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POSTULATION AND REDUCTION VECTORS OF MULTIGRADED FILTRATIONS OF IDEALS

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ABSTRACT. We study the relationship between postulation and reduction vectors of admissible multigraded filtrations $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ of ideals in Cohen-Macaulay local rings of dimension at most two. This is enabled by a suitable generalization of the Kirby-Mehran complex. An analysis of its homology leads to an analogue of Huneke's fundamental lemma which plays a crucial role in our investigations. We also clarify the relationship between the Cohen-Macaulay property of the multigraded Rees algebra of $\mathcal{F}$ and reduction vectors with respect to complete reductions of $\mathcal{F}$.

1. Introduction. The objective of this paper is to study properties of Hilbert functions and Hilbert polynomials of multigraded filtrations of ideals under certain cohomological conditions. Among the themes presented are:

(1) an analogue of Huneke's fundamental lemma in terms of the homology of the generalized Kirby-Mehran complex for multigraded filtrations of ideals using complete reductions, (2) the relationship between postulation vectors and reduction vectors for multigraded filtrations of ideals in Cohen-Macaulay local rings of dimension at most two, (3) providing necessary and sufficient conditions for the equality of multigraded Hilbert functions and polynomials in terms of reduction numbers with respect to complete reductions, and finally, (4) the relationship between the Cohen-Macaulay property of the Rees algebra of multigraded filtrations of ideals and reduction numbers in two dimensional Cohen-Macaulay local rings.

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Hilbert functions of mutigraded filtrations of ideals were found useful in the work of Teissier **[17]**, who used them in his investigations of Milnor numbers of singularities of complex analytic hypersurfaces. To wit, let $(R, \mathfrak{m})$ be a d-dimensional local ring of an isolated singularity of a complex analytic hypersurface, and let f be the defining polynomial. Then, the Jacobian ideal $J := J(f)$ is an $\mathfrak{m}$-primary ideal, and the function $B(r,s) = \lambda(R/\mathfrak{m}^r J^s)$ is a polynomial $P(r,s)$ of degree d in r and s. Here, λ denotes length. Teissier proved that the normalized coefficients of monomials of degree d in $P(r,s)$ are the Milnor numbers of linear sections of the isolated singularity. Joint reductions and the Bhattacharya function $B(r,s) = \lambda(R/I^r J^s)$ for $\mathfrak{m}$-primary ideals I and J were used by Rees in several contexts. For example, he characterized pseudo-rational local rings of dimension 2 in terms of the constant term of the normal Hilbert polynomial for the normal Hilbert function $\lambda(R/\overline{I^r J^s})$.

We now describe the contents of the paper. We recall a few definitions and set up notation to explain the results of this paper.

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. For $s \geq 1$, we set $e = (1, \ldots, 1)$, $\underline{0} = (0, \ldots, 0) \in \mathbb{Z}^s$ and $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^s$ where 1 occurs at the ith position. Let $\underline{n} = (n_1, \ldots, n_s) \in \mathbb{Z}^s$. Then, we write $\underline{I}^{\underline{n}} = I_1^{n_1} \cdots I_s^{n_s}$ and $\underline{n}^+ = (n_1^+, \ldots, n_s^+)$, where

$$n_i^+ = \begin{cases} n_i & \text{if } n_i > 0, \\ 0 & \text{if } n_i \leq 0. \end{cases}$$

For $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_s$. Define $\underline{m} = (m_1, \ldots, m_s) \geq \underline{n} = (n_1, \ldots, n_s)$ if $m_i \geq n_i$ for all $i = 1, \ldots, s$. By the phrase "for all large $\underline{n}$" we mean $\underline{n} \in \mathbb{N}^s$ and $n_i \gg 0$ for all $i = 1, \ldots, s$.

Definition 1.1. A set of ideals $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ is called a $\mathbb{Z}^s$-graded $\underline{I} = (I_1, \ldots, I_s)$*-filtration* if, for all $\underline{m}, \underline{n} \in \mathbb{Z}^s$,

(i) $\underline{I}^{\underline{n}} \subseteq \mathcal{F}(\underline{n})$,
(ii) $\mathcal{F}(\underline{n})\mathcal{F}(\underline{m}) \subseteq \mathcal{F}(\underline{n} + \underline{m})$, and
(iii) if $\underline{m} \geq \underline{n}$, $\mathcal{F}(\underline{m}) \subseteq \mathcal{F}(\underline{n})$.

Let $t_1, \ldots, t_s$ be indeterminates. For $\underline{n} \in \mathbb{Z}^s$, set $\underline{t}^{\underline{n}} = t_1^{n_1} \cdots t_s^{n_s}$, and denote the $\mathbb{N}^s$-graded *Rees ring of* $\mathcal{F}$ by

$$\mathcal{R}(\mathcal{F}) = \bigoplus_{\underline{n} \in \mathbb{N}^s} \mathcal{F}(\underline{n}) \underline{t}^{\underline{n}}$$

and the $\mathbb{Z}^s$-graded *extended Rees ring of* $\mathcal{F}$ by

$$\mathcal{R}'(\mathcal{F}) = \bigoplus_{\underline{n} \in \mathbb{Z}^s} \mathcal{F}(\underline{n}) \underline{t}^{\underline{n}}.$$

For an $\mathbb{N}^s$-graded ring $S = \bigoplus_{\underline{n} \geq 0} S_{\underline{n}}$, we denote the ideal $\bigoplus_{\underline{n} \geq e} S_{\underline{n}}$ by S_{++}. For $\mathcal{F} = \{\underline{I}^{\underline{n}}\}_{\underline{n} \in \mathbb{Z}^s}$, we set $\mathcal{R}(\mathcal{F}) = \mathcal{R}(\underline{I})$, $\mathcal{R}'(\mathcal{F}) = \mathcal{R}'(\underline{I})$ and $\mathcal{R}(\underline{I})_{++} = \mathcal{R}_{++}$.

Definition 1.2. A $\mathbb{Z}^s$-graded $\underline{I} = (I_1, \ldots, I_s)$-filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ of ideals in R is called an $\underline{I} = (I_1, \ldots, I_s)$-*admissible filtration* if $\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n}^+)$ for all $\underline{n} \in \mathbb{Z}^s$ and $\mathcal{R}'(\mathcal{F})$ is a finite $\mathcal{R}'(\underline{I})$-module.

For an $\mathfrak{m}$-primary ideal I, the *Hilbert function* $H_I(n)$ is defined as $H_I(n) = \lambda(R/I^n)$ for all $n \in \mathbb{Z}$. Here, we adopt the convention that $I^n = R$ if $n \leq 0$. Samuel **[16]** showed that, for sufficiently large n, $H_I(n)$ coincides with a polynomial $P_I(n)$ of degree d, called the *Hilbert polynomial* of I. For all $n \in \mathbb{Z}$, $P_I(n)$ is often written in the form

$$P_I(n) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{n+d-1-i}{d-i}.$$

The coefficients $e_i(I)$ are integers for all $i = 0, 1, \ldots, d$, called the *Hilbert coefficients* of I. The leading coefficient $e_0(I)$ is sometimes denoted by $e(I)$ and called the multiplicity of I.

Let $s \geq 2$ and $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in a Noetherian local ring $(R, \mathfrak{m})$ of dimension d. For the *Hilbert function* $H_{\mathcal{F}}(\underline{n}) = \lambda(R/\mathcal{F}(\underline{n}))$ of $\mathcal{F}$, Rees **[15]** proved that there exists a polynomial of degree d, called the *Hilbert polynomial* of $\mathcal{F}$,

$$P_{\mathcal{F}}(\underline{n}) = \sum_{\substack{\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s \\ |\alpha| \leq d}} (-1)^{d-|\alpha|} e_{\alpha}(\mathcal{F}) \binom{n_1 + \alpha_1 - 1}{\alpha_1} \cdots \binom{n_s + \alpha_s - 1}{\alpha_s}$$

such that $P_{\mathcal{F}}(\underline{n}) = H_{\mathcal{F}}(\underline{n})$ for all large $\underline{n}$. Here, $e_\alpha(\mathcal{F})$ are integers called the *Hilbert coefficients* of $\mathcal{F}$. This was proved by Bhattacharya for the filtration $\mathcal{F} = \{I^r J^s\}_{r,s\in\mathbb{Z}}$ in [**1**], where I and J are $\mathfrak{m}$-primary ideals. Teissier [**17**] showed the existence of $P_{\mathcal{F}}(\underline{n})$ for the filtration $\mathcal{F} = \{\underline{I}^{\underline{n}}\}_{\underline{n}\in\mathbb{Z}^s}$.

Let I be an $\mathfrak{m}$-primary ideal in a local ring $(R, \mathfrak{m})$ of dimension $d \geq 1$. An integer $n(I)$ is called the *postulation number of* I if $P_I(n) = H_I(n)$ for all $n > n(I)$ and $P_I(n(I)) \neq H_I(n(I))$. An ideal $J \subseteq I$ is called a *reduction* of I if $JI^n = I^{n+1}$ for some n. We say J is a *minimal reduction* of I if, whenever $K \subseteq J$ and K is a reduction of I, then $K = J$. Let

$$r_J(I) = \min\{m : JI^n = I^{n+1} \text{ for } n \geq m\}$$

and

$$r(I) = \min\{r_J(I) : J \text{ is a minimal reduction of } I\}.$$

The analogues of reduction and postulation numbers for $\underline{I}$-admissible multigraded filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ are described as follows.

Definition 1.3. A vector $\underline{n} \in \mathbb{Z}^s$ is called a *postulation vector of* $\mathcal{F}$ if $H_{\mathcal{F}}(\underline{m}) = P_{\mathcal{F}}(\underline{m})$ for all $\underline{m} \geq \underline{n}$.

Rees [**15**] introduced the concept of complete reduction for $\{\underline{I}^{\underline{n}}\}_{\underline{n}\in\mathbb{Z}^s}$. In a similar manner, we define complete reduction of an $\underline{I}$-admissible filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$.

Definition 1.4. Let $\mathcal{F}$ be an $\underline{I}$-admissible filtration. A set of elements $\mathcal{A} = \{x_{ij} \in I_i : j = 1, \ldots, d; i = 1, \ldots, s\}$ is called a *complete reduction* of $\mathcal{F}$ if, for all large $\underline{n} \in \mathbb{N}^s$, $y_j = x_{1j} \cdots x_{sj}$ for all $j = 1, \ldots, d$ and $J = (y_1, \ldots, y_d)$,

$$J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e).$$

Definition 1.5. Let $\mathcal{F}$ be an $\underline{I}$-admissible filtration. A complete reduction $\mathcal{A} = \{x_{ij} \in I_i : j = 1, \ldots, d; i = 1, \ldots, s\}$ of $\mathcal{F}$ is called a *good complete reduction* if, for all large $\underline{m} \in \mathbb{N}^s$ and $y_1 = x_{11} \cdots x_{s1}$,

$$\mathcal{F}(\underline{m}) \cap (y_1) = y_1\mathcal{F}(\underline{m} - e).$$

Let $\mathcal{A} = \{x_{ij} \in I_i : j = 1, \dots, d; i = 1, \dots, s\}$ be a complete reduction $\mathcal{F}$, $y_j = x_{1j} \cdots x_{sj}$ for all $j = 1, \dots, d$ and $J = (y_1, \dots, y_d)$.

Definition 1.6. A vector $\underline{r} \in \mathbb{N}^s$ is called a *reduction vector of $\mathcal{F}$ with respect to $\mathcal{A}$* if, for all $\underline{n} \geq \underline{r}$, $J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$.

Definition 1.7. An integer $k \in \mathbb{N}$ is called the *complete reduction number of $\mathcal{F}$ with respect to $\mathcal{A}$* if $J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ for all $\underline{n} \geq ke$ and, whenever $k \neq 0$, there do not exist any $0 \leq t < k$ such that $J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ for all $\underline{n} \geq te$.

We use the following notation

(1) $\mathcal{P}(\mathcal{F}) = \{\underline{n} \in \mathbb{Z}^s \mid \underline{n}$ is a postulation vector of $\mathcal{F}\}$.
(2) $\mathcal{R}_{\mathcal{A}}(\mathcal{F}) = \{\underline{n} \in \mathbb{Z}^s \mid \underline{n}$ is a reduction vector of $\mathcal{F}$ with respect to $\mathcal{A}\}$.
(3) $r_{\mathcal{A}}(\mathcal{F})$ is the complete reduction number of $\mathcal{F}$ with respect to $\mathcal{A}$.

We now describe the main results proved in this paper. In Section 2, we prove some preliminary results regarding the coefficients of the Hilbert polynomial of a multigraded filtration of ideals which we use to prove our main results. Let $f(\underline{n}) : \mathbb{Z}^s \to \mathbb{Z}$ be an integer-valued function. Define the *first difference function* of $f(\underline{n})$ by $\Delta^1(f(\underline{n})) = f(\underline{n} + e) - f(\underline{n})$. For all $k \geq 2$, we define $\Delta^k(f(\underline{n})) = \Delta^{k-1}(\Delta^1(f(\underline{n})))$. In [**7**], Huneke proved the following fundamental lemma.

