

## QUASI-GORENSTEINNESS OF EXTENDED REES ALGEBRAS

YOUNGSU KIM

**ABSTRACT.** Let  $R$  be a Noetherian local ring and  $I$  an  $R$ -ideal. It is well known that, if the associated graded ring  $\text{gr}_I(R)$  is Cohen-Macaulay (Gorenstein), then so is  $R$ , but in general, the converse is not true. In this paper, we investigate the Cohen-Macaulayness and Gorensteinness of the associated graded ring  $\text{gr}_I(R)$  under the hypothesis that the extended Rees algebra  $R[It, t^{-1}]$  is quasi-Gorenstein or the associated graded ring  $\text{gr}_I(R)$  is a domain.

**1. Introduction.** A Noetherian ring with a canonical module is called *quasi-Gorenstein* if it is locally isomorphic to the canonical module. Clearly, a ring is Gorenstein if and only if it is quasi-Gorenstein and Cohen-Macaulay. Murthy [15] showed that a Cohen-Macaulay UFD having a canonical module is Gorenstein. In general, the UFD property implies quasi-Gorensteinness if the ring has a canonical module. There exists a complete UFD having a canonical module which is not Cohen-Macaulay, see [5, Theorem 5.8]. This shows that a quasi-Gorenstein ring need not be Gorenstein in general. Surprisingly, the quasi-Gorenstein property implies Gorensteinness for some classes of extended Rees algebras. In this regard, Heinzer, M.-K. Kim and Ulrich posed the following question.

**Question 1.1** ([6, Question 4.11]). *Let  $(R, \mathfrak{m})$  be a local Gorenstein ring, and let  $I$  be an  $\mathfrak{m}$ -primary ideal. Is the extended Rees algebra  $R[It, t^{-1}]$  Gorenstein (equivalently Cohen-Macaulay) if it is quasi-Gorenstein?*

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In the case where the dimension of  $R$  is one, Question 1.1 has an affirmative answer because  $R[It, t^{-1}]$  has dimension 2 in this case and quasi-Gorenstein rings satisfy Serre's condition  $(S_2)$ . The authors showed that Question 1.1 has an affirmative answer when  $R$  is a two dimensional pseudo-rational ring [6, Corollary 4.12]. The general case still remains open. However, if one removes the condition of  $I$  being  $\mathfrak{m}$ -primary, then there exists an extended Rees algebra which is a UFD (hence, quasi-Gorenstein), but not Gorenstein [12, Example 4.7].

In Section 3, we provide an affirmative answer to Question 1.1 if  $I$  is an almost complete intersection under the additional assumption that the index of nilpotency and the reduction number of  $I$  coincide (which is a necessary condition for  $R[It, t^{-1}]$  to be Cohen-Macaulay), see Theorem 3.19. We are also able to treat the case where  $I$  is a monomial ideal in a polynomial ring in  $d$ -variables and  $I$  has a  $d$ -generated monomial reduction, see Theorem 3.14. The latter condition,  $I$  having such a reduction, is equivalent to the condition that  $I$  has only one Rees valuation.

For monomial ideals in a polynomial ring, the normalization of the extended Rees algebra is Cohen-Macaulay [11]. Therefore, these normalizations are quasi-Gorenstein if and only if they are Gorenstein. In [7], the authors characterized the Gorenstein property of normalized extended Rees algebras of a monomial ideal when the ideal of finite colength has only one Rees valuation. Recall that the Rees valuations of such monomial ideals correspond to the bounded half spaces defining the Newton polyhedron of  $I$ . We are able to remove the condition of having one Rees valuation, and provide a characterization of the Gorenstein property in terms of the half spaces, defining the Newton polyhedron of the given ideal.

Since  $\text{gr}_I(R) \cong R[It, t^{-1}]/(t^{-1})$  and  $t^{-1}$  is a homogeneous non zero-divisor, the associated graded ring  $\text{gr}_I(R)$  is Cohen-Macaulay (or Gorenstein) if and only if so is the extended Rees algebra  $R[It, t^{-1}]$ . Hochster [11, page 55, Proposition] and Herzog, Simis and Vasconcelos [9, Proposition 1.1] showed that, if  $R$  is a local Gorenstein ring and the associated graded ring  $\text{gr}_I(R)$  is a domain, then  $R[It, t^{-1}]$  is quasi-Gorenstein. Hence, an affirmative answer to Question 1.1 would imply that  $\text{gr}_I(R)$  is Cohen-Macaulay if it is a domain. This version of the question is meaningful even when the ambient ring  $R$  is not Gorenstein.

**Question 1.2.** *Let  $(R, \mathfrak{m})$  be a local Cohen-Macaulay ring. Is  $\text{gr}_{\mathfrak{m}}(R)$  Cohen-Macaulay if it is a domain?*

Question 1.2 has an affirmative answer when  $R$  is a complete intersection ring of embedding codimension at most 2, i.e.,  $\widehat{R} \cong S/I$ , where  $(S, \mathfrak{n})$  is a regular local ring and  $I$  is a complete intersection ideal of height at most 2. It is natural to ask whether the question has an affirmative answer when the ideal  $I$  is generated by three elements. In Section 5, we prove that this is indeed the case if, in addition,  $I \not\subseteq \mathfrak{n}^5$  (Theorem 5.7).

We have shown that, if  $\text{gr}_I(R)$  is Cohen-Macaulay, then  $R$  is Cohen-Macaulay. Recall that the Cohen-Macaulayness of a ring can be characterized by Serre's condition  $(S_i)$  for every  $i$ .

**Question 1.3.** *Let  $R$  be a Noetherian ring and  $I$  be an  $R$ -ideal. If  $\text{gr}_I(R)$  satisfies Serre's condition  $(S_i)$  (or  $(R_i)$ ), then does  $R[It, t^{-1}]$  satisfy the same condition?*

We give a positive answer to Question 1.3 for proper ideals in a universally catenary equidimensional local ring (Theorem 6.2).

The outline of the paper is as follows. In Section 2, we provide the notation. In Section 3, we begin by introducing the graded canonical module of  $\mathbb{Z}$ -graded rings and study basic properties related to the quasi-Gorensteinness of extended Rees algebras. The two main theorems, Theorems 3.14, 3.19, are proved in this section. In addition, a result on the  $\mathfrak{a}$ -invariant of the extended Rees algebra (Theorem 3.24) and the core of powers of the ideal (Theorem 3.20) are presented. In Section 4, we provide a characterization of the Gorensteinness of normalized extended Rees algebras of finite colength monomial ideals in a polynomial ring. This is a generalization of [7, Theorem 5.6]. In Section 5, we discuss Question 1.2 in detail and provide a positive answer in the case of almost complete intersections of codimension 2 (Theorem 5.7). In Section 6, we give a positive answer to Question 1.3 when the ring in question is a universally catenary equidimensional local ring and the ideal is not a unit ideal (Theorem 6.2).

**2. Preliminaries.** All rings are commutative Noetherian with unity. Let  $(R, \mathfrak{m})$  be a local ring and  $M$  a finitely generated  $R$ -module. Let  $\mu(M)$  denote the minimal number of generators of  $M$ . By  $\text{Min}(M)$  and  $\text{Ass}(M)$ , we denote the set of minimal primes and associated primes of  $M$ , respectively. For  $R$ -ideals  $J \subseteq I$ , the ideal  $J$  is called a *reduction* of  $I$  if there exists a non-negative integer  $n$  such that  $JI^n = I^{n+1}$ . The smallest such integer is called the *reduction number* of  $I$  with respect to  $J$ , denoted by  $r_J(I)$ , and let  $r(I) := \min\{r_J(I) \mid J \text{ a reduction of } I\}$ . For an  $r \times s$  matrix  $\Phi$  with entries in  $R$ , we write  $I_n(\Phi)$  for the ideal generated by the  $n \times n$  minors of  $\Phi$ . By convention, we set  $I_n(\Phi) = R$  for  $n \leq 0$ , and we set  $I_n(\Phi) = 0$  for  $n > \min\{r, s\}$ .

For an ideal  $I \subset R$ , we write

$$R[It] = \bigoplus_{i \geq 0} I^i t^i, R[It, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} I^i t^i, \text{ and } \text{gr}_I(R) = \bigoplus_{i \geq 0} I^i / I^{i+1},$$

for the *Rees algebra*, the *extended Rees algebra* and the *associated graded ring* of  $R$  with respect to the ideal  $I$ , respectively. Sometimes, they are also called *blowup algebras*. These are the rings which appear in the construction of blowing up an affine variety along a closed subvariety in algebraic geometry.

Let  $\omega_R$  denote a *canonical module* of  $R$ , if it exists. Here the *canonical module* is the dualizing module in the sense of Grothendieck's local duality theorem. For instance, when  $(R, \mathfrak{m})$  is a complete local ring of dimension  $d$ , then a canonical module  $\omega_R$  of  $R$  is  $\text{Hom}_R(\text{H}_{\mathfrak{m}}^d(R), E_R(R/\mathfrak{m}))$ , where  $\text{H}_{\mathfrak{m}}^d(-)$  and  $E_R(-)$  denote the  $d$ th local cohomology with respect to  $\mathfrak{m}$  and the injective envelope, respectively. A local ring  $R$  with a canonical module  $\omega_R$  is called *Gorenstein* if  $R$  is Cohen-Macaulay and  $R$  is the canonical module  $\omega_R$ . The second property  $R \cong \omega_R$  can be isolated, and it is called the *quasi-Gorenstein property*, that is, a local ring is *quasi-Gorenstein* if it is isomorphic to the canonical module  $\omega_R$ .

Let  $S$  be a Noetherian  $\mathbb{Z}$ -graded ring with unique homogeneous maximal ideal  $\mathfrak{M}$ , and assume that  $\mathfrak{M}$  is maximal. For a graded module  $M$ , let  $[M]_i$  denote the  $i$ th graded piece of  $M$ . The  *$\mathfrak{a}$ -invariant* of  $S$ , denoted by  $a(S)$ , is

$$\max\{i \in \mathbb{Z} \mid [* \text{Soc}(\text{H}_{\mathfrak{M}}^d(S))]_i \neq 0\}.$$

Here  $*\text{Soc}(M) = 0 :_M \mathfrak{M}$  for any graded  $S$ -module  $M$ . If  $S$  has a

graded canonical module  $\omega_S$ , then, by graded local duality, we have

$$a(S) = -\min\{i \in \mathbb{Z} \mid [\omega_S/\mathfrak{M}\omega_S]_i \neq 0\}.$$

If  $S$  is positively graded, then this number is

$$\max\{i \in \mathbb{Z} \mid [\mathbb{H}_{\mathfrak{M}}^d(S)]_i \neq 0\}.$$

In the sequel, we use the book by Bruns and Herzog [2] as a reference for basic terminology and definitions.

### 3. Quasi-Gorensteinness of extended Rees algebras.

**Graded canonical modules.** Let  $R$  be a Noetherian  $\mathbb{Z}$ -graded ring with unique maximal homogeneous ideal  $\mathfrak{M}$ . Then the subring  $R_0$  is local. We write  $\mathfrak{m}$  for the maximal ideal of  $R_0$ . Let  $E_{R_0}(R_0/\mathfrak{m})$  be the injective envelope of the residue field of  $R_0$ . For a homogeneous prime ideal  $\mathfrak{p}$  of  $R$ , we write  $R_{(\mathfrak{p})} := S^{-1}R$  where  $S$  is the set of homogeneous elements of  $R$  which are not in  $\mathfrak{p}$ . Observe that  $R_{(\mathfrak{p})}$  is  $\mathbb{Z}$ -graded and has unique maximal homogeneous ideal  $\mathfrak{p}R_{(\mathfrak{p})}$ . For  $\mathbb{Z}$ -graded  $R$ -modules  $M$  and  $N$ , let  ${}^*\text{Hom}_R(M, N)$  denote the  $R$ -submodule of  $\text{Hom}_R(M, N)$  generated by the homogeneous  $R$ -linear maps of arbitrary degree from  $M$  to  $N$ . For a finitely generated  $\mathbb{Z}$ -graded  $R$ -module  $M$ , let  $\widehat{M}$  denote the tensor product  $M \otimes_{R_0} \widehat{R_0}^{\mathfrak{m}}$ . We say  $M$  is *\*-complete* if  $M \cong \widehat{M}$  by the natural isomorphism. In particular,  $\widehat{R}$  is *\*-complete*, and  $R$  is *\*-complete* if and only if  $R_0$  is complete.

**Definition 3.1.** Let  $d = \dim R_{\mathfrak{M}}$ . A finitely generated graded  $R$ -module  $\omega_R$  is called a *graded canonical module* of  $R$  if

$$\widehat{\omega_R} \cong {}^*\text{Hom}_{R_0}(\mathbb{H}_{\mathfrak{M}}^d(R), E_{R_0}(R_0/\mathfrak{m}))$$

as graded  $R$ -modules.

**Remark 3.2.** Let  $R$  be a Noetherian  $\mathbb{Z}$ -graded ring with a unique maximal homogeneous ideal.

- (a) If  $R$  is *\*-complete*, then  $R$  has a graded canonical module.
- (b) [4, Exercise 7.5]. For finitely generated  $\mathbb{Z}$ -graded  $R$ -modules  $M$  and  $N$ , if  $\widehat{M} \cong \widehat{N}$ , then  $M \cong N$ .

