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POWERS OF EDGE IDEALS OF REGULARITY THREE BIPARTITE GRAPHS

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ABSTRACT. In this paper, we prove that, if $I(G)$ is the edge ideal of a connected bipartite graph with regularity 3, then, for all $s \geq 2$, the regularity of $I(G)^s$ is exactly $2s+1$.

1. Introduction. Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K, and let $I \subset R$ be an ideal. It is well known that, if I is a homogeneous ideal generated in degree d, then $\operatorname{reg}(I^s) = ds + b$ for $s \gg 0$ and for some constant b **[6, 7, 13]**. One problem that arises here is to find the exact form of this linear function. Many authors have studied this question; however, the exact form of this linear function has been found for only a few classes of ideals **[2, 5]**.

One important class of ideals are the edge ideals of finite simple graphs. Since in this case $d = 2$, we have the next question.

Question 1.1. *Let G be a graph and $I(G)$ its edge ideal. Since* $\operatorname{reg}(I(G)^s)$ *is asymptotically a linear function for $s \gg 0$, there are values b and s_0 such that* $\operatorname{reg}(I(G)^s) = 2s + b$ *for $s \geq s_0$. What are the values of b and s_0?*

There are few classes of graphs for which Question 1.1 has been answered (for example, see **[3, 9, 12]**). In this paper, we have added another class of graphs to the list. Here, we study the higher powers of the edge ideals of regularity three bipartite graphs. We have shown:

Theorem 1.2. *Let G be a bipartite connected graph with edge ideal $I(G)$. If* $\operatorname{reg}(I(G)) = 3$, *then for all $s \geq 1$, we have* $\operatorname{reg}(I(G)^s) = 2s+1$.

For our proof, we use the combinatorial characterization of the three bipartite regularity graphs by Fernández-Ramos and Gimenez, proved in **[8]**, and techniques introduced by the second author in **[1]**.

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2. Preliminaries. Throughout this paper, we let G be a finite simple graph with vertex set $V(G)$. A subgraph $G' \subseteq G$ is called *induced* if uv is an edge of G' whenever u and v are vertices of G' and uv is an edge of G. The *complement* of a graph G, for which we write G^c, is the graph on the same vertex set in which uv is an edge of G^c if and only if it is not an edge of G. Finally, let C_k denote the cycle on k vertices; a *chord* is an edge which is not in the edge set of C_k. A cycle is called *minimal* if it has no chord.

Definition 2.1 (Bipartite graphs)**.** A graph G is called *bipartite* if there are two disjoint independent subsets X, Y of $V(G)$ whose union is $V(G)$. Note that $X \subset V(G)$ is called *independent* if there is no edge $e \in E(G)$ such that $e = xy$ for some $x, y \in X$.

We have the following theorem for classifying bipartite graphs. For the proof, see [**16**].

Theorem 2.2 (König, [**16,** Theorem 1.2.18])**.** *A graph G is bipartite if and only if G contains no odd cycle.*

We also need the next definition for bipartite graphs. We take this definition from [**8**].

Definition 2.3 (Bipartite complement)**.** The *bipartite complement* of a bipartite graph G is a bipartite graph G^{bc} over the same vertex set and same bipartition $V(G) = X \sqcup Y$ such that

$$E(G^{bc}) = \{xy : x \in X, y \in Y, xy \notin G\}.$$

If G is a graph without isolated vertices, then let S denote the polynomial ring on the vertices of G over some fixed field K. Recall that the *edge ideal* of G is

$$I(G) = (xy : xy \text{ is an edge of } G).$$

Definition 2.4. Let S be a standard graded polynomial ring over a field K. The *Castelnuovo-Mumford regularity* of a finitely generated graded S module M, written $\operatorname{reg}(M)$, is given by

$$\operatorname{reg}(M) := \max\{j - i \mid \operatorname{Tor}_i(M, K)_j \neq 0\}.$$

Definition 2.5. For every s, we say that $I(G)^s$ is *k-steps linear* whenever the minimal free resolution of $I(G)^s$ over the polynomial ring is linear for k steps, i.e., $\operatorname{Tor}_i^S(I(G)^s, K)_j = 0$ for all $1 \leq i \leq k$ and all $j \neq i + 2s$.

In particular, we say $I(G)$ has *linear minimal free resolution* if the minimal free resolution is k-steps linear for all $k \geq 1$. We also say that $I(G)$ has *linear presentation* if its minimal free resolution is 1-step linear.

We proceed in this section by recalling a few well-known results. The reader is referred to [**1, 14**] for cogent references.

Observation 2.6. *Let $I(G)$ be the edge ideal of a graph G. Then $I(G)^s$ has linear minimal free resolution if and only if* $\operatorname{reg}(I(G)^s) = 2s$.

The next lemma follows from [**1,** Lemma 2.10].

Lemma 2.7. *Let $I \subseteq S$ be a monomial ideal, and let m be a monomial of degree d. Then,*

$$\operatorname{reg}(I) \leq \max\{\operatorname{reg}(I : m) + d, \operatorname{reg}(I, m)\}.$$

Moreover, if m is a variable x appearing in I, then $\operatorname{reg}(I)$ *is* equal *to one of these terms.*

The next theorem, due to Fröberg (see [**10,** Theorem 1] and [**14,** Theorem 1.1]) is used repeatedly throughout this paper. The anonymous referee notified us that this theorem was first proved by Wegner in a different language [**15**].