Lemma 1.8 ([**7,** Lemma 2.4]). *Let $(R, \mathfrak{m})$ be a two-dimensional local Cohen-Macaulay ring, and let $x, y \in \mathfrak{m}$ be any system of parameters of R. Let I be any ideal integral over (x, y). Then, for all $n \geq 1$,*

$$\lambda\left(\frac{I^{n+1}}{(x,y)I^n}\right) - \lambda\left(\frac{(I^n : (x,y))}{I^{n-1}}\right) = \Delta^2\Big(P_I(n-1) - H_I(n-1)\Big).$$

Huckaba [**5**] extended this result for dimension $d \geq 1$. In Section 3, for $l \geq 1$, $1 \leq k \leq d$ and $\underline{y}^{[l]} = ({y_1}^l, \dots, {y_k}^l)$, we introduce an analogue of the Kirby-Mehran complex [**9**] for multigraded filtrations:

$C\cdot(\underline{y}^{[l]},\mathcal{F}(\underline{n}))$:

$$0\longrightarrow\frac{R}{\mathcal{F}(\underline{n})}\xrightarrow{d_k}\left(\frac{R}{\mathcal{F}(\underline{n}+le)}\right)^{\binom{k}{1}}\xrightarrow{d_{k-1}}\cdots$$
$$\xrightarrow{d_2}\left(\frac{R}{\mathcal{F}(\underline{n}+(k-1)le)}\right)^{\binom{k}{k-1}}\xrightarrow{d_1}\frac{R}{\mathcal{F}(\underline{n}+kle)}\xrightarrow{d_0}0$$

and prove an analogue of Huneke's fundamental lemma for multigraded filtration of ideals.

Theorem 1.9. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d\geq 1$ with infinite residue field, let $I_1,\ldots,I_s$ be $\mathfrak{m}$-primary ideals of R and let $\mathcal{F}=\{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Let $\mathcal{A}=\{x_{ij}\in I_i : j=1,\ldots,d; i=1,\ldots,s\}$ be any complete reduction of $\mathcal{F}$, $y_j=x_{1j}\cdots x_{sj}$ for all $j=1,\ldots,d$. Let $\underline{y}=y_1,\ldots,y_d$ and $J=(\underline{y})$. Then, for all $\underline{n}\in\mathbb{Z}^s$,*

$$\Delta^d(P_{\mathcal{F}}(\underline{n})-H_{\mathcal{F}}(\underline{n}))=\lambda\left(\frac{\mathcal{F}(\underline{n}+de)}{J\mathcal{F}(\underline{n}+(d-1)e)}\right)-\sum_{i=2}^{d}(-1)^i\lambda(H_i(C\cdot(\underline{y},\mathcal{F}(\underline{n})))).$$

In Section 4, for $d\geq 2$, we compute the $\underline{n}$ degree component of local cohomology module $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}}$ for all $\underline{n}\in\mathbb{N}^s$ and give an equivalent criterion for the vanishing of $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}}$ for all $\underline{n}\in\mathbb{N}^s$. We discuss vanishing of Hilbert coefficients and generalize some results due to Marley **[10]** in the Cohen-Macaulay local ring of dimension $1\leq d\leq 2$.

Theorem 1.10. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $1\leq d\leq 2$ with infinite residue field, let $I_1,\ldots,I_s$ be $\mathfrak{m}$-primary ideals of R and let $\mathcal{F}=\{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Let $e_{(d-1)e_i}(\mathcal{F})=0$ for $i=1,\ldots,s$. Then,*

(i) *for $d=1$, $P_{\mathcal{F}}(\underline{n})=H_{\mathcal{F}}(\underline{n})$ for all $\underline{n}\in\mathbb{N}^s$.*
(ii) *For $d=2$, if $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}}=0$ for all $\underline{n}\in\mathbb{N}^s$, then $P_{\mathcal{F}}(\underline{n})=H_{\mathcal{F}}(\underline{n})$ for all $\underline{n}\in\mathbb{N}^s$ and $e_{\underline{0}}(\mathcal{F})=0$.*

Theorem 1.11. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $1\leq d\leq 2$ with infinite residue field, and let I,J be $\mathfrak{m}$-primary* *ideals of R. Let $\mathcal{F} = \{\mathcal{F}(r,s)\}_{r,s\in\mathbb{Z}}$ be a $\mathbb{Z}^2$-graded (I,J)-admissible filtration of ideals in R. Then, the following statements are equivalent.*

(i) $e_{(d-1)e_i}(\mathcal{F}) = 0$ *for* $i = 1, 2$.
(ii) *I and J are generated by a system of parameters, $P_{\mathcal{F}}(r,s) = H_{\mathcal{F}}(r,s)$ for all $r, s \in \mathbb{N}$ and $\mathcal{F}(r,s) = I^r J^s$ for all $r, s \in \mathbb{Z}$.*
(iii) $e_\alpha(\mathcal{F}) = 0$ *for* $|\alpha| \le d-1$.

In **[11]**, Marley proved that, for the Cohen-Macaulay local ring of dimension $d \ge 1$, $r(I) = n(I) + d$ under some depth condition of the associated graded ring of I. In his thesis **[10]**, Marley extended this result for $\mathbb{Z}$-graded I-admissible filtrations. We generalize this result for multigraded filtration of ideals when $d = 1, 2$. In Section 5, we prove the next theorem.

Theorem 1.12. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 1 with infinite residue field, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R and $\mathcal{A} = \{a_i \in I_i : i = 1, \ldots, s\}$ a complete reduction of $\mathcal{F}$. Then,*

$$\mathcal{P}(\mathcal{F}) \subseteq \mathbb{N}^s \quad \textit{and} \quad \mathcal{P}(\mathcal{F}) = \mathcal{R}_{\mathcal{A}}(\mathcal{F}).$$

Moreover, the set $\mathcal{R}_{\mathcal{A}}(\mathcal{F})$ is independent of any complete reduction $\mathcal{A}$ of $\mathcal{F}$.

We also show that, for the one-dimensional Cohen-Macaulay local ring $(R, \mathfrak{m})$, $r_{\mathcal{A}}(\mathcal{F})$ is independent of any complete reduction $\mathcal{A}$ of $\mathcal{F}$.

In Section 6, we provide a relation between reduction vectors of good complete reductions and postulation vectors of multigraded filtration of ideals in two-dimensional Cohen-Macaulay local rings. For bigraded filtration, we prove a result which relates the Cohen-Macaulayness of the bigraded Rees algebra, the complete reduction number, reduction numbers and the joint reduction number.

Theorem 1.13. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 2 with infinite residue field, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R and $s \ge 2$. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R and $\mathcal{A} = \{x_{ij} \in I_i : j = 1, 2; i = 1, \ldots, s\}$ a good complete reduction of $\mathcal{F}$. Let $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \ge \underline{0}$. Then,*

$\mathcal{P}(\mathcal{F}) \subseteq \mathbb{N}^s$, and there exists a one-to-one correspondence

$$f : \mathcal{P}(\mathcal{F}) \longleftrightarrow \{\underline{r} \in \mathcal{R}_{\mathcal{A}}(\mathcal{F}) \mid \underline{r} \geq e\}$$

defined by $f(\underline{n}) = \underline{n} + e$ where $f^{-1}(\underline{r}) = \underline{r} - e$.

Theorem 1.14. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 2 with infinite residue field, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R and $s \geq 2$. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R and $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \geq \underline{0}$. Then, the following statements are equivalent.*

(i) *$\mathcal{P}(\mathcal{F}) = \mathbb{N}^s$, i.e., $P_{\mathcal{F}}(\underline{n}) = H_{\mathcal{F}}(\underline{n})$ for all $\underline{n} \geq \underline{0}$.*
(ii) *$r_{\mathcal{A}}(\mathcal{F}) \leq 1$ for any good complete reduction $\mathcal{A}$ of $\mathcal{F}$.*
(ii$'$) *There exists a good complete reduction $\mathcal{A}$ of $\mathcal{F}$ such that $r_{\mathcal{A}}(\mathcal{F}) \leq 1$.*

In order to state the final result we recall the definition of joint reduction of multigraded filtrations **[12]**. The *joint reduction* of $\mathcal{F}$ of type $\mathbf{q} = (q_1, \ldots, q_s) \in \mathbb{N}^s$ is a collection of q_i elements $x_{i1}, \ldots, x_{iq_i} \in I_i$ for all $i = 1, \ldots, s$ such that $q_1 + \cdots + q_s = d$ and

$$\sum_{i=1}^{s} \sum_{j=1}^{q_i} x_{ij} \mathcal{F}(\underline{n} - e_i) = \mathcal{F}(\underline{n}) \quad \text{for all large } \underline{n}.$$

We say that the *joint reduction number of $\mathcal{F}$ with respect to a joint reduction $\{x_{ij} \in I_i : j = 1, \ldots, q_i; i = 1, \ldots, s\}$ of type q is zero* if

$$\sum_{i=1}^{s} \sum_{j=1}^{q_i} x_{ij} \mathcal{F}(\underline{n} - e_i) = \mathcal{F}(\underline{n}) \text{ for all } \underline{n} \geq \sum_{i \in A} e_i, \text{ where } A = \{i \mid q_i \neq 0\}.$$

We say that the *joint reduction number of $\mathcal{F}$ of type q is zero* if the joint reduction number of $\mathcal{F}$ with respect to any joint reduction of type q is zero.

Theorem 1.15. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension two with infinite residue field and I, J be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^2}$ be a $\mathbb{Z}^2$-graded (I, J)-admissible filtration of ideals in R. Then the following are equivalent.*

(i) *The Rees algebra $\mathcal{R}(\mathcal{F})$ is Cohen-Macaulay.*
(ii) *$\mathcal{P}(\mathcal{F}) = \mathbb{N}^2$, i.e., $P_{\mathcal{F}}(\underline{n}) = H_{\mathcal{F}}(\underline{n})$ for all $\underline{n} \geq \underline{0}$.*

(iii) *For any good complete reduction* $\mathcal{A}$ *of* $\mathcal{F}$, $r_{\mathcal{A}}(\mathcal{F}) \leq 1$ *and* $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ *for all* $\underline{n} \geq \underline{0}$.

(iii$'$) *There exists a good complete reduction* $\mathcal{A}$ *of* $\mathcal{F}$ *such that* $r_{\mathcal{A}}(\mathcal{F}) \leq 1$ *and* $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ *for all* $\underline{n} \geq \underline{0}$.

(iv) *For the filtrations* $\mathcal{F}^{(i)} = \{\mathcal{F}(ne_i)\}_{n\in\mathbb{Z}}$, $r(\mathcal{F}^{(i)}) \leq 1$ *where* $i = 1, 2$ *and the joint reduction number of* $\mathcal{F}$ *of type* e *is zero.*

2. Preliminary results. In this section, we discuss the existence of good complete reduction of an $\underline{I}$-admissible filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ and prove some results regarding Hilbert coefficients, which we will use in the following sections. For an admissible multigraded filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$, by Rees's lemma [**12,** Lemma 2.2] and [**15,** Lemma 1.2], we obtain elements $x_i \in I_i$ for all $i = 1, \ldots, s$, called *superficial elements* for $\mathcal{F}$ such that, for each i, there exist an integer r_i and $(x_i) \cap \mathcal{F}(\underline{n}) = x_i\mathcal{F}(\underline{n} - e_i)$ for all $\underline{n} \geq r_i e_i$. In [**15,** Theorem 1.3], Rees proved the existence of complete reduction of the filtration $\{\underline{I}^{\underline{n}}\}_{\underline{n}\in\mathbb{Z}^s}$. Using the same lines of proof of this theorem and existence of superficial elements we obtain the next theorem.

Theorem 2.1. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* d *with infinite residue field, and let* $I_1, \ldots, I_s$ *be* $\mathfrak{m}$*-primary ideals of* R. *Let* $\mathcal{F}$ *be an* $\underline{I}$*-admissible filtration of ideals in* R. *Then, there exists a good complete reduction of* $\mathcal{F}$.

Lemma 2.2. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$, *and let* $I_1, \ldots, I_s$ *be* $\mathfrak{m}$*-primary ideals of* R. *Put* $I = I_1 \cdots I_s$. *Then,*

$$\sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\in\mathbb{N}^s \\ |\alpha|=d}} \frac{d!e_\alpha(\underline{I})}{\alpha_1!\cdots\alpha_s!} = e_0(I) \quad \textit{and} \quad e_{\underline{0}}(\underline{I}) = e_d(I).$$

Proof. Since, for large n,

$$P_{\underline{I}}(ne) = H_{\underline{I}}(ne) = \lambda\left(\frac{R}{\underline{I}^{ne}}\right) = \lambda\left(\frac{R}{I^n}\right) = P_I(n),$$

comparing the coefficients of n^d and constant terms, the required result is obtained. $\square$

Proposition 2.3. *Let $s \geq 1$ be a fixed integer, and let $i_1, \ldots, i_s \in \mathbb{N}$ be such that $g = i_1 + \cdots + i_s \geq 1$. Then, $\Delta^g({n_1}^{i_1} \cdots {n_s}^{i_s}) = g!$, where $n_k \in \mathbb{Z}$ for all $k = 1, \ldots, s$.*

Proof. We use induction on g. Let $g = 1$. Then, without loss of generality, assume $i_1 = 1$ and $i_k = 0$ for all $k \neq 1$. Therefore, $\Delta^1(n_1) = (n_1 + 1) - n_1 = 1$. Let $g \geq 2$, and assume the result is true up to $g - 1$. Now,

$$\begin{aligned}\Delta^g({n_1}^{i_1} \cdots {n_s}^{i_s}) &= \Delta^{g-1}\Big[\Delta^1({n_1}^{i_1} \cdots {n_s}^{i_s})\Big] \\ &= \Delta^{g-1}\Bigg[\sum_{k=1}^{s} i_k({n_1}^{i_1} \cdots {n_k}^{i_k-1} \cdots {n_s}^{i_s})\Bigg] \\ &= \sum_{k=1}^{s} i_k \Delta^{g-1}({n_1}^{i_1} \cdots {n_k}^{i_k-1} \cdots {n_s}^{i_s}) \\ &= \sum_{k=1}^{s} i_k (g-1)! = g!. \qquad \square\end{aligned}$$

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then*:

(1) *$e_\alpha(\mathcal{F}) = e_\alpha(\underline{I})$ for all $\alpha \in \mathbb{N}^s$ where $|\alpha| = d$.*
(2) *$\Delta^d(P_{\mathcal{F}}(\underline{n})) = \Delta^d(P_{\underline{I}}(\underline{n})) = e_0(I_1 \cdots I_s)$.*
(3) *For an ideal $J \subseteq \mathcal{F}(e)$ such that $J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ for all large $\underline{n} \in \mathbb{N}^s$, we have $\Delta^d(P_{\mathcal{F}}(\underline{n})) = e_0(J) = e_0(I_1 \cdots I_s)$.*

Proof.