**Lemma 3.3** (cf., [2, Corollary 3.6.14]). *Let  $R$  be a  $\mathbb{Z}$ -graded Cohen-Macaulay ring with a unique maximal homogeneous ideal. Assume that  $R$  has a graded canonical module  $\omega_R$ . If  $\underline{x} = x_1, \dots, x_n$  form a homogeneous regular sequence on  $R$ , then  $\underline{x}$  form a regular sequence on  $\omega_R$ , and we have*

$$\omega_{R/(\underline{x})} \cong (\omega_R/\underline{x}\omega_R) \left( \sum_{i=1}^n \deg(x_i) \right).$$

*Proof.* It suffices to show the statement when  $n = 1$ . Let  $\mathfrak{M}$  be the unique maximal homogeneous ideal,  $d = \dim R_{\mathfrak{M}}$  and  $\mathfrak{m}$  the maximal ideal of  $R_0$ . By Remark 3.2 (b), we may assume that  $R$  is  $*$ -complete (hence, so is  $R/xR$ ). Since  $R$  and  $R/(x)$  are Cohen-Macaulay, the exact sequence

$$0 \longrightarrow R(-\deg(x)) \xrightarrow{\cdot x} R \longrightarrow R/(x) \longrightarrow 0$$

induces the exact sequence

$$0 \longrightarrow H_{\mathfrak{M}}^{d-1}(R/(x)) \longrightarrow H_{\mathfrak{M}}^d(R)(-\deg(x)) \xrightarrow{\cdot x} H_{\mathfrak{M}}^d(R) \longrightarrow 0.$$

Taking  ${}^* \text{Hom}_{R_0}(-, E_{R_0}(R_0/\mathfrak{m}))$ , we obtain the exact sequence (3.1)

$$0 \longrightarrow \omega_R \xrightarrow{\cdot x} \omega_R(\deg(x)) \longrightarrow {}^* \text{Hom}_{R_0}(H_{\mathfrak{M}}^{d-1}(R/(x)), E_{R_0}(R_0/\mathfrak{m})) \longrightarrow 0.$$

This shows that  $x$  is a nonzerodivisor of  $\omega_R$ . Hence, we are done once we have shown the isomorphism

$$\omega_{R/(x)} \cong {}^* \text{Hom}_{R_0}(H_{\mathfrak{M}}^{d-1}(R/(x)), E_{R_0}(R_0/\mathfrak{m})).$$

From the ring homomorphism  $R \rightarrow R/(x)$ , we obtain a surjective ring homomorphism  $R_0 \rightarrow [R/(x)]_0$  of local rings. We write  $(\overline{R_0}, \overline{\mathfrak{m}})$  for the local ring  $[R/(x)]_0$ . By [13, Exercise 13], we have

$$E_{\overline{R_0}}(\overline{R_0}/\overline{\mathfrak{m}}) \cong \text{Hom}_{R_0}(\overline{R_0}, E_{R_0}(R_0/\mathfrak{m})).$$

Therefore, by the hom-tensor adjointness, we obtain

$$\begin{aligned} {}^* \text{Hom}_{\overline{R_0}}(-, E_{\overline{R_0}}(\overline{R_0}/\overline{\mathfrak{m}})) &\cong {}^* \text{Hom}_{\overline{R_0}}(-, {}^* \text{Hom}_{R_0}(\overline{R_0}, E_{R_0}(R_0/\mathfrak{m}))) \\ &\cong {}^* \text{Hom}_{R_0}(- \otimes_{\overline{R_0}} \overline{R_0}, E_{R_0}(R_0/\mathfrak{m})) \\ &\cong {}^* \text{Hom}_{R_0}(-, E_{R_0}(R_0/\mathfrak{m})), \end{aligned}$$

for any  $R/(x)$ -module in the first variable. Since

$$H_{\mathfrak{M}}^{d-1}(R/(x)) \cong H_{\mathfrak{M}/(x)}^{d-1}(R/(x)),$$

this shows the statement. □

The next lemma is a partial converse of Lemma 3.3.

**Lemma 3.4.** *Let  $R$  be a  $\mathbb{Z}$ -graded Cohen-Macaulay ring with a unique maximal homogeneous ideal. Assume that  $\underline{x} = x_1, \dots, x_n$  forms a homogeneous regular sequence, and  $R/(x_1, \dots, x_n)$  has a graded canonical module  $\omega_{R/(x_1, \dots, x_n)}$ . Write  $\rho = \sum_{i=1}^n \deg(x_i)$ . If  $\omega_{R/(x_1, \dots, x_n)} \cong (R/(x_1, \dots, x_n))(a)$  for some  $a \in \mathbb{Z}$ , then  $\omega_R \cong R(a - \rho)$ .*

*Proof.* It suffices to show the statement when  $n = 1$ . By Remark 3.2 (a),  $\omega_{\widehat{R}}$  exists, and by Remark 3.2 (b), it suffices to show that  $\widehat{R(a - \rho)} \cong \omega_{\widehat{R}}$ . Hence, we may assume that  $R$  is  $*$ -complete. By Lemma 3.3, we have  $\omega_R/x\omega_R(\deg(x)) \cong \omega_{R/(x)} \cong R/(x)(a)$ . By Nakayama’s lemma, we see that  $\omega_R$  is a cyclic  $R$ -module. Consider the exact sequence

$$(3.2) \quad 0 \longrightarrow K \longrightarrow R(a - \deg(x)) \longrightarrow \omega_R \longrightarrow 0.$$

We tensor (3.2) with  $R/(x)$ . By Lemma 3.3,  $x$  is a nonzerodivisor on  $\omega_R$ . This implies that  $\text{Tor}_1^R(\omega_R, R/(x)) = 0$ , i.e., (3.2) remains exact after applying  $- \otimes_R R/(x)$ . Since  $R/(x)(a - \deg(x)) \cong \omega_R/x\omega_R$ , one has  $K/xK = 0$ . By Nakayama’s lemma, we obtain  $K = 0$ . Indeed, this implies

$$R(a - \deg(x)) \cong \omega_R. \quad \square$$

**Theorem 3.5** (cf., [8, Theorem 5.12]). *Let  $R$  and  $S$  be Noetherian  $\mathbb{Z}$ -graded rings with unique maximal homogeneous ideals  $\mathfrak{M}$  and  $\mathfrak{N}$ , respectively. Let  $\phi : S \rightarrow R$  be a graded ring homomorphism. Assume that  $R$  is a finitely generated  $S$ -module,  $\phi(S_0) = R_0$  and  $S$  is Cohen-Macaulay. We write  $\dim S_{\mathfrak{N}} = n$  and  $\dim R_{\mathfrak{M}} = d$ . If  $S$  has a graded canonical module  $\omega_S$ , then*

$$\omega_R \cong {}^* \text{Ext}_S^{n-d}(R, \omega_S).$$

*Proof.* By Remark 3.2 (b), it suffices to show the isomorphism after  $*$ -completions. Since  $\phi(S_0) = R_0$ , by  $*$ -completing both  $R$  and  $S$  as

$S$ -modules, we may assume that  $R$  and  $S$  are  $*$ -complete. Let  $\mathfrak{m}_{S_0}$  and  $\mathfrak{m}_{R_0}$  denote the maximal ideals of  $S_0$  and  $R_0$ , respectively. Set  $E' := E_{S_0}(S_0/\mathfrak{m}_{S_0})$  and  $E := E_{R_0}(R_0/\mathfrak{m}_{R_0})$ . By [13, Exercise 13], we have  $E \cong \text{Hom}_{S_0}(R_0, E')$ . From the hom-tensor adjunction and the graded version of the local duality theorem [2, Theorem 3.6.19(b)], we obtain the isomorphisms:

$$\begin{aligned} \omega_R &\cong {}^* \text{Hom}_{R_0}(H_{\mathfrak{M}}^d(R), E) \\ &\cong {}^* \text{Hom}_{R_0}(H_{\mathfrak{M}}^d(R), \text{Hom}_{S_0}(R_0, E')) \\ &\cong {}^* \text{Hom}_{S_0}(H_{\mathfrak{M}}^d(R) \otimes_{R_0} R_0, E') \\ &\cong {}^* \text{Hom}_{S_0}(H_{\mathfrak{M}}^d(R), E') \\ &\cong {}^* \text{Ext}_S^{n-d}(R, \omega_S), \end{aligned}$$

and this completes the proof. □

**Corollary 3.6.** *Let  $R$  be a Noetherian  $\mathbb{Z}$ -graded ring with a unique maximal homogeneous ideal  $\mathfrak{M}$ . Let  $S = A[X_1, \dots, X_n]$  be a  $\mathbb{Z}$ -graded polynomial ring over a Gorenstein local ring  $A$ . Assume that there exists a surjective graded ring homomorphism  $\phi : S \rightarrow R$  with  $\phi(A) = R_0$  and  $\mathfrak{M}$  is maximal. Let  $\mathfrak{N} := \phi^{-1}(\mathfrak{M})$ ,  $g = \text{ht ker}(\phi)$ , and  $\rho = \sum_{i=1}^n \text{deg}(X_i)$ . Then we obtain  $\omega_{S_{(\mathfrak{N})}} \cong S_{(\mathfrak{N})}(-\rho)$  and*

$$\begin{aligned} \omega_R &\cong {}^* \text{Ext}_{S_{(\mathfrak{N})}}^g(R, S_{(\mathfrak{N})})(-\rho) \\ &\cong {}^* \text{Ext}_S^g(R, S)(-\rho). \end{aligned}$$

*Proof.* Since  ${}^* \text{Ext}_S^g(R, S)$  is a graded  $R$ -module and  $R = R_{(\mathfrak{M})}$ , we have

$$\begin{aligned} {}^* \text{Ext}_S^g(R, S) &\cong {}^* \text{Ext}_S^g(R, S)_{(\mathfrak{M})} \cong {}^* \text{Ext}_S^g(R, S)_{(\mathfrak{N})} \\ &\cong {}^* \text{Ext}_{S_{(\mathfrak{N})}}^g(R_{(\mathfrak{N})}, S_{(\mathfrak{N})}) \cong {}^* \text{Ext}_{S_{(\mathfrak{N})}}^g(R_{(\mathfrak{M})}, S_{(\mathfrak{N})}) \\ &\cong {}^* \text{Ext}_{S_{(\mathfrak{N})}}^g(R, S_{(\mathfrak{N})}). \end{aligned}$$

Therefore, it suffices to show the first isomorphism.

First we show that we can reduce to the case where all  $X_i$  are in  $\mathfrak{N}$ . If  $\text{deg}(X_i) \neq 0$ , then  $\phi(X_i)$  is a homogeneous element of degree not equal to 0 in  $R$ . Since  $\mathfrak{M}$  is maximal, there is no homogeneous unit of degree not equal to 0. Hence,  $\phi(X_i) \in \mathfrak{M}$ , i.e.,  $X_i \in \mathfrak{N}$ . Therefore, if  $X_i \notin \mathfrak{N}$ ,



then  $\text{deg}(X_i) = 0$ . Suppose  $X_i \notin \mathfrak{N}$ . Since  $\phi(X_i) \in R_0 = \phi(A)$ , there exist  $z_i \in A \subseteq S_0$  such that  $z_i = \phi(X_i)$ . Since  $X_i - z_i$  is in  $\ker(\phi)$ ,  $X_i - z_i \in \mathfrak{N}$ . Replacing the variable  $X_i$  by  $X_i - z_i$ , we may assume that  $X_i \in \mathfrak{N}$ .

Since  $X_i \in \mathfrak{N}$  for all  $i$ , we have  $\mathfrak{N} = (\mathfrak{m}, X_1, \dots, X_n)$  where  $\mathfrak{m}$  is the maximal ideal of  $A$ . Let  $S' = S_{(\mathfrak{N})}$ . Since  $\mathfrak{N} = \phi^{-1}(\mathfrak{M})$ ,  $\phi$  factors through  $S'$ . Write  $\phi' : S' \rightarrow R$  for the ring homomorphism induced by  $\phi$ . Since  $S'$  and  $R$  have unique maximal homogeneous ideals,  $\phi'$  surjective, and  $\phi'(A) = R_0$ , by Theorem 3.5, we are done once we have shown that  $\omega_{S'} \cong S'(-\rho)$ . By Remark 3.2 (b), it suffices to show the isomorphism after  $*$ -completion. Hence, we may assume that  $S'$  is  $*$ -complete. Write  $\underline{\mathbf{X}} = X_1, \dots, X_n$ . Then  $\underline{\mathbf{X}}$  form a homogeneous regular sequence on  $S'$ , and

$$S'/(\underline{\mathbf{X}})S' \cong S/(\underline{\mathbf{X}})_{(\mathfrak{N})} \cong A_{(\mathfrak{N})} = A_{\mathfrak{m}} = A.$$

Since  $A$  is a Gorenstein local ring concentrated in degree 0, we have  $\omega_A \cong A$ . Then

$$S'/(\underline{\mathbf{X}})S' \cong A \cong \omega_A \cong \omega_{S'/(\underline{\mathbf{X}})S'}$$

and Lemma 3.4 imply the isomorphism  $\omega_{S'} \cong S'(-\rho)$ . This completes the proof. □

Recall that a Noetherian  $\mathbb{Z}$ -graded ring  $R$  with a unique maximal homogeneous ideal is called *quasi-Gorenstein* if  $\omega_R \cong R(a)$  for some  $a \in \mathbb{Z}$ . If the unique maximal ideal is maximal, the number  $a$  is well defined, and it is called the  $\mathbf{a}$ -invariant of the ring  $R$ . A Gorenstein  $\mathbb{Z}$ -graded ring is a quasi-Gorenstein ring which is Cohen-Macaulay. The ring  $S_{(\mathfrak{N})}$  in Remark 3.2 is a Gorenstein ring. Note that this definition agrees with [2, Theroem 3.6.19] when  $R$  is Cohen-Macaulay. However, we do not require a ring to be Cohen-Macaulay.

**Graded canonical modules of extended Rees algebras.**

**Proposition 3.7.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring having a canonical module  $\omega_R$ . Let  $I \subseteq \mathfrak{m}$  be an  $R$ -ideal. If  $R[It, t^{-1}]$  is quasi-Gorenstein, then  $R$  is Gorenstein.*

*Proof.* The canonical module of  $R[t, t^{-1}]$  is isomorphic to  $\omega_R[t, t^{-1}]$  [1, Proposition 4.1]. Since  $R[t, t^{-1}] = R[It, t^{-1}]_{(t^{-1})^{-1}}$  and canonical modules localize,

$$\omega_R[t, t^{-1}] \cong (\omega_{R[It, t^{-1}]})_{(t^{-1})^{-1}} \cong (R[It, t^{-1}])_{(t^{-1})^{-1}}(a) = R[t, t^{-1}](a),$$

where  $a$  is the  $\mathfrak{a}$ -invariant of  $R[It, t^{-1}]$ . Hence, we have  $R \cong \omega_R$ , i.e.,  $R$  is Gorenstein.  $\square$

**Lemma 3.8.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring with a canonical module  $\omega_R$  and  $I \subseteq \mathfrak{m}$  an ideal. If  $R[It, t^{-1}]$  is quasi-Gorenstein, then the canonical module  $\omega_{\text{gr}_I(R)}$  of  $\text{gr}_I(R)$  is of rank 1, and there exists an inclusion*

$$\text{gr}_I(R)(\rho) \subseteq \omega_{\text{gr}_I(R)}$$

of graded  $\text{gr}_I(R)$ -modules where  $\rho = a(R[It, t^{-1}]) - 1$ .

*Proof.* Write  $T = R[It, t^{-1}]$  and  $G = \text{gr}_I(R)$ . We need to show that  $(\omega_G)_{\mathfrak{p}} \cong G_{\mathfrak{p}}$  for all the associated primes of  $\omega_G$ . Since  $G$  is equidimensional and unmixed,  $\text{Ass}(\omega_G) = \text{Min}(G)$  [1, (1.7)]. Let  $\pi : T \xrightarrow{\text{nat}} G$  and  $\mathfrak{p} \in \text{Min}(G)$ . Write  $P = \pi^{-1}(\mathfrak{p})$ . Since  $T$  satisfies Serre’s condition  $(S_2)$  and  $P \in \text{Min}(T/(t^{-1}))$ ,  $P$  is of height 1. Hence,  $T_P$  is Gorenstein. Since  $G_{\mathfrak{p}} \cong T_P/(t^{-1})_P$ ,  $G_{\mathfrak{p}}$  is Gorenstein, and this shows that  $(\omega_G)_{\mathfrak{p}} \cong G_{\mathfrak{p}}$  is of rank 1.

