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DISCRETE VALUATION OVERRINGS OF A GRADED NOETHERIAN DOMAIN

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ABSTRACT. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integral domain graded by an arbitrary torsionless grading monoid Γ, M a homogeneous maximal ideal of R and $S(H) = R \setminus \bigcup_{P\in \text{h-Spec}(R)} P$. We show that R is a graded Noetherian domain with $\text{h-}\dim(R) = 1$ if and only if $R_{S(H)}$ is a one-dimensional Noetherian domain. We then use this result to prove a graded Noetherian domain analogue of the Krull-Akizuki theorem. We prove that, if R is a gr-valuation ring, then R_M is a valuation domain, $\dim(R_M) = \text{h-}\dim(R)$ and R_M is a discrete valuation ring if and only if R is discrete as a gr-valuation ring. We also prove that, if $\{P_i\}$ is a chain of homogeneous prime ideals of a graded Noetherian domain R, then there exists a discrete valuation overring of R which has a chain of prime ideals lying over $\{P_i\}$.

1. Introduction. Let D be an integral domain with quotient field K. An overring of D means a ring between D and K. As is standard, $\dim(D)$ denotes the (Krull) dimension of D and $\text{ht}(P) = \dim(D_P)$ for all prime ideals P of D. We say that a valuation domain V is a *discrete valuation ring* (DVR) if each primary ideal of V is a power of its radical. It is known that V is discrete if and only if each branched prime ideal of V is not idempotent [**8,** Theorem 17.3]. (A prime ideal P is branched if there exists a P-primary ideal distinct from P.) Also, if $\dim(V) < \infty$, then V is discrete if and only if QV_Q is principal for each prime ideal Q of V. It is well known that, if $(0) = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ is a chain of prime ideals in D, then there exists a valuation overring of D which has a chain of prime ideals lying over $(0) = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ [**8,**

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Corollary 19.7]. Moreover, in [**5,** Theorem], Cahen, Houston and Lucas showed that, if D is a Noetherian domain and $(0) = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ is a chain of prime ideals in D, then there is a rank n discrete valuation overring of D whose prime ideals contract to $\{P_i\}_{i=0}^n$. Chang and Oh generalized this result to an integral domain A with the property that A_P is a Noetherian domain for each prime ideal P of A with $\mathrm{ht}(P) < \infty$. Specifically, they showed that, if $\{P_k\}$ is a chain of prime ideals of A such that $\mathrm{ht}(P_k) < \infty$ for each k, then there exists a discrete valuation overring of A which has a chain of prime ideals lying over $\{P_k\}$ [**6,** Corollary 4]. The purpose of this paper is to study a graded Noetherian domain analogue of Cahen, Houston and Lucas's result [**5,** Theorem].

This paper consists of four sections, including the introduction. In Section 2, we review some basic notation and results on graded integral domains for the reading of this paper. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integral domain graded by an arbitrary torsionless grading monoid Γ, and let $S(H) = R \setminus \bigcup_{P\in \text{h-Spec}(R)} P$.

In Section 3, we show that R is a graded Noetherian domain with $\text{h-dim}(R) = 1$ if and only if $R_{S(H)}$ is a one-dimensional Noetherian domain. In this case, $\mathrm{Max}(R_{S(H)}) = \{PR_{S(H)} \mid P \in \text{h-Spec}(R)$ and $P \neq (0)\}$. We use this result to introduce a graded Noetherian domain analogue of the Krull-Akizuki theorem.

Let V be a homogeneous graded valuation overring of R. Finally, in Section 4, we show that, if M is a homogeneous maximal ideal of V, then V_M is a valuation domain, $\dim(V_M) = \text{h-dim}(V)$, and V is discrete as a graded valuation ring if and only if V_M is a DVR. We prove that, if $\{P_\lambda\}$ is a chain of homogeneous prime ideals of R, then there exists a homogeneous graded valuation overring of R with a chain of homogeneous prime ideals that contract to $\{P_\lambda\}$.

Let $(0) = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ be a chain of homogeneous prime ideals of a graded Noetherian domain R, and let V be a homogeneous graded valuation overring of R with a chain $\{Q_\alpha\}_{\alpha\in\Lambda}$ of (homogeneous) prime ideals such that $\{Q_\alpha \cap R\}_{\alpha\in\Lambda} = \{P_i\}_{i=0}^n$. We show in Theorem 4.5 that, if $\{P_i\}$ is saturated, then $\{Q_\alpha\}_{\alpha\in\Lambda} = \{(0) = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n\}$ and $V_{H\setminus Q_n}$ is discrete as a gr-valuation ring with $\text{h-dim}(V_{H\setminus Q_n}) = n$, where H is the set of nonzero homogeneous elements of R. As a corollary, in Corollary 4.6, we have that there exists a discrete valuation overring of R which has a chain of prime ideals lying over $\{P_i\}$.