(1) This follows from [**15,** Theorem 2.4].

(2) Using Proposition 2.3, we obtain

$$\begin{aligned}&\Delta^d(P_{\mathcal{F}}(\underline{n})) \\ &\quad = \Delta^d\Bigg(\sum_{|\alpha| \leq d} (-1)^{d-|\alpha|} e_\alpha(\mathcal{F}) \binom{n_1 + \alpha_1 - 1}{\alpha_1} \cdots \binom{n_s + \alpha_s - 1}{\alpha_s}\Bigg) \\ &\quad = \Delta^d\Bigg(\sum_{|\alpha| = d} \frac{e_\alpha(\mathcal{F})}{\alpha_1! \cdots \alpha_s!} {n_1}^{\alpha_1} \cdots {n_s}^{\alpha_s}\Bigg)\end{aligned}$$

$$= \sum_{|\alpha|=d} \frac{e_\alpha(\mathcal{F})}{\alpha_1! \cdots \alpha_s!}\left[\Delta^d({n_1}^{\alpha_1} \cdots {n_s}^{\alpha_s})\right]$$

$$= \sum_{|\alpha|=d} \frac{e_\alpha(\mathcal{F})}{\alpha_1! \cdots \alpha_s!}(\alpha_1 + \cdots + \alpha_s)! = \sum_{|\alpha|=d} d!\frac{e_\alpha(\mathcal{F})}{\alpha_1! \cdots \alpha_s!}.$$

In a similar manner, we obtain

$$\Delta^d(P_{\underline{I}}(\underline{n})) = \sum_{|\alpha|=d} d!\frac{e_\alpha(\underline{I})}{\alpha_1! \cdots \alpha_s!}.$$

By (1) and Lemma 2.2, $\Delta^d(P_{\mathcal{F}}(\underline{n})) = \Delta^d(P_{\underline{I}}(\underline{n})) = e_0(I_1 \cdots I_s)$.

(3) Since $J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ for all large $\underline{n}$, there exists an integer $k \in \mathbb{N}$ such that $J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ for all $\underline{n} \geq ke$. Now, for all $n \geq k$, $J^{n-k}\mathcal{F}(ke) = \mathcal{F}(ne)$, and hence, $J^n \subseteq \mathcal{F}(ne) \subseteq J^{n-k}$. This implies

$$\lambda\left(\frac{R}{J^{n-k}}\right) \leq \lambda\left(\frac{R}{\mathcal{F}(ne)}\right) \leq \lambda\left(\frac{R}{J^n}\right)$$

for all $n \geq k$. Therefore, for all $n \geq k$, we have

$$\lim_{n\to\infty} \frac{P_J(n-k)}{n^d/d!} \leq \lim_{n\to\infty} \frac{P_{\mathcal{F}}(ne)}{n^d/d!} \leq \lim_{n\to\infty} \frac{P_J(n)}{n^d/d!}$$

which implies $\Delta^d(P_{\mathcal{F}}(\underline{n})) = e_0(J)$. Hence, using part (2), we get the required result. □

Proposition 2.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$, $\operatorname{depth} R \geq 1$, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R and $\mathcal{A} = \{x_{ij} \in I_i : j = 1, \ldots, d; i = 1, \ldots, s\}$ be a complete reduction of $\mathcal{F}$. Then, $J_i = (x_{i1}, \ldots, x_{id})$ is a reduction of I_i for all $i = 1, \ldots, s$.*

Proof. For all large $\underline{n}$, $J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ where $y_j = x_{1j} \cdots x_{sj}$ for all $j = 1, \ldots, d$ and $J = (y_1, \ldots, y_d)$. Since $\mathcal{F}$ is an $\underline{I}$-admissible filtration, for each $i \in \{1, \ldots, s\}$, there exists an $r_i \in \mathbb{N}$ such that, for all $\underline{n} \geq r_i e_i$, $\mathcal{F}(\underline{n} + e_i) = I_i\mathcal{F}(\underline{n})$. Hence, for all large $\underline{n}$,

$$I_i\mathcal{F}(\underline{n}+e-e_i) \supseteq J_i\mathcal{F}(\underline{n}+e-e_i) \supseteq J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n}+e) \supseteq I_i\mathcal{F}(\underline{n}+e-e_i).$$

Now, by [**15,** Lemma 1.5], J_i is a reduction of I_i for all $i = 1, \ldots, s$. □

Proposition 2.6. *Let*

$$f(\underline{n}) : \mathbb{Z}^s \longrightarrow \mathbb{Z}$$

be an integer-valued function such that, for all large $\underline{n} \in \mathbb{N}^s$, $f(\underline{n}) = 0$. *Let*

$$\mathcal{B} = \{\underline{n} \in \mathbb{Z}^s : f(\underline{m}) = 0 \textit{ for all } \underline{m} \geq \underline{n}\}$$

and, for all $j \geq 0$,

$$\mathcal{C}_j = \{\underline{n} \in \mathbb{Z}^s : \Delta^j(f(\underline{m})) = 0 \textit{ for all } \underline{m} \geq \underline{n}\}.$$

Then, for $j \geq 0$, $\mathcal{B} = \mathcal{C}_j$.

Proof. For $j = 0$, the results hold due to the definition of $\mathcal{B}$. It is sufficient to prove the statement for $j = 1$. Let $\underline{n} \in \mathcal{B}$. Then, $\Delta^1(f(\underline{m})) = f(\underline{m} + e) - f(\underline{m}) = 0$ for all $\underline{m} \geq \underline{n}$. This implies that $\underline{n} \in \mathcal{C}_1$. Conversely, let $\underline{n} \in \mathcal{C}_1$. Then, for all $\underline{m} \geq \underline{n}$, $0 = \Delta^1(f(\underline{m})) = f(\underline{m} + e) - f(\underline{m})$. Let $\underline{k} \in \mathbb{N}^s$ be such that $f(\underline{r}) = 0$ for all $\underline{r} \geq \underline{k}$. Let $\underline{m} \geq \underline{n}$. For all $i = 1, \ldots, s$, define

$$u(m)_i = \begin{cases} k_i - m_i & \text{if } k_i > m_i, \\ 0 & \text{if } k_i \leq m_i. \end{cases}$$

Let $u(m) = \max\{u(m)_1, \ldots, u(m)_s\} + 1$. Then, for all $\underline{m} \geq \underline{n}$,

$$0 = f(\underline{m} + u(m)e) = \cdots = f(\underline{m} + e) = f(\underline{m}).$$

Hence, $\underline{n} \in \mathcal{B}$. □

Proposition 2.7. *Let*

$$R = \bigoplus_{\underline{n} \in \mathbb{N}^s} R_{\underline{n}}$$

be a standard Noetherian $\mathbb{N}^s$*-graded ring,* S *an* $\mathbb{N}^s$*-graded* R*-algebra and* $b \in R_e$. *Let*

$$S_{\underline{n}} \xrightarrow{\cdot b} S_{\underline{n}+e}$$

be an injective map for all large $\underline{n}$ *and* grade$(S_{++}) \geq 1$. *Then,* b *is a nonzerodivisor of* S.

Proof. Let $\underline{m} \in \mathbb{N}^s$ be such that, for all $\underline{n} \geq \underline{m}$, $S_{\underline{n}} \stackrel{\cdot b}{\to} S_{\underline{n}+e}$ is an injective map. Let $x \in (0 :_S b) \cap S_{\underline{k}}$ for some $\underline{k} \in \mathbb{N}^s$. We show that $x(S_{++})^{m+1} = 0$, where $m = \max\{m_1, \ldots, m_s\}$. Let $0 \neq z \in (S_{++})^{m+1} \cap S_{\underline{p}}$. Now, $xz \in S_{\underline{k}+\underline{p}}$ and $bxz = 0$. Since $\underline{k} + \underline{p} \geq (m+1)e$, $xz = 0$. Thus,

$$x \in (0 :_S (S_{++})^{m+1}) = 0. \qquad \square$$

3. An analogue of Huneke's fundamental lemma. Throughout this section, $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension $d \geq 1$ with infinite residue field, and $I_1, \ldots, I_s$ are $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. In **[8]**, Jayanthan and Verma generalized the Kirby-Mehran complex **[6, 9]** for the bigraded filtration $\{I^r J^s\}_{r,s \in \mathbb{Z}}$ where I, J are $\mathfrak{m}$-primary ideals and studied the relation between cohomology modules of the complex and local cohomology modules of $\mathcal{R}(I, J)$. We construct a multigraded analogue of the Kirby-Mehran complex and compute its homology modules. As a consequence of this we prove an analogue of Huneke's fundamental lemma **[5, 7]**.

Let $y_1, \ldots, y_k$ be elements in $I_1 \cdots I_s$ where $1 \leq k \leq d$. For $l \geq 1$ and

$$(\underline{yt})^{[l]} = {y_1}^l \underline{t}^{le}, \ldots, {y_k}^l \underline{t}^{le},$$

consider the Koszul complex $K \cdot ((\underline{yt})^{[l]}, \mathcal{R}'(\mathcal{F}))$:

$$\begin{aligned} 0 &\longrightarrow \mathcal{R}'(\mathcal{F}) \longrightarrow \mathcal{R}'(\mathcal{F})(le)^{\binom{k}{1}} \longrightarrow \cdots \longrightarrow \mathcal{R}'(\mathcal{F})((k-1)le)^{\binom{k}{k-1}} \\ &\longrightarrow \mathcal{R}'(\mathcal{F})(kle) \longrightarrow 0. \end{aligned}$$

This complex has a $\mathbb{Z}^s$-graded structure inherited from $\mathcal{R}'(\mathcal{F})$. The graded component of degree $\underline{n}$ of the above complex is $K \cdot_{\underline{n}} ((\underline{yt})^{[l]}, \mathcal{R}'(\mathcal{F}))$:

$$\begin{aligned} 0 &\longrightarrow \mathcal{F}(\underline{n}) \longrightarrow (\mathcal{F}(\underline{n} + le))^{\binom{k}{1}} \longrightarrow \cdots \longrightarrow (\mathcal{F}(\underline{n} + (k-1)le))^{\binom{k}{k-1}} \\ &\longrightarrow \mathcal{F}(\underline{n} + kle) \longrightarrow 0. \end{aligned}$$

Let $\underline{y}^{[l]} = {y_1}^l, \ldots, {y_k}^l$. We can consider the above complex as a subcomplex of the Koszul complex $K \cdot (y^{[l]}, R)$:

$$0 \longrightarrow R \longrightarrow R^{\binom{k}{1}} \longrightarrow \cdots \longrightarrow R^{\binom{k}{k-1}} \longrightarrow R \longrightarrow 0.$$

Hence, we have a chain map of complexes $0 \to K \cdot_{\underline{n}} ((\underline{yt})^{[l]}, \mathcal{R}'(\mathcal{F})) \to K \cdot (\underline{y}^{[l]}, R)$ which produces a quotient complex $C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n}))$:

$$0 \longrightarrow \frac{R}{\mathcal{F}(\underline{n})} \xrightarrow{d_k} \left(\frac{R}{\mathcal{F}(\underline{n}+le)}\right)^{\binom{k}{1}} \xrightarrow{d_{k-1}} \cdots$$
$$\xrightarrow{d_2} \left(\frac{R}{\mathcal{F}(\underline{n}+(k-1)le)}\right)^{\binom{k}{k-1}} \xrightarrow{d_1} \frac{R}{\mathcal{F}(\underline{n}+kle)} \xrightarrow{d_0} 0.$$

In the next proposition, we compute homology modules of the above complex.

Proposition 3.1. *For all $l \geq 1$, $\underline{n} \in \mathbb{Z}^s$ and $1 \leq k \leq d$,*

(1) $H_0(C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n}))) = R/(\mathcal{F}(\underline{n}+kle), \underline{y}^{[l]})$,
(2) $H_k(C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n}))) = (\mathcal{F}(\underline{n}+le) : (\underline{y}^{[l]}))/\mathcal{F}(\underline{n})$,
(3) *if $y_1, \ldots, y_k$ is a regular sequence then*

$$H_1(C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n}))) = \frac{((\underline{y}^{[l]}) \cap \mathcal{F}(\underline{n}+kle))}{(\underline{y}^{[l]})\mathcal{F}(\underline{n}+(k-1)le)}.$$

Proof.

