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SYSTEMS OF PARAMETERS AND THE COHEN-MACAULAY PROPERTY

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ABSTRACT. Let R be a commutative, Noetherian, local ring and M a finitely generated R-module. Consider the module of homomorphisms $\mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ where $\mathfrak{b} \subseteq \mathfrak{a}$ are parameter ideals of M. When $M = R$ and R is Cohen-Macaulay, Rees showed that this module of homomorphisms is isomorphic to $R/\mathfrak{a}$, and in particular, a free module over $R/\mathfrak{a}$ of rank one. In this work, we study the structure of such modules of homomorphisms for a not necessarily Cohen-Macaulay R-module M.

1. Introduction. Let R be a commutative, Noetherian, local ring. This work concerns the module of homomorphisms $\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$, where $\mathfrak{a}$ and $\mathfrak{b}$ are parameter ideals of R with $\mathfrak{b} \subseteq \mathfrak{a}$.

An immediate consequence of a result by Rees [**4**] is that, when R is Cohen-Macaulay, this module of homomorphisms is isomorphic to $R/\mathfrak{a}$. In particular, as an $R/\mathfrak{a}$-module, it is free of rank one. The focus of this work is to study the structure of this module of homomorphisms when R is not Cohen-Macaulay. Our main results identify circumstances under which it is decomposable and not free.

When R has dimension one and depth zero and $\mathfrak{a}$ and $\mathfrak{b}$ are in sufficiently high powers of the maximal ideal, we prove that $\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ is neither indecomposable nor free as an $R/\mathfrak{a}$-module.

We can extend the result regarding decomposability both to modules and to higher dimensions. In particular, for M a nonzero, finitely generated R-module, we consider the module $\mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$, where $\mathfrak{b} \subseteq \mathfrak{a}$ are parameter ideals of M. When M is not Cohen-Macaulay, we

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can show that the module of homomorphisms decomposes for any parameter ideal $\mathfrak{a}$ and for $\mathfrak{b}$ chosen to be generated by suitable powers of any system of parameters generating $\mathfrak{a}$. This result generalizes recent work of Bahmanpour and Naghipour [**1**] in the case where $M = R$. Specifically, when R is not Cohen-Macaulay, they showed that there exist some parameter ideals $\mathfrak{b} \subseteq \mathfrak{a}$ of R for which $\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ is acyclic.

2. Preliminaries. Throughout, R denotes a commutative Noetherian ring with unique maximal ideal $\mathfrak{m}$, and M denotes a finitely generated R-module. For any R-module L and any ideal $I \subseteq R$, the *I-torsion submodule* of L is

$$\Gamma_I(L) = \bigcup_{n=1}^{\infty} (0 :_L I^n).$$

A *system of parameters* of M is a set of $d = \dim M$ elements generating an ideal $\mathfrak{a}$ such that $M/\mathfrak{a}M$ has finite length. An ideal $\mathfrak{a}$ generated by a system of parameters is called a *parameter ideal*. We begin by reviewing the consequence of Rees's result in the Cohen-Macaulay case.

Remark 2.1. If M is a Cohen-Macaulay R-module of dimension d and $\mathfrak{b} \subseteq \mathfrak{a}$ are parameter ideals of M, then

$$\operatorname{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M) \cong M/\mathfrak{a}M.$$

In particular, when $M = R$, this is a free $R/\mathfrak{a}$-module of rank one, and hence, indecomposable.

Indeed, the elements of a system of parameters of M form an M-regular sequence. The above isomorphism is deduced from Rees's theorem [**2,** Lemma 1.2.4]:

$$\operatorname{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M) \cong \operatorname{Ext}^d_R(R/\mathfrak{a}, M) \cong \operatorname{Hom}_R(R/\mathfrak{a}, M/\mathfrak{a}M) \cong M/\mathfrak{a}M.$$

We now recall some well-known results.

Fact 2.2. Let $a_1, \ldots, a_d$ be elements of R and $\overline{a_1}, \ldots, \overline{a_d}$ their images in $R/\operatorname{ann}(M)$. Then $a_1, \ldots, a_d$ is a system of parameters of M if and only if $\overline{a_1}, \ldots, \overline{a_d}$ is a system of parameters of the ring $R/\operatorname{ann}(M)$.

The integer n appearing in the next result plays a key role in the main results. We include a proof for completeness.