2. Definitions related to graded integral domains. Let Γ be a nontrivial torsionless grading monoid, that is, Γ is a commutative cancellative monoid (written additively), $\Gamma \neq (0)$, and the quotient group

$$G := \{a - b \mid a, b \in \Gamma\}$$

of Γ is a torsion-free abelian group. It is well known that a cancellative monoid is torsionless if and only if it can be totally ordered [**14,** page 123]. By a (Γ)-graded integral domain

$$R = \bigoplus_{\alpha \in \Gamma} R_\alpha,$$

we mean an integral domain graded by an arbitrary torsionless grading monoid Γ, that is, each nonzero $x \in R_\alpha$ has degree α, i.e., $\deg(x) = \alpha$, and thus each nonzero $f \in R$ can be written uniquely as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$, where $\alpha_i \in \Gamma$, x_{α_i} is a nonzero homogeneous element with $\deg(x_{\alpha_i}) = \alpha_i$, and $\alpha_1 < \cdots < \alpha_n$. The most well-known example of a graded integral domain is the semigroup ring $D[\Gamma] = \bigoplus_{\alpha \in \Gamma} DX^\alpha$ over an integral domain D with $\deg(aX^\alpha) = \alpha$ for $0 \neq a \in D$ and $\alpha \in \Gamma$. Clearly, if Γ is the monoid of nonnegative integers, then $D[\Gamma] = D[X]$ is the polynomial ring over D.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded integral domain, and let H be the set of nonzero homogeneous elements of R; thus, H is a saturated multiplicative subset of R. Let

$$(R_H)_\alpha = \left\{ \frac{a}{b} \mid a \in R_\beta, 0 \neq b \in R_\gamma \text{ and } \alpha = \beta - \gamma \right\}$$

for each $\alpha \in G$. Then,

$$R_H = \bigoplus_{\alpha \in G} (R_H)_\alpha,$$

and hence, R_H, called the *homogeneous quotient field of* R, is a G-graded integral domain. Clearly, $(R_H)_0$ is a field, and each nonzero homogeneous element of R_H is a unit.

Note that, if we let $\operatorname{Supp}(\Gamma) = \{\alpha \in \Gamma \mid R_\alpha \neq (0)\}$, then $R = \bigoplus_{\alpha \in \operatorname{Supp}(\Gamma)} R_\alpha$, and $\operatorname{Supp}(\Gamma)$ is a submonoid of Γ since R is an integral domain. Hence, throughout this paper, we assume that $R_\alpha \neq (0)$ for all $\alpha \in \Gamma$. An ideal I of R is said to be *homogeneous* if I is generated by homogeneous elements in I; thus, I is homogeneous if and only if $I = \bigoplus_{\alpha\in\Gamma}(I \cap R_\alpha)$. A homogeneous prime ideal (respectively, homogeneous maximal ideal) means a homogeneous ideal that is a prime ideal (respectively, maximal among proper integral homogeneous ideals). Clearly, homogeneous prime ideals are prime, but homogeneous maximal ideals need not be maximal ideals. Let h-Spec(R) be the set of homogeneous prime ideals of R. The h-height of a homogeneous prime ideal P, denoted by h-ht(P), is defined to be the supremum of the lengths of chains of homogeneous prime ideals descending from P, and the h-dimension of R is defined by

$$\text{h-}\dim(R) = \sup\{\text{h-ht}(P) \mid P \in \text{h-Spec}(R)\}.$$

Clearly, h-ht$(P) \leq \text{ht}(P)$ and h-$\dim(R) \leq \dim(R)$.

An overring T of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is called a *homogeneous overring* if $R \subseteq T \subseteq R_H$ and $T = \bigoplus_{\alpha\in G}(T \cap (R_H)_\alpha)$. Thus, T is a G-graded integral domain. We call R a *graded valuation ring* (in short, gr-valuation ring) if, for every homogeneous element x of R_H, either $x \in R$ or $x^{-1} \in R$. A homogeneous gr-valuation overring of R means a homogeneous overring of R which is a gr-valuation ring. Clearly, R is a gr-valuation ring if and only if the homogeneous ideals of R are linearly ordered by set inclusion. In particular, a gr-valuation ring R is said to be *discrete* if each homogenous primary ideal of R is a power of its radical. By a minor change in the proof of the standard non-graded expression in [**8,** Theorem 17.3], we can show that a gr-valuation ring R is discrete if and only if each branched homogeneous prime ideal of R is not idempotent. It is easy to see that a gr-valuation ring R is discrete if and only if $PR_{H\setminus P}$ is principal for all branched homogeneous prime ideals P of R by using the fact that $PR_{H\setminus P}$ is the homogenous maximal ideal. We say that R is a *graded Noetherian domain* if R satisfies the ascending chain condition (a.c.c.) on homogeneous ideals; equivalently, each homogeneous prime ideal of R is finitely generated [**17,** Lemma 2.3]. Obviously, a Noetherian domain is a graded Noetherian domain, while graded Noetherian domains need not be Noetherian. (It is known that the monoid ring $A[\Gamma]$ over a commutative ring A with identity is a Noetherian ring, respectively, graded Noetherian ring, if and only if A is a Noetherian ring and Γ, respectively, each ideal of Γ, is finitely generated [**7,** Theorem 7.7], respectively, [**17,** Theorem 2.4]. Hence, if $\mathbb{Q}$ is the additive group of rational numbers and D is a Noetherian domain, the group ring $R = D[\mathbb{Q}]$ is a graded Noetherian domain but not a Noetherian domain.) Note that if Q is a homogeneous prime ideal of a graded Noetherian domain, then h-ht$(Q) < \infty$ [**15,** Theorem 3.6].

For each nonzero fractional ideal I of an integral domain D with quotient field K, let

- $I^{-1} = \{x \in K \mid xI \subseteq D\}$,
- $I_v = (I^{-1})^{-1}$,

and

- $I_t = \bigcup\{J_v \mid J$ is a nonzero finitely generated subideal of $I\}$.