(1) Since $\ker d_0 = R/\mathcal{F}(\underline{n}+kle)$ and $\operatorname{im} d_1 = ((\underline{y}^{[l]}) + \mathcal{F}(\underline{n}+kle))/\mathcal{F}(\underline{n}+kle)$, we obtain

$$H_0(C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n}))) = \ker d_0/\operatorname{im} d_1 = R/(\mathcal{F}(\underline{n}+kle), \underline{y}^{[l]}).$$

(2) Since $\operatorname{im} d_{k+1} = 0$,

$$\begin{aligned} H_k(C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n}))) &= \ker d_k \\ &= \{x + \mathcal{F}(\underline{n}) \in R/\mathcal{F}(\underline{n}) \mid x{y_i}^l \in \mathcal{F}(\underline{n}+le) \text{ for all } i = 1, \ldots, k\} \\ &= \{x + \mathcal{F}(\underline{n}) \in R/\mathcal{F}(\underline{n}) \mid x \in \cap_{i=1}^k (\mathcal{F}(\underline{n}+le) : ({y_i}^l))\} \\ &= \{x + \mathcal{F}(\underline{n}) \in R/\mathcal{F}(\underline{n}) \mid x \in (\mathcal{F}(\underline{n}+le) : (\underline{y}^{[l]}))\}. \end{aligned}$$

(3) Since $y_1, \ldots, y_k$ is a regular sequence, the following sequence is exact

$$R^{\binom{k}{k-2}} \xrightarrow{\phi_2} R^{\binom{k}{k-1}} \xrightarrow{\phi_1} (\underline{y}^{[l]}) \xrightarrow{\phi_0} 0.$$

Tensoring by $R/(\mathcal{F}(\underline{n}+(k-1)le))$, we obtain an exact sequence

$$\left(\frac{R}{\mathcal{F}(\underline{n}+(k-1)le)}\right)^{\binom{k}{k-2}} \xrightarrow{\overline{\phi}_2} \left(\frac{R}{\mathcal{F}(\underline{n}+(k-1)le)}\right)^{\binom{k}{k-1}}$$
$$\xrightarrow{\overline{\phi}_1} (\underline{y}^{[l]})/(\underline{y}^{[l]})\mathcal{F}(\underline{n}+(k-1)le) \xrightarrow{\overline{\phi}_0} 0.$$

Hence $\operatorname{im}\overline{\phi}_2 = \operatorname{im} d_2$, and we obtain the following commutative diagram of exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} d_2 \arrow[r] \arrow[d, "i"'] & \left(\frac{R}{\mathcal{F}(\underline{n}+(k-1)le)}\right)^{\binom{k}{k-1}} \arrow[r] \arrow[d, "I_1\cdots I_s"'] & \frac{(\underline{y}^{[l]})}{(\underline{y}^{[l]})\mathcal{F}(\underline{n}+(k-1)le)} \arrow[r] \arrow[d, "\theta"'] & 0 \\
0 \arrow[r] & \ker d_1 \arrow[r] & \left(\frac{R}{\mathcal{F}(\underline{n}+(k-1)le)}\right)^{\binom{k}{k-1}} \arrow[r] & \frac{R}{\mathcal{F}(\underline{n}+kle)},
\end{tikzcd}$$

where i is the inclusion map and id is the identity map. Then, by the Snake lemma,

$$H_1(C\cdot(\underline{y}^{[l]},\mathcal{F}(\underline{n}))) = \frac{\ker d_1}{\operatorname{im} d_2} \cong \ker\theta = \frac{((\underline{y}^{[l]})\cap\mathcal{F}(\underline{n}+kle))}{(\underline{y}^{[l]})\mathcal{F}(\underline{n}+(k-1)le)}. \quad \square$$

Theorem 3.2 (Analogue of Huneke's fundamental lemma)**.** *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d\geq 1$ with infinite residue field, let $I_1,\ldots,I_s$ be $\mathfrak{m}$-primary ideals of R and $\mathcal{F}=\{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ an $\underline{I}$-admissible filtration of ideals in R. Let $\mathcal{A}=\{x_{ij}\in I_i : j=1,\ldots,d; i=1,\ldots,s\}$ be any complete reduction of $\mathcal{F}$, $y_j=x_{1j}\cdots x_{sj}$ for all $j=1,\ldots,d$. Let $\underline{y}=y_1,\ldots,y_d$ and $J=(\underline{y})$. Then, for all $\underline{n}\in\mathbb{Z}^s$,*

$$\Delta^d(P_{\mathcal{F}}(\underline{n})-H_{\mathcal{F}}(\underline{n})) = \lambda\left(\frac{\mathcal{F}(\underline{n}+de)}{J\mathcal{F}(\underline{n}+(d-1)e)}\right)$$
$$-\sum_{i=2}^{d}(-1)^i\lambda(H_i(C\cdot(\underline{y},\mathcal{F}(\underline{n})))).$$

Proof. By Propositions 2.4 and 3.1, we obtain:

$$\Delta^d(P_{\mathcal{F}}(\underline{n})-H_{\mathcal{F}}(\underline{n})) = e_0(J)-\Delta^d(H_{\mathcal{F}}(\underline{n}))$$

$$
\begin{aligned}
&= e_0(J) - \sum_{i=0}^{d} (-1)^i \lambda(H_i(C \cdot (\underline{y}, \mathcal{F}(\underline{n})))) \\
&= \lambda\bigg(\frac{R}{(\underline{y})}\bigg) - \lambda\bigg(\frac{R}{(\mathcal{F}(\underline{n}+de), (\underline{y}))}\bigg) + \lambda\bigg(\frac{(\underline{y}) \cap \mathcal{F}(\underline{n}+de)}{(\underline{y})\mathcal{F}(\underline{n}+(d-1)e)}\bigg) \\
&\quad - \sum_{i=2}^{d} (-1)^i \lambda(H_i(C \cdot (\underline{y}, \mathcal{F}(\underline{n})))) \\
&= \lambda\bigg(\frac{\mathcal{F}(\underline{n}+de)}{(\underline{y})\mathcal{F}(\underline{n}+(d-1)e)}\bigg) - \sum_{i=2}^{d} (-1)^i \lambda(H_i(C \cdot (\underline{y}, \mathcal{F}(\underline{n})))) \\
&= \lambda\bigg(\frac{\mathcal{F}(\underline{n}+de)}{J\mathcal{F}(\underline{n}+(d-1)e)}\bigg) - \sum_{i=2}^{d} (-1)^i \lambda(H_i(C \cdot (\underline{y}, \mathcal{F}(\underline{n})))). \qquad \square
\end{aligned}
$$

4. Vanishing of Hilbert coefficients. In this section, we compute the local cohomology module $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}}$ for all $\underline{n} \geq \underline{0}$ **[2, 8]**. We discuss the vanishing of Hilbert coefficients of an $\underline{I}$-admissible filtration $\mathcal{F}$ and generalize some results due to Marley **[10]** in Cohen-Macaulay local ring of dimension $1 \leq d \leq 2$.

The filtration $\breve{\mathcal{F}} = \{\breve{\mathcal{F}}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ of ideals for a $\mathbb{Z}^s$-graded $\underline{I}$-filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ is called the *Ratliff-Rush closure filtration* of $\mathcal{F}$ where

$$\breve{\mathcal{F}}(\underline{n}) = \bigcup_{k \geq 1} (\mathcal{F}(\underline{n}+ke) : \mathcal{F}(e)^k)$$

for all $\underline{n} \in \mathbb{N}^s$ and

$$\breve{\mathcal{F}}(\underline{n}) = \breve{\mathcal{F}}(\underline{n}^+)$$

for all $\underline{n} \in \mathbb{Z}^s$ **[12]**. In order to compute $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}}$, we follow the lines of **[2,** Proof of Theorem 3.3].

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field, let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R and $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then, for all $\underline{n} \in \mathbb{N}^s$,*

$$H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} \cong \frac{\breve{\mathcal{F}}(\underline{n})}{\mathcal{F}(\underline{n})}.$$

Proof. Let $\mathcal{A} = \{x_{ij} \in I_i : j = 1, \dots, d; i = 1, \dots, s\}$ be any complete reduction of $\mathcal{F}$, $y_j = x_{1j} \cdots x_{sj}$ for all $j = 1, \dots, d$. For each $\underline{n}$, $l \geq 1$, $(\underline{yt})^{[l]} = {y_1}^l \underline{t}^{le}, \dots, {y_d}^l \underline{t}^{le}$ and $\underline{y}^{[l]} = {y_1}^l, \dots, {y_d}^l$, we have the following exact sequence

$$0 \longrightarrow K \cdot_{\underline{n}} ((\underline{yt})^{[l]}, \mathcal{R}'(\mathcal{F})) \longrightarrow K \cdot (\underline{y}^{[l]}, R) \longrightarrow C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n})) \longrightarrow 0.$$

For each $i \in \{1, \dots, d\}$, the commutative diagram of complexes:

$$\begin{array}{rcccccc} K \cdot ({y_i}^l, R): & 0 & \longrightarrow & R & \xrightarrow{{y_i}^l} & R & \longrightarrow & 0 \\ & & & \Big\downarrow \mathrm{id} & & \Big\downarrow y_i & & \\ K \cdot ({y_i}^{l+1}, R): & 0 & \longrightarrow & R & \xrightarrow{{y_i}^{l+1}} & R & \longrightarrow & 0 \end{array}$$

gives a map $K \cdot ({y_1}^l, \dots, {y_d}^l, R) = \otimes_{i=1}^d K \cdot ({y_i}^l, R) \to K \cdot ({y_1}^{l+1}, \dots, {y_d}^{l+1}, R) = \otimes_{i=1}^d K \cdot ({y_i}^{l+1}, R)$. The maps can be restricted to $K \cdot_{\underline{n}} ({y_1}^l \underline{t}^{le}, \dots, {y_d}^l \underline{t}^{le}, \mathcal{R}'(\mathcal{F}))$. Hence, for all $l \geq 1$, we get morphisms of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K \cdot \underline{n}((\underline{yt})^{[l]}, \mathcal{R}'(\mathcal{F})) & \longrightarrow & K \cdot (\underline{y}^{[l]}, R) & \longrightarrow & C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n})) & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & K \cdot_{\underline{n}} ((\underline{yt})^{[l+1]}, \mathcal{R}'(\mathcal{F})) & \longrightarrow & K \cdot (\underline{y}^{[l+1]}, R) & \longrightarrow & C \cdot (\underline{y}^{[l+1]}, \mathcal{F}(\underline{n})) & \longrightarrow & 0 \end{array}$$

which produce an inductive system of exact sequences of complexes. Applying $\varinjlim_l$ to the long exact sequence of cohomology modules, we obtain

$$\begin{aligned} 0 &\longrightarrow H^0_{(\underline{yt})}(\mathcal{R}'(\mathcal{F}))_{\underline{n}} \longrightarrow H^0_{(\underline{y})}(R) \longrightarrow \varinjlim_l H^0(C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n}))) \\ &\longrightarrow H^1_{(\underline{yt})}(\mathcal{R}'(\mathcal{F}))_{\underline{n}} \longrightarrow \cdots . \end{aligned}$$

Since $(R, \mathfrak{m})$ is Cohen-Macaulay, $H^i_{(\underline{y})}(R) = 0$ for $0 \leq i \leq d-1$. Hence,

$$\begin{aligned} H^1_{(\underline{yt})}(\mathcal{R}'(\mathcal{F}))_{\underline{n}} \cong \varinjlim_l H^0(C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n}))) &= \varinjlim_l H_d(C \cdot (\underline{y}^{[l]}, \mathcal{F}(\underline{n}))) \\ &= \varinjlim_l \frac{(\mathcal{F}(\underline{n}+le) : (\underline{y}^{[l]}))}{\mathcal{F}(\underline{n})}. \end{aligned}$$

Since $\sqrt{\mathcal{R}_{++}} = \sqrt{(\underline{yt})}$, by [**12,** Proposition 3.1, Proposition 4.2], for all $\underline{n} \in \mathbb{N}^s$,

$$H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} \cong H^1_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}} \cong \frac{\breve{\mathcal{F}}(\underline{n})}{\mathcal{F}(\underline{n})}. \qquad \square$$

For all $i = 1, \ldots, s$, we denote the *associated multigraded ring of* $\mathcal{F}$ *with respect to* $\mathcal{F}(e_i)$ by

$$G_i(\mathcal{F}) = \bigoplus_{\underline{n} \in \mathbb{N}^s} \frac{\mathcal{F}(\underline{n})}{\mathcal{F}(\underline{n} + e_i)}.$$

For $\mathcal{F} = \{\underline{I}^{\underline{n}}\}_{\underline{n} \in \mathbb{Z}^s}$, we set $G_i(\mathcal{F}) = G_i(\underline{I})$. In the next proposition, we give an equivalent criterion for the vanishing of $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}}$ for all $\underline{n} \geq \underline{0}$ in terms of $\operatorname{grade}(G_i(\mathcal{F})_{++})$ for all $i = 1, \ldots, s$.

Proposition 4.2. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* $d \geq 2$ *with infinite residue field,* $I_1, \ldots, I_s$ *the* $\mathfrak{m}$*-primary ideals of* R *and* $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ *an* $\underline{I}$*-admissible filtration of ideals in* R*. Then the following statements are equivalent.*

(1) *For all* $\underline{n} \in \mathbb{N}^s$, $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$.
(2) *For all* $i = 1, \ldots, s$, $\operatorname{grade}(G_i(\mathcal{F})_{++}) \geq 1$.

