JOURNAL OF COMMUTATIVE ALGEBRA
Volume 9, Number 4, Winter 2017

FROBENIUS BETTI NUMBERS AND MODULES OF FINITE PROJECTIVE DIMENSION

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ABSTRACT. Let $(R,\mathfrak{m},K)$ be a local ring, and let M be an R-module of finite length. We study asymptotic invariants, $\beta_i^F(M,R)$, defined by twisting with Frobenius the free resolution of M. This family of invariants includes the Hilbert-Kunz multiplicity ($e_{HK}(\mathfrak{m},R)=\beta_0^F(K,R)$). We discuss several properties of these numbers that resemble the behavior of the Hilbert-Kunz multiplicity. Furthermore, we study when the vanishing of $\beta_i^F(M,R)$ implies that M has finite projective dimension. In particular, we give a complete characterization of the vanishing of $\beta_i^F(M,R)$ for one-dimensional rings. As a consequence of our methods we give conditions for the non-existence of syzygies of finite length.

1. Introduction. Let $(R,\mathfrak{m},K)$ denote an F-finite local ring of dimension d and characteristic $p>0$, and let $\alpha=\log_p[K:K^p]$. Given an R-module M and an integer $e\geqslant 0$, ${}^eM$ denotes the R-module structure on M given by $r*m=r^{p^e}m$ for every $m\in {}^eM$ and $r\in R$. In addition, $\lambda_R(M)$, or simply $\lambda(M)$ when the ring is clear from the context, denotes the length of M as an R-module.

Let $q=p^e$ be a power of p. For an ideal $I\subseteq R$, let $I^{[q]}=(i^q\mid i\in I)$ be the ideal generated by the qth powers of elements in I. If I is $\mathfrak{m}$-primary, the *Hilbert-Kunz multiplicity of I in R* is defined by

$$e_{HK}(I,R)=\lim_{e\to\infty}\frac{\lambda(R/I^{[q]})}{q^d}.$$

2010 AMS *Mathematics subject classification.* Primary 13A35, 13D02, Secondary 13D07, 13H10.

Keywords and phrases. Hilbert-Kunz multiplicity, projective dimension, Krull dimension of syzygies, F-contributors.

The second author was supported by the National Science Foundation, grant No. DMS-1259142. The third author was supported by the National Council of Science and Technology of Mexico (CONACYT), grant No. 207063.

Received by the editors on December 13, 2014, and in revised form on September 18, 2015.

DOI:10.1216/JCA-2017-9-4-455

The existence of the previous limit was proven by Monsky [**25**]. Under mild conditions, $e_{HK}(\mathfrak{m}, R) = 1$ if and only if R is a regular ring [**34**]. The Hilbert-Kunz multiplicity can be interpreted as a measure of singularity: the smaller it is, the nicer is the ring. For instance, Aberbach and Enescu proved rings with small Hilbert-Kunz multiplicity are Gorenstein and F-regular [**1**] (see also [**7**]). We have that

$$\lambda(R/I^{[q]}) = q^{\alpha}\lambda(R/I \otimes_R {}^eR) = q^{\alpha}\lambda(\operatorname{Tor}_0^R(R/I, {}^eR)).$$

This gives rise to the following extension of the Hilbert-Kunz multiplicity for higher Tor functors. Let N be a finitely generated R-module, and let M be an R-module of finite length. For an integer $i \geqslant 0$, define

$$\beta_i^F(M, N) = \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}_i^R(M, {}^eN))}{q^{(d+\alpha)}}.$$

We denote $\beta_i^F(K, R)$ by $\beta_i^F(R)$ and call it the *ith Frobenius Betti number of R*.

These higher invariants also detect regularity, namely, Aberbach and Li [**3**] showed that R is a regular ring if and only if $\beta_i^F(R) = 0$ for some $i \geqslant 1$. Note that R is regular if and only if K has finite projective dimension as R-module.

In this manuscript, we seek an answer to the following question.

Question 1.1. *Let M be an R-module of finite length. What vanishing conditions on $\beta_i^F(M, R)$ imply that M has finite projective dimension?*

Miller [**23**] showed that, if R is a complete intersection and M is an R-module of finite length, then the vanishing of $\beta_i^F(M, R)$ for some $i \geqslant 1$ implies that M has finite projective dimension. We refer to [**13**] for related results for Gorenstein rings. In Section 4, we answer this question for rings that have small regular algebras, and for rings that have F-contributors. Later, we focus on one-dimensional rings and give the following characterization for the vanishing of $\beta_i^F(M, R)$.

Theorem (see Theorem 4.7)**.** *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring of positive characteristic p, and let M be an R-module of finite length. Let $(G_j, \varphi_j)_{j\geqslant 0}$ be a minimal free resolution of M. Then the following are equivalent*:

(i) $\operatorname{Im}(\varphi_{i+1}) \subseteq H^0_{\mathfrak{m}}(G_i)$.
(ii) $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ *for all* $e \geqslant 0$, *for all* $\mathfrak{p} \in \operatorname{Min}(R)$.
(iii) $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ *for all* $e \gg 0$, *for all* $\mathfrak{p} \in \operatorname{Min}(R)$.
(iv) $\beta_i^F(M, R) = 0$.

Assume, in addition, that R *is complete and* K *is algebraically closed. If* V *denotes the integral closure of* R *in its ring of fractions, then the conditions above are equivalent to*

(v) $\operatorname{Tor}_i^R(M, V) = 0$.

As a consequence of this theorem, we show that, if R is a one dimensional Cohen-Macaulay local ring and $\lambda(M) < \infty$, then $\beta_i^F(M,R) = 0$ for any $i \geqslant 1$ implies that M has finite projective dimension (see Corollary 4.8). Furthermore, we prove that the vanishing of two consecutive $\beta_i^F(M,R)$ implies that M has finite projective dimension in every one-dimensional local ring (see Corollary 4.9).

From the above theorem we have that $\beta_i^F(M,R) = 0$ if and only the $(i+1)$-syzygy has finite length. On the other hand, there are modules of infinite projective dimension over one-dimensional rings which have second syzygies of finite length (see Example 5.1). Motivated by Iyengar's question about the eventual stability of dimensions of syzygies and by our results regarding $\beta_i^F(M,R)$, we ask the following question.

Question 1.2. *Let* R *be a* d*-dimensional local ring, and let* M *be a finitely generated* R*-module such that* $\operatorname{pd}_R(M) = \infty$ *and* $\lambda(M) < \infty$. *If* $i > d+1$, *then must the length of the* i*th syzygy be infinite?*

In Section 5, we study this question, mainly for one-dimensional rings. In particular, we show that the answer to Question 1.2 is positive for one-dimensional Buchsbaum rings (see Proposition 5.3). We also obtain a partial answer for modules whose Betti numbers are eventually non-decreasing (see Proposition 5.7). Furthermore, we show that the first and third syzygies of M are either zero or have infinite length for every finite length module M over a one-dimensional ring (see Corollary 5.10). The assumption of M having finite length is necessary, as shown in Example 5.11. Aside from the study of projective dimension, we study basic properties of the higher invariants that resemble the Hilbert-Kunz multiplicity in other aspects.

2. Notation and terminology. Throughout this article, $(R, \mathfrak{m}, K)$ will denote a local ring of Krull dimension $\dim(R) = d$. For a finitely generated R-module M, we define $\dim(M) = \dim(R/(0 :_R M))$, where $0 :_R M = \{x \in R \mid xM = 0\}$. When $M = 0$, we set $\dim(M) = -1$. An element $x \in R$ such that $\dim(R/(x)) = d - 1$ will be called a *parameter of* R. Given a finitely generated R-module M, a *minimal free resolution* $(G_\bullet, \varphi_\bullet)$ of M is an exact sequence

$$\cdots \to G_{i+1} \xrightarrow{\varphi_{i+1}} G_i \xrightarrow{\varphi_i} \cdots \to G_1 \xrightarrow{\varphi_1} G_0 \to M \to 0$$

such that $G_i \cong R^{\beta_i(M)}$ are free R-modules and $\operatorname{Im}(\varphi_{i+1}) \subseteq \mathfrak{m}G_i$. The integers $\beta_i(M) = \operatorname{rk}(G_i) = \lambda(\operatorname{Tor}_j^R(M, K))$ are called the *Betti numbers of* M. If $\beta_i(M) = 0$ for some i, we say that M has *finite projective dimension*, and that it is equal to $\operatorname{pd}_R(M) = \max\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$. We adopt the convention that $\operatorname{pd}_R(M) = -\infty$, when $M = 0$. For all $i \geqslant 0$, we set $\Omega_i(M) = \operatorname{Coker}(\varphi_i)$, and we call it the *ith syzygy of the module* M. Note that $\Omega_0(M) = M$. When no confusion may arise, we will denote $\Omega_i(M)$ simply by Ω_i.

Herein, we often use local cohomology tools. For every $k \in \mathbb{N}$, the quotient map $R/\mathfrak{m}^{k+1} \to R/\mathfrak{m}^k$ induces maps of functors

$$\operatorname{Ext}_R^i(R/\mathfrak{m}^k, -) \longrightarrow \operatorname{Ext}_R^i(R/\mathfrak{m}^{k+1}, -).$$

For an R-module M, we define the *ith local cohomology of* M *with support on* $\mathfrak{m}$ by

$$H_\mathfrak{m}^i(M) = \lim_{k\to\infty} \operatorname{Ext}_R^i(R/\mathfrak{m}^k, M).$$

In particular,

$$H_\mathfrak{m}^0(M) = \bigcup_{k\in\mathbb{N}} 0 :_M \mathfrak{m}^k = \{v \in M \mid \mathfrak{m}^k v = 0 \text{ for some } k \in \mathbb{N}\}.$$

For a non-zero finitely generated R-module M, $\operatorname{depth}(M)$ denotes the smallest integer j such that $H_\mathfrak{m}^j(R) \neq 0$. When $\operatorname{depth}(M) = \dim(M)$, the module is called *Cohen-Macaulay*, and M is called *maximal Cohen-Macaulay* if $\operatorname{depth}(M) = \dim(R)$.

We now review some basic facts regarding integral closures. For an ideal $I \subseteq R$ and an element $x \in R$, we say that x is integral over I if it satisfies an equation of the form $x^n + r_1x^{n-1} + \cdots + r_n = 0$, where $r_j \in I^j$ for all $j = 1, \ldots, n$. The set of elements integral over I forms an ideal, which is called the *integral closure of* I, and denoted $\overline{I}$. For an ideal $J \subseteq I$, we say that J is a *reduction of* I if $\overline{J} = \overline{I}$. We say that J is a *minimal reduction of* I if it is a reduction of I which is minimal with respect to containment. We refer the reader to [**32,** Chapter 8] for more details about reductions. For a domain R, let V be the integral closure of R in its field of fractions L. We define the *conductor of* R as the set of all elements $z \in L$ such that $zV \subseteq R$, and we denote it by C. When V is finite over R, it can be shown that C is the largest ideal which is common to R and V, and that C contains a non-zero divisor for R [**32,** Exercise 2.11]. In particular, if $(R, \mathfrak{m}, K)$ is an excellent one-dimensional local domain, the conductor is $\mathfrak{m}$-primary. See [**32,** Chapter 12] for more results about conductors.

We also need the notion of *dualizing complex.* We refer to [**27,** page 51] or to [**15,** Chapter V] for more details.

Definition 2.1. Let $(S, \mathfrak{n}, L)$ be a local ring of dimension d. We say that a complex $D^\bullet$ is a dualizing complex of S, if

(i) $D^i = \bigoplus_{\dim S/\mathfrak{p} = d-i} E_S(S/\mathfrak{p})$.
(ii) The cohomology $H^i(D^\bullet)$ is finitely generated.