If $I = I_v$ (respectively, $I = I_t$), then I is called a *v-ideal* (respectively, *t-ideal*) of D. We say that a nonzero ideal of D is a *maximal t-ideal* if it is maximal among proper integral t-ideals, and let t-Max(D) denote the set of maximal t-ideals of D. Clearly, if $x \in D$ is a nonzero nonunit, then xD is a t-ideal, and it is well known that each prime ideal of R minimal over xR is a t-ideal and xD is contained in a maximal t-ideal of D. Thus, if D is not a field, then t-Max$(D) \neq \emptyset$. The v- and t-operations are examples of a star operation; for background on star operations, the reader is referred to [**8,** Sections 32, 34]. It is easy to see that, if I is a nonzero homogeneous ideal of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, then both I_v and I_t are also homogeneous. We say that R is a *graded Krull domain* if it is completely integrally closed and satisfies the a.c.c. on homogeneous v-ideals. For $f \in R_H$, let $C_R(f)$ (simply, $C(f)$) denote the fractional ideal of R generated by the homogeneous components of f. It is clear that $C(f)$ is a finitely generated homogeneous fractional ideal of R. Let

$$N(H) = \{f \in R \mid C(f)_v = R\}.$$

It is known that, if R is a nontrivial graded integral domain, then R is a graded Krull domain if and only if $R_{N(H)}$ is a principal ideal domain (PID) [**3,** Theorem 2.3]. Also, the integral closure of a graded Noetherian domain is a graded Krull domain [**16,** Theorem 2.10].

3. Graded Notherian domains. Let Γ be a torsionless grading monoid, G the quotient group of Γ,

$$R = \bigoplus_{\alpha\in\Gamma} R_\alpha$$

a (Γ-)graded integral domain and H the set of nonzero homogeneous elements of R; thus, R_H is a G-graded integral domain whose nonzero homogeneous elements are units.

We first recall a very useful result on homogeneous prime ideals. Let Q be a nonzero prime ideal of R, and let Q^* be the ideal of R generated by the homogeneous elements in Q. Then $Q^* \subseteq Q$, and either $Q^* = (0)$ or Q^* is a nonzero homogeneous prime ideal [**14,** page 124]. This also implies that a prime ideal minimal over a nonzero homogeneous ideal is homogeneous.

Lemma 3.1. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Noetherian domain, let P be a homogeneous prime ideal of R and let R' be the integral closure of R. If* h-ht$(P) = 1$, *then R_P is a one-dimensional Noetherian domain and $R'_{R\setminus P}$ is a semilocal* PID.

Proof. Recall that R' is a homogeneous overring of R [**16,** Lemmas 2.2, 2.3]. Let Q be a prime ideal of R' such that $Q \cap R = P$. Then, $PR' \subseteq Q$, and thus, if Q^* is the prime ideal of R' generated by the homogeneous elements in Q, then $PR' \subseteq Q^* \subseteq Q$. Clearly, $Q^* \cap R = Q \cap R$, and hence, $Q^* = Q$ [**8,** Corollary 11.2] since R' is integral over R. Hence, Q is homogeneous. Also, if Q_0 is a nonzero homogeneous prime ideal of R' with $Q_0 \subseteq Q$, then $Q_0 \cap R$ is homogeneous and $Q_0 \cap R \subseteq Q \cap R = P$. Therefore, since h-ht$(P) = 1$ and R' is integral over R, we have $Q_0 \cap R = P$; thus, $Q_0 = Q$. Hence, h-ht$(Q) = 1$, and since R' is a graded Krull domain, ht$(Q) = 1$ and R'_Q is a DVR [**1,** Proposition 5.5]. This implies that ht$(P) = 1$. Note that P is finitely generated, and thus, R_P is a one-dimensional Noetherian domain. Note also that $R'_{R\setminus P}$ is the integral closure of R_P, and hence, $R'_{R\setminus P}$ is a Dedekind domain with a finite number of maximal ideals [**12,** page 85, Corollary]. Thus, $R'_{R\setminus P}$ is a semilocal PID [**8,** Corollary 37.4]. □

Let x be a nonzero nonunit homogeneous element of a graded Noetherian domain R. It is known that, if P is a prime ideal of R minimal over xR, then h-ht$(P) = 1$ [**15,** Theorem 3.5]. Hence, by Lemma 3.1, we have the following.

Corollary 3.2. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Noetherian domain and x a nonzero nonunit homogeneous element of R. If P is a prime ideal of R minimal over xR, then* ht$(P) = 1$.

Proposition 3.3. *Let* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *be a graded integral domain and* $S(H) = R \setminus \bigcup_{P \in \text{h-Spec}(R)} P$. *Then,* R *is a graded Noetherian domain with* $\text{h-dim}(R) = 1$ *if and only if* $R_{S(H)}$ *is a one-dimensional Noetherian domain. In this case,*

$$\text{Max}(R_{S(H)}) = \{PR_{S(H)} \mid P \in \text{h-Spec}(R) \textit{ and } P \neq (0)\}.$$

Proof. Let $\Omega = \text{h-Spec}(R) \setminus \{(0)\}$.