Proof. Fix i. Denote

$$\frac{\mathcal{R}'(\mathcal{F})}{\mathcal{R}'(\mathcal{F})(e_i)}$$

by $G'_i(\mathcal{F})$. Consider the short exact sequence of $\mathcal{R}(\underline{I})$-modules,

$$0 \longrightarrow \mathcal{R}'(\mathcal{F})(e_i) \longrightarrow \mathcal{R}'(\mathcal{F}) \longrightarrow G'_i(\mathcal{F}) \longrightarrow 0.$$

This induces a long exact sequence of local cohomology modules,

$$0 \longrightarrow H^0_{\mathcal{R}_{++}}(G'_i(\mathcal{F}))_{\underline{n}} \longrightarrow H^1_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}+e_i} \longrightarrow \cdots . \tag{4.1}$$

Since $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{N}^s$, by [**12,** Proposition 4.2], we have $H^1_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{N}^s$. Hence, from the exact sequence (4.1) and [**12,** Proposition 4.2], for all $\underline{n} \in \mathbb{N}^s$, we obtain

$$H^0_{G_i(\mathcal{F})_{++}}(G_i(\mathcal{F}))_{\underline{n}} \cong H^0_{G_i(\mathcal{F})_{++}}(G'_i(\mathcal{F}))_{\underline{n}} \cong H^0_{\mathcal{R}_{++}}(G'_i(\mathcal{F}))_{\underline{n}} = 0.$$

Conversely, suppose that $\operatorname{grade}(G_i(\mathcal{F})_{++}) \geq 1$ for all $i = 1, \ldots, s$. By [**12,** Theorem 3.3], for all $\underline{n} \in \mathbb{N}^s$,

$$\breve{\mathcal{F}}(\underline{n} + e_i) \cap \mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e_i).$$

We show that, if $\breve{\mathcal{F}}(\underline{n}) = \mathcal{F}(\underline{n})$ for some $\underline{n} \geq \underline{0}$, then $\breve{\mathcal{F}}(\underline{m}) = \mathcal{F}(\underline{m})$ for all $\underline{m} \geq \underline{n}$. Let $t_i = m_i - n_i$ for all $i = 1, \ldots, s$. For each i,

$$\mathcal{F}(\underline{n} + e_i) = \breve{\mathcal{F}}(\underline{n} + e_i) \cap \mathcal{F}(\underline{n}) = \breve{\mathcal{F}}(\underline{n} + e_i) \cap \breve{\mathcal{F}}(\underline{n}) = \breve{\mathcal{F}}(\underline{n} + e_i).$$

Continuing this process t_i times, for each i, we get $\breve{\mathcal{F}}(\underline{m}) = \mathcal{F}(\underline{m})$. Since $\breve{\mathcal{F}}(\underline{0}) = \mathcal{F}(\underline{0})$, by Proposition 4.1, we obtain the required result. $\square$

Remark 4.3. Note that, if $(R, \mathfrak{m})$ is an analytically unramified local ring of dimension $d \geq 1$, $I_1, \ldots, I_s$ are $\mathfrak{m}$-primary ideals of R, then, for the filtration $\mathcal{F} = \{\overline{\underline{I}^{\underline{n}}}\}_{\underline{n} \in \mathbb{Z}^s}$, by [**12,** Corollary 3.4], $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{N}^s$.

Proposition 4.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R and $\mathcal{A} = \{x_{ij} \in I_i : j = 1, \ldots, d; i = 1, \ldots, s\}$ be a good complete reduction of $\mathcal{F}$. Set $y_1 = x_{11} \ldots x_{s1}$. Let $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{N}^s$. Then,*

1. *$y_{i1} = y_1 + \underline{I}^{e+e_i} \in G_i(\underline{I})_e$ is $G_i(\mathcal{F})$-regular for all $i = 1, \ldots, s$.*
2. *$(y_1) \cap \mathcal{F}(\underline{n}) = y_1 \mathcal{F}(\underline{n} - e)$ for all $\underline{n} \geq e$.*

Proof.

(1) Fix i. Let $\underline{m} \geq e$ be such that $(y_1) \cap \mathcal{F}(\underline{n}) = y_1 \mathcal{F}(\underline{n} - e)$ for all $\underline{n} \geq \underline{m}$. We show that $(G_i(\mathcal{F}))_{\underline{n}} \xrightarrow{\cdot y_{i1}} (G_i(\mathcal{F}))_{\underline{n}+e}$ is injective for all $\underline{n} \geq \underline{m}$. Let $(z + \mathcal{F}(\underline{n} + e_i)) y_{i1} = \mathcal{F}(\underline{n} + e + e_i)$. Then, $y_1 z \in \mathcal{F}(\underline{n} + e + e_i)$. Since $\underline{n} \geq \underline{m}$, $z \in \mathcal{F}(\underline{n} + e_i)$. Hence, by Propositions 2.7 and 4.2, y_{i1} is a nonzerodivisor of $G_i(\mathcal{F})$.

(2) For all $i = 1, \ldots, s$, consider the Koszul complex $K_i \cdot = K_i \cdot (y_{i1}, G_i(\mathcal{F}))$:

$$0 \longrightarrow G_i(\mathcal{F}) \longrightarrow G_i(\mathcal{F})(e) \longrightarrow 0.$$

The $\underline{n}$th component of this complex is $K_i \cdot (y_{i1}, G_i(\mathcal{F}), \underline{n})$:

$$0 \longrightarrow G_i(\mathcal{F})_{(\underline{n})} \longrightarrow G_i(\mathcal{F})_{\underline{n}+e} \longrightarrow 0.$$

Hence, for all $i = 1, \ldots, s$ and $\underline{n} \geq \underline{0}$, we have the exact sequence of complexes

$$0 \longrightarrow K_i.(y_{i1}, G_i(\mathcal{F}), \underline{n}) \longrightarrow C\cdot(y_1, \mathcal{F}(\underline{n}+e_i)) \longrightarrow C\cdot(y_1, \mathcal{F}(\underline{n})) \longrightarrow 0,$$

which gives a long exact sequence of homology modules

$$\begin{aligned} \cdots \longrightarrow H_j(K_i \cdot (y_{i1}, G_i(\mathcal{F}), \underline{n})) &\longrightarrow H_j(C \cdot (y_1, \mathcal{F}(\underline{n} + e_i))) \\ &\longrightarrow H_j(C \cdot (y_1, \mathcal{F}(\underline{n}))) \longrightarrow \cdots . \end{aligned}$$

Since y_{i1} is a nonzerodivisor of $G_i(\mathcal{F})$ for all $i = 1, \ldots, s$, we have $H_1(K_i \cdot (y_{i1}, G_i(\mathcal{F}), \underline{n})) = 0$ for all $\underline{n}$ and $i = 1, \ldots, s$. Due to the fact that $H_1(C \cdot (y_1, \mathcal{F}(\underline{0}))) = 0$, applying the above exact sequence several times for all $i = 1, \ldots, s$, we obtain $H_1(C \cdot (y_1, \underline{n})) = 0$ for all $\underline{n} \geq \underline{0}$. Hence, $\mathcal{F}(\underline{n}) \cap (y_1) = y_1\mathcal{F}(\underline{n} - e)$ for all $\underline{n} \geq e$. □

Proposition 4.5. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 2 and $I_1, \ldots, I_s$ the $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R and $\mathcal{A} = \{x_{ij} \in I_i : j = 1, 2; i = 1, \ldots, s\}$ a good complete reduction of $\mathcal{F}$. Let $y_j = x_{1j} \cdots x_{sj}$ for all $j = 1, 2$ and $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{N}^s$. Then $H_2(C \cdot (y_1, y_2, \mathcal{F}(\underline{n}))) = 0$ for all $\underline{n} \geq \underline{0}$.*

Proof. Fix i. Let $y_{ij} = y_j + \underline{I}^{e+e_i} \in G_i(\underline{I})_e$ for all $j = 1, 2$ and $i = 1, \ldots, s$. Consider the Koszul complex $K \cdot (y_{i1}, y_{i2}, G_i(\mathcal{F}))$:

$$0 \longrightarrow G_i(\mathcal{F}) \longrightarrow (G_i(\mathcal{F})(e))^2 \longrightarrow G_i(\mathcal{F})(2e) \longrightarrow 0$$

whose $\underline{n}$th component is

$$0 \longrightarrow G_i(\mathcal{F})_{\underline{n}} \longrightarrow (G_i(\mathcal{F})_{\underline{n}+e})^2 \longrightarrow G_i(\mathcal{F})_{\underline{n}+2e} \longrightarrow 0.$$

For all $\underline{n} \geq \underline{0}$, we have the exact sequence

$$\begin{aligned} 0 \longrightarrow K \cdot (y_{i1}, y_{i2}, G_i(\mathcal{F}), \underline{n}) &\longrightarrow C \cdot (y_1, y_2, \mathcal{F}(\underline{n} + e_i)) \\ &\longrightarrow C \cdot (y_1, y_2, \mathcal{F}(\underline{n})) \longrightarrow 0. \end{aligned}$$

This gives a long exact sequence of homology modules

$$\begin{aligned} \cdots \longrightarrow H_j(K \cdot (y_{i1}, y_{i2}, G_i(\mathcal{F}), \underline{n})) &\longrightarrow H_j(C \cdot (y_1, y_2, \mathcal{F}(\underline{n} + e_i))) \\ &\longrightarrow H_j(C \cdot (y_1, y_2, \mathcal{F}(\underline{n}))) \longrightarrow \cdots \end{aligned}$$

for all $\underline{n} \geq \underline{0}$. Since, by Proposition 4.4, y_{i1} is a regular element in $G_i(\mathcal{F})$ for all $i = 1, \ldots, s$, we have $H_2(K \cdot (y_{i1}, y_{i2}, G_i(\mathcal{F}))) = 0$ for all $\underline{n}$ and $i = 1, \ldots, s$. Due to the fact that $H_2(C \cdot (y_1, y_2, \mathcal{F}(\underline{0}))) = 0$, using the above exact sequence several times for all $i = 1, \ldots, s$, we obtain $H_2(C \cdot (y_1, y_2, \mathcal{F}(\underline{n}))) = 0$ for all $\underline{n} \geq \underline{0}$. $\square$

Theorem 4.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $1 \leq d \leq 2$ with infinite residue field, $I_1, \ldots, I_s$ the $\mathfrak{m}$-primary ideals of R and $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ an $\underline{I}$-admissible filtration of ideals in R. Let $e_{(d-1)e_i}(\mathcal{F}) = 0$ for $i = 1, \ldots, s$. Then,*

(1) *For $d = 1$, $P_{\mathcal{F}}(\underline{n}) = H_{\mathcal{F}}(\underline{n})$ for all $\underline{n} \in \mathbb{N}^s$.*
(2) *For $d = 2$, if $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{N}^s$, then $P_{\mathcal{F}}(\underline{n}) = H_{\mathcal{F}}(\underline{n})$ for all $\underline{n} \in \mathbb{N}^s$ and $e_{\underline{0}}(\mathcal{F}) = 0$.*

Proof.

(1) For $d = 1$, since $e_{\underline{0}}(\mathcal{F}) = 0$, we obtain $P_{\mathcal{F}}(\underline{0}) = H_{\mathcal{F}}(\underline{0})$. Therefore, by the difference formula [**12,** Theorem 4.3], we obtain

$$\lambda_R(H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{0}}) = 0.$$

Since $\dim R = 1$, by [**12,** Lemma 2.11], for all $\underline{n} \in \mathbb{N}^s$, $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$. Therefore, again using the difference formula [**12,** Theorem 4.3], we obtain $P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n}) = 0$ for all $\underline{n} \in \mathbb{N}^s$.

