. Therefore, we use the same notation unless it causes confusion.

By using these invariants, we can characterize some singularities.

Theorem 1.4 ([**1, 6, 14, 22**])**.** *Let $(R,\mathfrak{m},k)$ be a d-dimensional reduced F-finite Noetherian local ring with* $\operatorname{char}R=p>0$*. Then, we obtain*:

(1) *R is regular if and only if $s(R)=1$;*
(2) *R is strongly F-regular if and only if $s(R)>0$.*

In addition, we suppose that R is Cohen-Macaulay with the canonical module ω_R. Then,

(3) *R is F-rational if and only if $s(\omega_R) > 0$;*
(4) *$s(R) \leq s(\omega_R)$;*
(5) *$s(R) = s(\omega_R)$ if and only if R is Gorenstein.*

As the above theorem shows, the value of $s(R)$ and $s(\omega_R)$ contains some information regarding singularities. What about the value of the dual F-signature for other R-modules? The value of the dual F-signature is unknown, except in the case of two-dimensional Veronese subrings [**14**]. Therefore, in this paper, we determine the dual F-signature for a certain class of Cohen-Macaulay (CM) modules (so-called special CM modules) over cyclic quotient surface singularities.

The study of special CM modules was begun by the work of Wunram [**19, 20**] (the definition of special CM modules appears in Section 3). For a finite subgroup G of $\mathrm{SL}(2,k)$ such that the order of G is invertible in k, the McKay correspondence is very famous, that is, there is a one-to-one correspondence between non-trivial irreducible representations of G and irreducible exceptional curves on the minimal resolution of quotient surface singularity. When we intend to generalize this correspondence to a finite subgroup G of $\mathrm{GL}(2,k)$, this correspondence is no longer true. In fact, there are more irreducible representations than exceptional curves. However, if we choose some irreducible representations which are called special, then we again obtain one-to-one correspondence between irreducible special representations of G and exceptional curves [**20**], and a maximal CM module associated with a special representation is called a special CM module. For more about the special McKay correspondence, also see [**7, 8, 13**].

Remark 1.5. All irreducible representations of a finite subgroup of $\mathrm{SL}(2,k)$ are special; thus, we can recover the McKay correspondence in the original sense from the special one.

For a cyclic quotient singularity, a special CM module takes the following simple form. (For more details on terminology, see Sections 2 and 3.)

Suppose that R is the invariant subring of $S = k[[x,y]]$ under the action of a cyclic group $\frac{1}{n}(1,a)$. In this situation, a non-free indecomposable special CM R-module is described as $M_{i_t} = Rx^{i_t} +$

Ry^{j_t}, i.e., it is minimally 2-generated. The following theorem gives the value of the dual F-signature; note that they are all rational.

Theorem 1.6 (see Theorem 3.9). *For any non-free indecomposable special* CM *R-module M_{i_t}, we have*

$$s(M_{i_t}) = \begin{cases} \dfrac{\min(i_t, j_t)+1}{n} & \text{if } i_t \neq j_t \\ \dfrac{2i_t+1}{2n} & \text{if } i_t = j_t. \end{cases}$$

Moreover, by paying attention to special CM modules and their Auslander-Reiten translations, we characterize when the ring is Gorenstein.

Theorem 1.7 (see Theorem 4.2). *Let R be a quotient surface singularity. Note that we do not restrict to a cyclic case. Suppose that M is an indecomposable special CM R-module. Then, we have*

$$s(M) \leq s(\tau(M)).$$

Moreover, if $s(M) = s(\tau(M))$ for an indecomposable special CM *R-module M, then R is Gorenstein. Note that, if R is Gorenstein, then $s(M) = s(\tau(M))$ holds for all indecomposable* MCM *modules.*

Remark 1.8. Since $\tau(R) \cong \omega_R$ in our situation, this theorem is an analogue of Theorem 1.4 (4), (5). However, it states that this characterization is obtained by not only the comparison between R and ω_R but also by the comparison between a special CM module and its AR translation.

The structure of this paper is as follows. In order to determine the dual F-signature, we need the notion of the generalized F-signature and the Auslander-Reiten quiver. Thus, we prepare them in Section 2. In Section 3, we determine the dual F-signature of special CM modules over cyclic quotient surface singularities and give several examples. In Section 4, we compare a special CM module with its Auslander-Reiten translation by using the dual F-signature and characterize the Gorensteiness. Note that the statements appearing in Section 4 hold not only for cyclic quotient surface singularities but also for any quotient surface singularities.

2. Preliminaries.

2.1. Generalized F-signature of invariant subrings. Let G be a finite subgroup of $\mathrm{GL}(d,k)$ which contains no pseudo-reflections except the identity, and let $S := k[[x_1,\dots,x_d]]$ be a power series ring. We assume that the order of G is coprime to $p = \operatorname{char} k$. We denote the invariant subring of S under the action of G by $R := S^G$. In order to determine the dual F-signature of a finitely generated R-module M, we must know about the structure of ${}^eM$ (for instance, the direct sum decomposition of ${}^eM$, the asymptotic behavior of the multiplicities of direct summands, etc.). In order to achieve this, we use the results of the generalized F-signature of invariant subrings [**5**].

For a positive characteristic Noetherian ring, Smith and Van den Bergh introduced the notion of a finite F-representation type [**15**]. This notion is a characteristic p analogue of the notion of the finite representation type. The definition of a finite F-representation type is the following.

Definition 2.1 ([**15**])**.** We say that R has finite F-representation type (FFRT) by $\mathcal{N}$ if there exists a finite set $\mathcal{N}$ of isomorphism classes of indecomposable finitely generated R-modules such that, for every $e \in \mathbb{N}$, the R-module ${}^eR$ is isomorphic to a finite direct sum of elements of $\mathcal{N}$.

For example, a power series ring S has FFRT by $\{S\}$, cf., [**10,** Kunz's theorem], and FFRT is inherited by a direct summand [**15**]. Thus, an invariant subring R also has FFRT. More explicitly, we have the next proposition.