Now, we show the inclusion

$$\text{gr}_I(R)(\rho) \subseteq \omega_{\text{gr}_I(R)}.$$

Let  $S := R[X_1, \dots, X_n]$  be a  $\mathbb{Z}$ -graded polynomial ring over  $R$  such that  $\phi : S \rightarrow R[It, t^{-1}]$  is a surjective homogeneous ring homomorphism. Let  $H = \ker(\phi)$  and  $g = \text{ht } H$ . Observe that  $G = S/(H, h)$ , where  $\phi(h) = t^{-1}$ . From the exact sequence of graded  $T$ -modules

$$0 \longrightarrow T(1) \xrightarrow{t^{-1}} T \longrightarrow G \longrightarrow 0,$$

we have an exact sequence of graded Ext modules

$$0 \longrightarrow \text{Ext}_S^g(T, S) \xrightarrow{t^{-1}} \text{Ext}_S^g(T(1), S) \longrightarrow \text{Ext}_S^{g+1}(G, S).$$

By Corollary 3.6, this shows that

$$(3.3) \quad 0 \longrightarrow \omega_T \xrightarrow{t^{-1}} \omega_T(-1) \longrightarrow \omega_G,$$

is exact, and the result follows since  $\omega_T \cong T(a)$ , where  $a$  is the  $\mathbf{a}$ -invariant of  $T$ . □

**Theorem 3.9.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring having a canonical module  $\omega_R$  and  $I \subseteq \mathfrak{m}$  an ideal. Assume that a graded canonical module  $\omega_{R[It, t^{-1}]}$  of  $R[It, t^{-1}]$  satisfies Serre’s condition  $(S_3)$ . This gives us the following statements.*

- (a) *If  $R[It, t^{-1}]$  is quasi-Gorenstein, then  $(\omega_{R[It, t^{-1}]} / t^{-1}\omega_{R[It, t^{-1}]}) (1)$  is a graded canonical module of  $\text{gr}_I(R)$ .*
- (b)  *$R[It, t^{-1}]$  is quasi-Gorenstein if and only if  $\text{gr}_I(R)$  is quasi-Gorenstein.*

*Proof.* Let  $G = \text{gr}_I(R)$  and  $T = R[It, t^{-1}]$ . Let  $\omega_G$  be the graded canonical module of  $G$ .

(a) We have

$$0 \longrightarrow (\omega_T / t^{-1}\omega_T)(-1) \longrightarrow \omega_G \longrightarrow L \longrightarrow 0,$$

where  $L$  is the cokernel of the natural map in (3.3). We show that  $L = 0$ . This is equivalent to showing that  $L_{\mathfrak{p}} = 0$  for all  $\mathfrak{p} \in \text{Ass}(L)$ . Since  $\omega_T$  satisfies Serre’s condition  $(S_3)$ ,  $\omega_T / t^{-1}\omega_T$  satisfies Serre’s condition  $(S_2)$  as a  $G$ -module. Let  $\mathfrak{p} \in \text{Ass}(L)$ . Since  $T_{\mathfrak{p}}$  is Gorenstein for all prime ideals of height at most 2,  $G_{\mathfrak{p}}$  is Gorenstein for all prime ideals of height at most 1. Hence, if  $\mathfrak{p} \in \text{Ass}(L)$ , then  $\text{ht } \mathfrak{p} \geq 2$ . Since  $\mathfrak{p}$  is an associated prime of  $L$ ,  $\text{depth } L_{\mathfrak{p}} = 0$ . However,  $\text{depth}(\omega_T / t^{-1}\omega_T)_{\mathfrak{p}} \geq 2$  and  $\text{depth}(\omega_G)_{\mathfrak{p}} \geq 2$ . This implies that  $\text{depth } L_{\mathfrak{p}} \geq 1$ , and this is a contradiction.

(b) The forward direction follows immediately from part (a). For the other direction, we only need to show that  $\omega_T$  is a cyclic faithful  $T$ -module. From the isomorphism  $\omega_G \cong \omega_T / t^{-1}\omega_T$  and  $\mu_T(\omega_T) = \lambda_T(\omega_T / \mathfrak{M}_T \omega_T) = \mu_G(\omega_T / t^{-1}\omega_T) = 1$ , where  $\mathfrak{M}_T$  is the maximal homogeneous ideal of  $T$ , we conclude that  $\omega_T$  is cyclic. Here,  $\lambda(-)$  denotes the length of a module. Also,  $\omega_T$  is faithful since  $T$  is unmixed, cf., [1, (1.8)(a)(c)]. Therefore,  $\omega_T \cong T(a(G) + 1)$ . □

**Main theorems.** It is interesting to see under which conditions quasi-Gorenstein extended Rees algebras are Gorenstein. To this end,

Heinzer, M.-K. Kim and Ulrich posed the next question. In this section, we present two cases which give an affirmative answer to the question.

**Question 3.10** ([6, Question 4.11]). *Let  $(R, \mathfrak{m})$  be a local Gorenstein ring. Let  $I$  be an  $\mathfrak{m}$ -primary ideal. If the extended Rees algebra  $R[It, t^{-1}]$  is quasi-Gorenstein, then is it Gorenstein?*

In the same paper, the authors characterized the quasi-Gorenstein property of the extended Rees algebra in terms of colon ideals.

**Proposition 3.11** ([6, Theorem 4.1]). *Let  $(R, \mathfrak{m})$  be a Gorenstein local ring of dimension  $d$ , and let  $I$  be an  $\mathfrak{m}$ -primary ideal. Assume that  $J \subseteq I$  is a reduction of  $I$  with  $\mu(J) = d$ . Let  $r := r_J(I)$  be the reduction number of  $I$  with respect to  $J$ , and let  $k$  be an integer such that  $k \geq r$ . Then the graded canonical module  $\omega_{R[It, t^{-1}]}$  of  $R[It, t^{-1}]$  has the form*

$$\omega_{R[It, t^{-1}]} \cong \bigoplus_{i \in \mathbb{Z}} (J^{i+k} :_R I^k) t^{i+d-1}.$$

In particular, for  $a \in \mathbb{Z}$ , the following are equivalent:

- (a)  $R[It, t^{-1}]$  is quasi-Gorenstein with  $\mathfrak{a}$ -invariant  $a$ .
- (b)  $J^i :_R I^r = I^{i+a-(r-d+1)}$  for every  $i \in \mathbb{Z}$ .

**Definition 3.12.** Let  $R$  be a Noetherian local ring. Let  $I$  be an ideal and  $J$  a minimal reduction of  $I$ . Then the *index of nilpotency*, denoted by  $s_J(I)$ , is  $\min\{i \mid I^{i+1} \subset J\}$ , and  $s(I) = \max\{s_J(I) \mid J \text{ is a minimal reduction of } I\}$ .

**Lemma 3.13** ([6, Remark 4.4]). *We use the setting of Proposition 3.11. In addition, assume that  $R[It, t^{-1}]$  is quasi-Gorenstein. Then,*

- (a)  $s_J(I) - d + 1 \leq a(R[It, t^{-1}]) \leq r_J(I) - d + 1$ , and
- (b)  $\max\{n \mid I^{r_J(I)} \subseteq J^n\} = r_J(I) - d + 1 - a(R[It, t^{-1}])$ .

Now, we are ready to provide a positive answer to Question 3.10 for a class of monomial ideals in a polynomial ring.

**Theorem 3.14.** *Let  $R$  be a polynomial ring in  $d$ -variables over a field and  $I$  a monomial ideal of height  $d$ . Assume that  $I$  has a  $d$ -generated monomial reduction. If  $R[It, t^{-1}]$  is quasi-Gorenstein, then  $R[It, t^{-1}]$  is Gorenstein.*

*Proof.* We show that  $\text{gr}_I(R)$  is Cohen-Macaulay. Let  $J = (g_1, \dots, g_d)$  be the monomial reduction of  $I$ , and write  $r := r_J(I)$ . Let  $u = r - d + 1 - a(R[It, t^{-1}])$ , where  $a(R[It, t^{-1}])$  is the  $\mathfrak{a}$ -invariant of  $R[It, t^{-1}]$ . Since  $R[It, t^{-1}]$  is quasi-Gorenstein, we have  $J^i : I^r = I^{i-u}$  for all  $i$  by Proposition 3.11 (b). The Cohen-Macaulayness of  $\text{gr}_I(R)$  follows by the Valabrega-Valla criterion [21, Theorem 1.1] once we have shown that

$$J \cap I^i \subseteq JI^{i-1} \quad \text{for } 0 \leq i \leq r.$$

Recall that all of the ideals in question are monomial. Let  $a$  be an arbitrary monomial in  $J \cap I^i$ . Since  $a \in J$  and  $J$  is a monomial ideal generated by the  $g_i$ 's, we can write  $a = a'g$  where  $a' \in R$  and  $g = g_j$  for some  $j$ . We want to show that  $a' \in I^{i-1}$ . For an arbitrary element  $z \in I^r$ , we have  $az \in I^{i+r} = J^i I^r \subseteq J^i J^u$ , where the last inclusion follows from Lemma 3.13 (b). Then  $az = a'zg \in J^{i+u}$ , and this implies  $a'z \in J^{i+u-1}$  since  $g + J^2$  is a non zero-divisor on  $\text{gr}_J(R)$ . Our choice of  $z \in I^i$  was arbitrary. Hence, we conclude that  $a' \in J^{i+u-1} : I^r = I^{i-1}$ .  $\square$

In the rest of this section we use the setting of Proposition 3.11 and study quasi-Gorenstein extended Rees algebras under the condition that  $s_J(I) = r_J(I)$  for some  $d$ -generated minimal reduction  $J$  of  $I$ . This is a necessary condition if the associated graded ring  $\text{gr}_I(R)$  is Cohen-Macaulay by [21, Theorem 1.1].

**Remark 3.15.** Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Gorenstein local ring and  $I$  an  $\mathfrak{m}$ -primary ideal. Assume that  $s_J(I) = r_J(I)$  for some  $d$ -generated minimal reduction  $J$  of  $I$ . Then,  $R[It, t^{-1}]$  is quasi-Gorenstein if and only if  $J^i : I^r = I^i$  for all  $i \in \mathbb{Z}$ .

*Proof.* This follows directly from Proposition 3.11 and Lemma 3.13 (a).  $\square$

**Definition 3.16** ([19]). Let  $(R, \mathfrak{m})$  be a Noetherian local ring and  $I$  an  $\mathfrak{m}$ -primary ideal. The ideal  $I$  called *n-standard* if  $J \cap I^i = JI^{i-1}$  for all  $i \leq n$ .

**Proposition 3.17.** *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring having a canonical module and  $I$  an  $\mathfrak{m}$ -primary ideal. Assume that  $R[It, t^{-1}]$  is quasi-Gorenstein and  $s_J(I) = r_J(I)$  for some  $d$ -generated minimal reduction  $J$  of  $I$ . Then  $I$  is 2-standard.*

*Proof.* Since  $R[It, t^{-1}]$  is quasi-Gorenstein and  $s_J(I) = r_J(I)$ , we have  $J^i : I^r = I^i$  for all  $i \in \mathbb{Z}$  by Remark 3.15. We show that  $J \cap I^2 \subseteq JI$ . Write  $J = (x_1, \dots, x_d)$ , and let  $a \in J \cap I^2$ . Then, we may write  $a = \sum a_i x_i$  for some  $a_i$  in  $R$ . For any  $z \in I^r$ ,  $az \in I^r I^2 = I^{r+2} = J^2 I^r \subseteq J^2$ , that is,

$$z \sum a_i x_i = \sum z a_i x_i \in J^2.$$

Since  $J/J^2$  is a free  $R/J$ -module with basis  $x_1 + J^2, \dots, x_d + J^2$ , we obtain  $z a_i \in J$ . This implies that, indeed,  $a_i \in J : I^r = I$ .  $\square$

**Lemma 3.18.** *Let  $(R, \mathfrak{m}, \mathbf{k})$  be a  $d$ -dimensional Cohen-Macaulay local ring having a canonical module with infinite residue field  $\mathbf{k}$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal. Assume that  $R[It, t^{-1}]$  is quasi-Gorenstein and  $s_J(I) = r_J(I)$  for some minimal reduction  $J$  of  $I$ .*

- (a) *It is obvious that  $r_J(I) = r(I)$  and  $s_J(I) = s(I)$ ; in particular,  $s(I) = r(I)$ .*
- (b)  *$a(\text{gr}_I(R)) = a(R[It, t^{-1}]) - 1 = d - r_J(I)$  where  $a(\text{gr}_I(R))$  and  $a(R[It, t^{-1}])$  are the  $\mathbf{a}$ -invariants of  $\text{gr}_I(R)$  and  $R[It, t^{-1}]$ , respectively.*

*Proof.*

- (a) By Lemma 3.13 (a), we have

$$s_J(I) - d + 1 = a := a(R[It, t^{-1}]) = r_J(I) - d + 1.$$

Let  $L, K \subseteq I$  be minimal reductions with  $r_L(I) = r(I)$  and  $s_K(I) = s(I)$ , respectively. Apply Lemma 3.13 (a) to obtain

$$s(I) - d + 1 \leq a \leq r(I) - d + 1.$$

This shows that  $s(I) \leq s_J(I)$  and  $r(I) \geq r_J(I)$ . The other direction of the inequalities follows from the definition of  $s(I)$  and  $r(I)$ .

(b) From the exact sequence (3.3) in the proof of Lemma 3.8, we have  $a(\text{gr}_I(R)) \geq a(R[It, t^{-1}]) - 1 = r - d$ , and the other inequality follows from a result of Trung [20, Proposition 3.2].  $\square$

**Theorem 3.19.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring having a canonical module with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal. Assume that  $I$  is an almost complete intersection ideal, i.e.,  $\mu(I) \leq \text{ht } I + 1$ . The following are equivalent:*

- (a)  $R[It, t^{-1}]$  is Gorenstein.
- (b)  $R[It, t^{-1}]$  is quasi-Gorenstein and  $s_J(I) = r_J(I)$  for some minimal reduction  $J$  of  $I$ .