(2) Let $d = 2$,

$$\mathcal{A} = \{x_{ij} \in I_i : j = 1, 2; i = 1, \ldots, s\}$$

be any good complete reduction of $\mathcal{F}$, $y_j = x_{1j} \cdots x_{sj}$ for $j = 1, 2$ and $J = (y_1, y_2)$. Fix i. Let $R' = R/(x_{i1})$ and $'$ denote the image of an ideal in R'. For all large $\underline{n}$, consider the following exact sequence

$$0 \longrightarrow \frac{(\mathcal{F}(\underline{n}) : (x_{i1}))}{\mathcal{F}(\underline{n} - e_i)} \longrightarrow \frac{R}{\mathcal{F}(\underline{n} - e_i)} \xrightarrow{\cdot x_{i1}} \frac{R}{\mathcal{F}(\underline{n})} \longrightarrow \frac{R}{(x_{i1}, \mathcal{F}(\underline{n}))} \longrightarrow 0.$$

Since $\mathcal{A}$ is a good complete reduction, for all large $\underline{n}$, $(\mathcal{F}(\underline{n}) : (x_{i1})) = \mathcal{F}(\underline{n} - e_i)$ and

$$\lambda\left(\frac{R}{(x_{i1}, \mathcal{F}(\underline{n}))}\right) = \lambda\left(\frac{R}{\mathcal{F}(\underline{n})}\right) - \lambda\left(\frac{R}{\mathcal{F}(\underline{n} - e_i)}\right).$$

Therefore, for the filtration $\mathcal{F}' = \{\mathcal{F}(\underline{n})R'\}_{\underline{n}\in\mathbb{Z}^s}$,we have $P_{\mathcal{F}'}(\underline{n}) = P_{\mathcal{F}}(\underline{n}) - P_{\mathcal{F}}(\underline{n}-e_i)$. This implies that the constant term of $P_{\mathcal{F}'}(\underline{n})$ is $e_{e_i}(\mathcal{F}) = 0$. Due to the fact that $\mathcal{A}' = \{x'_{i2} \in I_i : i = 1,\dots,s\}$ is a complete reduction of $\mathcal{F}'$, $J' = (y'_2)$ and $\dim R' = 1$, by Theorem 3.2, Proposition 2.6 and part (1), we have

$$\lambda\left(\frac{\mathcal{F}(\underline{n}+e)R'}{J'\mathcal{F}(\underline{n})R'}\right) = \Delta^1(P_{\mathcal{F}'}(\underline{n}) - H_{\mathcal{F}'}(\underline{n})) = 0 \quad \text{for all } \underline{n} \in \mathbb{N}^s.$$

Thus, we obtain $\mathcal{F}(\underline{n}+e) = y_2\mathcal{F}(\underline{n}) + ((x_{i1}) \cap \mathcal{F}(\underline{n}+e))$ for all $\underline{n} \in \mathbb{N}^s$. We show that $(x_{i1}) \cap \mathcal{F}(\underline{n}+e) = x_{i1}\mathcal{F}(\underline{n}+e-e_i)$ for all $\underline{n} \in \mathbb{N}^s$. It is clear that $x_{i1}\mathcal{F}(\underline{n}+e-e_i) \subseteq (x_{i1}) \cap \mathcal{F}(\underline{n}+e)$. Let $ax_{i1} \in \mathcal{F}(\underline{n}+e)$. Then, $ay_1 \in \mathcal{F}(\underline{n}+2e-e_i)$. Since $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{N}^s$ and $\mathcal{A}$ is a good complete reduction, by Proposition 4.4, $a \in \mathcal{F}(\underline{n}+e-e_i)$. Hence, we obtain

$$\mathcal{F}(\underline{n}+e) = y_2\mathcal{F}(\underline{n}) + x_{i1}\mathcal{F}(\underline{n}+e-e_i) \quad \text{for all } \underline{n} \in \mathbb{N}^s \text{ and } i = 1,\dots,s.$$

We show that $\mathcal{F}(\underline{n}+2e) = J\mathcal{F}(\underline{n}+e)$ for all $\underline{n} \in \mathbb{N}^s$. Let $\underline{n} \in \mathbb{N}^s$. Then,

$$\begin{aligned}
\mathcal{F}(\underline{n}+2e) &= y_2\mathcal{F}(\underline{n}+e) + x_{11}\mathcal{F}(\underline{n}+2e-e_1) \\
&= y_2\mathcal{F}(\underline{n}+e) + x_{11}(y_2\mathcal{F}(\underline{n}+e-e_1) \\
&\quad + x_{21}\mathcal{F}(\underline{n}+2e-e_1-e_2)) \\
&\subseteq y_2\mathcal{F}(\underline{n}+e) + x_{11}x_{21}\mathcal{F}(\underline{n}+2e-e_1-e_2) \\
&\qquad \vdots \\
&\subseteq y_2\mathcal{F}(\underline{n}+e) + x_{11}\cdots x_{s1}\mathcal{F}(\underline{n}+e) \\
&= (y_1,y_2)\mathcal{F}(\underline{n}+e) = J\mathcal{F}(\underline{n}+e).
\end{aligned}$$

Since $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{N}^s$, by Theorem 3.2 and Proposition 4.5, for all $\underline{n} \in \mathbb{N}^s$, we obtain

$$\Delta^2(P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n})) = \lambda\left(\frac{\mathcal{F}(\underline{n}+2e)}{J\mathcal{F}(\underline{n}+e)}\right),$$

and hence, by Proposition 2.6, we have $P_{\mathcal{F}}(\underline{n}) = H_{\mathcal{F}}(\underline{n})$ for all $\underline{n} \in \mathbb{N}^s$. Setting $\underline{n} = \underline{0}$ in the above equality, we obtain $e_{\underline{0}}(\mathcal{F}) = 0$. □

Theorem 4.7. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $1 \le d \le 2$ with infinite residue field and I, J the $\mathfrak{m}$-primary ideals of*

R. Let $\mathcal{F} = \{\mathcal{F}(r,s)\}_{r,s\in\mathbb{Z}}$ be a $\mathbb{Z}^2$-graded (I,J)-admissible filtration of ideals in R. Then, the following statements are equivalent.

(1) $e_{(d-1)e_i}(\mathcal{F}) = 0$ *for* $i = 1, 2$.
(2) *I and J are generated by a system of parameters,* $P_{\mathcal{F}}(r,s) = H_{\mathcal{F}}(r,s)$ *for all* $r,s \in \mathbb{N}$ *and* $\mathcal{F}(r,s) = I^rJ^s$ *for all* $r,s \in \mathbb{Z}$.
(3) $e_\alpha(\mathcal{F}) = 0$ *for* $|\alpha| \leq d-1$.

Proof.

(1) $\Rightarrow$ (2). Let $\mathcal{F}^{(1)} = \{\mathcal{F}(r,0)\}_{r\in\mathbb{Z}}$ and $\mathcal{F}^{(2)} = \{\mathcal{F}(0,s)\}_{s\in\mathbb{Z}}$. Since $\mathcal{F}$ is an (I,J)-admissible filtration, $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are I-admissible and J-admissible filtrations, respectively. By [**10,** Lemma 3.19], [**12,** Theorem 5.5] and [**13**], we have

$$0 \leq e_1(I) \leq e_1(\mathcal{F}^{(1)}) \leq e_{(d-1)e_1}(\mathcal{F}) = 0,$$
$$0 \leq e_1(J) \leq e_1(\mathcal{F}^{(2)}) \leq e_{(d-1)e_2}(\mathcal{F}) = 0.$$

Then, by [**10,** Theorem 3.21], we obtain that I and J are generated by a system of parameters, $\mathcal{F}(r,0) = I^r$ and $\mathcal{F}(0,s) = J^s$ for all $r,s \in \mathbb{Z}$. Let $d = 1$. Then, by Theorem 4.6, $P_{\mathcal{F}}(r,s) = H_{\mathcal{F}}(r,s)$ for all $r,s \in \mathbb{N}$. It is sufficient to prove that $\mathcal{F}(r,s) = I^rJ^s$ for all $r,s \geq 1$. Since I and J are generated by the system of parameters, for $r,s \geq 1$, we have

$$\begin{aligned}\lambda\left(\frac{R}{\mathcal{F}(r,s)}\right) &= P_{\mathcal{F}}(r,s) = \mathrm{re}(I) + \mathrm{se}(J)\\ &= \lambda\left(\frac{R}{I^r}\right) + \lambda\left(\frac{R}{J^s}\right) = \lambda\left(\frac{R}{I^r}\right) + \lambda\left(\frac{I^r}{I^rJ^s}\right) = \lambda\left(\frac{R}{I^rJ^s}\right).\end{aligned}$$

This implies that $\mathcal{F}(r,s) = I^rJ^s$ for all $r,s \geq 1$. Let $d = 2$. Since I and J are parameter ideals, we have

$$e(I) - e_{e_1}(\mathcal{F}) = e(I) = \lambda\left(\frac{R}{I}\right) = \lambda\left(\frac{R}{\mathcal{F}(1,0)}\right),$$
$$e(J) - e_{e_2}(\mathcal{F}) = e(J) = \lambda\left(\frac{R}{J}\right) = \lambda\left(\frac{R}{\mathcal{F}(0,1)}\right).$$

Therefore, by [**12,** Theorem 7.3], we obtain $P_{\mathcal{F}}(r,s) = H_{\mathcal{F}}(r,s)$ for all $r,s \in \mathbb{N}$. It is sufficient to prove that $\mathcal{F}(r,s) = I^rJ^s$ for all $r,s \geq 1$. By [**12,** Theorem 7.3], the joint reduction number of $\mathcal{F}$ of type e is zero.

Let (a,b) be a joint reduction of $\mathcal{F}$ of type e. Then,

$$\mathcal{F}(r,s) = a\mathcal{F}(r-1,s) + b\mathcal{F}(r,s-1) \quad \text{for all } r,s \geq 1.$$

We use induction on $r+s$. Let $r,s \geq 1$. If $r+s=2$, then $r=s=1$ and

$$\mathcal{F}(1,1) = a\mathcal{F}(0,1) + b\mathcal{F}(1,0) = aJ + bI \subseteq IJ \subseteq \mathcal{F}(1,1).$$

Let $r+s>2$. Then $r \geq 2$ or $s \geq 2$. Without loss of generality, assume that $r \geq 2$. If $s=1$, then using induction, we get

$$\mathcal{F}(r,1) = a\mathcal{F}(r-1,1) + b\mathcal{F}(r,0) = aI^{r-1}J + bI^r \subseteq I^rJ \subseteq \mathcal{F}(r,1).$$

Hence, we may assume that $s \geq 2$. Therefore,

$$\begin{aligned}\mathcal{F}(r,s) &= a\mathcal{F}(r-1,s) + b\mathcal{F}(r,s-1)\\ &= aI^{r-1}J^s + bI^rJ^{s-1} \subseteq I^rJ^s \subseteq \mathcal{F}(r,s).\end{aligned}$$

(2) $\Rightarrow$ (3). For $d=1$, putting $r=s=0$ in the equation $P_{\mathcal{F}}(r,s) = H_{\mathcal{F}}(r,s)$, we obtain the required result. Let $d=2$. Since $P_{\mathcal{F}}(r,s) = H_{\mathcal{F}}(r,s)$ for all $r,s \in \mathbb{N}$, we have $e_{\underline{0}}(\mathcal{F}) = 0$,

$$e(I) - e_{e_1}(\mathcal{F}) = \lambda\left(\frac{R}{\mathcal{F}(1,0)}\right)$$

and

$$e(J) - e_{e_2}(\mathcal{F}) = \lambda\left(\frac{R}{\mathcal{F}(0,1)}\right).$$

Since I and J are parameter ideals and $\mathcal{F}(r,s) = I^rJ^s$ for all $r,s \in \mathbb{Z}$, by [**12,** Theorem 5.5], we have

$$e_{e_1}(\mathcal{F}) = e_1(\mathcal{F}^{(1)}) = e_1(I) = 0 \quad \text{and} \quad e_{e_2}(\mathcal{F}) = e_1(\mathcal{F}^{(2)}) = e_1(J) = 0.$$

(3) $\Rightarrow$ (1). It follows directly. $\square$

Example 4.8. Let $R = k[|X,Y|]$. Then, R is a regular local ring of dimension 2. Let $I = (X, Y^2)$ and $J = (X^2, Y)$. Then, I and J are complete parameter ideals in R. Consider the filtration $\mathcal{F} = \{\overline{I^rJ^s}\}_{r,s\in\mathbb{Z}}$. Since I and J are complete ideals, by [**18,** Theorem 2′, Appendix 5], I^r, J^s and I^rJ^s are complete ideals as well. By [**14,**

Theorem 1.2],

$$e_{e_1}(\mathcal{F}) = \overline{e}_1(I) = e_1(I) = 0 \quad \text{and} \quad e_{e_2}(\mathcal{F}) = \overline{e}_1(J) = e_1(J) = 0.$$

Since

$$\begin{aligned} e(I) - e_{e_1}(\mathcal{F}) = e(I) = \lambda\left(\frac{R}{I}\right) = \lambda\left(\frac{R}{\overline{I}}\right), \\ e(J) - e_{e_2}(\mathcal{F}) = e(J) = \lambda\left(\frac{R}{J}\right) = \lambda\left(\frac{R}{\overline{J}}\right), \end{aligned}$$

by [**12,** Theorem 7.3], we obtain $P_{\mathcal{F}}(r,s) = H_{\mathcal{F}}(r,s)$ for all $r, s \in \mathbb{N}$.

5. Postulation and reduction vectors in dimension 1. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 1 with infinite residue field, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. In this section, we prove that the set of reduction vectors of $\mathcal{F}$ with respect to any complete reduction is the same as the set of postulation vectors of $\mathcal{F}$. Thus, the set of reduction vectors of $\mathcal{F}$ with respect to any complete reduction is independent of the choice of complete reduction. Then, we show that the complete reduction number of $\mathcal{F}$ with respect to any complete reduction is independent of the choice of complete reduction.

Theorem 5.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 1 with infinite residue field, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R and $\mathcal{A} = \{a_i \in I_i : i = 1, \ldots, s\}$ a complete reduction of $\mathcal{F}$. Then,*

$$\mathcal{P}(\mathcal{F}) \subseteq \mathbb{N}^s \quad \textit{and} \quad \mathcal{P}(\mathcal{F}) = \mathcal{R}_{\mathcal{A}}(\mathcal{F}).$$

Moreover, the set $\mathcal{R}_{\mathcal{A}}(\mathcal{F})$ is independent of any complete reduction $\mathcal{A}$ of $\mathcal{F}$.