Lemma 2.3. *Let $(R,\mathfrak{m})$ be a local ring and M a finitely generated R-module. There exists an integer n such that $\mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M) = (0)$.*

Proof. Since $\Gamma_{\mathfrak{m}}(M)$ is Artinian, the descending chain of submodules

$$(\mathfrak{m}M \cap \Gamma_{\mathfrak{m}}(M)) \supseteq (\mathfrak{m}^2 M \cap \Gamma_{\mathfrak{m}}(M)) \supseteq \cdots$$

must stabilize, that is, there is some $n \in \mathbb{N}$ such that

$$\mathfrak{m}^{n+i} M \cap \Gamma_{\mathfrak{m}}(M) = \mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M)$$

for all integers $i \geq 0$. Thus,

$$\mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M) = \bigcap_{i \geq n} (\mathfrak{m}^i M \cap \Gamma_{\mathfrak{m}}(M)) \subseteq \bigcap_{i \geq n} \mathfrak{m}^i M = (0).$$

The last equality is due to the Krull intersection theorem. □

Remark 2.4. In fact, given any finite-length submodule $L \subseteq M$, we have $\mathfrak{m}^n M \cap L = (0)$ where n is the integer of Lemma 2.3.

The next result is in the spirit of [**3,** Proposition 4.7.13]. We include a proof in order to obtain specific bounds on the powers of a in this special case.

Proposition 2.5. *Let A be any commutative ring, L an A-module and $a, b \in A$. Then, for arbitrary positive integers $p \leq q \leq r$, we have the equality*:

$$(ba^r L : a^p) = a^{r-q}(ba^q L : a^p) + (0 :_L a^p).$$

Proof. First, let $x \in (ba^r L : a^p)$. Then, $a^p x = ba^r y$ for some y in L. Now,

$$a^p(x - ba^{r-p} y) = 0,$$

so that $x - ba^{r-p} y \in (0 :_L a^p)$. Additionally,

$$ba^{r-p} y = a^{r-q} \cdot ba^{q-p} y \in a^{r-q}(ba^q L : a^p).$$

We now have

$$x = ba^{r-p} y + (x - ba^{r-p} y) \in a^{r-q}(ba^q L : a^p) + (0 :_L a^p).$$

The other inclusion is clear. □

The next result will be applied in Section 4 in the situation where $I = (a_1, \ldots, a_d)$ is a parameter ideal and J is of the form $(a_1^n, a_2, \ldots, a_d)$ for a positive integer n.

Lemma 2.6. *Let A be a Noetherian ring, $J \subseteq I$ proper ideals of A with $\sqrt{I} = \sqrt{J}$ and N an A-module. If $\operatorname{Hom}_A(A/J, N)$ is decomposable, then so is $\operatorname{Hom}_A(A/I, N)$.*

Proof. Suppose that $\operatorname{Hom}_A(A/J, N) = X \oplus Y$, where X and Y are nonzero A-modules. There are isomorphisms:

$$\begin{aligned} \operatorname{Hom}_A(A/I, N) &\cong \operatorname{Hom}_A((A/I) \otimes_A (A/J), N) \\ &\cong \operatorname{Hom}_A(A/I, \operatorname{Hom}_A(A/J, N)) \\ &\cong \operatorname{Hom}_A(A/I, X) \oplus \operatorname{Hom}_A(A/I, Y). \end{aligned}$$

By symmetry, it suffices to show that $\operatorname{Hom}_A(A/I, X) \neq 0$. It is clear that $JX = (0)$, since $X \subseteq \operatorname{Hom}_A(A/J, N)$. Choose $\mathfrak{p} \in \operatorname{Ass}_A X$, and note that $J \subseteq \mathfrak{p}$. Since $\sqrt{J} = \sqrt{I}$, we have $I \subseteq \mathfrak{p}$; thus, there are maps:

$$A/I \twoheadrightarrow A/\mathfrak{p} \hookrightarrow X.$$

The composition of these maps is nonzero, and thus, $\operatorname{Hom}_A(A/I, X) \neq (0)$ as desired. □

Remark 2.7.