Remark 2.2. If $(S, \mathfrak{n}, L)$ is a complete ring, then S has a dualizing complex, $D_S^\bullet$ [**15,** page 299]. If $\mathfrak{p}$ is a prime ideal such that $\dim S/\mathfrak{p} = \dim S$, we have that $S_\mathfrak{p}$ is Artinian, hence complete. In addition, $D_{S_\mathfrak{p}}^\bullet := D_S^\bullet \otimes S_\mathfrak{p}$ is a dualizing complex for $S_\mathfrak{p}$. Furthermore, $H^j(D_{S_\mathfrak{p}}^\bullet) = H^j(D_S^\bullet) \otimes S_\mathfrak{p} = 0$ for $j > 0$ and $\omega_{S_\mathfrak{p}} \cong H^0(D_{S_\mathfrak{p}}^\bullet) = E_{S_\mathfrak{p}}(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})$, since $S_\mathfrak{p}$ is Artinian, and thus it is Cohen-Macaulay.

We now introduce Buchsbaum rings. We study Question 1.2 in Section 5.

Definition 2.3. Let $(R, \mathfrak{m}, K)$ be a local ring of dimension d. We say that R is a *Buchsbaum ring* if, for any system of parameters $x_1, \ldots, x_d$, we have

$$(x_1, \ldots, x_{i-1}) : x_i = (x_1, \ldots, x_{i-1}) : \mathfrak{m}$$

for every $i = 1, \ldots, d$. When $i = 1$, the ideal $(x_1, \ldots, x_{i-1})$ is simply the zero ideal.

There are several equivalent ways for defining Buchsbaum rings, but that above is the most convenient for our purposes.

Remark 2.4. Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring. Suppose that R is not Cohen-Macaulay, so that $H^0_{\mathfrak{m}}(R) \neq 0$. Then there exists a parameter x of R such that $H^0_{\mathfrak{m}}(R) = 0 :_R x$. In fact, fix an integer $n \in \mathbb{N}$ such that $\mathfrak{m}^n H^0_{\mathfrak{m}}(R) = 0$, using that $H^0_{\mathfrak{m}}(R) \subseteq R$ is an ideal; hence, it is finitely generated. Take any parameter $y \in \mathfrak{m}$, and set $x = y^n$. With this choice, we have $xH^0_{\mathfrak{m}}(R) \subseteq \mathfrak{m}^n H^0_{\mathfrak{m}}(R) = 0$, so that $H^0_{\mathfrak{m}}(R) \subseteq 0 :_R x$. On the other hand, there exists a $k \in \mathbb{N}$ such that $\mathfrak{m}^k \subseteq (x)$. Therefore, if $r \in 0 :_R x$, we get $r\mathfrak{m}^k \subseteq r(x) = 0$, so that $r \in H^0_{\mathfrak{m}}(R)$. We conclude that $H^0_{\mathfrak{m}}(R) = 0 :_R x$.

Remark 2.5. Let $(R, \mathfrak{m}, K)$ be a one-dimensional Buchsbaum ring. By Remark 2.4, there exists a parameter $x \in R$ such that $0 :_R x = H^0_{\mathfrak{m}}(R)$. By the definition of the Buchsbaum ring, we have that

$$H^0_{\mathfrak{m}}(R) = 0 :_R x = 0 :_R \mathfrak{m}.$$

In particular, $\mathfrak{m}H^0_{\mathfrak{m}}(R) = 0$, that is, $H^0_{\mathfrak{m}}(R) \cong \bigoplus_{j=1}^t K$ is a finite-dimensional K-vector space.

For the rest of the section, assume that $(R, \mathfrak{m}, K)$ is a local ring of characteristic $p > 0$. For an integer $e \geqslant 1$, we consider the eth iteration of the Frobenius endomorphism $F^e : R \to R$, $F^e(r) = r^{p^e}$ for all $r \in R$. For an R-module M, we can consider M with the action induced by restriction of scalars, via F^e. We denote this module by ${}^eM$. More explicitly, for $r \in R$ and $m \in {}^eM$, we have $r * m = r^{p^e} m$.

Definition 2.6. We say that R is *F-finite* if ${}^1R$ is a finitely generated R-module.

Note that R is F-finite if and only if ${}^eR$ is a finitely generated R-module for any $e \geqslant 1$ or, equivalently, for all $e \geqslant 1$. Furthermore, F-finite rings are excellent [**20,** Theorem 2.5]. When R is F-finite, we have that $[K : K^p] < \infty$. In this case, we set $\alpha = \log_p[K : K^p]$.

3. Definition and properties of $\beta_i^F(M, N)$ and $\mu_i^F(M, N)$. We begin by defining the Frobenius Betti numbers and showing basic properties that resemble the Hilbert-Kunz multiplicity.

Definition 3.1 (see also [**22**])**.** Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, let M be an R-module of finite length, and let N be a finitely generated R-module. Define

$$\beta_{i,R}^F(M, N) = \lim_{e \to \infty} \frac{\lambda(\mathrm{Tor}_i^R(M, {}^eN))}{q^{(d+\alpha)}}.$$

We denote $\beta_{i,R}^F(K, R)$ by $\beta_{i,R}^F(R)$ and call it the *ith Frobenius Betti number of R*. If the ring is clear from the context, we only write $\beta_i^F(M, N)$. The above limit exists by the main result in [**29**].

We point out that Li [**22**] focused on $\beta_i^F(R/I, R)$, which he denoted by $t_i(I, R)$.

Example 3.2. Suppose that $R = S/fS$, where S is an F-finite regular local ring of characteristic $p > 0$, and $f \in S$. We write ${}^eR \cong R^{a_e} \oplus M_e$, where M_e has no free summands. The limit $s(R) := \lim_{e\to\infty}(a_e/q^{(d+\alpha)})$ exists [**33,** Theorem 4.9], and it is called the *F-signature of R*, which is an important invariant related to strong F-regularity [**2,** Theorem 0.2]. We consider the minimal free resolution of ${}^eR$:

$$\cdots \to R^{\beta_i({}^eR)} \to R^{\beta_{i-1}({}^eR)} \to \cdots \to R^{\beta_0({}^eR)} \to {}^eR \to 0.$$

We note that $\beta_0({}^eR) = a_e + \beta_0(M_e)$ and $\beta_i({}^eR) = \beta_i(M_e)$ for $i > 0$. Since M_e is a maximal Cohen-Macaulay module with no free summands, we have that $\beta_i(M_e) = \beta_0(M_e)$ for $i > 0$ [**14,** Proposition 5.3 and Theorem 6.1]. Then,

$$\begin{aligned} \beta_0^F(R) = e_{HK}(\mathfrak{m}, R) &= \lim_{e\to\infty} \frac{\beta_0({}^eR)}{q^{(d+\alpha)}} \\ &= \lim_{e\to\infty} \frac{a_e}{q^{(d+\alpha)}} + \lim_{e\to\infty} \frac{\beta_0(M_e)}{q^{(d+\alpha)}} \\ &= s(R) + \lim_{e\to\infty} \frac{\beta_0(M_e)}{q^{(d+\alpha)}}. \end{aligned}$$

Hence,

$$\begin{aligned} \beta_i^F(R) &= \lim_{e\to\infty} \frac{\beta_i({}^eR)}{q^{(d+\alpha)}} = \lim_{e\to\infty} \frac{\beta_i(M_e)}{q^{(d+\alpha)}} \\ &= \lim_{e\to\infty} \frac{\beta_0(M_e)}{q^{(d+\alpha)}} = e_{HK}(\mathfrak{m}, R) - s(R) \end{aligned}$$

for $i > 0$.

As for the Hilbert-Kunz multiplicity, the Frobenius Betti numbers also increase after taking the quotient by a nonzero divisor.

Proposition 3.3. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, M an R-module of finite length, and $x \in \operatorname{ann}(M)$ a nonzero divisor on R. Then,*

$$\beta^F_{i,R}(M,R) = \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}^R_i(M, {}^eR))}{q^{(d+\alpha)}} \leqslant \beta^F_{i,R/(x)}(N, R/(x))$$
$$= \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}^{R/(x)}_i(M, {}^e(R/(x))))}{q^{(d-1+\alpha)}},$$

where the subscripts indicate over which ring we are computing the Frobenius Betti numbers. In particular, $\beta^F_{i,R}(R) \leqslant \beta^F_{i,R/(x)}(R/(x))$.

Proof. Let $G_\bullet \to {}^eR$ be a minimal free resolution of ${}^eR$. Let $\overline{R}$ denote R/xR. We have that $\overline{G}_\bullet = G_\bullet \otimes_R \overline{R}$ is a free resolution for ${}^eR \otimes_R \overline{R}$ as an $\overline{R}$-module. Furthermore, we have that $H_0(\overline{G}_\bullet) = {}^eR \otimes_R \overline{R}$. This is a consequence of the fact that $H_i(\overline{G}_\bullet) = \operatorname{Tor}^R_i({}^eR, \overline{R}) = 0$ for $i > 0$ since x is a nonzero divisor on R and ${}^eR$.

Due to the fact that $x \in \operatorname{ann}(M)$, we have

$$\begin{aligned}\operatorname{Tor}^R_i(M, {}^eR) &= H_i(M \otimes_R G_\bullet) = H_i(M \otimes_{\overline{R}} \overline{R} \otimes_R G_\bullet)\\ &= H_i(M \otimes_{\overline{R}} \overline{G}_\bullet)\\ &= \operatorname{Tor}^{\overline{R}}_i(M, {}^eR \otimes_R \overline{R}).\end{aligned}$$

Since x is a nonzero divisor on R, there is a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_q = {}^eR \otimes_R \overline{R}$$

such that $L_{r+1}/L_r = {}^e(\overline{R})$. As a consequence, $\lambda(\operatorname{Tor}^{\overline{R}}_i(M, {}^eR \otimes_R \overline{R})) \leqslant q \cdot \lambda(\operatorname{Tor}^{\overline{R}}_i(M, {}^e\overline{R}))$. Then,

$$\lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}^R_i(M, {}^e R))}{q^{(d+\alpha)}} \leqslant \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}^{\overline{R}}_i(M, {}^e\overline{R}))}{q^{(d-1+\alpha)}} \qquad \square$$

We now introduce $\mu^F_i(M,N)$, a dual version of $\beta^F_i(M,N)$, which is defined in terms of Ext. In Proposition 3.11, we establish a relation between these asymptotic invariants.

Definition 3.4. Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, let M be an R-module of finite length, and let N be a finitely generated R-module. We define

$$\mu_i^F(M, N) = \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eN))}{q^{(d+\alpha)}},$$

Next, we prove that the numbers $\mu_i^F(M, N)$ are well defined. The proof is essentially the same as that for $\beta_i^F(M, N)$, as it uses the main result in [**29**]. Nonetheless, we include it here for completeness.