Proposition 2.2 ([**15**])**.** *Let $V_0 = k, V_1, \dots, V_{n-1}$ be the complete set of irreducible representations of G, and set $M_t := (S \otimes_k V_t)^G$, for $t = 0, 1, \dots, n-1$. Then, R has finite F-representation type by the finite set $\{M_0 \cong R, M_1, \dots, M_{n-1}\}$.*

Thus, we can write ${}^eR$ as follows.

$$ {}^eR \cong R^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_{n-1}^{\oplus c_{n-1,e}}. $$

Remark 2.3. We can see that each M_t is an indecomposable maximal Cohen-Macaulay (MCM) R-module and $M_s \not\cong M_t$, $s \neq t$, under the assumption that G contains no pseudo-reflections except the identity.

Also, the multiplicities $c_{i,e}$ are uniquely determined in that case. For more details, we refer the reader to [**5,** Section 2].

Moreover, since an invariant subring R has FFRT, $\lim_{e\to\infty} c_{t,e}/p^{de}$, $t=0,1,\dots,n-1$, exists [**15, 21**]. Therefore, we can define the limit $s(R,M_t):=\lim_{e\to\infty} c_{t,e}/p^{de}$ and call it the generalized F-signature of M_t with respect to R. The value of $s(R,M_t)$ was determined by Hashimoto and the author as follows.

Theorem 2.4 ([**5**])**.** *For $t=0,1,\dots,n-1$, we have*

$$s(R,M_t)=\frac{\operatorname{rank}_R M_t}{|G|}$$

Remark 2.5. The case of $t=0$ is also due to [**17**]. A similar result holds for a finite subgroup scheme of SL_2 [**4**].

We also obtain the next statement as a corollary.

Corollary 2.6 ([**5**])**.** *Suppose an* MCM *R-module M_t decomposes as*:

$${}^eM_t\cong R^{\oplus d^t_{0,e}}\oplus M_1^{\oplus d^t_{1,e}}\oplus\cdots\oplus M_{n-1}^{\oplus d^t_{n-1,e}}.$$

Then, for all $t,u=0,\dots,n-1$, we obtain

$$\begin{aligned} s(M_t,M_u) &:= \lim_{e\to\infty}\frac{d^t_{u,e}}{p^{de}}=(\operatorname{rank}_R M_t)\cdot s(R,M_u)\\ &=\frac{(\operatorname{rank}_R M_t)\cdot(\operatorname{rank}_R M_u)}{|G|}.\end{aligned}$$

Remark 2.7. In dimension two, it is known that an invariant subring R is of finite representation type, that is, it has only finitely many non-isomorphic indecomposable MCM R-modules $\{R,M_1,\dots,M_{n-1}\}$. From Corollary 2.6, every indecomposable MCM R-module appears in ${}^eM_t$ as a direct summand for sufficiently large e. Thus, the additive closure $\operatorname{add}_R({}^eM_t)$ coincides with the category of MCM R-modules $\mathrm{CM}(R)$. Thus, we apply several results in Auslander-Reiten theory to $\operatorname{add}_R({}^eM_t)$ (see the next subsection).

2.2. Auslander-Reiten quiver. In this subsection, we review some results of Auslander-Reiten theory. For details, consult the literature (e.g., [**2, 3, 11, 23**]). We only discuss such a theory for the case of an invariant subring $R=S^G$ in $\dim R=2$.

Definition 2.8 (Auslander-Reiten sequence)**.** Let R be an invariant subring and M, N be indecomposable MCM R-modules. We call a non-split short exact sequence

$$0 \longrightarrow N \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$$

the Auslander-Reiten (AR) sequence ending in M if, for all MCM modules X and for any morphism $\varphi : X \to M$ which is not a split surjection, there exists a $\phi : X \to L$ such that $\varphi = g \circ \phi$.

Since R is an isolated singularity, the AR sequence ending in M_t for each non-free indecomposable MCM R-module M_t exists, and it is unique up to isomorphism **[3]**. Concretely, the AR sequence ending in M_t, $t \neq 0$, is

$$0 \longrightarrow (S \otimes_k (\wedge^2 V \otimes_k V_t))^G \longrightarrow (S \otimes_k (V \otimes_k V_t))^G \longrightarrow M_t = (S \otimes_k V_t)^G \longrightarrow 0,$$

where V is a natural representation of G.

In the case of $t = 0$, there is the exact sequence

$$0 \longrightarrow \omega_R = (S \otimes_k \wedge^2 V)^G \longrightarrow (S \otimes_k V)^G \longrightarrow R = S^G \longrightarrow k \longrightarrow 0.$$

This exact sequence is called the *fundamental sequence* of R.

We call the left term of these sequences the Auslander-Reiten (AR) translation of M_t and denote it by $\tau(M_t)$. Sometimes we denote the middle term of the AR sequence by E_{M_t}. It is known that $\tau(M_t) \cong (M_t \otimes_R \omega_R)^{**}$, where $(-)^* = \mathrm{Hom}_R(-, R)$ is the R-dual functor **[2]**. Note that $\tau(M_t) = M_{t-a-1}$ and $E_{M_t} = M_{t-1} \oplus M_{t-a}$ for $t = 0, 1, \ldots, n-1$ in the case of subsection 2.3.

Next, we prepare some notions to define the Auslander-Reiten quiver.

Definition 2.9 (Irreducible morphism)**.** Let M and N be MCM R-modules. We decompose M and N into indecomposable modules as $M = \oplus_i M_i$, $N = \oplus_j N_j$ and decompose $\psi \in \mathrm{Hom}_R(M, N)$ along this decomposition as $\psi = (\psi_{ij} : M_i \to N_j)_{ij}$. Then, we define the submodule $\mathrm{rad}_R(M, N) \subset \mathrm{Hom}_R(M, N)$ as follows.

$$\psi \in \mathrm{rad}_R(M, N) \overset{\text{def}}{\Longleftrightarrow} \text{no } \psi_{ij} \text{ is an isomorphism}$$

In addition, we define the submodule $\mathrm{rad}_R^2(M, N) \subset \mathrm{Hom}_R(M, N)$. The submodule $\mathrm{rad}_R^2(M, N)$ consists of morphisms $\psi : M \to N$ such that ψ decomposes as $\psi = g \circ f$,

$$\begin{tikzcd} M \arrow[rr, "\psi"] \arrow[dr, "f"'] & & N \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

where Z is an MCM R-module, $f \in \mathrm{rad}_R(M, Z)$, $g \in \mathrm{rad}_R(Z, N)$. We call a morphism $\psi : M \to N$ *irreducible* if $\psi \in \mathrm{rad}_R(M, N) \setminus \mathrm{rad}_R^2(M, N)$. Set

$$\mathrm{Irr}_R(M, N) := \mathrm{rad}_R(M, N) \big/ \mathrm{rad}_R^2(M, N).$$

Then, $\mathrm{Irr}_R(M, N)$ is a vector space over k.