($\Rightarrow$). Since $\text{h-dim}(R) = 1$ and each maximal t-ideal of R intersecting H is homogeneous [**2,** Lemma 1.2], each prime ideal in Ω is a maximal t-ideal of R. Let $0 \neq f \in R$. If $C(f) = R$, then $f \notin P$ for all $P \in \Omega$. If $C(f) \neq R$, then each prime ideal of R minimal over $C(f)$ must be in Ω, and, since each $P \in \Omega$ is finitely generated, $C(f)$ (so f) is contained only in a finite number of prime ideals $P \in \Omega$ [**9,** Theorem 1.6]. Thus, the intersection $\bigcap_{P \in \Omega} R_P$ is locally finite, and hence,

$$\text{Max}(R_{S(H)}) = \{PR_{S(H)} \mid P \in \Omega\}$$

[**3,** Lemma 2.2, Proposition 1.4]. In addition, $\text{ht}(PR_{S(H)}) = 1$ by Lemma 3.1 and $PR_{S(H)}$ is finitely generated for all $P \in \Omega$. Thus, $R_{S(H)}$ is a one-dimensional Noetherian domain.

($\Leftarrow$). Clearly, if $P \in \Omega$, then $P \cap S(H) = \emptyset$, and hence, $PR_{S(H)}$ is a proper prime ideal of $R_{S(H)}$. Hence, by assumption, $\text{ht}(P) = \text{ht}(PR_{S(H)}) = 1$, and thus, $\text{h-dim}(R) = 1$. Next, note that $PR_{S(H)}$ is finitely generated and P is homogeneous. Hence, there is a finitely generated homogeneous subideal I of P such that $IR_{S(H)} = PR_{S(H)}$. If $f \in P$, then $f \in IR_{S(H)} \cap R$, whence $f = h/g$ for some $h \in I$ and $g \in S(H)$. Thus, there is an integer $n \geq 1$ such that

$$C(g)^{n+1}C(f) = C(g)^n C(fg) = C(g)^n C(h)$$

[**3,** Lemma 1.1]. Note that $g \in S(H) \Leftrightarrow g \notin P$ for all $P \in \Omega$, $\Leftrightarrow C(g) \nsubseteq P$ for all $P \in \Omega$, $\Leftrightarrow C(g) = R$. Hence, $C(f) = C(h) \subseteq I$, and therefore, $P = I$. Thus, R is a graded Noetherian domain. □

Following [**15**], we say that a graded R-module M is *h-irreducible* if M has no nontrivial homogeneous submodules, and, for a graded R-module M, a chain

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_r = (0)$$

of homogeneous R-submodules of M is an *h-composition series* of M if every M_i/M_{i+1} is h-irreducible; in this case, r is called the *h-length* of M, which is independent of the choice of h-composition series [**15,** Theorem 3.1]. The notion of h-length is a graded module analogue of the length of a module (see [**12,** page 12] for the definition of the length of a module).

In [**15,** Theorem 4.2], Park and Park generalized the Krull-Akizuki theorem [**12,** Theorem 11.7] to a graded integral domain as follows:

> Let $R \subseteq T$ be graded integral domains with homogeneous quotient fields $K \subseteq L$, respectively. Assume that R is graded Noetherian with h-dim$(R) = 1$ and L is finite over K. Then T is graded Noetherian with h-dim$(T) \leq 1$, and if J is a nonzero homogeneous ideal of T, then T/J is a graded R-module of finite h-length.

We next give another type of a graded integral domain analogue of the Krull-Akizuki theorem, where we denote by $qf(D)$ the quotient field of D. This result is stronger than the Park and Park's result because $qf(T)$ is finite over $qf(R)$ when L is finite over K.

Corollary 3.4. *Let $R \subseteq T$ be graded integral domains such that every homogeneous element of R is homogeneous in T. Assume that R is graded Noetherian with* $\text{h-}\dim(R) = 1$ *and $qf(T)$ is finite over $qf(R)$.*

(1) *T is a graded Noetherian domain with* $\text{h-}\dim(T) \leq 1$.

(2) *If J is a nonzero homogeneous ideal of T, then T/J is a graded R-module of finite h-length.*

(3) *If Q is a nonzero homogeneous maximal ideal of T, then T/Q is a finitely generated $R/(Q \cap R)$-module and $qf(T/Q)$ is finite over $qf(R/(Q \cap R))$.*

Proof. Let

$$S(H) = R \setminus \bigcup_{P \in \text{h-Spec}(R)} P$$

and

$$S(T) = T \setminus \bigcup_{Q \in \text{h-Spec}(T)} Q.$$

Note that $f \in S(H)$ if and only if $C(f) = R$ (see the proof of Proposition 3.3), and thus, $S(H) \subseteq S(T)$ since every homogeneous element of R is homogeneous in T. Thus, $R_{S(H)} \subseteq T_{S(H)} \subseteq T_{S(T)}$ and $R_{S(H)}$ is a one-dimensional Noetherian domain by Proposition 3.3.

(1) By the Krull-Akizuki theorem, $T_{S(T)}$ is a Noetherian domain with $\dim(T_{S(T)}) \leq 1$, and thus, by Proposition 3.3, T is a graded Noetherian domain with $\text{h-}\dim(T) \leq 1$.