Proposition 3.5. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, let M be an R-module of finite length, and let N be a finely generated R-module. Then, $\lim_{e\to\infty}[\lambda(\operatorname{Ext}^i_R(N, {}^e M))]/q^{(d+\alpha)}$ exists. Moreover, if*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

is a short exact sequence, then

$$\lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eN_2))}{q^{(d+\alpha)}} = \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eN_1))}{q^{(d+\alpha)}} + \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eN_3))}{q^{(d+\alpha)}}.$$

Proof. Let $G_\bullet \to M$ be a minimal free resolution of M, and define

$$g_e(N) = \lambda(H^i(\operatorname{Hom}_R(G_\bullet), {}^eN)).$$

Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of finitely generated R-modules. We have that $g_e(N_2) \leqslant g_e(N_1) + g_e(N_3)$, and equality holds if the sequence splits. Then,

$$\lim_{e\to\infty} \frac{g_e(N)}{q^{(d+\alpha)}} = \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eN))}{q^{(d+\alpha)}}$$

exists, and it is additive in short exact sequences [**29**]. □

Proposition 3.6. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, M an R-module of finite length, and N a finitely generated R-module. Let Λ be the set of all prime ideals $\mathfrak{p}$ such that $\dim R/\mathfrak{p} = \dim R$. We have that*

$$\beta_i^F(M, N) = \sum_{\mathfrak{p}\in\Lambda} \beta_i^F(M, R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(N_\mathfrak{p})$$

and

$$\mu_i^F(M,N) = \sum_{\mathfrak{p}\in\Lambda} \mu_i^F(M,R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(N_\mathfrak{p}).$$

Proof. We only prove the first statement, since the proof of the second is completely analogous. Let $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_h = N$ be a filtration for N such that $N_j/N_{j-1} \cong R/\mathfrak{p}_j$, where $\mathfrak{p}_j \subseteq R$ is a prime ideal; we have short exact sequences

$$0 \longrightarrow N_{j-1} \longrightarrow N_j \longrightarrow R/\mathfrak{p}_j \longrightarrow 0.$$

We deduce that

$$\beta_i^F(M,N) = \sum_{j=1}^{h} \beta_i^F(M,R/\mathfrak{p}_j)$$

[**29,** Proposition 1 (b)]. In addition, we have that $\beta_i^F(M,R/\mathfrak{p}_j) = 0$ whenever $\dim(R/\mathfrak{p}_j) < \dim(R)$ [**29,** Proposition 1 (a)]. In order to prove the result, we need to count the number of times that a prime $\mathfrak{p}$ such that $\dim R/\mathfrak{p} = \dim R$ appears in the prime filtration. This number is obtained by localizing the above filtration at $\mathfrak{p}$ and counting the length of the resulting chain. Since the localized chain is a composition series of the module $N_\mathfrak{p}$, we obtain that the number of times $\mathfrak{p}$ appears in the above prime filtration is given by $\lambda_{R_\mathfrak{p}}(N_\mathfrak{p})$. Then,

$$\beta_i^F(M,N) = \sum_{j=1\, \mathfrak{p}_j\in\Lambda}^{h} \beta_i^F(M,R/\mathfrak{p}_j) = \sum_{\mathfrak{p}\in\Lambda} \beta_i^F(M,R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(N_\mathfrak{p}). \quad \square$$

Remark 3.7. It follows from Proposition 3.6 that, if $\beta_i^F(M,R) = 0$ for some $i \in \mathbb{N}$, we have that $\beta_i^F(M,R/\mathfrak{p}) = 0$ for every minimal prime of R such that $\dim(R/\mathfrak{p}) = d$. Therefore, if this is the case, $\beta_i^F(M,N) = 0$ for every finitely generated R-module N, again using Proposition 3.6. A similar statement holds for $\mu_i^F(M,R)$.

The following theorem is related to results of Chang [**10,** Lemma 1.20, Corollary 2.4], and in some cases it follows from them. We present a different proof that does not use spectral sequences.

Theorem 3.8. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, M an R-module of finite length, and N a finitely generated R-module. Then*

$$\lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eN))}{q^{(i+1+\alpha)}} = 0$$

for $i < d$. In particular, $\mu_i^F(M, N) = 0$ for $i < d$.

Proof. Our proof follows by induction on $n = \dim(N)$.

If $n = 0$, we have that $h = \lambda(N)$ is finite. There is a filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_h = N$$

such that $N_j/N_{j-1} \cong K$. From the short exact sequences

$$0 \longrightarrow N_{j-1} \longrightarrow N_j \longrightarrow K \longrightarrow 0,$$

we have that

$$\lambda(\operatorname{Ext}^i_R(M, {}^eN_j))) \leqslant \lambda(\operatorname{Ext}^i_R(K, {}^eN_{j-1})) + \lambda(\operatorname{Ext}^i_R(K, {}^eK)).$$

Since

$$\begin{aligned} \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eK))}{q^{(i+1+\alpha)}} &= \lim_{e\to\infty} \frac{q^\alpha\lambda(\operatorname{Ext}^i_R(M, K))}{q^{(i+1+\alpha)}} \\ &= \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, K))}{q^{(i+1)}} = 0, \end{aligned}$$

we have that

$$\lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eN))}{q^{(i+1+\alpha)}} = 0$$

by an inductive argument.

Suppose that our claim is true for modules of dimension less than or equal to $n - 1$. There is a filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_h = N$$

such that $N_j/N_{j-1} \cong R/\mathfrak{p}_j$, where $\mathfrak{p}_j \subset R$ is a prime ideal. From the short exact sequences $0 \to N_{j-1} \to N_j \to R/\mathfrak{p}_j \to 0$, we have that

$$\lambda(\operatorname{Ext}^i_R(M, {}^eN_j)) \leqslant \lambda(\operatorname{Ext}^i_R(M, {}^eN_{j-1})) + \lambda(\operatorname{Ext}^i_R(M, {}^e(R/\mathfrak{p}_j))).$$

It suffices to show that

$$\lim_{e\to\infty} \frac{\lambda(\mathrm{Ext}^i_R(M, {}^e(R/\mathfrak{p}_j)))}{q^{(i+1+\alpha)}} = 0 \tag{3.1}$$

for primes $\mathfrak{p}_j$ such that $\dim_R(R/\mathfrak{p}_j) = n = \dim_R N$. Let $T = R/\mathfrak{p}_j$. Let $x \in \mathrm{Ann}_R M \setminus \mathfrak{p}_j$, which exists because $\dim_R T = \dim_R N > 0 = \dim_R M$. We have a short exact sequence

$$0 \longrightarrow {}^eT \xrightarrow{\ x\ } {}^eT \longrightarrow {}^eT/x({}^eT) \longrightarrow 0,$$

which induces a long exact sequence
(3.2)

$$\cdots \longrightarrow \mathrm{Ext}^i_R(M, {}^eT) \xrightarrow{0} \mathrm{Ext}^i_R(M, {}^eT) \longrightarrow \mathrm{Ext}^i_R(M, {}^eT/x({}^eT)) \longrightarrow \cdots .$$

Then, for every i,

$$\lambda(\mathrm{Ext}^i_R(M, {}^eT)) \leqslant \lambda(\mathrm{Ext}^{i-1}_R(M, {}^eT/x({}^eT))).$$

We have a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_q = {}^eT/x({}^eT)$$

such that $L_{r+1}/L_r = {}^e(T/xT)$ since x is not a zero divisor of T. From the induced long exact sequence by $\mathrm{Ext}^i_R(M, -)$, we have that

$$\lambda(\mathrm{Ext}^i_R(M, {}^eT/x({}^eT))) \leqslant q \cdot \lambda(\mathrm{Ext}^i_R(M, {}^e(T/xT)))).$$

Therefore,

$$\begin{aligned}
\lim_{e\to\infty} \frac{\lambda(\mathrm{Ext}^i_R(M, {}^eT))}{q^{(i+1+\alpha)}} &\leqslant \lim_{e\to\infty} \frac{\lambda(\mathrm{Ext}^{i-1}_R(M, {}^eT/x({}^eT)))}{q^{(i+1+\alpha)}} \\
&\leqslant \lim_{e\to\infty} \frac{q \cdot \lambda(\mathrm{Ext}^{i-1}_R(M, {}^e(T/xT)))}{q^{(i+1+\alpha)}} \\
&= \lim_{e\to\infty} \frac{\lambda(\mathrm{Ext}^{i-1}_R(K, {}^e(T/xT))))}{q^{(i+\alpha)}} \\
&= 0 \quad \text{since } \dim T/xT = n-1.
\end{aligned}$$

□

Corollary 3.9. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$. Let N be a finitely generated R-module, and let C be an R-module such that, for all $e \gg 0$, C^{θ_e} is a direct summand of ${}^eN$ for some $\theta_e \in \mathbb{N}$. Assume that $\theta = \limsup_{e\to\infty}(\theta_e/q^{(d+\alpha)}) > 0$. Then, for all R-modules*

M of finite length, and all integers i, we have

$$\mu_i^F(M,N) \geqslant \theta \cdot \lambda(\operatorname{Ext}_R^i(M,C)).$$

In particular, C is a maximal Cohen-Macaulay module.

Proof. We have

$$\mu_i^F(M,N) = \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}_R^i(M,{}^eN))}{q^{(d+\alpha)}} \geqslant \limsup_{e\to\infty} \frac{\theta_e \cdot \lambda(\operatorname{Ext}_R^i(M,C))}{q^{(d+\alpha)}}$$
$$= \theta \cdot \lambda(\operatorname{Ext}_R^i(M,C)).$$

Using $M = K$ in Theorem 3.8, we obtain that $\mu_i^F(K,N) = 0$ for all $i < d$. It follows from the inequality that $\operatorname{Ext}_R^i(K,C) = 0$ for all $i < d$, and then C is a maximal Cohen-Macaulay module. □

Remark 3.10. Let $(R,\mathfrak{m},K)$ be a local ring of characteristic $p > 0$, and let N be a finitely generated R-module. We say that an R-module C is an F-contributor of N if C^{θ_e} is a direct summand of ${}^eN$ for $e \gg 0$, and $\limsup_{e\to\infty}(\theta_e/q^{(d+\alpha)}) > 0$ [**35**]. Corollary 3.9 shows that every F-contributor is a maximal Cohen-Macaulay module. This was already noted by Yao [**35,** Lemma 2.2] when N has finite F-representation type.

The next proposition shows that taking limits with respect to Tor or Ext give the same invariants up to a shift in the homological degrees.

Proposition 3.11. *Let $(R,\mathfrak{m},K)$ be a local ring of characteristic $p > 0$ and M an R-module of finite length. Then,*

$$\beta_i^F(M,R) = \mu_{d+i}^F(M,R)$$

for every $i \in \mathbb{N}$.

Proof. Since $\beta_i^F(M,R)$ and $\mu_{d+i}^F(M,R)$ are not affected by completion at $\mathfrak{m}$, we may assume that R is a complete local ring. In this case, R has a dualizing complex $D_R^\bullet$ by Remark 2.2. We have that

$$\beta_i^F(M,R) = \mu_{d+i}^F(M,H^0(D_R^\bullet))$$

by [**10,** Proposition 2.3(2)]. Let Λ be the set of all prime ideals of R such that $\dim R/\mathfrak{p} = \dim R$. Let $\mathfrak{p} \in \Lambda$. We have that

$(H^0(D^\bullet_R))_\mathfrak{p} = H^0(D^\bullet_{R_\mathfrak{p}}) = \omega_{R_\mathfrak{p}}$ by Remark 2.2. We have that $\omega_{R_\mathfrak{p}} = \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))$ and $\lambda_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}}) = \lambda_{R_\mathfrak{p}}(R_\mathfrak{p})$. Finally, by Proposition 3.6,

$$\begin{aligned} \mu^F_{d+i}(M, H^0(D^\bullet_R)) &= \sum_{\mathfrak{p}\in\Lambda} \mu^F_{d+i}(M, R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(H^0(D^\bullet_{R_\mathfrak{p}})) \\ &= \sum_{\mathfrak{p}\in\Lambda} \mu^F_{d+i}(M, R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}}) \\ &= \sum_{\mathfrak{p}\in\Lambda} \mu^F_{d+i}(M, R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(R_\mathfrak{p}) \\ &= \mu^F_{d+i}(M, R). \end{aligned}$$ □

Remark 3.12. If R itself has an F-contributor C, then we get a relation involving the β^F_i's. In fact, by Proposition 3.11, we have $\beta^F_i(M,R) = \mu^F_{d+i}(M,R)$ for all $i \in \mathbb{N}$. Thus, in the notation of Corollary 3.9, we have $\beta^F_i(M,R) \geqslant \theta \cdot \lambda(\operatorname{Ext}^{d+i}_R(M,C))$.

We end this section with a proposition which shows how $\beta^F_i(M,N)$ behaves under some flat ring extensions. First, we need a different way of computing $\beta^F_i(M,N)$.