Using these notions, next we define the Auslander-Reiten quiver.

Definition 2.10 (Auslander-Reiten quiver)**.** The Auslander-Reiten (AR) quiver of R is an oriented graph whose vertices are indecomposable MCM R-modules $R, M_1, \ldots, M_{n-1}$ with $\dim_k \mathrm{Irr}_R(M_s, M_t)$ arrows from M_s to M_t, for $s, t = 0, 1, \ldots, n-1$.

We will give an example of an AR quiver in the next subsection.

2.3. The case of cyclic quotient surface singularities. Since one of the aims of this paper is to determine the dual F-signature of special CM modules over cyclic quotient surface singularities, we restate results in subsections 2.1 and 2.2 for the cyclic case. Thus, we suppose that G is a cyclic group as follows.

$$G := \left\langle \sigma = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^a \end{pmatrix} \right\rangle,$$

where ζ_n is a primitive nth root of unity, $1 \leq a \leq n-1$, and $\gcd(a, n) = 1$. We denote the cyclic group G as above by $\frac{1}{n}(1, a)$. Let $S := k[[x, y]]$ be a power series ring, and we assume that n is coprime to $p = \mathrm{char}\, k$. We denote the invariant subring of S under the action of G by $R := S^G$. Since G is an abelian group, any irreducible representations of G are described by:

$$V_t : \sigma \longmapsto \zeta_n^{-t}, \quad t = 0, 1, \ldots, n-1.$$

Then, we set

$$M_t := (S \otimes_k V_t)^G = \left\{ \sum_{i,j} a_{ij} x^i y^j \in S \mid a_{ij} \in k,\ i + ja \equiv t \ (\mathrm{mod}\, n) \right\},$$

$t = 0, 1, \dots, n-1$. These give all indecomposable MCM modules over R, and each has rank one.

From Corollary 2.6, $s(M_t, M_u) = 1/n$, for $u = 0, 1, \dots, n-1$. Thus, when we discuss the asymptotic behavior of ${}^eM_t$ on the order of p^{2e}, we may consider as

$$ {}^eM_t \approx (R \oplus M_1 \oplus \cdots \oplus M_{n-1})^{\oplus p^{2e}/n}. \tag{2.1} $$

In addition, the AR sequence ending in M_t, $t \neq 0$, is

$$ 0 \longrightarrow M_{t-a-1} \longrightarrow M_{t-1} \oplus M_{t-a} \longrightarrow M_t \longrightarrow 0. \tag{2.2} $$

In the case of $t = 0$, the fundamental sequence of R is

$$ 0 \longrightarrow \omega_R \longrightarrow M_{-1} \oplus M_{-a} \longrightarrow R \longrightarrow k \longrightarrow 0. \tag{2.3} $$

Thus, we have $\tau(M_t) = M_{t-a-1}$ and $E_{M_t} = M_{t-1} \oplus M_{t-a}$ for $t = 0, 1, \dots, n-1$. It is known that $\dim_k \mathrm{Irr}_R(M_s, M_t)$ is equal to the multiplicity of M_s in the decomposition of E_{M_t}. Therefore, by (2.2) and (2.3), there is an arrow from M_{t-1} to M_t, and from M_{t-a} to M_t for $t = 0, 1, \dots, n-1$, namely, we have $\dim_k \mathrm{Irr}_R(M_{t-1}, M_t) = 1$ and $\dim_k \mathrm{Irr}_R(M_{t-a}, M_t) = 1$.

Remark 2.11. Since $S \cong R \oplus M_1 \oplus \cdots \oplus M_{n-1}$, each MCM R-module M_t is an R-submodule of S, and we can take a morphism $\cdot x$, respectively, $\cdot y$, as a basis of one-dimensional vector space $\mathrm{Irr}_R(M_{t-1}, M_t)$, respectively, $\mathrm{Irr}_R(M_{t-a}, M_t)$.

$$ M_{t-1} = \{ f \in S \mid \sigma \cdot f = \zeta_n^{t-1} f \} \xrightarrow{x} M_t = \{ f \in S \mid \sigma \cdot f = \zeta_n^t f \} $$

$$ M_{t-a} = \{ f \in S \mid \sigma \cdot f = \zeta_n^{t-a} f \} \xrightarrow{y} M_t = \{ f \in S \mid \sigma \cdot f = \zeta_n^t f \}. $$

Example 2.12. Let $G = \frac{1}{7}(1,3)$ be a cyclic group of order 7. Irreducible representations of G are

$$ V_t : \sigma \longmapsto \zeta_7^{-t}, \quad t = 0, \dots, 6, $$

where ζ_7 is a primitive 7th root of unity. Then, the AR quiver of R is described as follows. For simplicity, we only describe subscripts as vertices, and all common numbers are identified.

$$\begin{tikzcd}
0 \arrow[r] & 3 \arrow[r] & 6 \arrow[r] & \cdots \arrow[r] & 1 \arrow[r] & 4 \arrow[r] & 0 \\
6 \arrow[r] \arrow[u] & 2 \arrow[r] \arrow[u] & 5 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 3 \arrow[r] \arrow[u] & 6 \arrow[u] \\
5 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & 4 \arrow[r] \arrow[u] & \cdots \arrow[r] & 6 \arrow[r] \arrow[u] & 2 \arrow[r] \arrow[u] & 5 \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
2 \arrow[r] \arrow[u] & 5 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & \cdots \arrow[r] & 3 \arrow[r] \arrow[u] & 6 \arrow[r] \arrow[u] & 2 \arrow[u] \\
1 \arrow[r] \arrow[u] & 4 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \arrow[r] & 2 \arrow[r] \arrow[u] & 5 \arrow[r] \arrow[u] & 1 \arrow[u] \\
0 \arrow[r] \arrow[u] & 3 \arrow[r] \arrow[u] & 6 \arrow[r] \arrow[u] & \cdots \arrow[r] & 1 \arrow[r] \arrow[u] & 4 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Remark 2.13. For each diagram, $\begin{tikzcd} a \arrow[r] & b \\ c \arrow[r] \arrow[u] & d \arrow[u] \end{tikzcd}$, if $b \neq 0$, then

$$0 \longrightarrow M_c \longrightarrow M_a \oplus M_d \longrightarrow M_b \longrightarrow 0$$

is the AR sequence ending in M_b, and any diagram commutes $\begin{tikzcd} a \arrow[r, "y"] & b \\ c \arrow[r, "y"'] \arrow[u, "x"] \arrow[ur, phantom, "\circlearrowleft"] & d \arrow[u, "x"'] \end{tikzcd}$ by Remark 2.11.