(2) Again, by the Krull-Akizuki theorem, $T_{S(H)}/JT_{S(H)}$ is an $R_{S(H)}$-module of finite length. Clearly, each homogeneous R-submodule of T/J is of the form M/J, where M is a homogeneous R-submodule of T containing J. Let $T/J \supseteq M/J \supsetneq N/J$ be homogeneous R-submodules of T/J. Then,

$$(T/J)_{S(H)+J/J} \supseteq (M/J)_{S(H)+J/J} \supseteq (N/J)_{S(H)+J/J}$$

are $R_{S(H)}$-submodules of $(T/J)_{S(H)+J/J}$. If

$$(M/J)_{S(H)+J/J} = (N/J)_{S(H)+J/J},$$

then $m + J \in (N/J)_{S(H)+J/J}$ for all $m \in M$, and hence,

$$m + J = \frac{n + J}{f + J}$$

for some $n \in N$ and $f \in S(H)$. Thus, there is a $g \in S(H)$ such that

$$(g + J)(f + J)(m + J) = (g + J)(n + J);$$

hence, $gfm \in N$. Thus, $m \in C(m) = C(m)C(fg) = C(mfg) \subseteq N$, and therefore, $M \subseteq N$, a contradiction. Hence,

$$(M/J)_{S(H)+J/J} \neq (N/J)_{S(H)+J/J}.$$

Note that $T_{S(H)}/JT_{S(H)} \cong (T/J)_{S(H)+J/J}$ as rings, and thus, as $R_{S(H)}$-modules. Hence, T/J is a graded R-module of finite h-length by the fact that $T_{S(H)}/JT_{S(H)}$ is an $R_{S(H)}$-module of finite length.

(3) This is an immediate consequence of (2). ∎

Corollary 3.5. *Let* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *be a graded Noetherian domain with* h-dim$(R) = 1$*. If* V *is a homogeneous* gr*-valuation overring of* R*, then* V *is discrete as a* gr*-valuation domain and* h-dim$(V) = 1$*.*

Proof. Since V is a homogeneous overring of R, V is also a G-graded integral domain. From Corollary 3.4, V is a graded Noetherian domain with h-dim$V \leq 1$. Thus, if M is the homogeneous maximal ideal of V, then M is finitely generated, and, since each generator of M is homogeneous, M must be principal. Thus, V is discrete as a gr-valuation ring. □

We conclude this section with some comments which are related to Lemma 3.1 and Proposition 3.3.

Remark 3.6.

(1) Let P be a homogeneous prime ideal of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, and assume that h-ht$(P) = 1$. While ht$(P) = 1$ when R is graded Noetherian by Lemma 3.1, in general, this is not true. For example, let (D, M) be a one dimensional quasi-local domain which is not a valuation domain, X an indeterminate over D and $R = D[X]$. Clearly, h-ht$(M[X]) = 1$ but ht$(M[X]) > 1$ since ht$(M[X]) = 1$ implies that D is a valuation domain [**8,** Theorem 19.15]. Thus, ht$(M[X]) >$ h-ht$(M[X])$.

(2) It is interesting to note that there is a graded integral domain R which has a homogeneous prime ideal P with $2 =$ h-ht$(P) <$ ht$(P) = 3$ [**10,** page 1579]. However, we do not know whether there is a graded Noetherian domain with a homogeneous prime ideal P with $2 =$ h-ht$(P) <$ ht(P).

(3) Let $R = D[X, X^{-1}]$ be the Laurent polynomial ring over an integral domain D. Then, R is a $\mathbb{Z}$-graded integral domain with $\deg(aX^n) = n$ for $0 \neq a \in D$ and an integer n. For $f = \sum a_i X^i \in R$, let $A_f = (\{a_i\})$ be the ideal of D generated by the coefficients of f. Let

$$S(H) = R \setminus \bigcup_{P \in \text{h-Spec}(R)} P.$$

Then, $S(H) = \{f \in R \mid C(f) = R\}$ by the proof of Proposition 3.3, and, since $X, X^{-1} \in R$, each homogeneous ideal of R is generated by a set of elements in D and $S(H) = \{f \in R \mid A_f = D\}$. Hence, $R_{S(H)} = D(X)$, the Nagata ring of D [**4,** Example 1]. In addition, R is graded Noetherian if and only if D is Noetherian, if and only if R is Noetherian, if and only if $R_{S(H)}$ is Noetherian.

4. Valuation overrings of a graded Noetherian domain. As in Section 3, G denotes the quotient group of a torsionless grading monoid Γ, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a (Γ-)graded integral domain and H is the set of nonzero homogeneous elements of R.

Lemma 4.1. *Let V be a homogeneous* gr*-valuation overring of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. Then, V is discrete as a* gr*-valuation ring if and only if PV_P is principal for all branched homogeneous prime ideals P of V.*

Proof.

($\Rightarrow$). This follows since $V_P = (V_{H \setminus P})_{P_{H \setminus P}}$.