(i) The hypothesis that $J \subseteq I$ is necessary. For any pair of ideals I, J, if we let $N = R/I$, then

$$\operatorname{Hom}_R(R/I, N) = \operatorname{Hom}_R(R/I, R/I) \cong R/I$$

is indecomposable. However, it is possible that $\operatorname{Hom}_R(R/J, N)$ is decomposable. For instance, let k be a field, and consider the ring $R = k[\![x, y]\!]/(x^2, xy)$ along with the ideals $I = (y^2)$, $J = (y)$ of R. Here, $\sqrt{I} = \sqrt{J} = (x, y)$, but $J \not\subseteq I$. However,

$$\operatorname{Hom}_R(R/J, R/I) = \operatorname{Hom}_R(R/(y), R/(y^2)) \cong \frac{(y)}{(y^2)} \oplus \frac{(x, y^2)}{(y^2)}.$$

(ii) The hypothesis that $\sqrt{I} = \sqrt{J}$ is also necessary. For example, let k be a field, $R = k[\![x, y, z]\!]/(x^2, xyz)$, $N = R/(y^2)$, $I = (y, z)$ and

$J = (y)$. We have $J \subseteq I$; however, $\sqrt{J} = (x, y) \subsetneq (x, y, z) = \sqrt{I}$. Using this notation, we obtain

$$\mathrm{Hom}_R(R/J, N) \cong \frac{(y^2) : y}{(y^2)} = \frac{(y)}{(y^2)} \oplus \frac{(xz, y^2)}{(y^2)}$$

decomposes, but

$$\mathrm{Hom}_R(R/I, N) \cong \frac{(y^2) : (y, z)}{(y^2)} = \frac{(xy, y^2)}{(y^2)}$$

is cyclic, and hence, indecomposable.

This final result will be used in Section 4 in an induction argument.

Lemma 2.8. *Let R be a local ring and M an R-module of dimension $d \geq 2$. If M is not Cohen-Macaulay, then for any system of parameters $a_1, \ldots, a_d$ of M, there exist positive integers i and s such that $M/a_i^s M$ is not Cohen-Macaulay.*

Proof. If some a_i is M-regular, then $M/a_i M$ is not Cohen-Macaulay; thus, we may assume that each a_i is a zero-divisor on M. Suppose, by way of contradiction, that $M/a_1^s M$ is Cohen-Macaulay for each $s \geq 1$. Then, $a_2, \ldots, a_d$ is a regular sequence on $M/a_1^s M$ for all integers $s \geq 1$. In particular, a_2 is $M/a_1^s M$-regular for all integers $s \geq 1$. We claim that this implies a_2 is M-regular, which is a contradiction. Indeed, suppose that $a_2 m = 0$ for some $m \in M$. Then, $a_2 \overline{m} = 0$ in $M/a_1^s M$ for all integers $s \geq 1$, so that $m \in a_1^s M$ for all integers $s \geq 1$. By the Krull intersection theorem, we have $m = 0$, implying that a_2 is M-regular, a contradiction. $\square$

The next example shows that, even when all of the parameters are zero divisors, M may have positive depth, and M/aM may be Cohen-Macaulay.

Example 2.9. Let k be a field. Consider the ring $R = k[\![x, y, z]\!]/(x^2, xyz)$ of dimension two and depth one along with the system of parameters y, z. Both y and z are zero-divisors in R, and both $R/(y)$ and $R/(z)$ are Cohen-Macaulay rings of dimension one.

3. Dimension one. We begin with results on modules of dimension one and depth zero since we are able to obtain stronger bounds in this case. We show that $\operatorname{Hom}_R(R/(a), M/bM)$ is decomposable if the parameter b is chosen to be in a sufficiently high power of the ideal generated by an arbitrary parameter a.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a nonzero finitely generated R-module of dimension one and depth zero. Choose an integer n such that $\mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M) = (0)$. For any parameter a of M, and any parameter b of M with $b \in (a^{n+1})$, the following R-module is decomposable*:

$$\operatorname{Hom}_R(R/(a), M/bM).$$

Remark 3.2. The integer n in the statement exists by Lemma 2.3. Note that $n \geq 1$ since $\Gamma_{\mathfrak{m}}(M) \neq (0)$.

Proof of Theorem 3.1. Set $S := R/\operatorname{ann}(M)$, and let $\overline{(\_)}$ denote the image in S. In light of Fact 2.2, $\overline{a}$ and $\overline{b}$ are parameters of S. Moreover, there is an R-module isomorphism

$$\operatorname{Hom}_S(S/\overline{a}S, M/\overline{b}M) \cong \operatorname{Hom}_R(R/aR, M/bM).$$

By replacing R with S, we may assume that M is faithful as an R-module.