3. Dual F-signature of special CM modules. In this section, we introduce the notion of special CM modules and determine the dual F-signature of them. Firstly, we recall the definition of special CM modules over an invariant subring R, and the properties of them.

Definition 3.1 ([**20**])**.** For an MCM R-module M, we call M special if $(M \otimes_R \omega_R)/\operatorname{tor}$ is also an MCM R-module.

In other words, let φ be the natural morphism $M \otimes_R \omega_R \to (M \otimes_R \omega_R)^{**}$. Then, $M \otimes_R \omega_R / \operatorname{Ker} \varphi$ is also an MCM R-module if and only if M is a special CM R-module. In that case, we have the following (cf., [**13,** Lemma 9]),

$$M \otimes_R \omega_R / \operatorname{Ker} \varphi \cong \tau(M) \cong (M \otimes_R \omega_R)^{**}.$$

Therefore, M is a special CM R-module if and only if φ is a surjection. Furthermore, there are several characterizations of special CM modules as follows (see [**9,** Theorem 2.7 and 3.6]).

Proposition 3.2. *Suppose that M is an* MCM *R-module. Then, the following are equivalent.*

(1) *M is a special CM module*;
(2) $\mathrm{Ext}^1_R(M,R)=0$;
(3) $(\Omega M)^* \cong M$ *where ΩM is the syzygy of M.*

Suppose that M is a special CM R-module. Then, we have the following exact sequence by condition (3). Here, $\mu_R(M)$ is the number of minimal generators of M.

$$0 \longrightarrow M^* \cong \Omega M \longrightarrow R^{\oplus \mu_R(M)} \longrightarrow M \longrightarrow 0.$$

Thus, we have $\mu_R(M) = 2\,\mathrm{rank}_R\, M$. The converse is true if $\mathrm{rank}_R\, M = 1$ (cf., [**20,** Theorem 2.1]). If $\mathrm{rank}_R\, M > 1$, the converse is no longer true (cf., [**12,** Example A.5] and [**9**]). Since each MCM module over cyclic quotient surface singularities has rank one, a special CM module is minimally 2-generated (see Theorem 3.5).

For a cyclic group $G = \frac{1}{n}(1,a)$, we can describe special CM-modules as follows.

Firstly, we consider the Hirzebruch-Jung continued fraction expansion of n/a,

$$\frac{n}{a} = \alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\cdots - \cfrac{1}{\alpha_r}}} := [\alpha_1, \alpha_2, \cdots, \alpha_r],$$

and then we introduce the notion of i-series and j-series (cf., [**18, 19**]).

Definition 3.3. For $n/a = [\alpha_1, \alpha_2, \ldots, \alpha_r]$, we define the i-series and the j-series as follows.

$$i_0 = n, \quad i_1 = a, \qquad i_t = \alpha_{t-1} i_{t-1} - i_{t-2}, \quad t = 2, \ldots, r+1,$$
$$j_0 = 0, \quad j_1 = 1, \qquad j_t = \alpha_{t-1} j_{t-1} - j_{t-2}, \quad t = 2, \ldots, r+1.$$

Remark 3.4. By the definition of the i-series and the j-series, it is easy to see

- $i_t \equiv j_t a \pmod n$,
- $i_0 = n > i_1 = a > i_2 > \cdots > i_r = 1 > i_{r+1} = 0$,
- $j_0 = 0 < j_1 = 1 < j_2 = \alpha_1 < \cdots < j_r < j_{r+1} = n$.

By using the i-series and the j-series, we can characterize special CM R-modules.

Theorem 3.5 (**[19]**). *For a cyclic group $G = \frac{1}{n}(1,a)$ with $n/a = [\alpha_1, \alpha_2, \ldots, \alpha_r]$, special* CM *$R$-modules are M_{i_t}, for $t = 0, 1, \ldots, r$. Moreover, minimal generators of M_{i_t} are x^{i_t} and y^{j_t} for $t = 1, \ldots, r$.*

Example 3.6. Let $G = \frac{1}{7}(1,3)$ be a cyclic group of order 7. The Hirzebruch-Jung continued fraction expansion of $7/3$ is

$$\frac{7}{3} = 3 - \frac{1}{2 - 1/2} = [3,\ 2,\ 2],$$

and the i-series and the j-series are described as follows.

$$\begin{aligned} &i_0 = 7, \quad i_1 = 3, \quad i_2 = 2, \quad i_3 = 1, \quad i_4 = 0, \\ &j_0 = 0, \quad j_1 = 1, \quad j_2 = 3, \quad j_3 = 5, \quad j_4 = 7. \end{aligned}$$

Thus, special CM modules are R, M_1, M_2, M_3, and these are explicitly described as:

$$\begin{aligned} R &= k[[x^7, x^4y, xy^2, y^7]] \\ M_1 &= Rx + Ry^5 \\ M_2 &= Rx^2 + Ry^3 \\ M_3 &= Rx^3 + Ry. \end{aligned}$$

We now show, using AR theory, how to investigate possible surjections ${}^eM_2 \twoheadrightarrow M_2^{\oplus b_e}$.

We take the MCM R-module M_2 as an example. From the AR quiver around the vertex ②, we can see that there are several morphisms ending in ② and obtain minimal generators x^2 and y^3 by following the morphisms described by double arrows in Figure 1.

Since each of the diagrams $\begin{smallmatrix} a & \to & b \\ \uparrow & & \uparrow \\ c & \to & d \end{smallmatrix}$ commute, morphisms from vertices which are denoted by $\bullet$ in Figure 2 to ② go through "0" (that

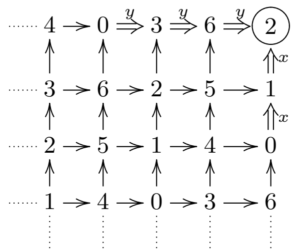

Figure 1.