*Proof.* Recall that, if  $I$  is a complete intersection, i.e.,  $\mu(I) = \text{ht } I$ , then  $R[It, t^{-1}]$  is a Cohen-Macaulay ring, and the equivalence immediately follows. Suppose that  $\mu(I) = \text{ht } I + 1$ . Implication (a)  $\Rightarrow$  (b) is obvious. Let  $d = \dim R$  and  $J$  a minimal reduction such that  $r = r_J(I) = s_J(I)$ . Choose a generating set  $x_1, \dots, x_d$  of  $J$ . Since  $J$  is a minimal reduction of  $I$ , a generating set of  $J$  can be extended to that of  $I$ . Hence, we may write  $I = J + (x)$  for some  $x$  in  $R$ . By [21, Theorem 1.1], it suffices to show that

$$J \cap I^i \subseteq JI^{i-1} \quad \text{for } 1 \leq i \leq r.$$

Since  $I = J + (x)$ ,

$$\begin{aligned} J \cap I^i \subseteq JI^{i-1} &\iff J \cap (J + (x))^i \subseteq JI^{i-1} \\ &\iff J \cap (J^i + xJ^{i-1} + \dots + (x)^i) \subseteq JI^{i-1} \\ &\stackrel{(\star)}{\iff} J^i + xJ^{i-1} + \dots + x^{i-1}J + J \cap (x)^i \subseteq JI^{i-1} \\ &\iff J \cap (x)^i \subseteq JI^{i-1}, \end{aligned}$$

where the equivalence  $(\star)$  follows from the containment  $J^i, \dots, x^{i-1}J \subseteq J$ . First, we claim that  $J \cap (x)^i \subseteq I^{i+1}$ . Observe that  $J \cap (x)^i =$

$J \cap (x^i) = x^i(J :_R x^i) = x^i(J :_R I^i)$ . This gives

$$\begin{aligned} I^r x^i(J : I^i) &\subseteq I^r I^i(J : I^i) \subseteq I^{r+i}(J : I^i) \\ &= J^i I^r(J : I^i) = J^i I^{r-i} I^i(J : I^i) \\ &\subseteq J^i I^{r-i} J \subseteq J^{i+1}. \end{aligned}$$

Therefore,  $x^i(J : I^i) \subseteq J^{i+1} : I^r = I^{i+1}$ , and this shows the claim.

Now, we apply decreasing induction on  $i$ . When  $i = r$ ,

$$J \cap (x)^r \subseteq I^{r+1} = JI^r \subseteq JI^{r-1}.$$

For  $i < r$ ,  $J \cap I^{i+1} = JI^i$  by the induction hypothesis. Hence, we have

$$J \cap (x)^i \subseteq J \cap I^{i+1} = JI^i \subseteq JI^{i-1}. \quad \square$$

**Results on the  $\mathbf{a}$ -invariant, and the core of an ideal.** Let  $R$  be a Noetherian local ring and  $I$  an  $R$ -ideal. In this section, we present results on the core of powers of an ideal  $I$  and the  $\mathbf{a}$ -invariant of the extended Rees algebras when the extended Rees algebra  $R[It, t^{-1}]$  is quasi-Gorenstein.

**Theorem 3.20.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring having a canonical module and  $\mathbf{k} = R/\mathfrak{m}$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal. Assume that  $R[It, t^{-1}]$  is quasi-Gorenstein and either characteristic of  $\mathbf{k}$  is 0 or greater than  $r(I)$ . Let  $a := a(R[It, t^{-1}])$  be the  $\mathbf{a}$ -invariant of  $R[It, t^{-1}]$ . Then  $\text{core}(I^u) = I^{du+a}$  for all  $u \in \mathbb{Z}$ .*

*Proof.* We may assume that the residue field is infinite. Let  $J$  be a minimal reduction of  $I$  with  $r := r_J(I) = r(I)$ . Fix a minimal generating set  $x_1, \dots, x_d$  of  $J$  where  $d = \dim R$ . Let  $J' = J^{[u]} := (x_1^u, \dots, x_d^u)$  and  $I' = I^u$ . Then  $J'$  is a minimal reduction of  $I'$ . By [16, Theorem 4.5],

$$\text{core}(I) = J^{n+1} : I^n \quad \text{for } n \geq r(I).$$

We compute the core of  $I'$  by using [17, Lemma 2.2], which shows

$$J^{n+1} : I^n = J^{[n]} : I^{dn} \quad \text{for } n \gg 0$$

in our setting. For  $n \gg 0$ , one has



$$\begin{aligned}
 \text{core}(I') &= J'^{n+1} : I'^n = J'^{[n+1]} : I'^{dn} \\
 &= (J^{[u]})^{[n+1]} : (I^u)^{dn} = J^{[nu+u]} : I^{udn} \\
 &= J^{[nu+u]} : I^{udn-r} I^r = J^{[nu+u]} : J^{udn-r} I^r \\
 &= (J^{[nu+u]} : J^{udn-r}) : I^r = J^{d(nu+u-1)+1-(udn-r)} : I^r \\
 &= J^{du-d+1+r} : I^r = I^{du+a},
 \end{aligned}$$

where the last equality follows from Proposition 3.11 (b). □

**Definition 3.21.** Let  $S$  be a  $\mathbb{Z}$ -graded ring and  $M$  a graded  $S$ -module. The initial degree of  $M$ , denoted by  $\text{indeg}_S(M)$ , is the  $\inf\{i \in \mathbb{Z} \mid [M]_i \neq 0\}$  if  $M \neq 0$  and 0 if  $M = 0$ .

**Remark 3.22.** The number  $\text{indeg}_S(M)$  can be  $-\infty$  in general. For a finitely generated  $\mathbb{Z}$ -graded module  $M$  over a Noetherian ring having a unique maximal homogeneous ideal  $\mathfrak{M}$ , which is maximal,  $\text{indeg}_S(M/\mathfrak{M}M)$  is a well-defined finite number. In this case,  $\text{indeg}_S(M/\mathfrak{M}M)$  is the minimum among the degrees of the elements in a minimal homogeneous generating set of  $M$ .

**Lemma 3.23.** *Let  $(R, \mathfrak{m})$  be an analytically unramified Cohen-Macaulay local ring having a canonical module with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal. Write  $T := R[It, t^{-1}]$ , and let  $\mathfrak{M}_T$  be the maximal homogeneous ideal of  $T$ . Let  $\overline{T}$  be the integral closure of  $T$  in  $R[t, t^{-1}]$  and  $\mathcal{C}$  the conductor ideal, i.e.,  $\mathcal{C} = T :_T \overline{T}$ . If  $T$  is quasi-Gorenstein, then*

$$a(\overline{T}) = a(T) - \text{indeg}(\mathcal{C}/\mathfrak{M}_T\mathcal{C}),$$

where  $a(-)$  denotes the  $\mathfrak{a}$ -invariant. Furthermore, we have  $\text{indeg}(\mathcal{C}/\mathfrak{M}_T\mathcal{C}) \leq 0$ , and equality happens if and only if  $T$  is integrally closed in  $R[t, t^{-1}]$ .

*Proof.* If  $T$  is integrally closed, then there is nothing to prove.

We assume that  $T$  is not integrally closed. Let  $d = \dim R$  and  $A := R[Jt, t^{-1}]$  where  $J$  is a minimal reduction of  $I$ . Then,  $A$  is a Cohen-Macaulay ring with  $A_0 = [\overline{T}]_0 = R$ , and, since  $R$  is analytically unramified,  $A \subseteq \overline{T}$  is module-finite extension. Hence, the

graded canonical modules  $\omega_T$  of  $T$  and  $\omega_{\overline{T}}$  of  $\overline{T}$  are  $\text{Hom}_A(T, \omega_A)$  and  $\text{Hom}_A(\overline{T}, \omega_A)$ , respectively, where  $\omega_A$  is the graded canonical module of  $A$ . We claim that  $\omega_{\overline{T}} \cong \text{Hom}_T(\overline{T}, \omega_T)$ . Observe that

$$\begin{aligned} \text{Hom}_T(\overline{T}, \omega_T) &\cong \text{Hom}_T(\overline{T}, \text{Hom}_A(T, \omega_A)) \\ &\cong \text{Hom}_A(\overline{T} \otimes_T T, \omega_A) \\ &\cong \text{Hom}_A(\overline{T}, \omega_A), \end{aligned}$$

where the last module is a graded canonical module of  $\overline{T}$ , see Theorem 3.5. Since  $\overline{T}$  has a unique maximal homogeneous ideal which is maximal, it has a well defined  $\mathfrak{a}$ -invariant

$$-\text{indeg}(\omega_{\overline{T}}/\mathfrak{M}_{\overline{T}}\omega_{\overline{T}}).$$

Since  $T \subset \overline{T}$  is birational, i.e., they have the same total quotient ring, we have  $\text{Hom}_T(\overline{T}, T) \cong (T :_T \overline{T})$ , where the last module is the conductor ideal  $\mathcal{C}$ . Recall that, since  $T$  is quasi-Gorenstein,  $\omega_T \cong T(a(T))$ . Therefore,  $\omega_{\overline{T}} \cong \mathcal{C}(a(T))$ . This implies that

$$\begin{aligned} a(\overline{T}) &= -\text{indeg}(\mathcal{C}/\mathfrak{M}_{\overline{T}}\mathcal{C}(a(T))) \\ &= -(\text{indeg}(\mathcal{C}/\mathfrak{M}_{\overline{T}}\mathcal{C}) - a(T)). \end{aligned}$$

It remains to show that

$$\text{indeg}(\mathcal{C}/\mathfrak{M}_{\overline{T}}\mathcal{C}) = \text{indeg}(\mathcal{C}/\mathfrak{M}_T\mathcal{C}).$$

Since  $R$  is analytically unramified, there exists a positive integer  $q$  such that  $\overline{T}^i = I^{i-q}\overline{T}^q$  for  $i \geq q \geq 0$  [18, Theorem 1.4]. This shows that  $t^{-q} \in \mathcal{C}$ . By choosing  $q$  the smallest positive integer with the property, we have  $\text{indeg}_T(\mathcal{C}/\mathfrak{M}_T\mathcal{C}) = -q \leq 0$ . Since  $[T]_i = [\overline{T}]_i$  for all  $i \leq 0$ , in particular,  $[\mathfrak{M}_T]_i = [\mathfrak{M}_{\overline{T}}]_i$  for  $i < 0$ , we obtain

$$\text{indeg}(\mathcal{C}/\mathfrak{M}_{\overline{T}}\mathcal{C}) = \text{indeg}(\mathcal{C}/\mathfrak{M}_T\mathcal{C}). \quad \square$$

**Theorem 3.24.** *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional analytically unramified Cohen-Macaulay local ring having a canonical module with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal. Let  $\{\mathcal{F}_i\}_{i \in \mathbb{Z}}$ , where  $\mathcal{F}_i = \overline{T}^i$ , be the integral closure filtration where  $\mathcal{F}_i = R$  when  $i \leq 0$ . Assume that  $\overline{T} = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i t^i$  is Cohen-Macaulay. If  $R[It, t^{-1}]$  is quasi-Gorenstein, then the index of nilpotency does not depend on a minimal reduction of  $I$ , and the  $\mathfrak{a}$ -invariant of  $R[It, t^{-1}]$  is  $s(I) - d + 1$ .*

*Proof.* Let  $T = R[It, t^{-1}]$ ,  $J$  a minimal reduction of  $I$  and  $q = a(\overline{T}) - a(T)$ , where  $a(-)$  denotes the  $\mathbf{a}$ -invariant. First, we claim that  $s_J(I) = s_J(\mathcal{F}) - q$ , where  $s_J(\mathcal{F}) := \min\{i \mid \mathcal{F}_{i+1} \subseteq J\}$ . Since  $\overline{T}$  is Cohen-Macaulay,  $a(\overline{T}) = s_J(\mathcal{F}) - d + 1$ , equivalently,  $a(T) = s_J(\mathcal{F}) - q - d + 1$ . Lemma 3.13 (a)  $s_J(I) - d + 1 \leq a(T)$  implies that  $s_J(I) + q \leq s_J(\mathcal{F})$ . Since  $t^{-q} \in T :_T \overline{T}$ , we have  $\overline{T}^{n+q} \subset I^n$ . Hence, we have  $J : \overline{T}^{n+q} \supseteq J : I^n$ , and, by definition, this implies the other inequality  $s_J(I) + q \geq s_J(\mathcal{F})$ . This proves the claim.

Since  $a(T) = s_J(I) - d + 1$  is independent of a minimal reduction  $J$ , we have  $s_J(I) = s(I)$ . This completes the proof.  $\square$

The ring  $\bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i t^i$  is not Cohen-Macaulay in general. However, Hochster [10] showed that it is Cohen-Macaulay when  $R$  is a polynomial ring (localized at the origin) over a field and  $\mathcal{F}_1$  is a monomial ideal.

**4. The Gorensteinness of extended Rees algebras of monomial normal ideals.** Let  $I$  be a monomial ideal in a polynomial ring  $R = k[x_1, \dots, x_d]$  over a field  $k$ . The integral closure of the extended Rees algebra  $\overline{R[It, t^{-1}]}$  is Cohen-Macaulay by a result of Hochster [10]. In [7, Theorem 5.6], the authors characterized the Gorensteinness of  $\overline{R[It, t^{-1}]}$  when  $I$  has a minimal reduction  $J$  which is generated by powers of variables, i.e.,  $J = (x_1^{a_1}, \dots, x_d^{a_d})$  for some  $a_i \in \mathbb{N}$ . This condition having such a minimal reduction is equivalent to the condition that  $I$  has only one Rees valuation. In this section, we generalize this result by removing the condition on the number of Rees valuations. We are able to interpret the reduction number that appears in [7, Proposition 5.4] in terms of the  $\mathbf{a}$ -invariant of  $\overline{R[It, t^{-1}]}$ , and this leads to a lower bound on the reduction number. We follow the notation of [2, Chapter 6].

**Setting 4.1.** Let  $R = k[x_1, \dots, x_d]$  be a polynomial ring in  $d$ -variables over a field  $k$  and  $\mathfrak{m} = (x_1, \dots, x_d)R$ . We assign a  $\mathbb{Z}^{d+1}$ -grading to the Laurent polynomial ring  $R[t, t^{-1}]$  by setting the *exponent function*, denoted by  $\exp$ , from the set of monomial ideals of  $R$  to  $\mathbb{Z}^d$  as

$$\exp(x_1^{a_1} \cdots x_d^{a_d} t^{a_{d+1}}) = (a_1, \dots, a_d, a_{d+1}).$$

This determines the grading, since  $\{\exp(x_1), \dots, \exp(x_d), \exp(t)\}$  forms an (orthonormal)  $\mathbb{Z}$ -basis for  $\mathbb{Z}^{d+1}$ . Let  $A$  be a  $\mathbb{Z}^{d+1}$ -graded subring of

$R[t, t^{-1}]$ . Then the semigroup

$$C_A := \{\exp(m) \mid m \text{ monomial of } R[t, t^{-1}] \text{ in } A\} \subseteq \mathbb{Z}^{d+1}$$

is called the *affine semigroup* of  $A$  if  $C_A$  is a finitely generated semigroup. For an affine semigroup  $C$ , let  $\text{relint}(C)$  denote the *relative interior* of  $C$  that is

$$\text{relint}(C) := C \cap \text{relint } \mathbb{R}_{\geq 0} C.$$

**Lemma 4.2** ([2, Proposition 6.1.5]). *With Setting 4.1, let  $C$  be the affine semigroup of  $R[\mathfrak{m}t, t^{-1}]$ . Let  $W = \mathbb{Z}^{d+1}$  and  $\{\mathbf{e}_i\}$  the standard basis of  $W$ . Let  $\phi \in \text{Aut}_{\mathbb{Z}}(W)$  be an automorphism of  $W$  defined as follows:  $\phi(\mathbf{e}_i) = \mathbf{e}_i + \mathbf{e}_{d+1}$  for  $i = 0, \dots, d$  and  $\phi(\mathbf{e}_{d+1}) = -\mathbf{e}_{d+1}$ . Then  $\phi|_C$  is an embedding of  $C$  into  $\mathbb{Z}_{\geq 0}^{d+1}$ .*

*Proof.* One can easily check that  $\phi$  is an automorphism on  $W$ . We show that  $\phi|_C$  is an embedding. Since  $C$  does not have any inverse (in the sense of affine semigroups),  $\ker(\phi|_C) = 0$ .