Write $b = ca^{n+1}$ with $c \in R$. Since M is faithful, we have $\sqrt{(a)} = \mathfrak{m}$, and thus,

$$(0 :_M a) \subseteq \Gamma_{(a)}(M) = \Gamma_{\mathfrak{m}}(M).$$

Therefore, we know that

$$(0 :_M a) \cap ca^n M \subseteq \Gamma_{\mathfrak{m}}(M) \cap \mathfrak{m}^n M = (0). \tag{3.1}$$

From Proposition 2.5, we have

$$(ca^{n+1}M : a) = a^n(caM : a) + (0 :_M a). \tag{3.2}$$

We now claim that

$$a^n(caM : a) = ca^n M.$$

Indeed, it is clear that elements of $ca^n M$ are also elements of $a^n(caM : a)$. For the reverse inclusion, let $x \in a^n(caM : a)$ and write $x = a^n m$

for some $m \in (caM : a)$. Thus, we have $am = cam'$ for some $m' \in M$. Then,

$$x = a^n m = a^{n-1} \cdot am = ca^n m' \in ca^n M.$$

Equation (3.2) now becomes

$$(bM : a) = ca^n M + (0 :_M a). \tag{3.3}$$

Since $(0 :_M a) \cap ca^n M = (0)$, then the equality

$$ca^{n+1} M = ca^n M \cap \big[(0 :_M a) + ca^{n+1} M\big] \tag{3.4}$$

follows by modular law. Now, there are isomorphisms

$$\begin{aligned} \operatorname{Hom}_R(R/aR, M/bM) &\cong \frac{(bM : a)}{bM} \\ &\cong \frac{ca^n M + (0 :_M a)}{ca^{n+1} M} && \text{by (3.3)} \\ &\cong \frac{ca^n M}{ca^{n+1} M} \oplus \frac{(0 :_M a) + ca^{n+1} M}{ca^{n+1} M} && \text{by (3.4).} \end{aligned}$$

All that remains to prove is that both summands are nonzero.

If the summand on the left were zero, then $ca^n M = (0)$ by Nakayama's lemma, a contradiction since $ca^{n+1} = b$ is a parameter of M.

If the summand on the right were zero, then

$$(0 :_M a) \subseteq ca^{n+1} M.$$

By equation (3.1), we have

$$(0 :_M a) = (0 :_M a) \cap ca^{n+1} M = (0).$$

This is also a contradiction since $\operatorname{depth}_R M = 0$. Thus, $\operatorname{Hom}_R(R/aR, M/bM)$ is decomposable, as desired. □

When R is a Cohen-Macaulay ring, we know from Remark 2.1 that the $R/\mathfrak{a}$-module $\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b}) \cong R/\mathfrak{a}$ is not only indecomposable, but also free. When R is one-dimensional and not Cohen-Macaulay, we can prove that, in addition to being decomposable, this module will be non-free if the parameters are chosen to be in sufficiently high powers of the maximal ideal.

Theorem 3.3. *Let* $(R, \mathfrak{m})$ *be a local ring of dimension one and depth zero, and* n *an integer such that* $\mathfrak{m}^n \cap \Gamma_{\mathfrak{m}}(R) = (0)$. *For any parameter* $a \in \mathfrak{m}^n$ *and any parameter* $b \in (a^2)$, *the* $R/(a)$*-module*

$$\mathrm{Hom}_R(R/(a), R/(b))$$

is decomposable and has a non-free summand.

Remark 3.4. Again, the integer n in the statement exists by Lemma 2.3 and must be positive since $\Gamma_{\mathfrak{m}}(R) \neq (0)$.