($\Leftarrow$). Let P be a branched homogeneous prime ideal of V, and assume that $PV_P = fV_P$ for some $f \in P$. Then, since the homogeneous ideals of V is linearly ordered by set inclusion, $PV_P = \alpha V_P$ for some homogeneous component α of f. If $a \in P$ is homogeneous, then $a = \alpha g/h$ for some $h \in V \setminus P$ and $g \in V$. Then, $ah = \alpha g$, and since $h \notin P$, there exists a homogeneous component m of h such that $m \in H \setminus P$. Hence, $am = \alpha x$, where x is a homogeneous component of g. Thus, $a \in \alpha V_{H \setminus P}$, and since P is homogeneous, $P \subseteq \alpha V_{H \setminus P}$. Therefore, $PV_{H \setminus P} = \alpha V_{H \setminus P}$. □

Lemma 4.2. *Let V be a homogeneous* gr*-valuation overring of $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. If $f, g \in R - \{0\}$, then $(C(f)C(g))V = C(fg)V$.*

Proof. From [**13**] or [**3,** Lemma 1.1], $C(f)^{n+1}C(g) = C(f)^n C(fg)$ for some integer $n \geq 1$. Note that $C(f)$ is a finitely generated homogenous ideal of R and V is a gr-valuation overring of R. Hence, $C(f)V$ is a nonzero principal ideal of V, and thus, $(C(f)C(g))V = C(fg)V$. □

Let D be an integral domain with quotient field K, V a valuation overring of D, M the maximal ideal of V and $R = D[X, X^{-1}]$ the Laurent polynomial ring over D. Then, R is a $\mathbb{Z}$-graded integral domain (see Remark 3.6 (3)), $V[X,X^{-1}]$ is a homogeneous gr-valuation overring of R with homogeneous maximal ideal $M[X,X^{-1}]$ and $V[X]_{M[X]}$ $= V(X)$ is the trivial extension of V to $K(X)$ [**8,** Section 18]. Note that $C(h)V[X,X^{-1}] = A_hV[X,X^{-1}]$ for all $h \in R$; thus, it is easy to see that, if we let

$$W = \left\{ \frac{f}{g} \mid f,g \in R, g \neq 0, \text{ and } C(f)V[X,X^{-1}] \subseteq C(g)V[X,X^{-1}] \right\},$$

then $W = V[X,X^{-1}]_{M[X,X^{-1}]}$, and, since $V[X,X^{-1}]_{M[X,X^{-1}]} = V[X]_{M[X]}$, we have $W = V(X)$. Hence, W is a valuation domain, $\dim(W) = \dim(V) = \text{h-}\dim(V[X,X^{-1}])$, and W is a DVR if and only if V is a DVR, if and only if $V[X,X^{-1}]$ is discrete as a gr-valuation ring.

Theorem 4.3. *Let V be a homogeneous* gr-*valuation overring of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, M the homogeneous maximal ideal of V and $\widehat{V} = \{f/g \mid f,g \in R,\ g \neq 0,\ \text{and } C(f)V \subseteq C(g)V\}$.*

(1) *$\widehat{V}$ is a (well-defined) valuation overring of R and $\widehat{V} \cap R_H = V$.*
(2) *$\widehat{V} = V_M$ and $\dim(\widehat{V}) = h\text{-}\dim(V)$.*
(3) *$\widehat{V}$ is a* DVR *if and only if V is discrete as a* gr-*valuation ring.*

Proof.

(1) Let $0 \neq f,g,h,k \in R$ be such that $C(f)V \subseteq C(g)V$ and $f/g = h/k$. Then, $fk = gh$. Since $C(f)V \subseteq C(g)V$, we have

$$C(g)C(h)V = C(gh)V = C(fk)V = C(f)C(k)V \subseteq C(g)C(k)V$$

by Lemma 4.2. Hence, $C(h)V \subseteq C(k)V$. Thus, $\widehat{V}$ is well defined. Let $f/g, h/k \in \widehat{V}$. Then, $f/g + h/k = (fk+gh)/gk$ and

$$C(fk+gh)V \subseteq C(fk)V + C(gh)V \subseteq C(gk)V.$$

Thus, $f/g + h/k \in \widehat{V}$. Also, $f/g \cdot h/k = fh/gk$ and

$$C(fh)V = C(f)C(h)V \subseteq C(g)C(k)V = C(gk)V.$$

Thus, $f/g \cdot h/k \in \widehat{V}$. Let u be a nonzero element of the quotient field of R. Then, $u = f/g$ for some $f,g \in R$. Recall that the homogeneous ideals of V are linearly ordered by set inclusion; therefore, either $C(f)V \subseteq C(g)V$ or $C(g)V \subseteq C(f)V$. Hence, u or u^{-1} is in $\widehat{V}$. Thus, $\widehat{V}$ is a valuation domain.

Finally, we claim that $\widehat{V} \cap R_H = V$. Let $f \in V$. Since $V \subseteq R_H$, we can write $f = f_1/\alpha$, where $f_1 \in R$ and $\alpha \in H$. Hence, $C(f_1)V = C(\alpha f)V = C(\alpha)C(f)V \subseteq C(\alpha)V$. Thus, $f_1/\alpha = f \in \widehat{V}$. Hence, $V \subseteq \widehat{V} \cap R_H$. For the reverse containment, let $g/\beta \in \widehat{V} \cap R_H$, where $g \in R$ and $\beta \in H$. Then, $C(g)V \subseteq C(\beta)V = \beta V$, and thus, $C(g/\beta)V \subseteq V$. Thus, $g/\beta \in V$.

(2) Let $0 \neq f, g \in R$ be such that $f/g \in \widehat{V}$. Then, $C(f)V \subseteq C(g)V$, and thus, if a and b are homogeneous components of f, g, respectively, such that $C(f)V = aV$ and $C(g)V = bV$, then

$$fV_M = aV_M \subseteq bV_M = gV_M.$$

Hence, $f/g \in V_M$, and thus, $V \subseteq \widehat{V} \subseteq V_M$. Note that $\widehat{V}_{V\setminus M} = \widehat{V}$ since $C(f)V = V$ for all $f \in V \setminus M$; thus, $V_M \subseteq \widehat{V} \subseteq V_M$. Therefore, $\widehat{V} = V_M$.