Remark 3.13. Let $(R,\mathfrak{m},K)$ be a local ring of characteristic $p > 0$, let M be an R-module of finite length, and let N be a finitely generated R-module. Let $G_\bullet = (G_j, \varphi_j)_{j\geqslant 0}$ denote a minimal free resolution of M. Let $G^{[q]}_\bullet$ be the complex $(G_j, \varphi^{[q]}_j)_{j\geqslant 0}$, where the matrix of $\varphi^{[q]}_j$ has as entries the qth powers of the entries in the matrix of φ_j. We have that

$$\lambda(\operatorname{Tor}^R_i(M, {}^eN)) = q^\alpha \lambda(H_i(G^{[q]}_\bullet \otimes_R N)).$$

Hence,

$$\beta^F_i(M,N) = \lim_{q\to\infty} \frac{\lambda(H_i(G^{[q]}_\bullet \otimes_R N))}{q^d}.$$

Proposition 3.14. *Let $(R,\mathfrak{m},K) \to (S,\mathfrak{n},L)$ be a flat extension of two F-finite local rings of characteristic $p > 0$. Let M be a finite length R-module. Let $\alpha = \log_p[K : K^p]$ and $\theta = \log_p[L : L^p]$. Suppose that*

$\mathfrak{m}S = \mathfrak{n}$. *Then,*

$$\begin{aligned}\beta^F_{i,R}(M,R) &= \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}^R_i(M, {}^eR))}{p^{e(d+\alpha)}} \\ &= \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}^S_i(M\otimes_R S, {}^eS))}{p^{e(d+\theta)}} \\ &= \beta^F_{i,S}(M\otimes_R S, S).\end{aligned}$$

In particular, we have that $\beta^F_{i,R}(M,R) = \beta^F_{i,\widehat{R}}(\widehat{M}, \widehat{R})$.

Proof. Let $q = p^e$. We have:

$$\begin{aligned}\frac{\lambda_R(\operatorname{Tor}^R_i(M, {}^eR))}{q^\alpha} &= \lambda_R(H_i(G_\bullet^{[q]})) \quad \text{by Remark 3.13} \\ &= \lambda_S(H_i(G_\bullet^{[q]}\otimes_R S)) \\ &\quad \text{since } S \text{ is flat and } \mathfrak{m}S=\mathfrak{n} \\ &= \lambda_S(H_i((G_\bullet\otimes_R S)^{[q]})) \\ &\quad \text{since } G_\bullet \text{ is free} \\ &= \frac{\lambda_S(\operatorname{Tor}^S_i(M\otimes_R S, {}^eS)}{q^\theta} \\ &\quad \text{by Remark 3.13 and since } S \text{ is flat.}\end{aligned}$$

After dividing by q^d and taking limits, we have that

$$\beta^F_{i,R}(M,R) = \beta^F_{i,S}(M\otimes_R S, S). \qquad \square$$

4. Relations with projective dimension. Let $(R, \mathfrak{m}, K)$ be a local F-finite ring of characteristic $p > 0$, and let M be an R-module of finite length. In this section, we investigate when the vanishing of $\beta^F_i(M,R)$ detects whether M has finite projective dimension.

First we recall known results in this direction. We have that R is a regular ring if and only if $\beta^F_i(R) = \beta^F_i(K,R) = 0$ for some $i \geqslant 1$ [**3,** Corollary 3.2]. Let M be a finitely generated R-module. If M has finite projective dimension, then $\operatorname{Tor}^R_i(M, {}^eR) = 0$ for all $i > 0$ and all $e \geqslant 0$ [**26,** Theorem 1.7]. Conversely, if $\operatorname{Tor}^R_i(M, {}^eR) = 0$ for infinitely many e and all $i > 0$, then M has finite projective dimension [**16,** Theorem 3.1]. In fact, even more is true: if $\operatorname{Tor}^R_i(M, {}^eR) = 0$ for depth$(R) + 1$

consecutive values of i and some $e \gg 0$, then M has finite projective dimension [**18,** Proposition 2.6] (see also [**24,** Theorem 2.2.8]). Now, suppose that R is a complete intersection. If $\beta_i^F(M,R) = 0$ for some $i > 0$, then M has finite projective dimension by [**23,** Corollary 2.5] (see also [**13,** Corollary 4.11]).

Proposition 4.1. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, and let M be an R-module of finite length. Suppose that there is a regular local ring $(A, \mathfrak{n}, L)$ and a map of local rings $\phi : R \to A$ such that A is finitely generated as an R-module, and* $\dim A = d$*. If*

$$\beta_j^F(M,R) = \beta_{j+1}^F(M,R) = \cdots = \beta_{j+d}^F(M,R) = 0,$$

then M has finite projective dimension.

Proof. We note that $\log_p[L : L^p] = \log_p[K : K^p] = \alpha < \infty$, and thus, A is F-finite. Since A is regular and local, ${}^e A \cong \bigoplus^{q^{(d+\alpha)}} A$. Let $x_1, \ldots, x_d \in A$ be a set of generators for $\mathfrak{n}$, and let $I_r := (x_1, \ldots, x_r)A$. By induction on r, we will show that

$$\operatorname{Tor}_{j+r}^R(M, A/I_r) = \cdots = \operatorname{Tor}_{j+d}^R(M, A/I_r) = 0 \tag{4.1}$$

for every r. If $r = 0$, we have that $\operatorname{Tor}_i^R(M, {}^e A) = \oplus^{q^{(d+\alpha)}} \operatorname{Tor}_i^R(M, A)$ for every $i \in \mathbb{N}$. Then, $\lambda(\operatorname{Tor}_i^R(M, {}^e A)) = q^{(d+\alpha)} \lambda(\operatorname{Tor}_i^R(M, A))$, and thus,

$$\beta_i^F(M, A) = \lambda(\operatorname{Tor}_i^R(M, A)).$$

Since A is finitely generated, and since $\beta_i^F(M,R) = 0$ for $i = j, \ldots, j+d$ by assumption, we have that $\beta_j^F(M,A) = \cdots = \beta_{j+d}^F(M,A) = 0$ by Remark 3.7. Hence, $\operatorname{Tor}_j^R(M,A) = \cdots = \operatorname{Tor}_{j+d}^R(M,A) = 0$. We suppose that (4.1) holds for $r-1$ and prove it for r. We have a short exact sequence

$$0 \longrightarrow A/I_{r-1} \xrightarrow{x_r} A/I_{r-1} \longrightarrow A/I_r \longrightarrow 0.$$

This induces a long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_i^R(M, A/I_{r-1}) \xrightarrow{x_r} \operatorname{Tor}_i^R(M, A/I_{r-1}) \longrightarrow \operatorname{Tor}_i^R(M, A/I_r) \longrightarrow \operatorname{Tor}_{i-1}^R(M, A/I_{r-1}) \longrightarrow \cdots.$$

Since $\operatorname{Tor}_{j+r-1}^R(M, A/I_{r-1}) = \cdots = \operatorname{Tor}_{j+d}^R(M, A/I_{r-1}) = 0$, we have that $\operatorname{Tor}_{j+r}^R(M, A/I_r) = \cdots = \operatorname{Tor}_{j+d}^R(M, A/I_r) = 0$, proving the claim.

In particular, we obtain $\operatorname{Tor}^R_{j+d}(M, A/I_d) = 0$. Since $L = A/I_d$ is a finite field extension of K, we have

$$0 = \lambda(\operatorname{Tor}^R_{j+d}(M, A/I_d)) = [L : K] \cdot \lambda(\operatorname{Tor}^R_{j+d}(M, K)).$$

Therefore, $\operatorname{Tor}^R_{j+d}(M, K) = 0$ and M has finite projective dimension. □

Lemma 4.2. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$. Suppose that there is an R-module N of dimension d that has an F-contributor C. Let M be an R-module of finite length. If $\beta^F_i(M, N) = 0$, then $\operatorname{Tor}^R_i(M, {}^eC) = 0$ for every $e \geqslant 0$. In particular, if R is strongly F-regular of positive dimension d, and $\beta^F_i(M, R) = 0$ for d consecutive values of i, then M has finite projective dimension.*

Proof. For $e' \gg 0$ and $q' = p^{e'}$, we have that $C^{\theta_{e'}}$ is a direct summand of ${}^{e'}N$, for some $\theta_{e'} \in \mathbb{N}$ such that $\limsup \theta_{e'}/q'^{(d+\alpha)} > 0$. We note that $({}^eC)^{\theta_{e'}}$ is a direct summand of ${}^{e+e'}N$ for all $e \geqslant 0$. Then,

$$\begin{aligned}\left(\limsup_{e'\to\infty} \frac{\theta_{e'}}{q'^{(d+\alpha)}}\right) \frac{\lambda(\operatorname{Tor}^R_i(M, {}^eC))}{q^{(d+\alpha)}} &\leqslant \lim_{e'\to\infty} \frac{\lambda(\operatorname{Tor}^R_i(M, {}^{e+e'}N))}{qq'^{(d+\alpha)}} \\ &= \beta^F_i(M, N) = 0.\end{aligned}$$

It follows that $\operatorname{Tor}^R_i(M, {}^eC) = 0$. If R is strongly F-regular, then R is an F-contributor of itself. In addition, R is Cohen-Macaulay and, if $\operatorname{Tor}_i(M, {}^eR) = 0$ for d consecutive values of i and for $e \gg 0$, we have that M has finite projective dimension [**18,** Proposition 2.6] (see also [**24,** Theorem 2.2.11]). □

Proposition 4.3. [**12,** Corollary 3.3] *Let $(R, \mathfrak{m}, K)$ be a local ring, let I be an integrally closed $\mathfrak{m}$-primary ideal, and let N be a finitely generated R-module. Then, $\operatorname{pd}_R(N) < i$ if and only if $\operatorname{Tor}^R_i(N, R/I) = 0$.*

In particular, Proposition 4.3 shows that, if $\operatorname{Tor}^R_i(R/I, {}^eR) = 0$ for some $e \geqslant 1$, then R is regular [**19,** Theorem 2.1].

We now present a similar result for Frobenius Betti numbers.

Proposition 4.4. *Let $(R, \mathfrak{m}, K)$ be a reduced local ring of characteristic $p > 0$. Suppose that there exists an R-module N of dimension d that* *has an F-contributor C. If I is an integrally closed $\mathfrak{m}$-primary ideal such that $\beta_i^F(R/I, N) = 0$ for some $i > 0$, then R is regular.*

Proof. By Lemma 4.2, we have that $\operatorname{Tor}_i(R/I, {}^eC) = 0$ for every $e \geqslant 0$, and thus, ${}^eC$ has finite projective dimension by [**12,** Corollary 3.3]. Since ${}^eC$ is a maximal Cohen-Macaulay module [**35,** Lemma 2.2], see Remark 3.10, we have that ${}^eC$ is a free module for every $e \geqslant 0$. In particular, ${}^1C \cong \bigoplus_n R$ and ${}^2C = {}^1(\bigoplus_n R) \cong \bigoplus_n {}^1R$ is free as well. Therefore, ${}^1R$ is free, and R is regular [**19,** Theorem 2.1]. □

We now focus on one-dimensional rings. In this case, we can find a characterization of the vanishing of $\beta_i^F(M, R)$. We first prove two lemmas.

Lemma 4.5. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional complete local domain of characteristic $p > 0$, with K algebraically closed. Then, there exists a parameter $x \in R$ such that $(x^q) = \mathfrak{m}^{[q]}$ for all $q = p^e \gg 0$. Furthermore, if V denotes the integral closure of R in its field of fractions, then ${}^eR \cong \bigoplus V$ for all $e \gg 0$ (as R-modules).*

Proof. Since R is a complete domain, we have that $(V, \mathfrak{m}_V, K)$ is a one-dimensional, integrally closed, local domain. Hence, V is a DVR. Let $x \in R$ be a minimal reduction of $\mathfrak{m}$, and let v denote the order valuation on V. Let $x, y_1, \ldots, y_n$ be a minimal generating set of the maximal ideal. We claim that we can choose the elements y_i's such that $v(x) < v(y_i)$ for all $i = 1, \ldots, n$. We have $v(x) \leqslant v(y_i)$ for all i since x is a minimal reduction of $\mathfrak{m}$ [**32,** Proposition 6.8.1]. If equality holds, say for $i = 1$, we have that $y_1/x = \alpha \in K_V = K$ since K is algebraically closed. Fix a lifting $u \in R$ of α. If we replace y_1 for $y_1' := y_1 - ux$, we have that $x, y_1', \ldots, y_n$ is still a minimal generating set of $\mathfrak{m}$. Now $v(x) < v(y_1')$, since $y_1'/x \in \mathfrak{m}_V$. Similarly, if necessary, we may replace each y_i to obtain our claim. Since the conductor C is $\mathfrak{m}_V$-primary, for all $e \gg 0$ and all $i = 1, \ldots, n$, we have that

$$(y_i/x)^q = r_i \in \mathfrak{m}_V^{[q]} \subseteq C \subseteq R.$$

Thus, $y_i^q = r_i x^q \in (x)^q$. This shows the first part of the lemma.