Proof of Theorem 3.3. We will first prove that the module decomposes. Both the proof of this fact and the decomposition obtained are similar to those found in the proof of Theorem 3.1. Write $I = \Gamma_{\mathfrak{m}}(R)$. For any $x \in \mathfrak{m}^n$, we know that $(x) \cap I = (0)$, and hence, $xI = (0)$. If $x \in \mathfrak{m}^n$ is also a parameter, then we know that $\Gamma_{(x)}(R) = I$, and $(0 : x) = I$ as well. Indeed, $\sqrt{(x)} = \mathfrak{m}$, and since $xI = 0$, we have

$$I \subseteq (0 : x) \subseteq \Gamma_{(x)}(R) = I.$$

Let $a \in \mathfrak{m}^n$ and $b \in (a^2)$ be parameters, and write $b = ca^2$. Applying Proposition 2.5 with $p = q = 1$ and $r = 2$, we obtain the equality

$$((ca^2) : a) = a((ca) : a) + (0 : a). \tag{3.5}$$

We now note that $a((ca) : a) = (ca)$. We may thus rewrite equation (3.5) as:

$$((b) : a) = (ca) + I. \tag{3.6}$$

Since $(ca) \cap I = (0)$, then the equality

$$(ca^2) = (ca) \cap \left[I + (ca^2)\right] \tag{3.7}$$

holds by modular law. Moreover, there are isomorphisms of $R/(a)$-modules:

$$\begin{aligned} \mathrm{Hom}_R(R/(a), R/(b)) &\cong \frac{((b) : a)}{(b)} \\ &\cong \frac{(ca) + I}{(ca^2)} && \text{by (3.6)} \\ &\cong \frac{(ca)}{(ca^2)} \oplus \frac{I + (ca^2)}{(ca^2)} && \text{by (3.7).} \end{aligned}$$

Next, we show that both summands are nonzero.

If the summand on the left were zero, then Nakayama's lemma would imply that $ca = 0$, a contradiction since $ca^2 = b$ is a parameter, and hence, nonzero.

If the summand on the right were zero, then $I \subseteq (ca^2)$ so that

$$I = I \cap (ca^2) \subseteq I \cap (a) = (0),$$

a contradiction as the depth of R is zero.

We now show that the summand on the left, that is, $(ca)/(ca^2)$, is not a free $R/(a)$-module. Toward that end, recall that $I \cap (a) = (0)$; however, $I \neq (0)$ so we can choose an element $y \in I \setminus (a)$. Since $aI = (0)$, $\overline{y}$ is a nonzero element of $R/(a)$ that annihilates $(ca)/(ca^2)$, and hence, $(ca)/(ca^2)$ cannot be free as an $R/(a)$-module. □

4. Higher dimensions. In higher dimensions, we can also prove a decomposition theorem. However, Example 5.4 shows that Theorem 3.1 is not strong enough to use the induction technique in Theorem 4.1 to prove that there is an integer N such that $\operatorname{Hom}_R(R/\mathfrak{a}, R/(a_1^{n_1}, \dots, a_d^{n_d}))$ decomposes for all $n_i \geq N$.

Theorem 4.1. *Let R be a local ring and M a finitely generated R-module of dimension d. If M is not Cohen-Macaulay, then, for any system of parameters $\mathbf{a} = a_1, \dots, a_d$ of M, there exist positive integers $n_1, \dots, n_d$ such that the following R-module is decomposable*:

$$\operatorname{Hom}_R(R/(\mathbf{a}), M/(a_1^{n_1}, \dots, a_d^{n_d})M).$$

Proof. As in the proof of Theorem 3.1, we may reduce to the case where M is a faithful module. We proceed by induction on d, the case $d = 1$ being covered by Theorem 3.1.

Assume, now, that $d \geq 2$. From Lemma 2.8, we can find some positive integer $i \leq d$ and a positive integer n_i such that $M/a_i^{n_i}M$ is not Cohen-Macaulay. We may harmlessly assume $i = 1$. Set

$$\overline{R} := R/(a_1^{n_1}), \qquad \overline{M} := M/a_1^{n_1}M, \qquad \overline{\mathfrak{a}} := (\overline{a_2}, \dots, \overline{a_d}).$$

Then, $\overline{\mathfrak{a}}$ is a parameter ideal of $\overline{M}$. Since $\overline{M}$ has dimension $d - 1$, by induction, there are natural numbers $n_2, \dots, n_d$ such that the R-module

$$U := \operatorname{Hom}_{\bar{R}}(\overline{R}/\overline{\mathfrak{a}}, \overline{M}/(\overline{a_2}^{n_2}, \dots, \overline{a_d}^{n_d})\overline{M})$$

is decomposable. Since there is an isomorphism of R-modules

$$U \cong \operatorname{Hom}_R(R/(a_1^{n_1}, a_2, \dots, a_d), M/(a_1^{n_1}, a_2^{n_2}, \dots, a_d^{n_d})M),$$

then applying Lemma 2.6 gives the desired decomposition. □

The result below is a version of Theorem 3.3 for rings of arbitrary dimension.