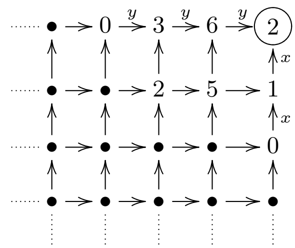

Figure 2.

is, "R"). Thus, the image of each morphism $\bullet \to ②$ is in $\mathfrak{m}M_2$, where $\mathfrak{m}$ is the maximal ideal of R. From Nakayama's lemma, such a morphism does not contribute to construct a surjection. Thus, we may ignore them. Also, there are morphisms from vertices which are denoted by $\bigstar$ in Figure 3 to ②. Minimal generators of $M_\bigstar$ are generated by morphisms from 0 (which are located outside of the dotted area in Figure 3) to $\bigstar$. Considering the composition of such a morphism and $\bigstar \to ②$

$$R \longrightarrow M_\bigstar \longrightarrow M_2, \quad 1 \mapsto \delta \mapsto x^{m_1}y^{m_2}\delta,$$

where δ is a minimal generator of $M_\bigstar$ and $m_1 \geq 1$, $m_2 \geq 1$. Then, it is easy to see that the image of the morphism $\bigstar \to ②$ is in $\mathfrak{m}M_2$. Thus, we may ignore them.

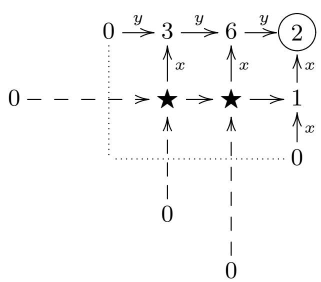

Figure 3.

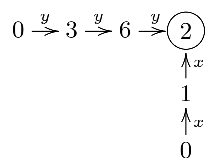

Figure 4.

Thus, in order to investigate a surjection ${}^eM_2 \twoheadrightarrow M_2^{\oplus b_e}$, we need only discuss the MCM R-modules located in the horizontal direction from M_2 to R and the vertical direction from M_2 to R (Figure 4).

In general, the number of minimal generators of a special CM R-module M_{i_t} is two, and minimal generators take a form like x^{i_t}, y^{j_t} by Theorem 3.5. Thus, it is equivalent that there is no "0" in the dotted vertices area of Figure 5. By the above arguments, in order to construct a surjection ${}^{e}M_{i_t} \twoheadrightarrow M_{i_t}^{\oplus b_e}$, we may only discuss horizontal direction arrows from R to M_{i_t} and vertical direction arrows from R to M_{i_t}. We consider sets of subscripts of vertices $\mathcal{F}_t = \{0, 1, \ldots, i_t - 1\}$ and $\mathcal{G}_t = \{i_t - a, \ldots, i_t - j_t a \equiv 0\}$ as in Figure 5. It is easy to see that $|\mathcal{F}_t| = i_t$, $|\mathcal{G}_t| = j_t$.

In order to determine the dual F-signature of special CM R-modules, we prepare some notation and lemmas.

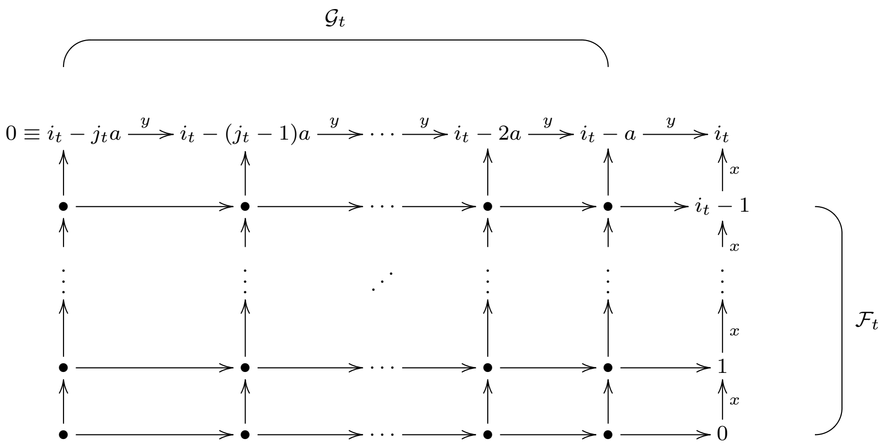

FIGURE 5.

For the i-series $(i_1, \ldots, i_r)$ associated with $\frac{1}{n}(1, a)$ and any $\beta \in \mathbb{Z}_{\geq 0}$ with $0 \leq \beta \leq n-1$, there are unique non-negative integers $d_1, \ldots, d_r \in \mathbb{Z}_{\geq 0}$ such that

$$\begin{aligned} &\beta = d_1 i_1 + h_1, && h_1 \in \mathbb{Z}_{\geq 0}, \quad 0 \leq h_1 < i_1, \\ &h_t = d_{t+1} i_{t+1} + h_{t+1}, && h_{t+1} \in \mathbb{Z}_{\geq 0},\ 0 \leq h_{t+1} < i_{t+1},\ t = 1, \ldots, r-1, \\ &h_r = 0. \end{aligned}$$

Thus, we can describe β as follows:

$$\beta = d_1 i_1 + d_2 i_2 + \cdots + d_r i_r.$$

For such β, there is the unique integer $\widetilde{\beta} \in \mathbb{Z}_{\geq 0}$ such that $a\widetilde{\beta} \equiv \beta \pmod{n}$, $0 \leq \widetilde{\beta} \leq n-1$.

Lemma 3.7 (**[19]**). *Let $\widetilde{\beta}$ be the same as above. Then, $\widetilde{\beta}$ is described as*

$$\widetilde{\beta} = d_1 j_1 + d_2 j_2 + \cdots + d_r j_r,$$

where $(j_1, \cdots, j_r)$ is the j-series associated with $\frac{1}{n}(1,a)$.