We now focus on the second part of the lemma. Since K is algebraically closed, R and K have the same residue field. It then follows that $R \subseteq V = R + \mathfrak{m}_V$. Since R is a domain, we can identify ${}^eR$ with $R^{1/q}$, the ring of qth roots of R. For $w \in V$, we can write $w = u + v$, for some $u \in R$ and $v \in \mathfrak{m}_V$. Therefore, we have that $\mathfrak{m}_V^{[q]} \subseteq C \subseteq R$ for $e \gg 0$, since C is $\mathfrak{m}_V$ primary. This shows that $w^q = u^q + v^q \in R$, that is, $w \in R^{1/q}$. Thus, for $e \gg 0$, we have $R \subseteq V \subseteq {}^eR$. Hence, ${}^eR$ is a V-module. Since V is a DVR, ${}^eR$ decomposes into a V-free part and a V-torsion part. However, ${}^eR$ is torsion free as a V-module because R is a domain. Thus,

$$ {}^eR \cong \bigoplus_q V. $$

Finally, the V-module structure on ${}^eR$ is compatible with the inclusion $R \subseteq V$; therefore, ${}^eR \cong \bigoplus V$ is also an isomorphism of R-modules. □

Lemma 4.6. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring of characteristic $p > 0$. Let $(G_j, \varphi_j)_{j \geqslant 0}$ be a minimal free resolution of a finite length R-module M. Suppose that there exists an $i \geqslant 0$ such that $\operatorname{Im}(\varphi_{i+1}) \nsubseteq \mathfrak{p}G_i$ for some $\mathfrak{p} \in \operatorname{Min}(R)$. Then*

$$ \beta_i^F(M, R) = \lim_{e \to \infty} \frac{\lambda(\operatorname{Tor}_i^R(M, {}^eR))}{q^\alpha} > 0. $$

Proof. We can write $\widehat{R} = K[[x_1, \ldots, x_n]]/I$ for some $n \in \mathbb{N}$ and some ideal $I \subseteq K[\![x_1, \ldots, x_n]\!]$ by the Cohen structure theorem. Let $S = L[\![x_1, \ldots, x_n]\!]/I'$, where L is the algebraic closure of K and $I' = I\ L[\![x_1, \ldots, x_n]\!]$. Every inclusion $K \to L$ gives a flat extension $R \to S$ such that $\mathfrak{m}S$ is the maximal ideal of S. If $\operatorname{Im}(\varphi_{i+1} \otimes_R 1_S) = \operatorname{Im}(\varphi_{i+1}) \otimes_R S$ is contained in a minimal prime of S, then $\operatorname{Im}(\varphi_{i+1})$ must be contained in the contraction of such a minimal prime to R. Then, we can assume that R is complete and that K is algebraically closed by Proposition 3.14.

Let $\overline{R}$ denote $R/\mathfrak{p}$, $\overline{x}$ the class of the element x modulo $\mathfrak{p}$ and V the integral closure of $\overline{R}$. Since $R/\mathfrak{p}$ is a one-dimensional complete local domain, by Lemma 4.5, we can choose $0 \neq \overline{x} \in \overline{R}$ a minimal reduction of $\overline{\mathfrak{m}} := \mathfrak{m}/\mathfrak{p}$ and $q_0 = p^{e_0}$ such that $\overline{\mathfrak{m}}^{[q]} = (\overline{x}^q)$ for $q \geqslant q_0$. We may also choose q_0 large enough such that $\overline{x}^q V \cap \overline{R} \subseteq \overline{x}\overline{R}$ by using the Artin-Rees lemma and the fact that the conductor from $\overline{R}$ to V is primary to the maximal ideal. In particular, $(\overline{x}^q V :_V \overline{r}) \subseteq \mathfrak{m}_V$ for every $\overline{r} \in \overline{R}$ such that $\overline{r} \notin \overline{x}\overline{R}$, where $\mathfrak{m}_V$ is the maximal ideal of V, which is a DVR.

Fix $q \geqslant q_0$, and consider the matrix associated to $\overline{\varphi}_{i+1}^{[q]} := \varphi_{i+1}^{[q]} \otimes 1_{\overline{R}}$. Since $q \geqslant q_0$, $\operatorname{Im}(\varphi_{i+1}^{[q]} \otimes 1_{\overline{R}}) \subseteq \overline{\mathfrak{m}}^{[q]} G_i = (\overline{x}^q) G_i$. Due to the fact that $\operatorname{Im}(\varphi_{i+1}) \nsubseteq \mathfrak{p} G_i$, by changing the basis for G_{i+1} if needed, we can assume that the matrix

$$\overline{\varphi}_{i+1}^{[q]} = \overline{x}^{q+j} \left[\begin{array}{c|ccc} \overline{r}_1 & * & \cdots & * \\ \hline \overline{r}_2 & * & \cdots & * \\ \hline \vdots & \vdots & & \vdots \\ \hline \overline{r}_n & * & \cdots & * \end{array}\right],$$

where we have factored out the biggest possible power of $\overline{x}$, so that $\overline{r}_1 \notin (\overline{x})$. Here, $n = \operatorname{rk}(G_i)$.

Let $q' = p^{e'}$, and consider the matrix associated to $\overline{\varphi}_{i+1}^{[qq']}$:

$$\overline{\varphi}_{i+1}^{[qq']} = \overline{x}^{(q+j)q'} \left[\begin{array}{c|ccc} \overline{r}_1^{q'} & * & \cdots & * \\ \hline \overline{r}_2^{q'} & * & \cdots & * \\ \hline \vdots & \vdots & & \vdots \\ \hline \overline{r}_n^{q'} & * & \cdots & * \end{array}\right]. \tag{4.2}$$

We claim that $[\overline{r}_1^{q'}, \overline{r}_2^{q'}, \ldots, \overline{r}_n^{q'}]^T \in \operatorname{Ker}(\overline{\varphi}_i^{[qq']})$. In fact, we have that

$$\overline{x}^{qq'+jq'} \begin{bmatrix} \overline{r}_1{}^{q'} \\ \overline{r}_2{}^{q'} \\ \vdots \\ \overline{r}_n{}^{q'} \end{bmatrix} \in \operatorname{Im}\left(\overline{\varphi}_{i+1}^{[qq']}\right) \subseteq \operatorname{Ker}\left(\overline{\varphi}_i^{[qq']}\right);$$

therefore,

$$\overline{\varphi}_i^{[qq']} \left(\overline{x}^{qq'+jq'} \cdot \begin{bmatrix} \overline{r}_1{}^{q'} \\ \overline{r}_2{}^{q'} \\ \vdots \\ \overline{r}_n{}^{q'} \end{bmatrix} \right) = \overline{x}^{qq'+jq'} \cdot \overline{\varphi}_i^{[qq']} \left(\begin{bmatrix} \overline{r}_1{}^{q'} \\ \overline{r}_2{}^{q'} \\ \vdots \\ \overline{r}_n{}^{q'} \end{bmatrix} \right) = 0.$$

Since $\overline{x}^{qq'+jq'}$ is a nonzero divisor in $\overline{R}$, we have

$$\overline{\varphi}_i^{[qq']}\left(\begin{bmatrix} \overline{r}_1{}^{q'} \\ \overline{r}_2{}^{q'} \\ \vdots \\ \overline{r}_n{}^{q'} \end{bmatrix}\right) = 0,$$

which proves the claim. Thus,

$$\begin{aligned}\lambda(\operatorname{Tor}_i^R(M, {}^{e+e'}(R/\mathfrak{p}))) &\geqslant \lambda\left(\frac{\overline{R}[\overline{r}_1^{q'}, \ldots, \overline{r}_n^{q'}]^T + \operatorname{Im}\left(\overline{\varphi}_{i+1}^{[qq']}\right)}{\operatorname{Im}\left(\overline{\varphi}_{i+1}^{[qq']}\right)}\right) \\ &\geqslant \lambda\left(\frac{(\overline{r}_1^{q'}) + (\overline{x}^{qq'})}{(\overline{x}^{qq'})}\right),\end{aligned}$$

since $\operatorname{Im}(\overline{\varphi}_{i+1}^{[qq']}) \subseteq (\overline{x}^{qq'})G_i$. This comes from the expression of $\overline{\varphi}_{i+1}^{[qq']}$ in (4.2). We also have projected onto the first component of G_i. This yields a cyclic module which is isomorphic to the quotient of $\overline{R}$ by the ideal $(\overline{x}^{qq'} : \overline{r}_1^{q'})$.

We claim that there exists an integer $q_1 = p^{e_1}$ such that, for all q',

$$(\overline{x}^{qq'} : \overline{r}_1^{q'}) \subseteq (\overline{x}^{q'/q_1}).$$

Assuming the claim, and lifting back to R, we obtain:

$$\begin{aligned}\lambda(\operatorname{Tor}_i^R(M, {}^{e+e'}\overline{R})) &\geqslant \lambda\left(\frac{(\overline{r}_1^{q'}) + (\overline{x}^{qq'})}{(\overline{x}^{qq'})}\right) \\ &\geqslant \lambda\left(\frac{\overline{R}}{(\overline{x}^{q'/q_1})}\right).\end{aligned}$$

Dividing by qq' and taking the limit as $e' \to \infty$, we get

$$\begin{aligned}\beta_i^F(M, \overline{R}) &= \lim_{e'\to\infty} \frac{\lambda(\operatorname{Tor}_i^R(M, {}^{e+e'}(\overline{R})))}{qq'} \\ &\geqslant \lim_{e'\to\infty} \frac{\lambda(\overline{R}/(\overline{x}^{q'/q_1}))}{qq'} \\ &= \frac{1}{qq_1}\, \mathrm{e}_{HK}(x, \overline{R}) > 0.\end{aligned}$$

Since $\dim(R/\mathfrak{q}) = \dim(R)$ for $\mathfrak{q} \in \operatorname{Spec}(R)$ if and only if $\mathfrak{q} \in \operatorname{Min}(R)$, we have

$$\beta_i^F(M,R) = \sum_{\mathfrak{q}\in\operatorname{Min}(R)} \left(\beta_i^F(M,R/\mathfrak{q})\lambda(R_{\mathfrak{q}})\right) \geqslant \beta_i^F(M,R/\mathfrak{p}) > 0,$$

by Proposition 3.6.

It remains to prove the claim. Suppose that $u \in (\overline{x}^{qq'} : \overline{r}_1^{q'})$. Then,

$$u \in (\overline{x}^{qq'} : \overline{r}_1^{q'})V \cap \overline{R} = (\overline{x}^q V :_V \overline{r}_1)^{[q']} \cap \overline{R} \subseteq \mathfrak{m}_V^{q'} \cap R,$$

by the choice of q. Since the conductor of $\overline{R}$ is primary to the maximal ideal, it follows that there exists a $q_1 = p^{e_1}$ such that $\mathfrak{m}_V^{q'} \cap \overline{R} \subseteq (\overline{x}^{q'/q_1})$, as claimed. □

Theorem 4.7. *Let $(R,\mathfrak{m},K)$ be a one-dimensional local ring of characteristic $p > 0$ and M an R-module of finite length. Let $(G_j,\varphi_j)_{j\geqslant 0}$ denote a minimal free resolution of M. Then the following are equivalent*:

(i) $\operatorname{Im}(\varphi_{i+1}) \subseteq H^0_{\mathfrak{m}}(G_i)$.
(ii) $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ *for all* $e \geqslant 0$, *for all* $\mathfrak{p} \in \operatorname{Min}(R)$.
(iii) $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ *for all* $e \gg 0$, *for all* $\mathfrak{p} \in \operatorname{Min}(R)$.
(iv) $\beta_i^F(M,R) = 0$.

Assume, in addition, that R is complete and K is algebraically closed. If V denotes the integral closure of R in its ring of fractions, then the conditions above are equivalent to:

(v) $\operatorname{Tor}_i^R(M,V) = 0$.