Theorem 4.2. *Let R be a local ring of dimension d. If R is not Cohen-Macaulay, then, for any system of parameters $a_1, \dots, a_d$ of R, there exist positive integers $n_1, \dots, n_d, N_1, \dots, N_d$ with $N_i \geq n_i$ for $i = 1, \dots, d$ such that the $R/(a_1^{n_1}, \dots, a_d^{n_d})$-module*

$$\operatorname{Hom}_R(R/(a_1^{n_1}, \dots, a_d^{n_d}), R/(a_1^{N_1}, \dots, a_d^{N_d}))$$

is decomposable and has a non-free summand.

Proof. We proceed by induction on d. If $d = 1$, then the result follows from Theorem 3.3 by choosing n_1 to be the n from Theorem 3.3 and $N_1 = 2n_1$.

Now suppose that $d \geq 2$. From Lemma 2.8, we can find integers i and n_i such that $R/(a_i^{n_i})$ is not Cohen-Macaulay. We may harmlessly assume $i = 1$. Set $S := R/(a_1^{n_1})$, and let $\overline{(\ )}$ denote the image in S. Then, $\overline{a_2}, \dots, \overline{a_d}$ is a system of parameters of S and, by induction, there exist integers $n_2, \dots, n_d, N_2, \dots, N_d$ such that the $S/(\overline{a_2}^{n_2}, \dots, \overline{a_d}^{n_d})$-module

$$U := \operatorname{Hom}_S(S/(\overline{a_2}^{n_2}, \dots, \overline{a_d}^{n_d}), S/(\overline{a_2}^{N_2}, \dots, \overline{a_d}^{N_d}))$$

decomposes and has a non-free summand. Note that

$$S/(\overline{a_2}^{n_2}, \dots, \overline{a_d}^{n_d}) \cong R/(a_1^{n_1}, \dots, a_d^{n_d}).$$

Setting $N_1 = n_1$, we then have

$$U \cong \operatorname{Hom}_R(R/(a_1^{n_1}, \dots, a_d^{n_d}), R/(a_1^{N_1}, \dots, a_d^{N_d})),$$

and this gives the desired decomposition and non-free summand. □

5. Examples. In this section, we focus on examples. In particular, we investigate the structure of the $R/\mathfrak{a}$-module $\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ for concrete examples of R, $\mathfrak{a}$, and $\mathfrak{b}$.

We take $M = R$ in Theorem 4.1. If we take $n_i = 1$ for each i, then

$$\mathrm{Hom}_R(R/(a_1, \dots, a_d), R/(a_1^{n_1}, \dots, a_d^{n_d})) \cong R/(a_1, \dots, a_d) \tag{5.1}$$

is a free $R/(a_1, \dots, a_d)$-module of rank one. Our first example shows that equation (5.1) sometimes holds even when R is not Cohen-Macaulay and at least one of the n_i's is greater than one.

Example 5.1. Let k be a field. Consider the parameter y of the ring $R = k[\![x, y]\!]/(x^2, xy^2)$. Then, we have

$$\mathrm{Hom}_R(R/(y), R/(y^2)) \cong \frac{(y^2) :_R y}{(y^2)} = \frac{(y)}{(y^2)} \cong R/(y).$$

The next example shows that the bound in Theorem 3.1 is close to being optimal.

Example 5.2. Let k be a field, $m \geq 2$ an integer and $R = k[\![x, y]\!]/(x^2, xy^m)$. Then, y is a parameter of R, and, setting

$$U_t := \mathrm{Hom}_R(R/(y), R/(y^t)),$$

we see that U_t is cyclic (and hence, indecomposable) for $t \leq m$ and decomposable for $t \geq m+1$. Since $\mathfrak{m}^i \cap \Gamma_{\mathfrak{m}}(R) = (0)$ precisely when $i \geq m+1$, Theorem 3.1 gives that U_t is decomposable for $t \geq m+2$. Thus, the bound obtained in Theorem 3.1 is, at worst, one away from a tight bound.

The next example shows that the module $\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ can neither be cyclic nor decomposable and also that the bound in Theorem 3.1 may be quite far from optimal.