Lemma 3.8. *Let the notation be the same as above. Then, $\mathcal{F}_t \cap \mathcal{G}_t = \{0\}$ as a set of subscripts of vertices.*

Proof. It is trivial that $0 \in \mathcal{F}_t \cap \mathcal{G}_t$ by the definition of $\mathcal{F}_t$ and $\mathcal{G}_t$. Thus, it suffices to show that there is no pair $(m_1, m_2) \in \mathbb{Z}^2_{>0}$ such that $m_1 \equiv m_2 a \pmod{n}$, where $1 \leq m_1 \leq i_t - 1$ and $1 \leq m_2 \leq j_t - 1$. Assume that there exists such a pair (m_1, m_2). Then, there are non-negative integers $d_1, \ldots, d_r$ such that $m_1 = d_1 i_1 + d_2 i_2 + \cdots + d_r i_r$. Since $1 \leq m_1 \leq i_t - 1$ and $i_t > i_{t+1}$, cf., Remark 3.4, we have $d_1 = \cdots = d_t = 0$ and there exists a λ such that $t+1 \leq \lambda \leq r$ and $d_\lambda \neq 0$. From Lemma 3.7, we obtain $m_2 = d_1 j_1 + d_2 j_2 + \cdots + d_r j_r$. Thus,

$$m_2 = d_{t+1} j_{t+1} + \cdots + d_r j_r \geq j_\lambda > j_t.$$

This contradicts $m_2 \leq j_t - 1$. $\square$

We are now ready to state the main theorem.

Theorem 3.9. *Let the notation be the same as above. Then, for any non-free special* CM *R-module M_{i_t}, we have*

$$s(M_{i_t}) = \begin{cases} \dfrac{\min(i_t, j_t) + 1}{n} & \text{if } i_t \neq j_t \\ \dfrac{2i_t + 1}{2n} & \text{if } i_t = j_t. \end{cases}$$

Proof. In order to determine the value of the dual F-signature of M_{i_t}, we must find the maximum number b_e such that there is a surjection ${}^e M_{i_t} \twoheadrightarrow M_{i_t}^{\oplus b_e}$. Note that we may consider ${}^e M_{i_t}$ as

$${}^e M_{i_t} \approx (R \oplus M_1 \oplus \cdots \oplus M_{n-1})^{\oplus p^{2e}/n}$$

by (2.1); hence, we may assume the number of each indecomposable MCM module in ${}^{e}M_{i_t}$ is the same on the order of p^{2e}.

Let $\mathcal{F}_t$, $\mathcal{G}_t$ be the sets of vertices as in Figure 5. By the above observations, MCM modules which contribute to construct a surjection are M_{i_t} itself and modules corresponding to elements in $\mathcal{F}_t$ or $\mathcal{G}_t$. Since an indecomposable MCM module which is not isomorphic to R and M_{i_t} could construct at most one generator of M_{i_t}, we should first combine MCM modules corresponding to elements in $\mathcal{F}_t \setminus \{0\}$ with those in $\mathcal{G}_t \setminus \{0\}$ for efficiently constructing surjections, and then we should use R and M_{i_t}. Therefore, in what follows, we will find disjoint sets of summands of ${}^{e}M_{i_t}$ which surject onto M_{i_t} as much as possible along this strategy.

Firstly, we show the case of $i_t > j_t$. Thus, $|\mathcal{F}_t| > |\mathcal{G}_t|$. We choose elements f_1 and g_1 from $\mathcal{F}_t \setminus \{0\}$ and $\mathcal{G}_t \setminus \{0\}$, respectively, and consider corresponding indecomposable MCM R-modules M_{f_1} and M_{g_1}. Here, we remark that $f_1 \neq g_1$ by Lemma 3.8. Then, we can construct a surjection $M_{f_1} \oplus M_{g_1} \twoheadrightarrow M_{i_t}$.

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & g_1 \arrow[r] & \cdots \arrow[r] & i_t \\
&&&& \vdots \arrow[u] \\
&&&& f_1 \arrow[u] \\
&&&& \vdots \arrow[u] \\
&&&& 0 \arrow[u]
\end{tikzcd}$$

Next, we consider the sets $\mathcal{F}_t \setminus \{0, f_1\}$ and $\mathcal{G}_t \setminus \{0, g_1\}$. Similarly, we choose elements f_2 and g_2 from the sets $\mathcal{F}_t \setminus \{0, f_1\}$ and $\mathcal{G}_t \setminus \{0, g_1\}$, respectively, and construct a surjection $M_{f_2} \oplus M_{g_2} \twoheadrightarrow M_{i_t}$. By repeating the same process, we finally arrive at $\mathcal{G}_t \setminus \{0, g_1, \ldots, g_{j_t-1}\} = \emptyset$ and have $j_t - 1$ surjections. Since we still do not use $0 \in \mathcal{G}_t$, that is, R, we construct a surjection by combining R and an indecomposable MCM module corresponding to an element

$$f' \in \mathcal{F}_t \setminus \{0, f_1, \ldots, f_{j_t-1}\} \neq \emptyset.$$

In addition, there is a trivial surjection $M_{i_t} \twoheadrightarrow M_{i_t}$. Thus, through these processes, we could obtain disjoint sets of summands

$$\{M_{f_1}, M_{g_1}\}, \ldots, \{M_{f_{j_t-1}}, M_{g_{j_t-1}}\}, \{M_{f'}, R\}, \{M_{i_t}\}$$

which surject onto M_{i_t}. Thus, we have a surjection

$$(R \oplus M_1 \oplus \cdots \oplus M_{n-1})^{\oplus p^{2e}/n} \twoheadrightarrow M_{i_t}^{\oplus p^{2e}(j_t+1)/n}.$$

Therefore, the dual F-signature of M_{i_t} is

$$s(M_{i_t}) = \frac{j_t + 1}{n}.$$

Similarly, we obtain $s(M_{i_t}) = (i_t + 1)/n$ for the case of $i_t < j_t$.

For the case of $i_t = j_t$, we repeat the same process until we have $\mathcal{F}_t \setminus \{0, f_1, \ldots, f_{i_t-1}\} = \emptyset$ and $\mathcal{G}_t \setminus \{0, g_1, \ldots, g_{j_t-1}\} = \emptyset$, and then we have a surjection

$$(M_1 \oplus \cdots \oplus M_{i_t-1} \oplus M_{i_t+1} \oplus \cdots \oplus M_{n-1})^{\oplus p^{2e}/n} \twoheadrightarrow M_{i_t}^{\oplus p^{2e}(i_t-1)/n}.$$

In addition, there is a trivial surjection $M_{i_t} \twoheadrightarrow M_{i_t}$. For now, we do not use R, and, by using two free summands, we also construct the surjection:

$$R \oplus R \xrightarrow{(x^{i_t}\ y^{j_t})} M_{i_t}.$$

Thus, the dual F-signature of M_{i_t} is

$$s(M_{i_t}) = \frac{i_t - 1}{n} + \frac{1}{n} + \frac{1}{2n} = \frac{2i_t + 1}{2n}. \qquad \square$$

Example 3.10. Let the notation be as in Example 3.6. Then, the dual F-signature of special CM modules are

$$s(M_1) = \frac{2}{7}, \qquad s(M_2) = \frac{3}{7}, \qquad s(M_3) = \frac{2}{7}.$$

Next, we give an example for the case $i_t = j_t$.