Proof. We will show that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i). We assume (i). Let $\mathfrak{p} \in \operatorname{Min}(R)$. Since M has finite length we have $M_{\mathfrak{p}} = 0$, and thus,

$$\operatorname{Tor}_j^R(M, {}^e(R/\mathfrak{p}))_{\mathfrak{p}} = \operatorname{Tor}_j^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, {}^e(R/\mathfrak{p})_{\mathfrak{p}}) = 0$$

for all $j \geqslant 0$. In particular, the complex

$$(G_\bullet \otimes {}^e(R))_{\mathfrak{p}} : \cdots \longrightarrow (G_{i+1})_{\mathfrak{p}} \xrightarrow{(\varphi_{i+1}^{[q]})_{\mathfrak{p}}} (G_i)_{\mathfrak{p}} \xrightarrow{(\varphi_i^{[q]})_{\mathfrak{p}}} (G_{i-1})_{\mathfrak{p}} \longrightarrow \cdots$$
$$\longrightarrow (G_0)_{\mathfrak{p}} \longrightarrow 0$$

is split exact. All the entries in a matrix associated to φ_{i+1} are in $H^0_{\mathfrak{m}}(R)$, and in particular, they are nilpotent. We choose $q_0 = p^{e_0}$ such that $\operatorname{Im}(\varphi_{i+1}^{[q]}) = 0$ for all $q \geqslant q_0$. For such a q, we have $(\varphi_{i+1}^{[q]})_{\mathfrak{p}} \equiv 0$; therefore, $(G_i)_{\mathfrak{p}}$ splits inside $(G_{i-1})_{\mathfrak{p}}$ via $(\varphi_i^{[q]})_{\mathfrak{p}}$. This means that

$$b_i := \operatorname{rk}((G_i)_{\mathfrak{p}}) = \operatorname{rk}(G_i) = \operatorname{rk}((\varphi_i^{[q]})_{\mathfrak{p}}) \quad \text{and} \quad I_{b_i}(\varphi_i^{[q]}) \not\subseteq \mathfrak{p}, \tag{4.3}$$

where $I_r(\psi)$ denotes the Fitting ideal of a homomorphism $\psi : G \to H$ of rank r between two free modules, G and H. Note that localizing and taking powers only decreases the rank of φ_i, and b_i is already the maximal possible rank. Thus, $b_i = \operatorname{rk}(\varphi_i^{[q]})$ for all $q \geqslant 1$. Furthermore, if $I_{b_i}(\varphi_i)$ were contained in $\mathfrak{p}$, then so would be $I_{b_i}(\varphi_i^{[q]})$. Hence, (4.3) holds in fact for all $q = p^e$.

Consider the complex

$$0 \longrightarrow G_i \otimes R/\mathfrak{p} \xrightarrow{\varphi_i^{[q]} \otimes 1_{R/\mathfrak{p}}} G_{i-1} \otimes R/\mathfrak{p} \longrightarrow C_q \longrightarrow 0,$$

where C_q is the cokernel. By the Buchsbaum-Eisenbud theorem **[8]**, the two conditions (4.3) ensure that it is acyclic for all q. Then,

$$\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = \operatorname{Tor}_1^R(C_q, R/\mathfrak{p}) = 0,$$

for all $e \geqslant 0$. This holds for all $\mathfrak{p} \in \operatorname{Min}(R)$, proving (ii).

Clearly, (ii) implies (iii). We now show (iii) $\Rightarrow$ (iv). Since, for all $\mathfrak{p} \in \operatorname{Min}(R)$, we have $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ for $e \gg 0$, in particular, $\beta_i^F(M, R/\mathfrak{p}) = 0$. Hence,

$$\beta_i^F(M,R) = \sum_{\mathfrak{p} \in \operatorname{Min}(R)} \left[\beta_i^F(M, R/\mathfrak{p}) \lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})\right] = 0.$$

We now prove (iv) $\Rightarrow$ (i). Suppose that $\beta_i^F(M,R) = 0$. By Lemma 4.6, we have

$$\operatorname{Im}(\varphi_{i+1}) \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Min}(R)} \mathfrak{p} G_i = \sqrt{0}\, G_i.$$

Since the image is nilpotent, as noticed above in (4.3) while taking $q = 1$, we have

$$b_i = \operatorname{rk}(G_i) = \operatorname{rk}(\varphi_i) \quad \text{and} \quad I_{b_i}(\varphi_i) \not\subseteq \mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Min}(R)$. Localizing the resolution at any $\mathfrak{p} \in \operatorname{Min}(R)$ gives a split exact complex

$$(G_\bullet)_\mathfrak{p} : 0 \longrightarrow (G_i)_\mathfrak{p} \xrightarrow{(\varphi_i)_\mathfrak{p}} \cdots \longrightarrow (G_0)_\mathfrak{p} \longrightarrow 0.$$

In particular, $\operatorname{Im}((\varphi_{i+1})_\mathfrak{p}) = (\operatorname{Im}(\varphi_{i+1}))_\mathfrak{p} = 0$. This holds for all minimal primes $\mathfrak{p}$ of R, proving that $\operatorname{Im}(\varphi_{i+1}) \subseteq H^0_\mathfrak{m}(G_i)$.

Finally, assume that R is complete and K is algebraically closed, and let V be an integral closure of R in its ring of fractions. Let $\mathfrak{p} \in \operatorname{Min}(R)$, and let $V(\mathfrak{p})$ be the integral closure of $R/\mathfrak{p}$, which is a DVR. By Lemma 4.5, we have that ${}^e(R/\mathfrak{p}) \cong \bigoplus V(\mathfrak{p})$ for all $e \gg 0$. Condition (iii) implies that

$$\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) \cong \bigoplus \operatorname{Tor}_i^R(M, V(\mathfrak{p})) = 0;$$

therefore, $\operatorname{Tor}_i^R(M, V(\mathfrak{p})) = 0$ for all $\mathfrak{p} \in \operatorname{Min}(R)$. Since $V \cong \bigoplus_{\mathfrak{p} \in \operatorname{Min}(R)} V(\mathfrak{p})$, we see that (iii) implies (v).

Conversely, if $\operatorname{Tor}_i^R(M, V) = 0$, by the same argument, we get that $\operatorname{Tor}_i^R(M, V(\mathfrak{p})) = 0$ implies $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ for all $e \gg 0$ and for all $\mathfrak{p} \in \operatorname{Min}(R)$. Then, (v) implies (iii). $\square$

Corollary 4.8. *Let $(R, \mathfrak{m}, K)$ be a one dimensional Cohen-Macaulay local ring of characteristic $p > 0$ and M an R-module of finite length. Then the following are equivalent*:

(i) $\beta_i^F(M, R) = 0$ *for all* $i \geqslant 1$.
(ii) $\beta_i^F(M, R) = 0$ *for some* $i \geqslant 1$.
(iii) $\operatorname{pd}_R(M) < \infty$.

Proof. Clearly (i) implies (ii). Now assume (ii). We want to show that (iii) holds. By assumption, there exists an integer $i \geqslant 1$ such that $\beta_i^F(M, R) = 0$. Then, Theorem 4.7 implies that $\operatorname{Im}(\varphi_{i+1}) \subseteq H^0_\mathfrak{m}(G_i)$, where $(G_j, \varphi_j)_{j \geqslant 0}$ is a minimal free resolution of M. However, R has positive depth, and hence,

$$\operatorname{Im}(\varphi_{i+1}) = H^0_\mathfrak{m}(\operatorname{Im}(\varphi_{i+1})) \subseteq H^0_\mathfrak{m}(G_i) = 0,$$

since G_i is a free module. Thus, $\operatorname{Im}(\varphi_{i+1}) = 0$ and $\operatorname{pd}_R(M) < \infty$. Finally, if (iii) holds, we have $\operatorname{Tor}_i^R(M, {}^eR) = 0$ for all $i \geqslant 1$ and $e \geqslant 0$ [**26,** Theorem 1.7]. In particular, $\beta_i^F(M, {}^eR) = 0$ for all $i \geqslant 1$. $\square$

Corollary 4.9. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring of characteristic $p > 0$, and let M be a finite length R-module. If $\beta_i^F(M, R) = \beta_{i+1}^F(M, R) = 0$ for some $i \geqslant 1$, then $\mathrm{pd}_R(M) < \infty$. In particular, for any parameter x, if $\beta_2^F(R/(x), R) = 0$, then R is Cohen-Macaulay.*

Proof. Let $(G_j, \varphi_j)_{j \geqslant 0}$ be a minimal free resolution of M. Since $\beta_i^F(M, R) = 0$, we have that $\mathrm{Im}(\varphi_{i+1})$ has finite length, and it is nilpotent. Take $q = p^e \gg 0$ such that $\mathrm{Im}(\varphi_{i+1}^{[q]}) = 0$. For such a q, we have $\mathrm{Ker}(\varphi_{i+1}) = G_{i+1}$. Since the resolution is minimal, we obtain

$$\lambda(\mathrm{Tor}_{i+1}^R(M, {}^eR)) = q^\alpha \lambda\left(\frac{G_{i+1}}{\mathrm{Im}(\varphi_{i+2}^{[q]})}\right) \geqslant q^\alpha \lambda\left(\frac{R}{\mathfrak{m}^{[q]}}\right),$$

where the last inequality comes from projecting onto one of the components of G_{i+1}. Dividing by q and taking limits, we get

$$\beta_{i+1}^F(M, R) = \lim_{e \to \infty} \frac{\lambda(\mathrm{Tor}_{i+1}^R(M, {}^eR))}{q^{(1+\alpha)}} \geqslant \lim_{e \to \infty} \frac{\lambda(R/\mathfrak{m}^{[q]})}{q} = \mathrm{e}_{HK}(\mathfrak{m}, R) > 0,$$

which is a contradiction.

The last claim follows from the fact that, for any parameter x, we have

$$\beta_1^F(R/(x), R) \leqslant \lim_{e \to \infty} \frac{\lambda(H_1(x^q; R))}{q} = 0,$$

where H_1 denotes the first Koszul homology, see **[28]** and **[17,** Theorem 6.2]). □

Lemma 4.10. *Let $(R, \mathfrak{m}, K)$ be a local ring of positive characteristic $p > 0$ and $\mathfrak{p} \in \mathrm{Spec}(R)$. If $\mathrm{pd}_R(\mathfrak{p}) < \infty$, then R is a domain.*

Proof. Since $\mathfrak{p}$ has finite projective dimension, given a minimal free resolution

$$0 \longrightarrow L_t \xrightarrow{\psi_t} \cdots \longrightarrow L_0 \longrightarrow R/\mathfrak{p} \longrightarrow 0$$

of $R/\mathfrak{p}$ over R, we have that

$$0 \longrightarrow L_t \xrightarrow{\psi_t^{[q]}} \cdots \longrightarrow L_0 \longrightarrow R/\mathfrak{p}^{[q]} \longrightarrow 0$$

is a minimal free resolution of $R/\mathfrak{p}^{[q]}$ over R **[26,** Exemples 1.3 d)]. Then, $\operatorname{Ass}_R(R/\mathfrak{p}^{[q]}) = \{\mathfrak{p}\}$, and thus, $\mathfrak{p}^{[q]}$ is $\mathfrak{p}$-primary for all $q = p^e$.