It remains to show that  $\phi(C) \subseteq \mathbb{Z}^{d+1}$ . Since  $\phi(\exp(x_i)) = (0, \dots, 0, 1, 0, \dots, 0, 1)$  and  $\phi(\exp(t^{-1})) = (0, \dots, 0, 1)$ , it suffices to show that, for  $a_i \in \mathbb{Z}_{\geq 0}$  and  $b \in \mathbb{Z}$  such that  $\sum_{i=0}^d a_i \geq b$ , the image  $\phi(\exp(x_1^{a_1} \cdots x_d^{a_d} t^b))$  is in  $\mathbb{Z}_{\geq 0}^{d+1}$ . Indeed, we have

$$\begin{aligned} \phi(\exp(x_1^{a_1} \cdots x_d^{a_d} t^b)) &= \left( \sum_{i=0}^d a_i \phi(\exp(x_i)) \right) + b\phi(\exp(t)) \\ &= \left( \sum_{i=0}^d a_i \phi(\exp(x_i)) \right) - b\phi(\exp(t^{-1})) \\ &= \left( a_1, \dots, a_d, \sum_{i=0}^d a_i \right) - b(0, \dots, 0, 1) \\ &= \left( a_1, \dots, a_d, \sum_{i=0}^d a_i - b \right) \in \mathbb{Z}_{\geq 0}^{d+1}. \quad \square \end{aligned}$$

**Corollary 4.3.** *With Setting 4.1, let  $\mathcal{F} = \{\mathcal{F}_i\}_{i \in \mathbb{Z}}$  be a filtration where  $\mathcal{F}_i = R$  when  $i \leq 0$  and  $\mathcal{F}_i$  are monomial ideals contained in  $\mathfrak{m}^i$ . Then, the affine semigroup of  $\bigoplus_{i \in \mathbb{Z}} \mathcal{F}_i t^i$  can be embedded into  $\mathbb{Z}_{\geq 0}^{d+1}$ .*

In particular, the affine semigroup of  $\overline{R[It, t^{-1}]}$  can be embedded into  $\mathbb{Z}_{\geq 0}^{d+1}$  when  $I$  is a monomial ideal.

For a monomial ideal  $I$  in  $R$ , its integral closure  $\overline{I}$  can be determined by the Newton polyhedron of the ideal  $I$ . Here, the *Newton polyhedron*  $\text{NP}(I)$  of a monomial ideal  $I$  is the cone generated (over  $\mathbb{R}_{\geq 0}$ ) by the exponent vectors of the monomials in  $I$  in  $\mathbb{R}^d$ . We would like to describe the ring  $\overline{R[It, t^{-1}]}$  using half spaces in  $\mathbb{R}^{d+1}$  that correspond to those which determine  $\overline{I}$  in  $\mathbb{R}^d$ . Let  $\langle \cdot, \cdot \rangle$  denote the inner product in  $\mathbb{R}^n$ .

**Lemma 4.4.** *With Setting 4.1, let  $I$  be a monomial ideal in  $R$ . Let  $H_i^+ = \{v \in \mathbb{R}^d \mid \langle (a_{i1}, \dots, a_{id}), v \rangle \geq h_i\}$ , where  $h_i \in \mathbb{Z}_{\geq 0}$  are the half spaces in  $\mathbb{Z}^d$  that determine the Newton polyhedron of  $I$ . Define the half spaces  $\tilde{H}_i^+$  that correspond to each  $H_i^+$  in  $\mathbb{R}^{d+1}$  as*

$$\tilde{H}_i^+ := \{v \in \mathbb{R}^{d+1} \mid \langle (a_{i1} - h_i, \dots, a_{id} - h_i, h_i), v \rangle \geq 0\}.$$

Let  $C$  be the affine semigroup of  $\overline{R[It, t^{-1}]}$ . Then the intersection  $\cap \tilde{H}_i^+$  is the cone generated by the affine semigroup  $\phi(C)$  where  $\phi$  is the embedding in Lemma 4.2.

*Proof.* Let  $H := H_i^+$  for some  $i$ , and write  $H = \{v \in \mathbb{R}^d \mid \langle (a_1, \dots, a_d), v \rangle \geq h\}$ . An exponent vector  $(z_1, \dots, z_{d+1})$  in  $\phi(C)$  is the image of  $(z_1, \dots, z_d, -z_{d+1} + (z_1 + \dots + z_d))$  under the map  $\phi$ . This gives  $(z_1, \dots, z_d, -z_{d+1} + (z_1 + \dots + z_d)) \in C$  if and only if the monomial  $x^{z_1} \dots x^{z_d}$  is in  $\overline{I^{-z_{d+1} + (z_1 + \dots + z_d)}}$ . In terms of half spaces, this corresponds to the condition  $\langle (a_1, \dots, a_d), (z_1, \dots, z_d) \rangle \geq (-z_{d+1} + (z_1 + \dots + z_d))h$  for the half space  $H_i$  defining  $\text{NP}(I)$ . By setting  $\tilde{H} = \{v \in \mathbb{R}^{d+1} \mid \langle (a_1 - h, \dots, a_d - h, +h), v \rangle \geq 0\}$ , we obtain

$$\cap \tilde{H}_i^+ = \mathbb{R}_{\geq 0} \phi(C). \quad \square$$

**Example 4.5.** Let  $R = \mathbb{C}[x_1, \dots, x_d]$  and  $I = (x_1, \dots, x_d)$ . Let  $\{e_i\}_{i=1}^{d+1}$  be the standard base of  $\mathbb{Z}^{d+1}$ . Then, one can easily see that

$$\begin{aligned} \phi(C) &= \{(z_1, \dots, z_{d+1}) \in \mathbb{Z}^{d+1} \mid z_i \geq 0 \text{ for all } i\} \\ &= \cap \{(z_1, \dots, z_{d+1}) \in \mathbb{Z}^{d+1} \mid \langle e_i, v \rangle \geq 0\}, \end{aligned}$$

where  $\phi$  as in Lemma 4.2.

**Lemma 4.6.** *With Setting 4.1, assume that  $I$  is an  $\mathfrak{m}$ -primary monomial ideal. Let  $C$  be the affine semigroup of  $\overline{R[It, t^{-1}]}$  and  $\phi$  the embedding in Lemma 4.2. Then there exists an exponent vector of the form  $(1, \dots, 1, q)$  in  $\phi(C)$  for some integer  $q$ , with  $1 \leq q \leq d + 1$ , which is part of a minimal generating set for the canonical ideal for  $\overline{R[It, t^{-1}]}$ .*

*Proof.* Since  $I$  is  $\mathfrak{m}$ -primary, the half spaces of the form  $\{v \in \mathbb{R}^d \mid (0, \dots, 1, \dots, 0) \cdot v \geq 0\}$ , where the  $i$ th entry is 1, are part of the boundary of the Newton polyhedron of  $I$ . By Lemma 4.4, these will be part of the boundary half spaces in  $\phi(C)$ . For instance, the half space  $\{v \in \mathbb{R}^d \mid (1, \dots, 0) \cdot v \geq 0\}$  corresponds to the half space  $\{v \in \mathbb{R}^{d+1} \mid (1, \dots, 0) \cdot v \geq 0\}$ . Hence, if  $(z_1, \dots, z_d, z_{d+1}) \in \text{relint}(C)$ , then  $z_i \geq 1$  for  $i = 0, \dots, d$ .

By [2, Theorem 6.3.5 (b)], it suffices to show that there exists an integer  $q$  in  $1 \leq q \leq d + 1$  such that, if  $(1, \dots, 1, q)$  is in  $\text{relint } \phi(C)$ , then  $(1, \dots, 1, q - 1)$  is not in  $\phi(C)$ . Since  $x_1 \cdots x_d \in R \subseteq \overline{R[It, t^{-1}]}$ ,  $\phi(1, \dots, 1, 0) = (1, \dots, 1, d)$  is in  $\phi(C)$ . If  $(1, \dots, 1, d)$  is on the boundary, then  $(1, \dots, 1, d + 1)$  is in  $\text{relint } \phi(C)$ . In this case, we set  $q = d + 1$ . If  $(1, \dots, 1, d)$  is not on the boundary, we can choose  $q \leq d$  to be the minimal in the last component since we have  $\phi(t^{-1}) = (0, \dots, 0, 1)$  in  $\phi(C)$ . □

**Theorem 4.7.** *Let  $R = k[x_1, \dots, x_d]$  be a polynomial ring in  $d$ -variables over a field  $k$  and  $\mathfrak{m} = (x_1, \dots, x_d)R$ . Let  $I$  be an  $\mathfrak{m}$ -primary monomial ideal and  $H_i$  the half spaces that determine the Newton polyhedron of  $I$  where*

$$H_i = \{v \in \mathbb{R}^d \mid \langle (a_{i1}, \dots, a_{id}), v \rangle \geq h_i\} \quad \text{for } h_i \in \mathbb{Z}_{\geq 0}.$$

Let  $q$  be as in Lemma 4.6 and

$$w_i := \langle (a_{i1} - h_i, \dots, a_{id} - h_i, h_i), (1, \dots, 1, q) \rangle.$$

Define  $N_i^+ := \{v \in \mathbb{R}^{d+1} \mid \langle (a_{i1} - h_i, \dots, a_{id} - h_i, h_i), v \rangle \geq w_i\}$ . Let  $C$  be the affine semigroup of  $\overline{R[It, t^{-1}]}$  and  $\phi$  the embedding in Lemma 4.2. Then  $\overline{R[It, t^{-1}]}$  is Gorenstein if and only if the relative interior of  $\phi(C)$  is contained in  $\cap N_i^+$ , equivalently,

$$\text{relint}(\phi(C)) = (\cap N_i^+) \cap \phi(C).$$

*Proof.* For  $v \in N_i^+$  and, for any  $i$ , since  $\langle (a_{i1} - h_i, \dots, a_{id} - h_i, h_i), v \rangle \geq w_i = \langle (a_{i1} - h_i, \dots, a_{id} - h_i, h_i), (1, \dots, 1, q) \rangle$ , we have

$$\langle (a_{i1} - h_i, \dots, a_{id} - h_i, h_i), v - (1, \dots, 1, q) \rangle \geq 0.$$

Hence,  $v - (1, \dots, 1, q)$  is in  $\mathbb{R}_{\geq 0}\phi(C)$  by Lemma 4.4. In other words,  $\cap N_i^+ = (1, \dots, 1, q) + \mathbb{R}_{\geq 0}\phi(C)$ . Since  $(1, \dots, 1, q) \in \text{relint}(C)$ , we have  $(\cap N_i^+) \cap \phi(C) \subseteq \text{relint}(C)$ .

By [2, Theorem 6.3.5.(b)],  $\overline{R[It, t^{-1}]}$  is Gorenstein if and only if the relative interior of  $\phi(C)$  is principal. The previous paragraph shows that  $(\cap N_i^+) \cap \phi(C)$  is principally generated by  $(1, \dots, 1, q)$  and contained in the relative interior of  $\phi(C)$ , and by Lemma 4.6,  $(1, \dots, 1, q)$  is a part of the minimal generating set for  $\text{relint } \phi(C)$ . Hence, the relative interior of  $\phi(C)$  is principally generated by  $(1, \dots, 1, q)$  if and only if  $(\cap N_i^+) \cap \phi(C) = \text{relint } \phi(C)$ . This proves the statement.  $\square$

The next example illustrates Theorem 4.7 when there is exactly one bounded half space among the half spaces defining the Newton polyhedron of the ideal  $I$ . This will help to understand and prove 4.10, which is [7, Theorem 5.6].

**Example 4.8.** Let  $R = \mathbb{C}[x, y, z]$  and  $I = (x^2, y^2, z^4)$ . Then the integral closure of  $I$  is determined by the half spaces  $\{v \in \mathbb{R}^3 \mid \langle (2, 2, 1), v \rangle \geq 4\}$  and  $\{v \in \mathbb{R}^3 \mid \langle \mathbf{e}_i, v \rangle \geq 0\}$  for  $i = 1, 2, 3$ , where  $\{\mathbf{e}_i\}$  denote the standard bases of  $\mathbb{R}^3$ . Let  $\nu$  be the valuation corresponding to the bounded half space  $\{v \in \mathbb{R}^3 \mid \langle (2, 2, 1), v \rangle \geq 4\}$ ; then, we have  $\nu(x) = 2, \nu(y) = 2$  and  $\nu(z) = 1$ . Let  $\overline{R[It, t^{-1}]}$  and  $C$  be the corresponding affine semigroup. Then  $\phi(C)$  is determined by the half spaces represented as a matrix

$$M = \begin{bmatrix} -2 & -2 & -3 & 4 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

in the sense that an exponent vector  $(z_1, \dots, z_4) \in \phi(C)$  if and only if all the entries of  $M(z_1, \dots, z_4)^{\text{tr}} \geq 0$ . Here  $M \geq c$ , where  $c \in \mathbb{R}$  if all of the entries of  $M$  are  $\geq c$ . It can easily be checked that

$(1, 1, 1, 2) \in \phi(C)$ , but  $(1, 1, 1, 1) \notin \phi(C)$ . Hence,  $q = 2$ . Furthermore,  $M \cdot (1, 1, 1, 2)^{\text{tr}} = (1, 1, 1, 1)^{\text{tr}}$ .