Proof. First, we prove that $\mathcal{P}(\mathcal{F}) \subseteq \mathbb{N}^s$. Suppose that there exists an $\underline{n} \in \mathbb{Z}^s \setminus \mathbb{N}^s$ such that $\underline{n} \in \mathcal{P}(\mathcal{F})$. Then, there exists at least one $i \in \{1, \ldots, s\}$ such that $n_i < 0$. Therefore,

$$P_{\mathcal{F}}(\underline{n} + e_i) = \lambda\left(\frac{R}{\mathcal{F}(\underline{n} + e_i)}\right) = \lambda\left(\frac{R}{\mathcal{F}(\underline{n})}\right) = P_{\mathcal{F}}(\underline{n})$$

implies $e_0(I_i) = P_{\mathcal{F}}(\underline{n}+e_i) - P_{\mathcal{F}}(\underline{n}) = 0$. This contradicts the fact that $e_0(I_i) > 0$. Thus, $\mathcal{P}(\mathcal{F}) \subseteq \mathbb{N}^s$. Let $J = (a_1 \cdots a_s)$. By Theorem 3.2, for all $\underline{n} \geq \underline{0}$,

$$\Delta^1(P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n})) = \lambda\left(\frac{\mathcal{F}(\underline{n}+e)}{J\mathcal{F}(\underline{n})}\right).$$

Hence, by Proposition 2.6, we obtain the required result. □

Example 5.2. Let $R = k[|t^3, t^4, t^5|]$. Then, R is a one dimensional Cohen-Macaulay local ring with unique maximal ideal $\mathfrak{m} = (t^3, t^4, t^5)$. Consider $I = (t^3, t^4)$ and $J = (t^3)$. Then, $JI^2 = I^3$. Since $(t^6)(IJ)^2 = (IJ)^3$, $\mathcal{A} = \begin{pmatrix} t^3 \\ t^3 \end{pmatrix}$ is a complete reduction for the filtration $\underline{I} = \{I^r J^s\}_{r,s\in\mathbb{Z}}$. We have $\lambda(R/J^n) = 3n$ for all $n \in \mathbb{N}$.

Now, $\lambda(R/I) = 2$, $\lambda(R/I^2) = 4$ and, for $n \geq 3$,

$$I^n = J^{n-2}I^2 = (t^{3n-6})(t^6, t^7, t^8) = (t^{3n}, t^{3n+1}, t^{3n+2}).$$

Hence, for all $n \geq 2$, $\lambda(R/I^n) = 3n - 2$. Let

$$P_I(n) = ne_0(I) - e_1(I),$$
$$P_J(n) = ne_0(J) - e_1(J),$$
$$P_{IJ}(n) = ne_0(IJ) - e_1(IJ)$$

and

$$P_{\underline{I}}(\underline{n}) = n_1 e_0(I) + n_2 e_0(J) - e_{\underline{0}}(\underline{I})$$

denote the Hilbert polynomials of I, J, IJ and $\underline{I}$, respectively, where $n \in \mathbb{Z}$ and $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$. Then, $e_0(I) = e_0(J) = 3$, $e_1(I) = 2$ and $e_1(J) = 0$. Now, by Lemma 2.2, $e_{\underline{0}}(\underline{I}) = e_1(IJ)$. For large n,

$$P_{IJ}(n) = ne_0(IJ) - e_1(IJ) = \lambda\left(\frac{R}{(IJ)^n}\right) = \lambda\left(\frac{R}{I^{2n}}\right) = 6n - 2.$$

Hence, $e_{\underline{0}}(\underline{I}) = e_1(IJ) = 2$. This implies $P_{\underline{I}}(\underline{n}) = 3n_1 + 3n_2 - 2$. Since $(t^6)(IJ) = IJ^3 \neq I^2J^2$ and $(t^6)(IJ)^2 = I^2J^4 = I^3J^3$, we have $r_{\mathcal{A}}(\underline{I}) = 2$.

Note that $(t^6)I^2 = J^2I^2 = JI^3$ and $(t^6)I = J^2I \neq I^2J$. Let $(1, n) \in \mathbb{Z}^2$ be such that $n \geq 1$. Then,

$$(t^6)IJ^n = IJ^{n+2} = (t^3, t^4)(t^{3n+6}) = (t^{3n+9}, t^{3n+10})$$
$$\neq (t^{3n+9}, t^{3n+10}, t^{3n+11}) = I^2J^{n+1}.$$

Hence, $\mathcal{R}_{\mathcal{A}}(\underline{I}) = \{\underline{m} \in \mathbb{N}^2 \mid \underline{m} \geq (2,0)\}$.

For all $n_1 \geq 2$ and $n_2 \geq 0$,

$$P_{\underline{I}}(n_1,n_2) = 3n_1 + 3n_2 - 2 = \lambda\left(\frac{R}{I^{n_1+n_2}}\right) = \lambda\left(\frac{R}{I^{n_1}J^{n_2}}\right) = H_{\underline{I}}(n_1,n_2)$$

and

$$P_{\underline{I}}(1,0) = 1 \neq 2 = \lambda\left(\frac{R}{I}\right).$$

Let $(1,n) \in \mathbb{Z}^2$ be such that $n \geq 1$. Then, $P_{\underline{I}}(1,n) = 3n+1$ and

$$H_{\underline{I}}(1,n) = \lambda\left(\frac{R}{IJ^n}\right) = \lambda\left(\frac{R}{(t^{3n+3}, t^{3n+4})}\right) = 3n+2.$$

Hence, for all $\underline{n} = (1,n)$ where $n \geq 0$, $P_{\underline{I}}(1,n) \neq H_{\underline{I}}(1,n)$. Thus, $\mathcal{R}_{\mathcal{A}}(\underline{I}) = \mathcal{P}(\underline{I})$.

In the next example, we show that we cannot drop the condition of Cohen-Macaulayness in Theorem 5.1.

Example 5.3. Let

$$R = \frac{k[|X,Y|]}{(X^2, XY)}.$$

Then, R is a one-dimensional local ring which is not Cohen-Macaulay. Consider the ideals $I = (x,y)$ and $J = (y)$ of R. Then, $JI = I^2$. Since $(y^2)(IJ) = I^2J^2$, $\mathcal{A} = \binom{y}{y}$ is a complete reduction for the filtration $\underline{I} = \{I^rJ^s\}_{r,s\in\mathbb{Z}}$. We have

$$\lambda\left(\frac{R}{J^n}\right) = n+1 \quad \text{for all } n \geq 1.$$

Now $\lambda(R/I) = 1$ and, for $n \geq 2$, $I^n = J^{n-1}I = (y^{n-1})(x,y) = (y^n)$. Hence, for all $n \geq 2$, $\lambda(R/I^n) = n+1$. Let

$$P_I(n) = ne_0(I) - e_1(I),$$
$$P_J(n) = ne_0(J) - e_1(J),$$
$$P_{IJ}(n) = ne_0(IJ) - e_1(IJ)$$

and

$$P_{\underline{I}}(\underline{n}) = n_1e_0(I) + n_2e_0(J) - e_{\underline{0}}(\underline{I})$$

denote the Hilbert polynomials with respect to I, J, IJ and $\underline{I}$, respectively, where $n \in \mathbb{Z}$ and $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$. Then, $e_0(I) = e_0(J) = 1$. Now, by Lemma 2.2, $e_{\underline{0}}(\underline{I}) = e_1(IJ)$. For large n,

$$P_{IJ}(n) = ne_0(IJ) - e_1(IJ) = \lambda\left(\frac{R}{(IJ)^n}\right) = \lambda\left(\frac{R}{I^{2n}}\right) = 2n+1.$$

Hence, $e_{\underline{0}}(\underline{I}) = e_1(IJ) = -1$. This implies $P_{\underline{I}}(\underline{n}) = n_1 + n_2 + 1$. Now $IJ = (x,y)(y) = (y^2)$. Hence, $r_{\mathcal{A}}(\underline{I}) = 0$. This implies $\mathcal{R}_{\mathcal{A}}(\underline{I}) = \mathbb{N}^2$; however, $P_{\underline{I}}(0,0) = 1 \neq 0 = H_{\underline{I}}(0,0)$. Therefore, $\mathcal{R}_{\mathcal{A}}(\underline{I}) \neq \mathcal{P}(\underline{I})$.

Theorem 5.4. *Let $(R, \mathfrak{m})$ be a one dimensional Cohen-Macaulay local ring, and let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then, the complete reduction number of $\mathcal{F}$ with respect to any complete reduction is independent of the choice of complete reduction of $\mathcal{F}$.*

Proof. Let $\mathcal{A} = \{a_i \in I_i : i = 1, \ldots, s\}$ be a complete reduction of $\mathcal{F}$, $J = (a_1 \cdots a_s)$ and $r_{\mathcal{A}}(\mathcal{F}) = k$. First, we show that

$$k = \min\{\max\{t_1, \ldots, t_s : \underline{t} = (t_1, \ldots, t_s) \in \mathcal{R}_{\mathcal{A}}(\mathcal{F})\}\}.$$

If $k = 0$, then it is true. Suppose that $k \geq 1$. Let $\underline{n} \in \mathbb{N}^s$ be such that $n_i < k$ for all $i = 1, \ldots, s$ and $\underline{n} \in \mathcal{R}_{\mathcal{A}}(\mathcal{F})$. Let $u = \max\{n_1, \ldots, n_s\}$. Then, $u < k$ and $\underline{n} \leq ue \leq (k-1)e$. Hence, $J\mathcal{F}(\underline{m}) = \mathcal{F}(\underline{m} + e)$ for all $\underline{m} \geq (k-1)e$. This contradicts the fact that k is the complete reduction number of $\mathcal{F}$ with respect to $\mathcal{A}$. Thus, $\underline{t} \in \mathcal{R}_{\mathcal{A}}(\mathcal{F})$ implies $t_i \geq k$ for at least one i. Since $J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ for all $\underline{n} \geq ke$, there exists an $\underline{r} \in \mathcal{R}_{\mathcal{A}}(\mathcal{F})$ such that $\max\{r_1, \ldots, r_s\} = k$. Hence,

$$k = \min\{\max\{t_1, \ldots, t_s : \underline{t} = (t_1, \ldots, t_s) \in \mathcal{R}_{\mathcal{A}}(\mathcal{F})\}\}.$$

Therefore, by Theorem 5.1, we obtain the required result. $\square$

6. Postulation and reduction vectors in dimension 2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 2 with infinite residue field and $I_1, \ldots, I_s$ the $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. In this section, we provide a relation between the reduction vectors of $\mathcal{F}$ with respect to any good complete reduction and the postulation vectors of $\mathcal{F}$. For a bigraded filtration $\mathcal{F}$, we prove a result which relates the Cohen-

Macaulayness of the bigraded Rees algebra, the complete reduction number, reduction numbers and the joint reduction number.

Theorem 6.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 2 with infinite residue field, and let $I_1, \dots, I_s$ be $\mathfrak{m}$-primary ideals of R and $s \geq 2$. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R and*

$$\mathcal{A} = \{x_{ij} \in I_i : j = 1, 2; i = 1, \dots, s\}$$

a good complete reduction of $\mathcal{F}$. Let $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \geq \underline{0}$. Then, $\mathcal{P}(\mathcal{F}) \subseteq \mathbb{N}^s$, and there exists a one-to-one correspondence

$$f : \mathcal{P}(\mathcal{F}) \longleftrightarrow \{\underline{r} \in \mathcal{R}_{\mathcal{A}}(\mathcal{F}) \mid \underline{r} \geq e\}$$

defined by $f(\underline{n}) = \underline{n} + e$ where $f^{-1}(\underline{r}) = \underline{r} - e$.

Proof. First, we prove that $\mathcal{P}(\mathcal{F}) \subseteq \mathbb{N}^s$. Suppose that there exists an $\underline{n} \in \mathbb{Z}^s \setminus \mathbb{N}^s$ such that $\underline{n} \in \mathcal{P}(\mathcal{F})$. Then, there exists at least one $i \in \{1, \dots, s\}$ such that $n_i < 0$. Therefore, for any $j \in \{1, \dots, s\}$ with $j \neq i$ and $l \geq 0$,

$$P_{\mathcal{F}}(\underline{n}+le_j+e_i) = \lambda\left(\frac{R}{\mathcal{F}(\underline{n}+le_j+e_i)}\right) = \lambda\left(\frac{R}{\mathcal{F}(\underline{n}+le_j)}\right) = P_{\mathcal{F}}(\underline{n}+le_j).$$

Thus, for $l = 1$, we obtain

$$\begin{aligned} 0 &= P_{\mathcal{F}}(\underline{n}+e_j+e_i) - P_{\mathcal{F}}(\underline{n}+e_j) \\ &= (n_i+1)e_0(I_i) + \sum_{k \neq i,j} n_k e_{e_k+e_i}(\mathcal{F}) \\ &\quad + (n_j+1)e_{e_j+e_i}(\mathcal{F}) - e_{e_i}(\mathcal{F}). \end{aligned}$$

Then, for $l = 0$, we obtain

$$\begin{aligned} 0 &= P_{\mathcal{F}}(\underline{n}+e_i) - P_{\mathcal{F}}(\underline{n}) \\ &= (n_i+1)e_0(I_i) + \sum_{k \neq i} n_k e_{e_k+e_i}(\mathcal{F}) - e_{e_i}(\mathcal{F}) \\ &= -e_{e_j+e_i}(\mathcal{F}). \end{aligned}$$

This contradicts the fact that $e_{e_j+e_i}(\mathcal{F}) > 0$ [**15,** Theorem 2.4]. Hence, $\mathcal{P}(\mathcal{F}) \subseteq \mathbb{N}^s$.