Next, note that $\dim(\widehat{V}) = \dim(V_M) \geq \text{h-}\dim(V)$; thus, to prove the equality of $\dim(\widehat{V}) = \text{h-}\dim(V)$, it suffices to show that, if Q is a nonzero prime ideal of V with $Q \subseteq M$, then Q is homogeneous. Let

$$f = x_{\alpha_1} + \cdots + x_{\alpha_n} \in Q,$$

where each x_{α_i} is a homogeneous component of f. Since V is a gr-valuation ring, there is an x_{α_k} such that $x_{\alpha_i} \in x_{\alpha_k}V$ for all x_{α_i}. Hence, $f/x_{\alpha_k} \in V \setminus M$, and thus, $x_{\alpha_k} \cdot f/x_{\alpha_k} = f \in Q$ implies $x_{\alpha_k} \in Q$. Thus, $x_{\alpha_i} \in x_{\alpha_k}V \subseteq Q$, whence Q is homogeneous.

(3) This follows directly from (2) and Lemma 4.1. □

Lemma 4.4. *Let $\{P_\lambda\}$ be a chain of homogeneous prime ideals of $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$. Then, there is a homogeneous* gr*-valuation overring of R with a set of homogeneous prime ideals that contract to $\{P_\lambda\}$.*

Proof. From **[11,** Theorem], there is a valuation overring W with a chain $\{N_\lambda\}$ of prime ideals such that $N_\lambda \cap R = P_\lambda$. Let

$$V = \sum_{\alpha\in G} (W \cap (R_H)_\alpha)$$

and

$$Q_\lambda = \sum_{\alpha \in G} (N_\lambda \cap (R_H)_\alpha).$$

Then, it is routine to check that V is a homogeneous gr-valuation overring of R and $\{Q_\lambda\}$ is a chain of homogeneous prime ideals of V such that $Q_\lambda \cap R = P_\lambda$. □

Next, we give the main result of this section, which is a graded Noetherian domain analogue of [**6,** Theorem 1], and its proof heavily depends on that of [**6,** Theorem 1].

Theorem 4.5. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded Noetherian domain, and let*

$$(0) = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

be a saturated chain of homogeneous prime ideals of R. If V is a homogeneous gr*-valuation overring of R with a chain $\{Q_\alpha\}_{\alpha \in \Lambda}$ of homogeneous prime ideals such that*

$$\{Q_\alpha \cap R\} = \{P_i\}_{i=0}^n$$

as in Lemma 4.4*, then*

(1) $\{Q_\alpha\}_{\alpha \in \Lambda} = \{(0) = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n\}$;
(2) $V_{H \setminus Q_n}$ *is discrete as a* gr*-valuation ring and* h-$\dim(V_{H \setminus Q_n}) = n$;
(3) *the homogeneous quotient field of V/Q_i is finite over the homogeneous quotient field of R/P_i for $i = 1, \ldots, n$.*

Proof. We prove it by induction on n. First, assume $n = 1$. Set

$$Q = \bigcup_{\alpha \in \Lambda} Q_\alpha.$$

Clearly, $Q \cap R = P_1$ and $V_{H \setminus Q}$ is a homogeneous gr-valuation overring of $R_{H \setminus P_1}$ with a chain $\{Q_\alpha V_{H \setminus Q}\}_{\alpha \in \Lambda}$ of homogeneous prime ideals such that

$$\{Q_\alpha V_{H \setminus Q} \cap R_{H \setminus P_1}\}_{\alpha \in \Lambda} = \{(0) \subsetneq P_1 R_{H \setminus P_1}\}.$$

Since the given chain is saturated, $R_{H \setminus P_1}$ is a graded Noetherian domain of h-dimension one. Thus, by Corollary 3.5, $V_{H \setminus Q}$ is discrete as a gr-valuation ring and $\text{h-}\dim(V_{H\backslash Q}) = 1$; thus,

$$\{Q_\alpha V_{H\backslash Q}\}_{\alpha\in\Lambda} = \{(0) \subsetneq QV_{H\backslash Q}\}.$$

Moreover, $V_{H\backslash Q}/QV_{H\backslash Q}$ and $R_{H\backslash P_1}/P_1R_{H\backslash P_1}$ are isomorphic to the homogeneous quotient fields of V/Q and R/P_1, respectively. Hence, we may assume that R is a graded Noetherian domain such that $\text{h-}\dim(R) = 1$, P_1 is the unique nonzero homogeneous prime ideal of R and V is discrete as a gr-valuation ring with $\text{h-}\dim(V) = 1$ and homogeneous maximal ideal Q. In addition, by Corollary 3.4 (3), V/Q is finite over R/P_1. Note that R/P_1 (respectively, V/Q) is the homogeneous quotient field of R/P_1 (respectively, V/Q) since P_1 (respectively, Q) is a homogeneous maximal ideal of R (respectively, V).