Example 5.3. Let k be a field, and consider the parameter y^2 of $R = k[\![x, y]\!]/(x^2, xy^m)$ for $m \geq 3$. Then,

$$U_t := \mathrm{Hom}_R(R/(y^2), R/(y^t))$$

is

$$\begin{cases} \text{cyclic,} & \text{if } t < m+1, \\ \text{indecomposable, but not cyclic,} & \text{if } t = m+1, \text{ and} \\ \text{decomposable,} & \text{if } t > m+1. \end{cases}$$

However, Theorem 3.1 only predicts that U_t decomposes for $t \geq 2m+4$ since $\mathfrak{m}^n \cap \Gamma_{\mathfrak{m}}(R) \neq 0$ for $n < m+1$.

In order to show the claim that U_t is indecomposable for $t = m+1$, the key step is showing that $\operatorname{End}_R(U_t)$ is a non-commutative local ring. This can be done by identifying which k-linear endomorphisms of U_t are also R-linear, and then showing that those with non-invertible matrix representations form a two-sided ideal.

Theorems 3.1 and 3.3, which give bounds on the powers necessary for making $\operatorname{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ decomposable and non-free, apply only in dimension one. However, examples seem to indicate that the $R/\mathfrak{a}$-module

$$\operatorname{Hom}_R(R/\mathfrak{a}, R/(a_1^{n_1}, \ldots, a_d^{n_d}))$$

is neither free nor indecomposable if the n_i are large enough. One such example is explained next.

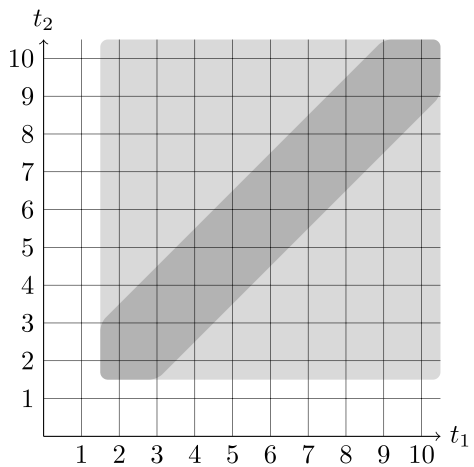

FIGURE 1. The modules corresponding to lattice points in the light grey regions are known to decompose due to Theorem 3.1. The modules corresponding to lattice points in the middle dark grey region are known to decompose by direct computation. The modules corresponding to lattice points where $t_1 = 1$ or $t_2 = 1$ are indecomposable since $R/(y)$ and $R/(z)$ are both Cohen-Macaulay rings.

Figure 1 shows a lattice point (t_1, t_2) which corresponds to the module $\mathrm{Hom}_R(R/(y,z), R/(y^{t_1}, z^{t_2}))$ from Example 5.4.

Example 5.4. Let k be a field. Consider the ring $R = k[\![x,y,z]\!]/(x^2, xyz)$ of dimension two and depth one along with the system of parameters y, z. If $n_1 \geq 2$, then $S_{n_1} := R/(y^{n_1})$ is not Cohen-Macaulay. Indeed, the one-dimensional ring S_{n_1} has depth zero since the non-zero element xy^{n_1-1} is in the socle. Letting $\mathfrak{m}$ be the maximal ideal of S_{n_1}, we have that $\mathfrak{m}^i \cap \Gamma_{\mathfrak{m}}(S_{n_1}) = 0$ if and only if $i \geq n_1 + 2$. By symmetry, the same holds for the ring $T_{n_2} := R/(z^{n_2})$. Thus, Theorem 3.1 gives that

$$U_{n_1,n_2} := \mathrm{Hom}_R(R/(y,z), R/(y^{n_1}, z^{n_2})$$

decomposes for all $n_1, n_2 \geq 2$ with $|n_1 - n_2| > 2$. However, direct computation shows that U_{n_1,n_2} actually decomposes as

$$U_{n_1,n_2} \cong \frac{(xy^{n_1-1}, y^{n_1}, z^{n_2})}{(y^{n_1}, z^{n_2})} \oplus \frac{(xz^{n_2-1}, y^{n_1}, z^{n_2})}{(y^{n_1}, z^{n_2})} \oplus \frac{(y^{n_1-1}z^{n_2-1}, y^{n_1}, z^{n_2})}{(y^{n_1}, z^{n_2})}$$

for all $n_1, n_2 \geq 2$. Figure 1 shows a visual representation of this.

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