Let $x \notin \mathfrak{p}$, and assume $xy = 0$ for $y \in R$. This implies that, for any q, we have $xy \in \mathfrak{p}^{[q]}$. We conclude that $y \in \mathfrak{p}^{[q]}$ since $x \notin \mathfrak{p}$. Thus,

$$y \in \bigcap_{q \geqslant 1} \mathfrak{p}^{[q]} = (0).$$

In particular, the localization map $R \to R_{\mathfrak{p}}$ is injective. We have that $\operatorname{pd}_R(R/\mathfrak{p}) < \infty$ implies $\operatorname{pd}_{R_{\mathfrak{p}}}(k(\mathfrak{p})) < \infty$. Then, $R_{\mathfrak{p}}$ is a regular local ring; in particular, it is a domain. Therefore, R is a domain. $\square$

Proposition 4.11. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring of characteristic $p > 0$, and let I be an $\mathfrak{m}$-primary integrally closed ideal. If $\beta_i^F(R/I, R) = 0$ for some $i > 0$, then R is regular.*

Proof. Let $\mathfrak{p}$ be a minimal prime of R. Since $\beta_i^F(R/I, R) = 0$, by Theorem 4.7, we have that $\operatorname{Tor}_i^R(R/I, R/\mathfrak{p}) = 0$. By Proposition 4.3, it follows that $\operatorname{pd}_R(R/\mathfrak{p}) < \infty$, and thus, R is a domain by Lemma 4.10. Since one-dimensional local domains are Cohen-Macaulay, by Corollary 4.8, we have that $\operatorname{pd}_R(R/I) < \infty$. In particular, $\operatorname{Tor}_j^R(R/I, K) = 0$ for $j \gg 0$. We conclude that $\operatorname{pd}_R(K) < \infty$ because R/I tests finite projective dimension **[9,** Theorem 5 (ii)]. Hence, R is regular. $\square$

5. Syzygies of finite length. We now present several characteristic-free results. In particular, we do not always assume that the rings have positive characteristic. We focus on Question 1.2. Specifically, we give support to the claim that a finite length R-module M of infinite projective dimension cannot have a finite length syzygy Ω_i for $i > \dim(R)+1$. As a consequence of our methods, we describe, in some cases, the dimension of the syzygies.

It follows from Theorem 4.7 that, if $\dim(R) = 1$ and R has positive characteristic, then an affirmative answer to Question 1.2 is equivalent to the statement: for every M of finite length, $\beta_i^F(M, R) = 0$ for some $i > 1$ implies $\operatorname{pd}_R(M) < \infty$.

We now provide an example that shows that the requirement of $i > \dim(R) + 1$ in Question 1.2 is necessary for a positive answer.

Example 5.1. Let $R = \mathbb{F}_p[\![x,y]\!]/(x^2,xy)$ and $M = R/(x)$. Then $\dim(R) = 1$. In addition, $\mathrm{pd}_R(M) = \infty$ since R is not Cohen-Macaulay. We have that $\Omega_2 \cong H^0_{(x,y)}(R) = (x)$ has finite length.

Lemma 5.2. *Let $(R, \mathfrak{m}, K)$ be a local ring, and let M be a finite length R-module that has a finite length syzygy Ω_{i+1}, for some fixed $i > 0$. Then,*

$$\mathrm{Tor}_i^R\left(M, R/H^0_{\mathfrak{m}}(R)\right) = 0.$$

If R has positive characteristic p, then for all $e \geqslant 0$,

$$\mathrm{Tor}_i^R\left(M, {}^e\!\left(R/H^0_{\mathfrak{m}}(R)\right)\right) = 0.$$

Proof. Set $H := H^0_{\mathfrak{m}}(R)$. Let $(G_\bullet, \varphi_\bullet)$ be a minimal free resolution of M:

$$G_\bullet : \cdots \to G_{i+1} \xrightarrow{\varphi_{i+1}} G_i \xrightarrow{\varphi_i} G_{i-1} \xrightarrow{\varphi_{i-1}} G_{i-2} \to \cdots \to G_0 \to M \to 0.$$

Tensor $G_\bullet$ with R/H and denote by $\overline{G}_\bullet$ its residue class modulo H:

$$\overline{G}_{i+1} \xrightarrow{\overline{\varphi}_{i+1}} \overline{G}_i \xrightarrow{\overline{\varphi}_i} \overline{G}_{i-1}$$

Since $\mathrm{Im}(\varphi_{i+1}) = \Omega_{i+1}$ has finite length, by assumption, we have $\overline{\varphi}_{i+1} = 0$. We want to show that $\mathrm{Ker}(\overline{\varphi}_i) = 0$ as well. For any $\mathfrak{p} \in \mathrm{Spec}(R) \smallsetminus \{\mathfrak{m}\}$, the complex $(G_\bullet)_{\mathfrak{p}}$ is split exact:

$$0 \to (G_i)_{\mathfrak{p}} \xrightarrow{(\varphi_i)_{\mathfrak{p}}} (G_{i-1})_{\mathfrak{p}} \xrightarrow{(\varphi_{i-1})_{\mathfrak{p}}} (G_{i-2})_{\mathfrak{p}} \to \cdots \to (G_0)_{\mathfrak{p}} \to 0,$$

since M and Ω_{i+1} have finite length. We have that $\mathrm{rk}((\varphi_i)_{\mathfrak{p}})$ is maximal, due to the fact that $\mathrm{rk}(G_i) \leqslant \mathrm{rk}(G_{i-1})$ as the localized complex is split exact, and localizing only decreases the rank of a map. Thus, $r := \mathrm{rk}(G_i) = \mathrm{rk}((\varphi_i)_{\mathfrak{p}}) = \mathrm{rk}(\varphi_i)$. Furthermore, $I_r(\varphi_i) \not\subseteq \mathfrak{p}$, by split exactness. Since this holds for all $\mathfrak{p} \in \mathrm{Spec}(R) \smallsetminus \{\mathfrak{m}\}$, in particular, we have $\mathrm{depth}(I_r(\overline{\varphi}_i)) \geqslant 1$. By the Buchsbaum-Eisenbud criterion, we have that

$$0 \longrightarrow \overline{G}_i \xrightarrow{\overline{\varphi}_i} \overline{G}_{i-1} \longrightarrow \overline{\Omega}_{i-1} = \Omega_{i-1}/H\Omega_{i-1} \longrightarrow 0.$$

is an exact complex. Therefore $\mathrm{Ker}(\overline{\varphi}_i) = 0$, and hence, $\mathrm{Tor}_i^R(M, R/H) = 0$. For the second part of the Lemma, when R has positive characteristic, the argument is the same: just notice that the complex ${}^e(G_\bullet)_{\mathfrak{p}}$

is again split exact for all primes $\mathfrak{p} \neq \mathfrak{m}$ and apply the same argument as above to the map $\overline{\varphi}_i^{[q]}$. $\square$

We now give results that support an affirmative answer to Question 1.2 for one-dimensional rings. Over Buchsbaum rings, the modules $H^i_{\mathfrak{m}}(R)$ are K-vector spaces for $i < \dim(R)$. Because of this fact, we can prove the following proposition using Lemma 5.2.

Proposition 5.3. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional Buchsbaum ring. Then the answer to Question 1.2 is positive.*

Proof. Assume that there exists a finite length R-module M such that $\Omega_{i+1}(M)$ has finite length for some $i \geqslant 2$. By Lemma 5.2, we have

$$0 = \operatorname{Tor}_i^R\left(M, R/H^0_{\mathfrak{m}}(R)\right) \cong \operatorname{Tor}_{i-1}^R\left(M, H^0_{\mathfrak{m}}(R)\right),$$

where $i - 1 \geqslant 1$ for dimension shifting. By Remark 2.5, we have that $H^0_{\mathfrak{m}}(R) \cong \bigoplus_{j=1}^t K$. Therefore,

$$0 = \operatorname{Tor}_{i-1}^R\left(M, H^0_{\mathfrak{m}}(R)\right) = \bigoplus_{j=1}^{t} \operatorname{Tor}_{i-1}^R\left(M, K\right),$$

which implies $\operatorname{Tor}_{i-1}^R(M, K) = 0$. Hence, $\operatorname{pd}_R(M) \leqslant i - 2$. $\square$

We now present two results about the dimension of syzygies of a finite-length module. These results will be used in Proposition 5.7 to give a case in which a finite-length module cannot have infinitely many syzygies of finite length.

Proposition 5.4. *Let $(R, \mathfrak{m}, K)$ be a local ring of dimension d, and let M be a finite length R-module. Let $i \geqslant 1$, and let Ω_i be the ith syzygy of M. Then, either $\dim(\Omega_i) = d$ or Ω_i has finite length.*

Proof. By way of contradiction, we suppose $\dim(\Omega_i) = k$ with $0 < k < d$. Let $G_\bullet \to M \to 0$ be a minimal free resolution of M. By our assumption on $\dim(\Omega_i)$, we can choose $\mathfrak{p} \in \operatorname{Min}(\operatorname{ann}(\Omega_i)) \smallsetminus (\{\mathfrak{m}\} \cup \operatorname{Min}(R))$ and localize $G_\bullet$ at $\mathfrak{p}$. The resulting complex is split exact, because $M_{\mathfrak{p}} = 0$. In particular, $(\Omega_i)_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module. By our choice of $\mathfrak{p}$, we have that $(\Omega_i)_{\mathfrak{p}}$ has finite length, and $\dim(R_{\mathfrak{p}}) > 0$, a contradiction. $\square$

Proposition 5.5. *Let $(R, \mathfrak{m}, K)$ be a local ring of positive dimension. Suppose that there exists an R-module M of infinite projective dimension and finite length which has a finite length syzygy Ω_{i+1}, for some fixed $i > 0$. If $\beta_i(M) \geqslant \beta_{i-1}(M)$, then Ω_{i-1} has finite length as well and R is one-dimensional.*

Proof. Let $(G_\bullet, \varphi_\bullet)$ be a minimal free resolution of M:

$$\begin{array}{ccccccccc} G_{i+1} & \xrightarrow{\varphi_{i+1}} & R^{\beta_i(M)} & \xrightarrow{\varphi_i} & R^{\beta_{i-1}(M)} & \xrightarrow{\varphi_{i-1}} & G_{i-2} & \longrightarrow & \cdots . \\ & \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & & \\ & \Omega_{i+1} & & \Omega_i & & \Omega_{i-1} & & & \end{array}$$

Let $\mathfrak{p} \in \operatorname{Spec}(R) \smallsetminus \{\mathfrak{m}\}$. We localize $G_\bullet$ at $\mathfrak{p}$. Since both M and Ω_{i+1} have finite length, we have a split exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_{\mathfrak{p}}^{\beta_i(M)} & \longrightarrow & R_{\mathfrak{p}}^{\beta_{i-1}(M)} & \longrightarrow & (G_{i-2})_{\mathfrak{p}} & \longrightarrow & \cdots \\ & & & \searrow^{\cong} \quad \nearrow & & \searrow & & & \\ & & & (\Omega_i)_{\mathfrak{p}} & & (\Omega_{i-1})_{\mathfrak{p}} . & & & \end{array}$$

In particular, this implies $\beta_i(M) \leqslant \beta_{i-1}(M)$. Since the opposite inequality holds by our assumption, equality is obtained. Set $\beta = \beta_i(M) = \beta_{i-1}(M)$. From the above split exact sequence, we also get that $R_{\mathfrak{p}}^\beta \cong (\Omega_i)_{\mathfrak{p}}$; therefore, $(\Omega_{i-1})_{\mathfrak{p}} = 0$. Since $\mathfrak{p}$ is an arbitrary prime in $\operatorname{Spec}(R) \smallsetminus \{\mathfrak{m}\}$, we have that Ω_{i-1} has finite length. Thus, we have a free complex $0 \to F_1 = R^\beta \to F_0 = R^\beta \to 0$ with finite length homology. We conclude that R has dimension 1 by the New intersection theorem **[28]**. □

Remark 5.6. If, in Proposition 5.5, it is assumed that the sequence of Betti numbers $\{\beta_i(M)\}$ is non-decreasing, then the argument above may be repeated to show that i is necessarily odd, and $\beta_i(M) = \beta_{i-1}(M), \beta_{i-2}(M) = \beta_{i-3}(M), \ldots, \beta_1(M) = \beta_0(M)$. In addition, $\Omega_j(M)$ has finite length for all even j, $0 \leqslant j \leqslant i+1$. In particular, the typical situation to study would be $(R, \mathfrak{m}, K)$ a one-dimensional ring and a resolution

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \Omega_4 & \longrightarrow & R^\beta & \longrightarrow & R^\beta & \longrightarrow & R^\alpha & \longrightarrow & R^\alpha & \longrightarrow & M \longrightarrow 0 \\ & & & & & & & \searrow \quad \nearrow & & & & & \\ & & & & & & & \Omega_2 & & & & & \end{array}$$

with Ω_4 and Ω_2 of finite length.