For simplicity, we view elements of vector spaces as column vectors. If  $w \in \text{relint } \phi(C)$ , then  $M \cdot w \geq 0$ , i.e.,  $M \cdot w \geq 1$ . Since  $M \cdot w \geq 1$  and  $M \cdot (1, 1, 1, 2)^{\text{tr}} = (1, 1, 1, 1)^{\text{tr}}$ , we have  $M \cdot w - M \cdot (1, 1, 1, 2)^T \geq 0$ , i.e.,  $M \cdot (w - (1, 1, 1, 2)^{\text{tr}}) \geq 0$ . Hence,  $w - (1, 1, 1, 2)^{\text{tr}} \in \phi(C)$ . This implies the exponent vector  $(1, 1, 1, 2)$  generates  $\text{relint } \phi(C)$ . Therefore,  $\overline{R[It, t^{-1}]}$  is Gorenstein by Theorem 4.7. Furthermore, by [2, Corollary 6.3.6], the ideal corresponding to  $\text{relint } \phi(C)$  is the graded canonical ideal of  $\overline{R[It, t^{-1}]}$ , and it is generated by  $xyzt$ , which corresponds to  $(1, 1, 1, 2)$ . This gives

$$a = a(\overline{R[It, t^{-1}]}) = -1 = 2 - (1 + 1 + 1) = q - d,$$

where  $d = \dim R = 3$ . Since  $\overline{R[It, t^{-1}]}$  is Cohen-Macaulay,  $a = r - d + 1$  where  $r = r(\mathcal{F})$  is the reduction number of the filtration  $\mathcal{F} = \{\overline{I^i}\}_{i \in \mathbb{Z}}$ . Therefore,  $r = -1 + 3 - 1 = 1$ .

**Corollary 4.9.** *With the setting of Theorem 4.7, we obtain  $a(\overline{R[It, t^{-1}]}) \geq q - d$  and  $r(\mathcal{F}) \geq q - 1$ . Furthermore, if  $\overline{R[It, t^{-1}]}$  is Gorenstein, then  $a(\overline{R[It, t^{-1}]}) = q - d$  and  $r(\mathcal{F}) = q - 1$ .*

*Proof.* Note that  $a(\overline{R[It, t^{-1}]}) = r(\mathcal{F}) - d + 1$  since  $\overline{R[It, t^{-1}]}$  is Cohen-Macaulay. The inequality  $a(\overline{R[It, t^{-1}]}) \geq q - d$  shows the first part of the statement. The second part follows from the fact that  $a(\overline{R[It, t^{-1}]}) = r - d + 1$  if  $\overline{R[It, t^{-1}]}$  is Gorenstein.  $\square$

**Corollary 4.10** ([7, Theorem 5.6]). *With the setting of Theorem 4.7, assume that  $I = (x_1^{a_1}, \dots, x_d^{a_d})$ . Let  $L = \text{lcm}(a_1, \dots, a_d)$ . Write  $L/a_1 + \dots + L/a_d = jL + p$ , where  $j \geq 0$  and  $1 \leq p \leq L$ . Then,  $\overline{R[It, t^{-1}]}$  is Gorenstein if and only if  $p = 1$ .*

*Proof.* Let  $\{\mathbf{e}_i\}$  be the standard basis of  $\mathbb{R}^d$ . Let  $H_i^+ = \{v \in \mathbb{R}^d \mid \langle \mathbf{e}_i, v \rangle \geq 0\}$  for  $i = 1, \dots, d$ . Observe that the half spaces which determine the integral closure of  $I$  are the bounded half space  $\{v \in \mathbb{Z}^d \mid \langle (L/a_1, \dots, L/a_d), v \rangle \geq L\}$  and the  $H_i^+$ 's, and the bounded half space corresponds to the Rees valuation of  $I$ . Now, we proceed as in Example 4.8. The affine semigroup  $\phi(C)$  is determined by the half



spaces represented as a matrix:

$$M = \begin{bmatrix} L/a_1 - L & L/a_2 - L & \cdots & L/a_d - L & L \\ \hline 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & 1 & \cdots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

i.e.,  $\phi(\mathcal{C}) = \{(x_1, \dots, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} \mid M \cdot (x_1, \dots, x_{d+1}) \geq 0\}$ . Therefore, we have

$$\text{relint } \phi(\mathcal{C}) = \{(x_1, \dots, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} \mid M \cdot (x_1, \dots, x_{d+1}) \geq 0\}.$$

Let  $q$  be as in Lemma 4.6. Then we have

$$\begin{aligned} (\cap N_i^+) \cap \phi(\mathcal{C}) &= \{(x_1, \dots, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} \mid M \cdot (x_1, \dots, x_{d+1}) \\ &\geq M \cdot (1, \dots, 1, q)\}. \end{aligned}$$

By Theorem 4.7  $R[It, t^{-1}]$  is Gorenstein if and only if  $\text{relint } \phi(\mathcal{C}) \subseteq \cap N_i^+$ , that is,

$$\begin{aligned} \{(x_1, \dots, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} \mid M(x_1, \dots, x_{d+1})^{\text{tr}} \geq 0\} \\ \subseteq \{(x_1, \dots, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} \mid M(x_1, \dots, x_{d+1})^{\text{tr}} \\ \geq M(1, \dots, 1, q)^{\text{tr}}\}. \end{aligned}$$

Observe that

$$M \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d+1} \end{bmatrix} = \begin{bmatrix} \frac{L}{a_1}x_1 + \cdots + \frac{L}{a_d}x_d + L(x_{d+1} - (x_1 + \cdots + x_d)) \\ x_1 \\ \vdots \\ x_d \end{bmatrix}$$

and

$$M \begin{bmatrix} 1 \\ \vdots \\ 1 \\ q \end{bmatrix} = \begin{bmatrix} \frac{L}{a_1} + \cdots + \frac{L}{a_d} + L(q - d) \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Hence, for any  $(x_1, \dots, x_{d+1})$  in the set  $\{(x_1, \dots, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} \mid M(x_1, \dots, x_{d+1})^{\text{tr}} \succeq 0\}$ , we have that  $x_i \geq 1$  for  $1 \leq i \leq d$ . Let

$$\rho := \min \left\{ \frac{L}{a_1}x_1 + \dots + \frac{L}{a_d}x_d + L(x_{d+1} - (x_1 + \dots + x_d)) \mid \right. \\ \left. \text{for integers } x_1, \dots, x_d \geq 1 \text{ and } x_{d+1} \geq 0 \right\},$$

and write  $\eta = L/a_1 + \dots + L/a_d + L(q - d)$ . It suffices to show that  $\eta \leq \rho$  if and only if  $p = 1$ . Since  $(1, \dots, 1, q)$  is in  $\text{relint}(\phi(\mathcal{C}))$ , we have  $\eta \succeq 0$ , i.e.,  $\eta \geq 1$ . Assume that  $\rho = 1$ . Then  $\text{relint} \phi(\mathcal{C}) \subseteq \cap N_i^+$  if and only if  $\eta = 1$ , and the last condition is equivalent to that of  $p = 1$ . Hence, it suffices to show that  $\rho = 1$ .

First, we claim that  $\gcd(L/a_1 - L, L/a_2 - L, \dots, L/a_d - L, L) = 1$ . Recall that  $L = \text{lcm}(a_1, \dots, a_d)$ . Since  $\delta := \gcd(L/a_1 - L, L/a_2 - L, \dots, L/a_d - L, L) = \gcd(L/a_1, \dots, L/a_d, L) = \gcd(L/a_1, \dots, L/a_d)$ , we have  $\delta \mid (L/a_i)$  for all  $i$ . This implies that  $a_i \mid (L/\delta)$  for all  $i$  since  $\delta$  divides  $L$ . Hence, we see that  $L/\delta \geq \text{lcm}(a_1, \dots, a_d) = L$ , and this implies that  $\delta = 1$ .

Next, we claim that there exists  $(y_1, \dots, y_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1}$  where  $y_i > 0$  for  $i = 1, \dots, d$  such that  $\langle (L/a_1 - L, L/a_2 - L, \dots, L/a_d - L, L), (y_1, \dots, y_{d+1}) \rangle = 1$ . Since  $\gcd(L/a_1 - L, L/a_2 - L, \dots, L/a_d - L, L) = 1$ , there exist such  $(y_1, \dots, y_{d+1})$  in  $\mathbb{Z}^{d+1}$ .

We show that one can modify the  $y_i$ 's so that  $y_i > 0$  for all  $i = 1, \dots, d$ . Suppose that  $i \leq d$  is the least index where  $y_i \leq 0$ . Let  $n$  be an integer such that  $y_i + nL > 0$ . By replacing  $y_i$  by  $y_i + nL$  and  $y_{d+1}$  by  $y_{d+1} - n(L/a_i - L)$ , we may assume that  $y_i > 0$ . This shows that the first row of  $M \cdot (y_1, \dots, y_{d+1})^{\text{tr}}$  is 1 and all the other rows are positive. This indeed shows that  $\rho = 1$ , and this completes the proof.  $\square$

**Remark 4.11.** One may ask if the numbers  $w_i$  in Theorem 4.7 are the minimum in  $\{\langle (a_1 - h_i, \dots, a_d - h_i, h_i), v \rangle \mid v \in \mathbb{Z}_{\geq 0}^{d+1}\}$  for each  $i$ . The following example in the next paragraph shows that it can happen that  $w_i$  is not the minimum for each halfspace, but it is the minimum in the intersection, i.e., it is the minimum in the relative interior.

Let  $R = \mathbb{C}[x, y]$ ,  $I = (x^3, xy, y^3)$  and  $T = \overline{R[It, t^{-1}]}$ . Then  $T$  is Gorenstein, and  $\phi(C_T)$  is determined by the half spaces

$$\begin{bmatrix} -1 & -2 & 3 \\ -2 & -1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easily seen that  $q = 2$ . But,  $\langle(-1, -2, 3), (3, 1, 2)\rangle = 1$ , whereas  $w_1 = \langle(-1, -2, 3), (1, 1, 2)\rangle = 3$ .

**5. Cohen-Macaulayness of the associated graded ring.** Let  $(R, \mathfrak{m})$  be a regular local ring and  $I$  an ideal. Let  $J$  be a proper ideal. We define a function  $\text{ord}_J : R \rightarrow \mathbb{N}_{\geq 0} \cup \{\infty\}$  as follows:

$$\text{ord}_J(x) := \sup\{i \mid x \in J^i\} \quad \text{for } x \in R$$

and

$$\text{ord}_J(I) := \inf\{\text{ord}_J(x) \mid x \in I\}.$$

By Krull’s intersection theorem, this number is finite if  $x \neq 0$ . In general, we have  $\text{ord}_{\mathfrak{m}}(xy) \geq \text{ord}_{\mathfrak{m}}(x) + \text{ord}_{\mathfrak{m}}(y)$ . Since  $\text{gr}_{\mathfrak{m}}(R)$  is a domain,  $\text{ord}_{\mathfrak{m}}$  is a valuation, i.e.,

$$\text{ord}_{\mathfrak{m}}(xy) = \text{ord}_{\mathfrak{m}}(x) + \text{ord}_{\mathfrak{m}}(y) \quad \text{for } x, y \in R.$$

For  $x \in R$ , let

$$x^* := x + \mathfrak{m}^{\text{ord}_{\mathfrak{m}}(x)+1} \in \mathfrak{m}^{\text{ord}_{\mathfrak{m}}(x)} / \mathfrak{m}^{\text{ord}_{\mathfrak{m}}(x)+1}$$

denote its image in  $\text{gr}_I(R)$ . We call  $x^*$  the *leading form* of  $x$  and  $I^* := (x^* \mid x \in I)$  the *leading ideal* of  $I$ , respectively.

**Remark 5.1.** Since  $(R, \mathfrak{m})$  is a regular local ring, so is  $(\widehat{R}, \widehat{\mathfrak{m}})$  where  $\widehat{\phantom{x}}$  denotes  $\mathfrak{m}$ -adic completion. Because  $\text{gr}_{\mathfrak{m}}(R) = \text{gr}_{\widehat{\mathfrak{m}}}(\widehat{R})$ , we have  $\text{ord}_{\mathfrak{m}}(x) = \text{ord}_{\widehat{\mathfrak{m}}}(x)$  and  $\text{ord}_{\mathfrak{m}}(I) = \text{ord}_{\widehat{\mathfrak{m}}}(\widehat{I})$ . In particular, we have  $I^* = (\widehat{I})^*$  in  $\text{gr}_{\mathfrak{m}}(R) = \text{gr}_{\widehat{\mathfrak{m}}}(\widehat{R})$ . Since  $\widehat{R}$  is complete, we may write

$$x = \sum_{i \geq 0}^{\infty} [x]_i = \sum_{i \geq \text{ord}_{\mathfrak{m}}(x)}^{\infty} [x]_i,$$

where  $[x]_i \in \widehat{\mathfrak{m}}^i \setminus \widehat{\mathfrak{m}}^{i+1} \cup \{0\}$  and  $x^* = ([x]_{\text{ord}_{\mathfrak{m}}(x)})^*$ .

**Lemma 5.2.** *Let  $(R, \mathfrak{m})$  be a regular local ring with maximal ideal  $\mathfrak{m}$ . Let  $I$  be an ideal which is minimally generated by the  $2 \times 2$  minors of the  $2 \times 3$  matrix*

$$M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$$

*whose entries are in  $\mathfrak{m}$ . Write  $G := \text{gr}_{\mathfrak{m}}(R)$ . Let  $L = (b^*w^* - c^*v^*, -(a^*w^* - c^*u^*), a^*v^* - b^*u^*)$ , and let  $C_{\bullet}$  be the complex*

$$0 \longrightarrow G^2 \xrightarrow{\begin{bmatrix} a^* & u^* \\ b^* & v^* \\ c^* & w^* \end{bmatrix}} G^3 \xrightarrow{\begin{bmatrix} b^*w^* - c^*v^* & -(a^*w^* - c^*u^*) & a^*v^* - b^*u^* \end{bmatrix}} G.$$

*If  $L$  is of height 2, then  $C_{\bullet}$  is acyclic. Furthermore, if any two elements in the generating set of  $L$  are homogeneous, then so is the third and  $I^* = L = ((bw - cv)^*, (aw - cu)^*, (av - bu)^*)$ .*

*Proof.* The acyclicity of  $C_{\bullet}$  follows by the Buchsbaum-Eisenbud acyclicity criterion [3, Corollary 1]. If any two elements of the generating set of  $L$  are homogeneous, then it is easy to see that the columns of the matrix representing the differential map  $G^2 \rightarrow G^3$  have the same degree. Hence,  $L$  is homogeneous. Since  $C_{\bullet}$  acyclic and all entries of the maps are in the maximal homogenous ideal of  $\text{gr}_{\mathfrak{m}}(R)$ , the ideal  $(b^*w^* - c^*v^*, -(a^*w^* - c^*u^*), a^*v^* - b^*u^*)$  is minimally generated by these three elements. In particular, these elements are not zero. This implies that

$$(bw - cv)^* = b^*w^* - c^*v^*, (aw - cu)^* = a^*w^* - c^*u^*, (av - bu)^* = a^*v^* - b^*u^*.$$

We shall show that  $I^* = L^*$ . Let  $I = (f_1, f_2, f_3)$ , where  $f_1 = bw - cv, f_2 = -(aw - cu), f_3 = av - bu$ . By Remark 5.1, we may assume that  $R$  is complete. By definition,  $(f_1^*, f_2^*, f_3^*) \subseteq I^*$ . Suppose that  $(f_1^*, f_2^*, f_3^*) \neq I^*$ . Then there exists an  $x \in I$  such that  $x^* \in I^* \setminus (f_1^*, f_2^*, f_3^*)$ . Since  $x \in I$ , we can write  $x = g_1f_1 + g_2f_3 + g_3f_3$  for some  $g_i \in R$ . Since we are in a complete local ring, we can write  $x = \sum_{i \geq 0} [x]_i$  as in Remark 5.1. Observe that

$$\text{ord}_{\mathfrak{m}}(x) = \text{ord}_{\mathfrak{m}}(g_1f_1 + g_2f_2 + g_3f_3) \geq \min\{\text{ord}_{\mathfrak{m}}(g_i f_i)\}_{i=1,2,3}.$$

Since  $x^* \notin (f_1^*, f_2^*, f_3^*)$ , the inequality is strict. We shall show that this leads to a contradiction.