Example 3.11. Let $G = \frac{1}{8}(1, 5)$ be a cyclic group of order 8. The Hirzebruch-Jung continued fraction expansion of $8/5$ is

$$\frac{8}{5} = 2 - \frac{1}{3 - 1/2} = [2, 3, 2],$$

and the i-series and the j-series are described as follows.

$$\begin{array}{lllll} i_0 = 8, & i_1 = 5, & i_2 = 2, & i_3 = 1, & i_4 = 0, \\ j_0 = 0, & j_1 = 1, & j_2 = 2, & j_3 = 5, & j_4 = 8. \end{array}$$

Thus, special CM modules are R, M_1, M_2, M_5. In this case, we have $i_2 = j_2$, and there are surjections as follows.

$$\begin{tikzcd} 0 \arrow[r, "y"] & 5 \arrow[r, "y"] & 2 \\ & & 1 \arrow[u, "x"'] \\ & & 0 \arrow[u, "x"'] \end{tikzcd} \qquad \begin{array}{lcl} M_2 & \twoheadrightarrow & M_2 \\ M_1 \oplus M_5 & \twoheadrightarrow & M_2 \\ R \oplus R & \twoheadrightarrow & M_2. \end{array}$$

Thus, the dual F-signature of M_2 is

$$s(M_2) = \frac{1}{8} + \frac{1}{8} + \frac{1}{16} = \frac{5}{16}.$$

Example 3.12. Let $G = \frac{1}{n}(1, n-1) \subset \mathrm{SL}(2,k)$ be a cyclic group of order n, that is, Dynkin type A_{n-1}. The Hirzebruch-Jung continued fraction expansion of $n/(n-1)$ is

$$\frac{n}{n-1} = 2 - \cfrac{1}{2 - \cfrac{1}{\cdots - 1/2}} = [\underbrace{2, 2, \cdots, 2}_{n-1}],$$

and the i-series and the j-series are described as follows:

$$\begin{array}{llllll} i_0 = n, & i_1 = n-1, & i_2 = n-2, & \ldots, & i_{n-1} = 1, & i_n = 0, \\ j_0 = 0, & j_1 = 1, & j_2 = 2, & \ldots, & j_{n-1} = n-1, & j_n = n, \end{array}$$

namely, $i_t = n - t$, $j_t = t$, for $t = 1, 2, \ldots, n-1$. As we mentioned in Remark 1.5, all irreducible representations of $G = \frac{1}{n}(1, n-1) \subset \mathrm{SL}(2,k)$ are special. Thus, any M_t is a special CM module, and the dual F-signature of M_t is obtained by Theorem 3.9.

$$s(M_{i_t}) = \begin{cases} \dfrac{1}{n} + \dfrac{j_t}{n} = \dfrac{t+1}{n} & \text{if } t < \dfrac{n}{2} \\ \dfrac{1}{n} + \dfrac{t-1}{n} + \dfrac{1}{2n} = \dfrac{2t+1}{2n} & \text{if } t = \dfrac{n}{2} \\ \dfrac{1}{n} + \dfrac{i_t}{n} = \dfrac{n-t+1}{n} & \text{if } t > \dfrac{n}{2}. \end{cases}$$

For other Dynkin types, i.e., D_n, E_6, E_7, E_8, see **[12]**.

4. Comparison with the Auslander-Reiten translation. In this section, we compare the dual F-signature of a special CM module with its AR translation. It will give us a characterization of Gorensteiness (see Theorem 4.2). As was mentioned in Section 1, it is an analogue of Theorem 1.4 (4), (5).

The statements appearing in this section are valid for any quotient surface singularities. Therefore, we suppose that G is a finite subgroup of $\mathrm{GL}(2,k)$ which contains no pseudo-reflections except the identity, and $S := k[[x,y]]$ a power series ring. We assume that the order of G is coprime to $p = \operatorname{char} k$. We denote the invariant subring of S under the action of G by $R := S^G$. Let $V_0 = k$, $V_1, \dots, V_n$ be the complete set of irreducible representations of G and set indecomposable MCM R-modules $M_t := (S \otimes_k V_t)^G$, for $t = 0, 1, \dots, n$.

Lemma 4.1. *Let M_t be an* MCM *R-module as above. Then, we have*

$$ {}^{e}M_t \approx (R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}})^{\oplus p^{2e}/n} \approx {}^{e}\tau(M_t) \tag{4.1}$$

on the order of p^{2e}, $e \gg 0$, where $d_{i,t} = (\operatorname{rank}_R M_t)\cdot(\operatorname{rank}_R M_i)$ and τ stands for the AR translation. Furthermore, we have

$$R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}} \cong \tau(R)^{\oplus d_{0,t}} \oplus \tau(M_1)^{\oplus d_{1,t}} \oplus \cdots \oplus \tau(M_n)^{\oplus d_{n,t}}.$$

Proof. From Corollary 2.6, we may write

$$\begin{aligned} {}^{e}M_t &\approx (R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}})^{\oplus p^{2e}/n}, \\ {}^{e}\tau(M_t) &\approx (R^{\oplus d'_{0,t}} \oplus M_1^{\oplus d'_{1,t}} \oplus \cdots \oplus M_n^{\oplus d'_{n,t}})^{\oplus p^{2e}/n}, \end{aligned}$$

where $d'_{i,t} = (\operatorname{rank}_R \tau(M_t))\cdot(\operatorname{rank}_R M_i)$. Since $\operatorname{rank}_R M_t = \operatorname{rank}_R \tau(M_t)$, it follows that $d_{i,t} = d'_{i,t}$, for $i = 0, 1, \dots, n$. This implies (4.1).