As a consequence of these results, we give a partial answer to Question 1.2 in the case where M has eventually non-decreasing Betti numbers. It is a conjecture of Avramov that every finitely generated module over a local ring has eventually non-decreasing Betti numbers [**4**]. The conjecture is known to be true in several cases [**5, 11, 14, 21, 30, 31**], in particular, for Golod rings [**21,** Corollaire 6.5].

Proposition 5.7. *Let* $(R, \mathfrak{m}, K)$ *be a local ring, and let* M *be a finite length* R*-module of infinite projective dimension with eventually non-decreasing Betti numbers. Then, for all* $i \gg 0$*, there exists a* $\mathfrak{p} \in \operatorname{Min}(R)$ *such that* $\dim(\Omega_i) = \dim(R/\mathfrak{p})$*. In particular,* M *cannot have arbitrarily high syzygies of finite length.*

Proof. If $\operatorname{Supp}(\Omega_i) \cap \operatorname{Min}(R) \neq \emptyset$ for all $i \gg 0$, then we are done. By way of contradiction, assume that there exist infinitely many syzygies Ω_i of M such that $\operatorname{Supp}(\Omega_i) \cap \operatorname{Min}(R) = \emptyset$. Note that, by Proposition 5.4, such syzygies must have finite length. By replacing M with a high enough syzygy, we can then assume that M is a module of finite length with non-decreasing Betti numbers, and with infinitely many syzygies of finite length. We have that R is one-dimensional by Proposition 5.5. Furthermore, by Remark 5.6, we have $\beta_{2i} = \beta_{2i+1}$ for all $i \geqslant 0$. For $i \geqslant 0$, consider the short exact sequence

$$0 \longrightarrow \Omega_{2i+2} \longrightarrow R^{\beta} \xrightarrow{\varphi} R^{\beta} \longrightarrow \Omega_{2i} \longrightarrow 0,$$

where $\beta := \beta_{2i} = \beta_{2i+1}$. Let $S := R[\varphi]$. Then R^{β} becomes an S-module. The above exact sequence shows that $\Omega_{2i} \cong R^{\beta} \otimes_S S/(\varphi)$ and $\Omega_{2i+2} \cong (0 :_{R^{\beta}} \varphi)$. Then, by [**32,** Proposition 11.1.9 (2)],

$$\lambda(\Omega_{2i}) - \lambda(\Omega_{2i+2}) = e(\varphi; R^{\beta}),$$

where $e(\varphi; -)$ denotes the Hilbert-Samuel multiplicity with respect to the ideal (φ) in S. Since such a multiplicity is always positive, we have that $\lambda(\Omega_{2i+2}) < \lambda(\Omega_{2i})$, for all $i \geqslant 0$. Since there cannot be an infinite strictly decreasing sequence of such lengths, we obtain a contradiction. $\square$

Remark 5.8. Proposition 5.7 also follows from [**6,** Theorem 8], and it gives another proof of the fact that, when M is a module of finite length with eventually non-decreasing Betti numbers and R is equidimensional, then the sequence of integers $\{\dim(\Omega_i)\}_{i=0}^{\infty}$ is constant for $i \gg 0$ (see [**6,** Corollary 2]).

Proposition 5.9. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring. Suppose that there exists a finite length module M of infinite projective dimension that has a finite length syzygy Ω_{i+1}, for some fixed $i \geqslant 2$. Then,*

$$\lambda(\Omega_{i+1}) = \sum_{j=0}^{i} (-1)^{i-j+1} \lambda(\operatorname{Tor}_j^R(M, R/(x))),$$

where x is a suitable parameter.

Proof. Consider a minimal free resolution of M:

$$\begin{array}{ccccccccccc} G_i & \longrightarrow & G_{i-1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow M \longrightarrow 0. \\ & \searrow & \nearrow & & & & & \searrow & \nearrow & \\ & \Omega_i & & & & & & \Omega_1 & & \end{array}$$

For all $j = 1, \ldots, i+1$, this can be broken into short exact sequences:

$$0 \longrightarrow \Omega_j \longrightarrow G_{j-1} \longrightarrow \Omega_{j-1} \longrightarrow 0,$$

where $\Omega_0 := M$. These give two exact sequences:

$$0 \longrightarrow \Omega_{i+1} \longrightarrow H^0_{\mathfrak{m}}(G_i) \longrightarrow H^0_{\mathfrak{m}}(\Omega_i) \longrightarrow 0$$

and

$$0 \longrightarrow H^0_{\mathfrak{m}}(\Omega_j) \longrightarrow H^0_{\mathfrak{m}}(G_{j-1}) \longrightarrow H^0_{\mathfrak{m}}(\Omega_{j-1}).$$

The first short exact sequence comes from the fact that Ω_{i+1} has finite length, and thus, $H^1_{\mathfrak{m}}(\Omega_{i+1}) = 0$. Furthermore, the cokernel of the rightmost map in the second exact sequence, which may be proved to be the kernel of the leftmost map in

$$\Omega_j \otimes_R H^1_{\mathfrak{m}}(R) \longrightarrow G_{j-1} \otimes_R H^1_{\mathfrak{m}}(R) \longrightarrow \Omega_{j-1} \otimes_R H^1_{\mathfrak{m}}(R) \longrightarrow 0$$

is then $\operatorname{Tor}_1^R(\Omega_{j-1}, H^1_{\mathfrak{m}}(R))$. For simplicity, we denote $\omega_j := \lambda(H^0_{\mathfrak{m}}(\Omega_j))$, $g_j := \lambda(H^0_{\mathfrak{m}}(G_j))$ and $\alpha_j := \lambda(\operatorname{Tor}_1^R(\Omega_j, H^1(R)))$. Then, we have relations

$$\begin{aligned}
\omega_{i+1} &= g_i - \omega_i \\
\omega_i &= g_{i-1} - \omega_{i-1} + \alpha_{i-1} \\
&\vdots \\
\omega_2 &= g_1 - \omega_1 + \alpha_1 \\
\omega_1 &= g_0 - \lambda(M) + \lambda(\operatorname{Tor}_1(M, H^1_{\mathfrak{m}}(R))).
\end{aligned}$$

After localizing the resolution $G_\bullet$ at any minimal prime $\mathfrak{p}$, since $(\Omega_{i+1})_{\mathfrak{p}} = 0$, we obtain that $\sum_{j=0}^{i}(-1)^j \beta_j(M) = 0$. Then, $\sum_{j=0}^{i}(-1)^j g_j = 0$ because $g_j = \beta_j(M) \cdot \lambda(H^0_m(R))$. Therefore,

$$\begin{aligned}
\omega_{i+1} &= \lambda(\Omega_{i+1}) \\
&= \sum_{j=1}^{i-1}(-1)^{i-j}\alpha_j + (-1)^i \lambda(\operatorname{Tor}_1(M, H^1(R))) + (-1)^{i-1}\lambda(M).
\end{aligned}$$

Choose a parameter x such that $H^0_{\mathfrak{m}}(R) = 0 :_R x$, as in Remark 2.4. By similar considerations, we can also assume that $xM = 0$. From this choice, we have that $xH^0_{\mathfrak{m}}(\Omega_j) = 0$ for all $j = 0, \dots, i+1$, since $\Omega_j \subseteq G_{j-1}$ is a free R-module. Since the Tor modules can be computed using flat resolutions, we have an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(R) \longrightarrow R \longrightarrow R_x \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow 0.$$

Completion is produced on the left to obtain a flat resolution of $H^1_{\mathfrak{m}}(R)$:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow R^{\mu(H^0_{\mathfrak{m}}(R))} & \longrightarrow & R & & \longrightarrow & & R_x & \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow 0. \\
& & & \searrow & & \nearrow & & \\
& & & & R/H^0_{\mathfrak{m}}(R) & & &
\end{array}$$

By our choice of x, we have that a free resolution of R/x begins as

$$\cdots \longrightarrow R^{\mu(H^0_{\mathfrak{m}}(R))} \longrightarrow R \longrightarrow R \longrightarrow R/(x) \longrightarrow 0.$$

For all $j = 1, \dots, i-1$, we obtain

$$\operatorname{Tor}^R_1(\Omega_j, H^1_{\mathfrak{m}}(R)) \cong \operatorname{Tor}^R_1(\Omega_j, R/(x)) \cong \operatorname{Tor}^R_{j+1}(M, R/(x)),$$

where the last isomorphism comes from dimension shifting. In addition,

$$\operatorname{Tor}_1^R(M, H^1_{\mathfrak{m}}(R)) \cong \operatorname{Tor}_1^R(M, R/(x)).$$

Finally, since $xH^0_{\mathfrak{m}}(\Omega_0) = xM = 0$, we get

$$M \cong M/xM \cong \operatorname{Tor}_0^R(M, R/(x)),$$

and the proposition then follows. □

Corollary 5.10. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional ring, and let M be a finite length module of infinite projective dimension. Then $\lambda(\Omega_1) = \lambda(\Omega_3) = \infty$.*

Proof. Note that $\lambda(\Omega_1) = \infty$; otherwise, we would have a short exact sequence

$$0 \longrightarrow \Omega_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

in which both Ω_1 and M have finite lengths. This cannot occur since $G_0 \neq 0$ is free and $\dim(R) = 1$.

Now, let us assume by way of contradiction that $\lambda(\Omega_3) < \infty$. Let $(G_\bullet, \varphi_\bullet)$ be a minimal free resolution of M:

$$0 \longrightarrow \Omega_3 \longrightarrow G_2 \xrightarrow{\varphi_2} G_1 \xrightarrow{\varphi_1} G_0 \longrightarrow M \longrightarrow 0.$$

Let $x \in R$ be a parameter such that $xM = xH^0_{\mathfrak{m}}(R) = 0$. Consider the short exact sequence

$$0 \longrightarrow (x) \longrightarrow R \longrightarrow R/(x) \longrightarrow 0.$$

By our choice of x we have $0 :_R x = H^0_{\mathfrak{m}}(R)$; hence, $(x) \cong R/H^0_{\mathfrak{m}}(R)$. After tensoring the sequence with M, we obtain that

$$0 \longrightarrow \operatorname{Tor}_1^R(M, R/(x)) \longrightarrow M/H^0_{\mathfrak{m}}(R)M \longrightarrow M \longrightarrow M/xM \longrightarrow 0.$$

Since $xM = 0$, we obtain

$$\lambda(\operatorname{Tor}_1^R(M, R/(x))) = \lambda(M/H^0_{\mathfrak{m}}(R)M).$$

Then, by Proposition 5.9, we have

$$\begin{aligned}\lambda(\Omega_3) &= -\lambda(\operatorname{Tor}_2^R(M, R/(x))) + \lambda(\operatorname{Tor}_1^R(M, R/(x))) - \lambda(M)\\ &\leqslant \lambda(\operatorname{Tor}_1^R(M, R/(x))) - \lambda(M) = \lambda(M/H^0_{\mathfrak{m}}(R)M) - \lambda(M) \leqslant 0,\end{aligned}$$

which gives a contradiction since $\Omega_3 \neq 0$, since M has infinite projective dimension. $\square$

The next example is due to the second author and is taken from **[6]**. It shows the assumption that M has finite length is needed in Corollary 5.10.

Example 5.11. Let $S = \mathbb{Q}[x,y,z,u,v]$, and let $I \subseteq S$ be the ideal

$$I = (x^2, xz, z^2, xu, zv, u^2, v^2, zu + xv + uv, yu, yv, yx - zu, yz - xv).$$

Let $R = S/I$, which is a one-dimensional ring of depth 0. In this case, y is a parameter, $0 :_R y = (u, v, z^2)$ and $(y) = 0 :_R (0 :_R y)$. Let M be the cokernel of the rightmost map in the exact complex:

$$\cdots \longrightarrow R^3 \xrightarrow{\begin{bmatrix} u & v & z^2 \end{bmatrix}} R \xrightarrow{\ y\ } R \xrightarrow{\begin{bmatrix} u \\ v \\ z^2 \end{bmatrix}} R^3.$$

Then M is a one-dimensional module with first and third syzygies $\Omega_1 \cong R/(y)$ and $\Omega_3 \cong 0 :_R y$. Both are modules of finite length since y is a parameter.

Acknowledgments. We thank the referee for helpful comments and suggestions.

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