The set

$$\Gamma = \left\{ \min\{\text{ord}_m(g_i f_i)_{i=1,2,3}\} \mid \text{for a triple } g_i \text{ such that } x = \sum g_i f_i \right\}$$

is finite since, for any  $n \in \Gamma$ ,  $n \leq \text{ord}_m(x)$ . We choose  $g_i$ 's such that the number  $\min\{\text{ord}_m(g_i f_i)\}_{i=1,2,3}$  is the maximum in  $\Gamma$ . We will construct  $g'_i$ 's such that

$$x = g'_1 f_1 + g'_2 f_2 + g'_3 f_3$$

and

$$\min\{\text{ord}_m(g'_i f_i)\} \geq \min\{\text{ord}_m(g_i f_i)\}.$$

This will contradict the maximality.

Let  $n := \min\{\text{ord}_m(g_i f_i)\}$ . If  $\text{ord}_m(x) = n$ , then  $x^* \in (f_1^*, f_2^*, f_3^*)$ . Therefore, we may assume that  $n < \text{ord}_m(x)$ . Let  $\delta_i := \text{ord}_m(f_i)$  for  $i = 1, 2, 3$ . We have

$$\begin{aligned} ([g_1 f_1 + g_2 f_2 + g_3 f_3]_n)^* &= ([g_1]_{n-\delta_1})^*(f_1)^* \\ &\quad + ([g_2]_{n-\delta_2})^*(f_2)^* \\ &\quad + ([g_3]_{n-\delta_3})^*(f_3)^* \\ &= 0, \end{aligned}$$

that is,  $([g_1]_{n-\delta_1})^*, ([g_2]_{n-\delta_2})^*, ([g_3]_{n-\delta_3})^*$  is a relation on  $f_1^*, f_2^*, f_3^*$ . Since the complex  $C_\bullet$  is acyclic, there exist  $s, t$  in  $R$  such that

$$\begin{aligned} &(( [g_1]_{n-\delta_1})^*, ([g_2]_{n-\delta_2})^*, ([g_3]_{n-\delta_3})^*) \\ &= s^*(a^*, b^*, c^*) + t^*(u^*, v^*, w^*) \\ &= (s^* a^* - t^* u^*, s^* b^* - t^* v^*, s^* c^* - t^* w^*). \end{aligned}$$

Let  $g'_1 = g_1 - (sa + tu)$ ,  $g'_2 = g_2 + (sb + tv)$  and  $g'_3 = g_3 - (sc + tw)$ . Then,  $x = g_1 f_1 + g_2 f_2 + g_3 f_3 = g'_1 f_1 + g'_2 f_2 + g'_3 f_3$  and  $\min\{\text{ord}_m(g_i f_i)\} \leq \min\{\text{ord}_m(g'_i f_i)\}$ . This is a contradiction.  $\square$

We now state a couple of folklore remarks which will be useful for the proofs of Lemmas 5.5 and 5.6.

**Remark 5.3.** Let  $(R, m)$  be a Noetherian local ring. Let  $x, y, u, v$  be elements in  $m$  such that  $xy - uv \neq 0$ . If  $(xy - uv)^*$  is a prime element in  $\text{gr}_m(R)$ , then  $\text{ord}_m(xy) = \text{ord}_m(uv)$ .

*Proof.* Assume, to the contrary,  $\text{ord}_{\mathfrak{m}}(xy) \neq \text{ord}_{\mathfrak{m}}(uv)$ . Without loss of generality, we may assume that  $\text{ord}_{\mathfrak{m}}(xy) < \text{ord}_{\mathfrak{m}}(uv)$ . Then  $(xy - uv)^* = (xy)^* = x^*y^*$ . Since  $x^*, y^*$  are in the maximal homogeneous ideal of  $\text{gr}_{\mathfrak{m}}(R)$ , neither is a unit. Therefore, the product  $x^*y^*$  is not a prime element. This is a contradiction.  $\square$

**Remark 5.4.** Let  $(R, \mathfrak{m})$  be a Noetherian local ring. Let  $x, y$  be elements in  $\mathfrak{m}$ . If  $\text{ord}_{\mathfrak{m}}(x) = \text{ord}_{\mathfrak{m}}(y)$  and  $x^* + y^* \neq 0$ , then  $(x + y)^* = x^* + y^*$ . In particular,  $\text{ord}_{\mathfrak{m}}(x + y) = \text{ord}_{\mathfrak{m}}(x)$ .

*Proof.* Let  $G = \text{gr}_{\mathfrak{m}}(R)$ . Recall that  $\text{ord}_{\mathfrak{m}}(x + y) = \text{ord}_{G_+}((x + y)^*)$  where  $G_+ = \bigoplus_{i>0} G_i$ . We have

$$\text{ord}_{G_+}(x^* + y^*) \geq \min\{\text{ord}_{G_+}(x^*), \text{ord}_{G_+}(y^*)\}.$$

If this is a strict inequality, then  $(x + y)^* = x^*$  or  $y^*$ , and this is a contradiction. Also, if  $\text{ord}_{G_+}(x^*) \neq \text{ord}_{G_+}(y^*)$ , then  $(x + y)^* = x^*$  or  $y^*$ , and this is a contradiction.  $\square$

**Lemma 5.5.** *Let  $(R, \mathfrak{m})$  be a regular local ring with maximal ideal  $\mathfrak{m}$ . Let  $I$  be an ideal of height 2 which is minimally generated by the  $2 \times 2$  minors of the  $2 \times 3$  matrix  $M$  with entries in  $\mathfrak{m}$ . Assume that the leading forms of any two minors form part of a minimal generating set of  $I^*$ . If  $I^*$  is a prime ideal and  $I_1(M) \not\subseteq \mathfrak{m}^2$ , then we can find a matrix*

$$\widetilde{M} = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$$

*such that  $I = I_2(\widetilde{M})$  and  $I^* = ((bw - cv)^*, (aw - cu)^*, (av - bu)^*)$  is perfect of height 2.*

*Proof.* Write

$$M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}.$$

By switching rows and columns, which preserves the minors up to sign, we may assume that  $a$  is in  $\mathfrak{m} \setminus \mathfrak{m}^2$ . Henceforth, we will only use the assumption that  $(av - bu)^*$  and  $(aw - cv)^*$  form part of a minimal generating set of  $I^*$ . We shall modify  $M$  by applying row and column operations to obtain  $\widetilde{M}$  with the desired properties. Observe that by adding a multiple of a row to the other does not change  $2 \times 2$  minors,

but adding a multiple of a column to another changes one minor while it does not change the other two in general. In both cases the ideal  $I_2(-)$  does not change. In the proof, the only type of column operations we perform is adding a multiple of the first to the second or the third column. This does not change the minors  $av - bu, aw - cu$ , but the minor  $bw - cv$  will be changed to  $bw - cv + f(av - bu) + g(aw - cu)$  for some  $f, g$  in  $R$ .

We claim that, after performing row and column operations on  $M$ , we may assume that  $(av - bu)^* = a^*v^* - b^*u^*$ . Assume to the contrary that  $(av - bu)^* \neq a^*v^* - b^*u^*$ . Since  $(av - bu)^*$  is part of a minimal generating set of a prime ideal  $I^*$ ,  $(av - bu)^*$  is a prime element. By Remark 5.3, we have  $\text{ord}_m(av) = \text{ord}_m(bu)$ . Therefore, by Remark 5.4, one has  $a^*v^* - b^*u^* = 0$ , equivalently,  $a^*v^* = b^*u^*$ . Since  $\text{gr}_m(R)$  is a UFD and  $a^*$  is of degree 1, it is a prime element. Hence, we have  $a^* \mid b^*$  or  $a^* \mid u^*$ . Suppose  $a^* \mid b^*$ . Then there exists a  $\delta$  in  $R$  such that  $a^*\delta^* = b^*$ , and this implies  $\text{ord}_m(b) \leq \text{ord}_m(b - \delta a)$ . We subtract the first column multiplied by  $\delta$  from the second column to obtain

$$M' = \begin{pmatrix} a & b - \delta a & c \\ u & v - \delta u & w \end{pmatrix}.$$

Since  $\text{ord}_m(b) \leq \text{ord}_m(b - \delta a)$ , we have  $\text{ord}_m(ub) \leq \text{ord}_m(u(b - \delta a))$ . We replace  $M$  by  $M'$ . The column operation changes the minors  $av - bu, aw - cu, bw - cv$  to  $av - bu, aw - cu, bw - cv + \delta(aw - cu)$ . If  $a^* \mid u^*$ , then we perform a row operation to obtain new  $u, v$  and  $w$ . Note that the row operation does not change the minors. We claim that this process terminates. Each time we replace  $M$  by  $M'$ , either  $\text{ord}_m(b)$  or  $\text{ord}_m(u)$  strictly increases whereas  $\text{ord}_m(av - bu)$  is fixed. By Remark 5.3, we have  $\text{ord}_m(av) = \text{ord}_m(bu)$ , and this implies

$$\text{ord}_m(av - bu) \geq \min\{\text{ord}_m(av), \text{ord}_m(bu)\} = \text{ord}_m(bu).$$

The number  $\text{ord}_m(av - bu)$  is fixed, whereas  $\text{ord}_m(bu)$  is strictly increasing after each process. Therefore this will terminate, and we obtain  $(av - bu)^* = a^*v^* - b^*u^*$ .

We shall show that, by subtracting a multiple of the first column from the third column, we can obtain a matrix  $M$  such that  $(av - bu)^* = a^*v^* - b^*u^*, (aw - cu)^* = a^*w^* - c^*u^*$ . In particular, this will not change the entries  $a, b, u, v$  of the matrix  $M$ ; hence, we preserve the property  $(av - bu)^* = a^*v^* - b^*u^*$ . Suppose that  $(aw - cu)^* \neq a^*w^* - c^*u^*$ .

Since  $(aw - cu)^*$  is part of a minimal generating set of  $I^*$ , it is a prime element. By Remark 5.3, we have  $\text{ord}_{\mathfrak{m}}(aw) = \text{ord}_{\mathfrak{m}}(cu)$  and  $a^*w^* = c^*u^*$ . If  $a^* \mid u^*$ , then the prime element  $(av - bu)^* = a^*v^* - b^*u^*$  is divisible by  $a^*$ . This contradicts the fact that  $a^*v^* - b^*u^*$  is a prime element. Therefore, we have  $a^* \mid c^*$ , and there exists a  $\delta$  in  $R$  such that  $a^*\delta^* = c^*$ . We subtract the first column multiplied by  $\delta$  from the third. In particular, this does not change the entries  $a, b, u, v$  of  $M$ . This process terminates, and we have

$$(av - bu)^* = a^*v^* - b^*u^* \quad \text{and} \quad (aw - cu)^* = a^*w^* - c^*u^*.$$

By Lemma 5.2, it suffices to show that  $\text{ht}(a^*w^* - c^*u^*, a^*v^* - b^*u^*) = 2$ . The images  $(av - bu)^*, (aw - cu)^*$  form a part of a minimal generating set of  $I^*$ . This implies that  $((av - bu)^*)$  is a prime ideal in  $\text{gr}_{\mathfrak{m}}(R)$  and  $(aw - cu)^* \notin ((av - bu)^*)$ . Hence, the height of the ideal  $\text{ht}((av - bu)^*, (aw - cu)^*)$  is 2. Indeed, our new  $M$  is  $\widetilde{M}$  in the statement.  $\square$

**Lemma 5.6.** *Let  $(R, \mathfrak{m})$  be a regular local ring with maximal ideal  $\mathfrak{m}$ . Let  $I$  be an ideal of height 2 which is minimally generated by the  $2 \times 2$  minors of the  $2 \times 3$  matrix  $M$  with entries in  $\mathfrak{m}$ , where*

$$M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}.$$

*Assume that  $(av - ub)^*, (aw - cu)^*$  is part of a minimal generating set of  $I^*$ . If  $I^*$  is a prime ideal and  $(av - bu)^* = a^*v^* - b^*u^*$ , then we have  $I^* = ((bw - cv + f(av - bu))^*, (aw - cu)^*, (av - bu)^*)$  for some  $f$  in  $R$ , and this ideal is perfect of height 2.*

*Proof.* Suppose that  $(aw - cu)^* \neq a^*w^* - c^*u^*$ . Since  $(av - bu)^*, (aw - cu)^*$  form part of a minimal generating set of a prime ideal, they are prime elements. By Remark 5.3, we have  $\text{ord}_{\mathfrak{m}}(aw) = \text{ord}_{\mathfrak{m}}(cu)$ . Therefore, we have  $a^*w^* - c^*u^* = 0$ . Since  $a^*v^* - b^*u^*$  is a prime element and  $\text{gr}_{\mathfrak{m}}(R)$  is a UFD,  $\text{gcd}(a^*, u^*) \sim 1$ . This implies that  $a^* \mid c^*$  and  $u^* \mid w^*$ . Hence, we may write  $c^* = \delta^*a^*$  and  $w^* = \delta^*u^*$  for some  $\delta$  in  $R$ . Let  $M'$  be a matrix modified by subtracting the first column of  $M$  multiplied by  $\delta$  from the third column. This column operation does not change the entries  $a, b, u$  and  $v$  of the matrix  $M$ . Note that  $\text{ord}_{\mathfrak{m}}(c - \delta a) \geq \text{ord}_{\mathfrak{m}}(c)$  and  $\text{ord}_{\mathfrak{m}}(w - \delta u) \geq \text{ord}_{\mathfrak{m}}(w)$ . As in the proof of Lemma 5.5, this process will terminate. Hence,



$(aw - cu)^* = a^*w^* - c^*u^*$ . Note that the  $2 \times 2$  minors of  $M'$  are  $av - bu, aw - cu, bw - cv + f(av - bu)$  for some  $f$  in  $R$ .