We next assume that the result is true for all saturated chains of homogeneous prime ideals of length $n-1$. Let

$$(0) = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

be a saturated chain of homogeneous prime ideals in R, and let V be a homogeneous gr-valuation overring of R with a chain $\{Q_\alpha\}_{\alpha\in\Lambda}$ of homogeneous prime ideals such that $\{Q_\alpha \cap R\}_{\alpha\in\Lambda} = \{P_i\}_{i=0}^n$. From the same argument as in the case $n = 1$, we may assume that R is a graded Noetherian domain with a unique homogeneous maximal ideal P_n and $\bigcup_{\alpha\in\Lambda} Q_\alpha$ is the homogeneous maximal ideal of V. By the induction hypothesis,

$$\begin{aligned}\{Q_\alpha\}_{\alpha\in\Lambda} = \{&0 = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_{n-1}\}\\ &\cup \{Q_\alpha \mid Q_{n-1} \subsetneq Q_\alpha \text{ and } Q_\alpha \cap R = P_n\}\end{aligned}$$

and the homogeneous quotient field of V/Q_i is finite over that of R/P_i for $i = 1, 2, \ldots, n-1$. Note that R/P_{n-1} is a graded Noetherian domain, $\text{h-}\dim(R/P_{n-1}) = 1$, and the homogeneous quotient field of V/Q_{n-1} is finite over the homogeneous quotient field of R/P_{n-1}; thus, by Corollary 3.4 (1), V/Q_{n-1} is a graded Noetherian domain of h-dimension one. Thus, V/Q_{n-1} is discrete as a gr-valuation ring of h-dimension one. Therefore, V is discrete as a gr-valuation ring and $\text{h-}\dim V = n$. Moreover, since $\text{h-}\dim(V/Q_{n-1}) = 1$, we have

$$|\{Q_\alpha \mid Q_{n-1} \subsetneq Q_\alpha \text{ and } Q_\alpha \cap R = P_n\}| = 1;$$

thus, let such a $Q_\alpha = Q_n$. By Corollary 3.4 (3),

$$V/Q_n \cong (V/Q_{n-1})/(Q_n/Q_{n-1})$$

is finite over $R/P_n \cong (R/P_{n-1})/(P_n/P_{n-1})$. $\square$

Corollary 4.6. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Noetherian domain, and let $\{P_i\}$ be a chain of homogeneous prime ideals of R. Then, there exists a discrete valuation overring of R whose prime ideals contract to $\{P_i\}$.*

Proof. Since R is a graded Noetherian domain, there exists a saturated chain $\{P_\beta\}$ of homogeneous prime ideals of R containing $\{P_i\}$. Hence, by Theorems 4.3 and 4.5, there exists a discrete valuation overring of R whose prime ideals contract to $\{P_\beta\}$, and thus to $\{P_i\}$. $\square$

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Noetherian domain,

$$(0) = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

a chain of homogeneous prime ideals of R, and V a discrete valuation overring of R whose prime ideals contract to $\{P_i\}$ (Corollary 4.6). It is known that, if R is Noetherian, then we can choose V as a rank n DVR [**5**, Theorem] even though the given chain is not saturated, while we do not know if the dimension of V can be n when R is not Noetherian.

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REFERENCES

1. D.D. Anderson and D.F. Anderson, *Divisibility properties of graded domains*, Canad. J. Math. **34** (1982), 196–215.

2. D.F. Anderson and G.W. Chang, *Homogeneous splitting sets of a graded integral domain*, J. Alg. **288** (2005), 527–544.

3. ______, *Graded integral domains and Nagata rings*, J. Alg. **387** (2013), 169–184.

4. ______, *Graded integral domains whose nonzero homogeneous ideals are invertible*, Int. J. Alg. Comp. **26** (2016), 1361–1368.

5. P.J. Cahen, E.G. Houston and T.G. Lucas, *Discrete valuation overrings of Noetherian domains*, Proc. Amer. Math. Soc. **124** (1996), 1719–1721.

6. G.W. Chang and D.Y. Oh, *Valuation overrings of a Noetherian domain*, J. Pure Appl. Alg. **218** (2014), 1081–1083.

7. R. Gilmer, *Commutative semigroup rings*, University of Chicago Press, Chicago, 1984.

8. ———, *Multiplicative ideal theory*, Queen's Papers Pure Appl. Math. **90**, Ontario, 1992.

9. R. Gilmer and W. Heinzer, *Primary ideals with finitely generated radical in a commutative ring*, Manuscr. Math. **78** (1993), 201–221.

10. W. Heinzer and M. Roitman, *The homogeneous spectrum of a graded commutative ring*, Proc. Amer. Math. Soc. **130** (2002), 1573–1580.

11. B.G. Kang and D.Y. Oh, *Lifting up an infinite chain of prime ideals to a valuation ring*, Proc. Amer. Math. Soc. **126** (1998), 645–646.

12. H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

13. D.G. Northcott, *A generalization of a theorem on the content of polynomials*, Proc. Cambridge Philos. Soc. **55** (1959), 282–288.

14. ———, *Lessons on rings, modules, and multiplicities*, Cambridge University Press, Cambridge, 1968.

15. C.H. Park and M.H. Park, *Integral closure of a graded Noetherian domain*, J. Korean Math. Soc. **48** (2011), 449–464.

16. M.H. Park, *Integral closure of graded integral domains*, Comm. Alg. **35** (2007), 3965–3978.

17. D.E. Rush, *Noetherian properties in monoid rings*, J. Pure Appl. Alg. **185** (2003), 259–278.

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