Since the AR translation τ gives a bijection from the set of finitely many indecomposable MCM R-modules to itself, we set $\tau(M_i) = M_{\sigma(i)}$, for $i = 0, 1, \dots, n$, where σ is an element of symmetric group $\mathfrak{S}_{n+1}$. Then, we have

$$R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}} = M_{\sigma(0)}^{\oplus d_{\sigma(0),t}} \oplus M_{\sigma(1)}^{\oplus d_{\sigma(1),t}} \oplus \cdots \oplus M_{\sigma(n)}^{\oplus d_{\sigma(n),t}},$$

and

$$\begin{aligned} d_{\sigma(i),t} &= (\operatorname{rank}_R M_t)\cdot(\operatorname{rank}_R M_{\sigma(i)}) = (\operatorname{rank}_R M_t)\cdot(\operatorname{rank}_R \tau(M_i)) \\ &= (\operatorname{rank}_R M_t)\cdot(\operatorname{rank}_R M_i) = d_{i,t}. \end{aligned}$$

Thus,

$$\begin{aligned} &M_{\sigma(0)}^{\oplus d_{\sigma(0),t}} \oplus M_{\sigma(1)}^{\oplus d_{\sigma(1),t}} \oplus \cdots \oplus M_{\sigma(n)}^{\oplus d_{\sigma(n),t}} \\ &\qquad = \tau(R)^{\oplus d_{0,t}} \oplus \tau(M_1)^{\oplus d_{1,t}} \oplus \cdots \oplus \tau(M_n)^{\oplus d_{n,t}}. \quad \square \end{aligned}$$

Theorem 4.2. *For any indecomposable special* CM *R-module M_t, we have*

$$s(M_t) \le s(\tau(M_t)).$$

Moreover, if $s(M_t) = s(\tau(M_t))$ for an indecomposable special CM *R-module M_t, then R is Gorenstein. Note that, if R is Gorenstein, then $s(M) = s(\tau(M))$ holds for all indecomposable* MCM *modules.*

Proof. From Lemma 4.1, we may write

$${}^eM_t \approx {}^e\tau(M_t) \approx (R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}})^{\oplus p^{2e}/n}$$

when we discuss the asymptotic behavior on the order of p^{2e} where $d_{i,t} = (\operatorname{rank}_R M_t)\cdot(\operatorname{rank}_R M_i)$.

Let $b_e := b_e(M_t)$ be the eth F-surjective number of M_t; hence, there exists a surjection ${}^eM_t \twoheadrightarrow M_t^{\oplus b_e}$. Since M_t is special, the number of minimal generators of M_t is equal to $u := 2\operatorname{rank}_R M_t$. Thus, there exists a surjection $R^{\oplus b_e u} \twoheadrightarrow M_t^{\oplus b_e}$ which induces the following commutative diagram.

$$\begin{tikzcd} {}^eM_t \arrow[r, two heads] & M_t^{\oplus b_e} \\ R^{\oplus b_e u} \arrow[u, dotted] \arrow[ur] & \end{tikzcd}$$

Applying the functor $(-\otimes_R \omega_R)^{**}$ to this commutative diagram, we then obtain the commutative diagram.

$$\begin{tikzcd} {}^e\tau(M_t) \approx ({}^eM_t \otimes_R \omega_R)^{**} \arrow[r, twoheadrightarrow, "\psi_2"] & \tau(M_t)^{\oplus b_e} \\ \omega_R^{\oplus b_e u} \arrow[u] \arrow[ur, "\psi_1"'] & \end{tikzcd}$$

Note that the morphism ψ_1 is surjective because the surjection $R^{\oplus b_e u} \twoheadrightarrow M_t^{\oplus b_e}$ induces

$$\begin{tikzcd} \omega_R^{\oplus b_e u} \arrow[r, twoheadrightarrow] \arrow[d, "\cong"'] & (M_t \otimes_R \omega_R)^{\oplus b_e} \arrow[d, twoheadrightarrow, "\oplus\varphi"] \\ ((\omega_R)^{**})^{\oplus b_e u} \arrow[r, "\psi_1"'] & ((M_t \otimes_R \omega_R)^{**})^{\oplus b_e} \end{tikzcd}$$

and $\varphi : M_t \otimes_R \omega_R \to (M_t \otimes_R \omega_R)^{**}$ is surjective. This implies ψ_2 is also surjective, and we have $s(M_t) \leq s(\tau(M_t))$.

If R is Gorenstein, then $M \cong \tau(M)$ for all indecomposable MCM modules. Thus $s(M) = s(\tau(M))$ holds. In the rest, we assume that R is not Gorenstein, and hence, $R \not\cong \omega_R$. Therefore, we also have $M \not\cong \tau(M)$ for all indecomposable MCM modules. For any indecomposable special CM module M_t, we have the following surjection in the same manner as above

$$\omega_R^{\oplus b_e u} \longrightarrow {}^e\tau(M_t) \approx \left(R^{\oplus d_{0,t}} \oplus \bigoplus_{i=1}^{n} M_i^{\oplus d_{i,t}} \right)^{\oplus p^{2e}/n} \overset{\psi_2}{\twoheadrightarrow} \tau(M_t)^{\oplus b_e}.$$

In this surjection, morphisms which go through R do not contribute to the construction of a surjection by Nakayama's lemma. Thus, in addition to a surjection $\omega_R^{\oplus b_e u} \twoheadrightarrow \tau(M_t)^{\oplus b_e}$, we also construct a surjection

$$R^{\oplus p^{2e} d_{0,t}/n} \longrightarrow\!\!\!\!\!\rightarrow \tau(M_t)^{\oplus (p^{2e}/n)(d_{0,t}/v)},$$

where v is the number of minimal generators of $\tau(M_t)$. Therefore, we obtain

$$b_e(\tau(M_t)) \geq b_e + \frac{d_{0,t}p^{2e}}{vn},$$

where $b_e(\tau(M_t))$ is the eth F-surjective number of $\tau(M_t)$. Thus,

$$s(\tau(M_t)) \geq s(M_t) + \frac{d_{0,t}}{vn} > s(M_t). \qquad \square$$

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Nagoya University, Graduate School Of Mathematics, Furocho, Chikusaku, Nagoya, 464-8602 Japan and The University of Tokyo, Kavli Institute for the Physics and Mathematics of the Universe (WPI), UTIAS, Kashiwa, Chiba 277-8583, Japan

Email address: **m06022z@math.nagoya-u.ac.jp, yusuke.nakajima@ipmu.jp**