Since  $(av - ub)^*, (aw - cu)^*$  form a part of a minimal generating set of  $I^*$  and  $(av - ub)^*$  is a prime element, we have

$$\text{ht}((av - ub)^*, (aw - cu)^*) = 2.$$

The result follows by applying Lemma 5.2 to the matrix  $M'$ . □

**Theorem 5.7.** *Let  $S \cong R/I$ , where  $(R, \mathfrak{m})$  is a regular local ring and  $I$  is a height 2 perfect ideal. Assume that  $\text{gr}_{\mathfrak{n}}(S)$  is an integral domain where  $\mathfrak{n} = \mathfrak{m}/I$ . If  $\mu(I) \leq 2$  or  $\mu(I) = 3$  and  $I \not\subseteq \mathfrak{m}^5$ , then  $\text{gr}_{\mathfrak{n}}(S)$  is Cohen-Macaulay.*

*Proof.* Since  $\text{gr}_{\mathfrak{n}}(S) \cong \text{gr}_{\mathfrak{m}}(R)/I^*$  [4, Exercise 5.3], it suffices to show that  $I^*$  is a Cohen-Macaulay ideal. There exists a minimal generating set of  $I$  such that its leading forms are part of a minimal generating set of  $I^*$ . We fix a generating set of  $I$  with this property.

*Case 1.* When  $\mu(I) = 2$ . Let  $I = (f, g)$  for some  $f$  and  $g$  in  $S$ . We claim that  $I^* = (f^*, g^*)$ . Since  $I^*$  is a prime ideal and  $f^*, g^*$  form a part of minimal generating set of  $I^*$ ,  $f^*$  is a prime element. Since  $f^*$  and  $g^*$  are part of a minimal generating set of  $I^*$ ,  $g^* \notin (f^*)$ . Since  $G/(f^*)$  is a domain, the image of  $g^*$  in this ring is a non-zerodivisor. By [4, Exercise 5.2], the image of  $I^*$  is generated by the image of  $g^*$ . Hence,  $I^* = (f^*, g^*)$ .

*Case 2.* When  $\mu(I) = 3$ : Since  $I$  is a height 2 perfect ideal, by the Hilbert-Burch theorem [2],  $I$  can be generated by the 2 by 2 minors of a 2 by 3 matrix  $M$  where the minors are the chosen generators. Write

$$M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}.$$

If  $\text{ord}_{\mathfrak{m}}(I) \leq 3$ ,  $I_2(M) = I \not\subseteq \mathfrak{m}^4$ , hence  $I_1(M) \not\subseteq \mathfrak{m}^2$ . Now, the result follows by applying Lemma 5.5 to the matrix  $M$ .

Suppose  $\text{ord}_{\mathfrak{m}}(I) = 4$ . If there exists an entry of  $M$  which has order 1, then we may apply Lemma 5.5. We may assume that no entry of  $M$  has order 1. Since  $\text{ord}_{\mathfrak{m}}(I) = 4$ , at least one of the  $2 \times 2$  minors of  $M$  has order 4. Without loss of generality, we may assume that  $\text{ord}_{\mathfrak{m}}(av - bu) = 4$ . By assumption on the orders  $\text{ord}_{\mathfrak{m}}(a), \text{ord}_{\mathfrak{m}}(b)$ ,

$\text{ord}_{\mathfrak{m}}(u)$ , and  $\text{ord}_{\mathfrak{m}}(v)$  are at least 2. Since  $4 = \text{ord}_{\mathfrak{m}}(av - bu) \geq \min\{\text{ord}_{\mathfrak{m}}(av), \text{ord}_{\mathfrak{m}}(bu)\} \geq 4$ , we have

$$\text{ord}_{\mathfrak{m}}(a) = \text{ord}_{\mathfrak{m}}(b) = \text{ord}_{\mathfrak{m}}(u) = \text{ord}_{\mathfrak{m}}(v) = 2.$$

This implies  $(av - bu)^* = a^*v^* - b^*u^*$ . Therefore, we are done by Lemma 5.6.  $\square$

**Remark 5.8.** One cannot relax the condition of  $\text{gr}_I(R)$ , a domain. Let  $R = \mathbb{C}[[a, b, c, d, e]]$  and  $I$  be the  $2 \times 2$  minors of the matrix  $M$ ,

$$\begin{bmatrix} a^2 + c^3 & 0 & ad + c^3 \\ ab + c^3 & ae + a^3 & 0 \end{bmatrix}.$$

Then  $I^*$  is not prime, and  $G/I^*$  is not Cohen-Macaulay.

We can see from Lemmas 5.5, 5.6 that, once we find a minor which commutes with taking  $*$ , then we can find a matrix  $M$  where the images of the minors generate the leading ideal.

The next theorem analyzes the case where none of the minors commute with taking  $*$ .

**Theorem 5.9.** *Let  $(R, \mathfrak{m})$  be a regular local ring with maximal ideal  $\mathfrak{m}$ . Let  $I$  be an ideal which is minimally generated by the  $2 \times 2$  minors of a  $2 \times 3$  matrix  $M$  with entries in  $\mathfrak{m}$ , where*

$$M = \begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}.$$

Let

$$M^* = \begin{pmatrix} a^* & b^* & c^* \\ u^* & v^* & w^* \end{pmatrix}.$$

Suppose that  $(av - bu)^*$ ,  $(aw - cu)^*$  form part of a minimal generating set of  $I^*$  and  $I^*$  is a prime ideal. If

$$(av - bu)^* \neq a^*v^* - b^*u^*$$

and

$$(aw - cu)^* \neq a^*w^* - c^*u^*,$$

then either  $\text{ht } I_1(M^*) \leq 2$  or one of the rows of  $M^*$  divides the other.

*Proof.* Since the images of the minors form part of a minimal generating set of an prime ideal  $I^*$ , we have  $a^*v^* = b^*u^*, a^*w^* = c^*u^*$ . Let  $p^* = \gcd(a^*, u^*)$ , where  $p$  in  $R$ . Write  $a^* = p^*(a')^*$  and  $u^* = p^*(u')^*$ , where  $a', u'$  in  $R$ . Then,  $(a')^*v^* = b^*(u')^*$  where  $\gcd((a')^*, (u')^*) \sim 1$ . Therefore,  $(a')^* \mid b^*$  and  $(u')^* \mid v^*$ . Let  $q$  in  $R$  be such that  $(a')^*q^* = b^*$  and  $(u')^*q^* = v^*$ . From  $a^*w^* = c^*u^*$ , we have  $(a')^*w^* = c^*(u')^*$ . Hence, an  $r$  in  $R$  exists such that  $(a')^*r^* = c^*$  and  $(u')^*r^* = w^*$ . Now, we have

$$M^* = \begin{pmatrix} (a')^*p^* & (a')^*q^* & (a')^*r^* \\ (u')^*p^* & (u')^*q^* & (u')^*r^* \end{pmatrix}.$$

Therefore,  $I_1(M^*) \subseteq ((a')^*, (u')^*)$ , and this implies  $\text{ht } I_1(M^*) \leq 2$  if  $((a')^*, (u')^*)$  is not a unit ideal. If  $((a')^*, (u')^*)$  is a unit ideal, then either  $(a')^*$  or  $(u')^*$  is a unit. Without loss of generality, assume that  $(a')^*$  is a unit. Then we can easily see that, indeed, the first row divides the second row. □

**6. Serre’s conditions.** In this section, we show that, when a ring  $R$  is local, equidimensional and universally catenary, if  $\text{gr}_I(R)$  satisfies Serre’s condition  $(S_i)$  (or  $(R_i)$ ), then  $R[It, t^{-1}]$  satisfies Serre’s condition  $(S_i)$  (or  $(R_i)$ ), and  $R$  satisfies Serre’s condition  $(S_i)$ .

When  $R$  is a  $\mathbb{Z}$ -graded ring and  $\mathfrak{p}$  a prime ideal, let  $\mathfrak{p}^*$  denote the ideal generated by all homogeneous elements in  $\mathfrak{p}$ . This is a homogeneous prime ideal which has height exactly one less than that of  $\mathfrak{p}$  if  $\mathfrak{p}$  is not a homogenous ideal.

Recall that a Noetherian ring  $R$  satisfies *Serre’s condition*  $(S_i)$  if, for every prime ideal  $\mathfrak{p}$  of  $R$ ,  $\text{depth } R_{\mathfrak{p}} \geq \min\{i, \dim R_{\mathfrak{p}}\}$ . A Noetherian ring  $R$  satisfies *Serre’s condition*  $(R_i)$  if, for every prime ideal  $\mathfrak{p}$  of  $R$  with  $\dim R_{\mathfrak{p}} \leq i$ , the ring  $R_{\mathfrak{p}}$  is regular.

**Lemma 6.1** ([2, Theorem 1.5.9 and Exercises 2.1.27, 2.2.24]). *Let  $R$  be a Noetherian  $\mathbb{Z}$ -graded ring.*

- (a) *For  $\mathfrak{p} \in \text{Spec}(R)$ , the localization  $R_{\mathfrak{p}}$  is regular (Cohen-Macaulay) if and only if so is  $R_{\mathfrak{p}^*}$ .*
- (b) *Let  $\mathfrak{p} \in \text{Spec}(R)$ . If  $\mathfrak{p}$  is not homogeneous, then  $\text{depth } R_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}^*} + 1$ .*

**Theorem 6.2.** *Let  $(R, \mathfrak{m})$  be a local equidimensional universally catenary ring. Let  $I \subseteq \mathfrak{m}$  be an  $R$ -ideal. Consider the following conditions:*

- (a) *The ring  $\text{gr}_I(R)$  satisfies Serre's condition  $(S_i)$  (or  $(R_i)$ ).*
- (b) *The ring  $R[It, t^{-1}]$  satisfies Serre's condition  $(S_i)$  (or  $(R_i)$ ).*
- (c) *The ring  $R$  satisfies Serre's condition  $(S_i)$ .*

We have (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c).

*Proof.*

(a)  $\Rightarrow$  (b). Let  $\pi : R[It, t^{-1}] \rightarrow \text{gr}_I(R)$  be the natural surjective ring homomorphism. By Lemma 6.1(a), (b), it suffices to show Serre's condition  $(S_i)$  (or  $(R_i)$ ) for homogenous prime ideals. Let  $P \subseteq R[It, t^{-1}]$  be a homogeneous prime ideal. Since  $R$  is universally catenary and equidimensional, so is  $R[It, t^{-1}]$ . This implies that  $\text{ht}(P + (t^{-1})) \leq \text{ht} P + 1$ . We can choose a minimal prime  $Q$  of  $P + (t^{-1})$  of height  $\text{ht}(P + (t^{-1}))$ . We first show the statement for Serre's condition  $(S_i)$ . Since  $\text{gr}_I(R)$  satisfies Serre's condition  $(S_i)$ ,  $\text{depth} \text{gr}_I(R)_{\pi(Q)} \geq \min\{i, \dim \text{gr}_I(R)_{\pi(Q)}\}$ . Since  $\text{depth} \text{gr}_I(R)_{\pi(Q)} = \text{depth} R[It, t^{-1}]_Q - 1$  and  $\dim \text{gr}_I(R)_{\pi(Q)} = \dim R[It, t^{-1}]_Q - 1$ , we have

$$(6.1) \quad \text{depth} R[It, t^{-1}]_Q \geq \min\{i + 1, \dim R[It, t^{-1}]_Q\}.$$

If  $P = Q$ , i.e.,  $t^{-1} \in P$ , then we are done. Suppose  $P \subsetneq Q$ . We need to show that  $\text{depth} R[It, t^{-1}]_P \geq \min\{i, \dim R[It, t^{-1}]_P\}$ . Since  $\dim R[It, t^{-1}]_Q = \dim R[It, t^{-1}]_P - 1$ , by (6.1), it suffices to show that  $\text{depth} R[It, t^{-1}]_P \geq \text{depth} R[It, t^{-1}]_Q + 1$ . This follows immediately once we have shown that

$$\begin{aligned} \text{Ext}_{R[It, t^{-1}]_Q}^j((R[It, t^{-1}]/P)_Q, R[It, t^{-1}]_Q) &= 0 \\ \text{for } j < \text{depth} R[It, t^{-1}]_Q - 1. \end{aligned}$$

Since  $\dim(R[It, t^{-1}]/P)_Q = 1$ , this follows from [14, Lemma 2, (15.E)].

Now, suppose that  $\text{gr}_I(R)$  satisfies Serre's condition  $(R_i)$ . When  $\text{ht} P \leq i$ , since  $\text{ht} \pi(Q) \leq \text{ht} P \leq i$ ,  $\text{gr}_I(R)_{\pi(Q)}$  is regular. Since  $R[It, t^{-1}]/(t^{-1}) \cong \text{gr}_I(R)$  and  $t^{-1}$  is a regular element,  $R[It, t^{-1}]_Q$  is regular. Since  $P \subseteq Q$ ,  $R[It, t^{-1}]_P$  is regular.

(b)  $\Rightarrow$  (c). Let  $\mathfrak{p}$  be a prime ideal of  $R$  of height  $c$ . Recall that  $R[It, t^{-1}]/(t^{-1} - 1) \cong R$ . Let  $P$  be the pre-image of  $\mathfrak{p}$  in  $R[It, t^{-1}]$ . Since  $R[It, t^{-1}]$  is equidimensional and universally catenary,

ht  $P = c + 1$ . Since  $P$  contains  $t^{-1} - 1$ , it is a non homogeneous prime ideal of  $R[It, t^{-1}]$ . Therefore,  $P^*$  is a homogeneous prime ideal of height  $c$ . Since  $R[It, t^{-1}]$  satisfies Serre's condition  $(S_i)$ , we have  $\text{depth } R[It, t^{-1}]_{P^*} \geq \min\{i, \dim R[It, t^{-1}]_{P^*}\}$ . Recall that  $t^{-1} - 1$  is a regular element in  $R[It, t^{-1}]$  and  $R \cong R[It, t^{-1}]/(t^{-1} - 1)$ . We have  $\text{depth } R[It, t^{-1}]_{P^*} = \text{depth } R[It, t^{-1}]_P - 1 = \text{depth } R_{\mathfrak{p}} + 1 - 1 = \text{depth } R_{\mathfrak{p}}$  and  $\dim R[It, t^{-1}]_{P^*} = \dim R[It, t^{-1}]_P - 1 = \dim R_{\mathfrak{p}} + 1 - 1 = \dim R_{\mathfrak{p}}$ , where the first equality follows from Lemma 6.1 (b). Therefore,  $\text{depth } R_{\mathfrak{p}} \geq \min\{i, \dim R_{\mathfrak{p}}\}$ .  $\square$

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UNIVERSITY OF CALIFORNIA, RIVERSIDE, DEPARTMENT OF MATHEMATICS, RIVERSIDE, CA 92521

**Email address:** [youngsu.kim@ucr.edu](mailto:youngsu.kim@ucr.edu)