Let $y_j = x_{1j} \cdots x_{sj}$ for $j = 1, 2$ and $J = (y_1, y_2)$. Then, by Theorem 3.2 and Proposition 4.5, for all $\underline{n} \geq \underline{0}$,

$$\Delta^2(P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n})) = \lambda\left(\frac{\mathcal{F}(\underline{n} + 2e)}{J\mathcal{F}(\underline{n} + e)}\right).$$

Hence, by Proposition 2.6, we get the required result. $\square$

Theorem 6.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 2 with infinite residue field and $I_1, \ldots, I_s$ the $\mathfrak{m}$-primary ideals of R and $s \geq 2$. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R and $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \geq \underline{0}$. Then, the following statements are equivalent.*

(1) *$\mathcal{P}(\mathcal{F}) = \mathbb{N}^s$, i.e., $P_{\mathcal{F}}(\underline{n}) = H_{\mathcal{F}}(\underline{n})$ for all $\underline{n} \geq \underline{0}$.*
(2) *$r_{\mathcal{A}}(\mathcal{F}) \leq 1$ for any good complete reduction $\mathcal{A}$ of $\mathcal{F}$.*
(2′) *There exists a good complete reduction $\mathcal{A}$ of $\mathcal{F}$ such that $r_{\mathcal{A}}(\mathcal{F}) \leq 1$.*

Proof.

(1) $\Rightarrow$ (2). Let $P_{\mathcal{F}}(\underline{n}) = H_{\mathcal{F}}(\underline{n})$ for all $\underline{n} \geq \underline{0}$ and $\mathcal{A} = \{x_{ij} \in I_i : j = 1, 2; i = 1, \ldots, s\}$ be any good complete reduction of $\mathcal{F}$. Let $y_j = x_{1j} \cdots x_{sj}$ for $j = 1, 2$ and $J = (y_1, y_2)$. Then, by Theorem 3.2, Propositions 2.6 and 4.5, for all $\underline{n} \geq \underline{0}$,

$$\lambda\left(\frac{\mathcal{F}(\underline{n} + 2e)}{J\mathcal{F}(\underline{n} + e)}\right) = 0 \Longrightarrow J\mathcal{F}(\underline{n} + e) = \mathcal{F}(\underline{n} + 2e).$$

Hence, $r_{\mathcal{A}}(\mathcal{F}) \leq 1$.

The implication (2) $\Rightarrow$ (2′) is trivial.

(2′) $\Rightarrow$ (1). Suppose that there exists a good complete reduction $\mathcal{A} = \{x_{ij} \in I_i : j = 1, 2; i = 1, \ldots, s\}$ of $\mathcal{F}$ such that $r_{\mathcal{A}}(\mathcal{F}) \leq 1$. Let $y_j = x_{1j} \cdots x_{sj}$ for $j = 1, 2$ and $J = (y_1, y_2)$. Then, again by Theorem 3.2 and Proposition 4.5, for all $\underline{n} \geq \underline{0}$,

$$\Delta^2(P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n})) = 0.$$

Now, using Proposition 2.6, we obtain $P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n}) = 0$ for all $\underline{n} \geq \underline{0}$. Since $\mathcal{P}(\mathcal{F}) \subseteq \mathbb{N}^s$, we obtain the required result. $\square$

In the next example, we show that we cannot drop the condition on $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}}$ in Theorem 6.2.

Example 6.3. Let $R = k[|X,Y|]$. Then R is a two-dimensional Cohen-Macaulay local ring with unique maximal ideal $\mathfrak{m} = (X,Y)$. Let $I = \mathfrak{m}^2$ and $J = (X^2, Y^2)$. Since $(X^4, Y^4)IJ = (X^4, Y^4)\mathfrak{m}^4 = \mathfrak{m}^8 = I^2J^2$, we have

$$\mathcal{A} = \begin{pmatrix} X^2 & Y^2 \\ X^2 & Y^2 \end{pmatrix}$$

is a complete reduction for the filtration $\underline{I} = \{I^r J^s\}_{r,s\in\mathbb{Z}}$. By **[4,** Proposition 1.2.2], for all large $\underline{n} = (n_1, n_2) \in \mathbb{N}^2$, we obtain

$$(X^4) \cap I^{n_1}J^{n_2} = X^4(I^{n_1}J^{n_2} : (X^4)) = X^4 I^{n_1-1}J^{n_2-1}.$$

Hence, $\mathcal{A}$ is a good complete reduction for the filtration $\underline{I}$. Note that, since $\underbrace{\underline{I}^{e_2}} = \breve{J} = (I^k J^{1+k} : I^k J^k)$ for some large k, $JI = I^2$ and $\mathfrak{m}$ is parameter ideal, we have

$$\breve{J} = (\mathfrak{m}^{4k+2} : \mathfrak{m}^{4k}) = \mathfrak{m}^2 \neq J,$$

and hence, $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\underline{I}))_{(0,1)} \neq 0$. Since $(X^4, Y^4)IJ = I^2J^2$, we obtain $r_{\mathcal{A}}(\underline{I}) \leq 1$. For $n \geq 1$,

$$\lambda\left(\frac{R}{I^n}\right) = \lambda\left(\frac{R}{\mathfrak{m}^{2n}}\right) = \binom{2n+1}{2} = 4\binom{n+1}{2} - n = P_I(n),$$

Since J is parameter ideal and $JI = I^2$,

$$\lambda\left(\frac{R}{J^n}\right) = 4\binom{n+1}{2} = P_J(n),$$
$$\lambda\left(\frac{R}{(IJ)^n}\right) = \lambda\left(\frac{R}{\mathfrak{m}^{4n}}\right) = \binom{4n+1}{2} = 16\binom{n+1}{2} - 6n = P_{IJ}(n)$$

and

$$\lambda\left(\frac{R}{I^{2n}J^n}\right) = \lambda\left(\frac{R}{\mathfrak{m}^{6n}}\right) = \binom{6n+1}{2} = 36\binom{n+1}{2} - 15n = P_{I^2J}(n).$$

Hence, $e_0(I) = e_0(J) = 4$. Now, for large n, $P_{IJ}(n) = \lambda(R/(IJ)^n) = P_{\underline{I}}(ne)$ and $P_{I^2J}(n) = \lambda(R/(I^{2n}J^n))) = P_{\underline{I}}(2n, n)$. Comparing the coefficients on both sides, we obtain

$$P_{\underline{I}}(r,s) = 4\binom{r+1}{2} + 4\binom{s+1}{2} + 4rs - r - s.$$

Then $P_{\underline{I}}(0,1) = 3 \neq 4 = \lambda(R/J)$.

Theorem 6.4. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 2 with infinite residue field and I, J the $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^2}$ be a $\mathbb{Z}^2$-graded (I,J)-admissible filtration of ideals in R. Then, the following statements are equivalent.*

(1) *The Rees algebra $\mathcal{R}(\mathcal{F})$ is Cohen-Macaulay.*
(2) *$\mathcal{P}(\mathcal{F}) = \mathbb{N}^2$, i.e., $P_{\mathcal{F}}(\underline{n}) = H_{\mathcal{F}}(\underline{n})$ for all $\underline{n} \geq \underline{0}$.*
(3) *For any good complete reduction $\mathcal{A}$ of $\mathcal{F}$, $r_{\mathcal{A}}(\mathcal{F}) \leq 1$ and $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \geq \underline{0}$.*
(3′) *There exists a good complete reduction $\mathcal{A}$ of $\mathcal{F}$ such that $r_{\mathcal{A}}(\mathcal{F}) \leq 1$ and $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \geq \underline{0}$.*
(4) *For the filtrations $\mathcal{F}^{(i)} = \{\mathcal{F}(ne_i)\}_{n\in\mathbb{Z}}$, $r(\mathcal{F}^{(i)}) \leq 1$ where $i = 1, 2$, and the joint reduction number of $\mathcal{F}$ of type e is zero.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1) and (3) $\Rightarrow$ (3′) $\Rightarrow$ (2) follow from [**12,** Theorem 7.3] and Theorem 6.2, respectively. It is sufficient to show (1) $\Rightarrow$ (3). Suppose that $\mathcal{R}(\mathcal{F})$ is Cohen-Macaulay. Then, by [**12,** Theorem 2.15, Proposition 7.2], we obtain $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\mathcal{F}))_{\underline{n}} = 0$ for all $\underline{n} \geq \underline{0}$. Since (1) implies (2), by Theorem 6.2, we obtain the required result. $\square$

The next example illustrates Theorems 6.1, 6.2 and 6.4.

Example 6.5. Let $R = K[|X,Y|]$. Then, R is a two dimensional Cohen-Macaulay local ring with unique maximal ideal $\mathfrak{m} = (X,Y)$. Let $I = \mathfrak{m}^2$ and $J = \mathfrak{m}^3$. Since

$$(X^5, Y^5)IJ = (X^5, Y^5)\mathfrak{m}^5 = \mathfrak{m}^{10} = I^2J^2,$$

we have

$$\mathcal{A} = \begin{pmatrix} X^2 & Y^2 \\ X^3 & Y^3 \end{pmatrix}$$

is a complete reduction for filtrations $\underline{I} = \{I^r J^s\}_{r,s\in\mathbb{Z}}$ and $r_{\mathcal{A}}(\underline{I}) \leq 1$. By [**4,** Proposition 1.2.2], for all large $\underline{n} = (n_1, n_2) \in \mathbb{N}^2$, we obtain

$$(X^5) \cap I^{n_1} J^{n_2} = X^5 (I^{n_1} J^{n_2} : (X^5)) = X^5 I^{n_1 - 1} J^{n_2 - 1}.$$

Hence, $\mathcal{A}$ is a good complete reduction for the filtration $\underline{I}$.

We show that $H^1_{\mathcal{R}_{++}}(\mathcal{R}(\underline{I}))_{\underline{n}} = 0$ for all $\underline{n} \geq \underline{0}$. For $\underline{n} = (n_1, n_2) \geq \underline{0}$, we have

$$\widetilde{\underline{I}^{\underline{n}}} = (\underline{I}^{\underline{n}+ke} : \underline{I}^{ke}) = (\mathfrak{m}^{2n_1+3n_2+5k} : \mathfrak{m}^{5k})$$

for some large k. Since $\mathfrak{m}$ is a parameter ideal, $\widetilde{\underline{I}^{\underline{n}}} = \mathfrak{m}^{2n_1+3n_2} = \underline{I}^{\underline{n}}$ for all $\underline{n} \geq \underline{0}$. For $n \geq 1$,

$$\lambda\left(\frac{R}{I^n}\right) = \lambda\left(\frac{R}{\mathfrak{m}^{2n}}\right) = \binom{2n+1}{2} = 4\binom{n+1}{2} - n = P_I(n),$$

$$\lambda\left(\frac{R}{J^n}\right) = \lambda\left(\frac{R}{\mathfrak{m}^{3n}}\right) = \binom{3n+1}{2} = 9\binom{n+1}{2} - 3n = P_J(n),$$

$$\lambda\left(\frac{R}{(IJ)^n}\right) = \lambda\left(\frac{R}{\mathfrak{m}^{5n}}\right) = \binom{5n+1}{2} = 25\binom{n+1}{2} - 10n = P_{IJ}(n)$$

and

$$\lambda\left(\frac{R}{I^n J^{2n}}\right) = \lambda\left(\frac{R}{\mathfrak{m}^{8n}}\right) = \binom{8n+1}{2} = 64\binom{n+1}{2} - 28n = P_{IJ^2}(n).$$

Hence, $e_0(I) = 4$ and $e_0(J) = 9$.

Now, for large n,

$$P_{IJ}(n) = \lambda\left(\frac{R}{(IJ)^n}\right) = P_{\underline{I}}(ne)$$

and

$$P_{IJ^2}(n) = \lambda\left(\frac{R}{I^n J^{2n}}\right) = P_{\underline{I}}(n, 2n).$$

Comparing the coefficients on both sides, we obtain

$$P_{\underline{I}}(n_1, n_2) = 4\binom{n_1+1}{2} + 9\binom{n_2+1}{2} + 6n_1 n_2 - n_1 - 3n_2. \tag{6.1}$$

Hence, by [**12,** Theorem 6.2], the joint reduction number of $\underline{I}$ of type e is zero. Since $(X^2, Y^2)I = I^2$ and $(X^3, Y^3)J = J^2$, we have

$r(\underline{I}^{(i)}) \leq 1$ for $i = 1, 2$, where $\underline{I}^{(1)} = \{I^r\}_{r\in\mathbb{Z}}$ and $\underline{I}^{(2)} = \{J^s\}_{s\in\mathbb{Z}}$. By [**3,** Corollaries 2.3, 2.4], $\mathcal{R}(\underline{I}^{(1)})$ and $\mathcal{R}(\underline{I}^{(2)})$ are Cohen-Macaulay. Hence, by [**12,** Theorem 7.1], $\mathcal{R}(\underline{I})$ is Cohen-Macaulay.

Using (6.1), we obtain $P_{\underline{I}}(n_1, n_2) = H_{\underline{I}}(n_1, n_2)$ for all $\underline{n} = (n_1, n_2) \in \mathbb{N}^2$. Thus, $\mathcal{P}(\underline{I}) = \mathbb{N}^2$. Since $(X^5, Y^5)IJ = I^2J^2$, $(X^5, Y^5)I \neq I^2J$, $(X^5, Y^5)J \neq IJ^2$, $(X^5, Y^5)I^2 = I^3J$ and$(X^5, Y^5)J^2 = IJ^3$, we have

$$\mathcal{R}_{\mathcal{A}}(\underline{I}) = \{\underline{m} \in \mathbb{N}^2 \mid \underline{m} \geq e\} \cup \{\underline{m} \in \mathbb{N}^2 \mid \underline{m} \geq 2e_1\} \cup \{\underline{m} \in \mathbb{N}^2 \mid \underline{m} \geq 2e_2\}.$$

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