Theorem 2.8 (Fröberg theorem)**.** *The minimal free resolution of $I(G)$ is linear if and only if the complement graph G^c is chordal, that is, no induced cycle in G^c has length greater than* 3.

The next results are due to Francisco, Hà and Van Tuyl [**11**].

Theorem 2.9 ([**14,** Proposition 1.3])**.** *Let G be a graph and $I(G)$ its edge ideal. Then, $I(G)$ has a linear presentation if and only if G^c has no induced* 4*-cycle.*

Theorem 2.10 ([**14,** Theorem 1.8])**.** *If $I(G)^s$ has a linear minimal resolution for a graph G and for some $s \geq 1$, then $I(G)$ has a linear presentation.*

The next theorem due to Herzog, Hibi and Zheng [**12**] is necessary for our results.

Theorem 2.11 ([**12,** Theorem 3.2])**.** *Let $I(G)$ be the edge ideal of a graph G. If $I(G)$ has linear minimal free resolution, then so has $I(G)^s$ for each $s \geq 2$.*

Definition 2.12. For any graph G, we write $\operatorname{reg}(G)$ as shorthand for $\operatorname{reg}(I(G))$.

The next proposition is necessary for proving one of our results.

Proposition 2.13 ([**8,** Theorem 4.1])**.** *Let G be a connected bipartite graph. The edge ideal $I(G)$ has regularity 3 if and only if G^c has at least one induced cycle of length ≥ 4 and G^{bc} does not contain any induced cycle of length ≥ 6.*

Finally, we mention the next theorem from [**1**] without proof. This will be the most important structural tool for our proof. This theorem shows that the powers of edge ideals have a very special property regarding short exact sequences, which makes the task of finding upper bounds for regularity easier.

Theorem 2.14 ([**1,** Theorem 5.2])**.** *For any finite simple graph G and any $s \geq 1$, let the set of minimal monomial generators of $I(G)^s$ be $\{m_1, \dots, m_k\}$. Then*

$$\operatorname{reg}(I(G)^{s+1}) \leq \max_{1 \leq l \leq k} \{\operatorname{reg}(I(G)^{s+1} : m_l) + 2s, \operatorname{reg}(I(G)^s)\}.$$

In order to analyze the generators of $(I(G)^{s+1} : e_1 \dots e_s)$, we recall the notion of *even-connectedness* with respect to s-fold products from [**1**].

Definition 2.15. Two vertices u and v (u may be the same as v) are said to be *even-connected* with respect to an s-fold product $e_1 \cdots e_s$ if there is a path $p_0 p_1 \cdots p_{2k+1}$, $k \geq 1$ in G such that:

(i) $p_0 = u, p_{2k+1} = v$.
(ii) For all $0 \leq l \leq k-1$, $p_{2l+1} p_{2l+2} = e_i$ for some i.
(iii) For all i,

$$|\{l \geq 0 | p_{2l+1} p_{2l+2} = e_i\}| \leq |\{j | e_j = e_i\}|.$$

(iv) For all $0 \leq r \leq 2k$, $p_r p_{r+1}$ is an edge in G.

If these properties are satisfied, then $p_0, \dots, p_{2k+1}$ is said to be an even-connection between u and v with respect to $e_1 \cdots e_s$.

We make an observation which follows directly from the definition:

Observation 2.16. *If u, v are even connected with respect to $e_1 \cdots e_s$, then they are even connected with respect to $e_{i_1} \cdots e_{i_t}$ for any $\{1, \ldots, s\} \subset \{i_1, \ldots, i_t\}$.*

By using the concept of even connection, the second author gave a description of $(I(G)^{s+1} : e_1 \ldots e_s)$ for each s-fold product.

Theorem 2.17 ([**1**, Theorem 6.7]). *Every generator uv (u may be equal to v) of $(I(G)^{s+1} : e_1 \cdots e_s)$ is either an edge of G or even-connected with respect to $e_1 \cdots e_s$, for $s \geq 1$.*

We next prove a result regarding bipartite graphs that is useful for our purposes.

Proposition 2.18. *If G is a bipartite graph, then its complement does not have any induced cycle of length > 4. In particular, an edge ideal of a bipartite graph has linear presentation, if and only if all its powers have linear resolution.*

Proof. Let G be a bipartite graph and $V(G) = X \sqcup Y$ a bipartition of G. Note that, since G^c contains the complete graph over X and Y, then we can say that every cycle in G^c of length ≥ 5 has at least three xs or three ys. Hence, it cannot be induced. The second part follows directly from Theorem 2.8, Theorem 2.9 and Theorem 2.11. $\square$

3. Bounding the regularity: The results. In this section, we give some new bounds on $\operatorname{reg}(I(G)^s)$ for biparite graphs G for which $\operatorname{reg}(I(G)) = 3$. The main idea is to use Theorem 2.10 and Proposition 2.18, as well as the analysis of the ideal $(I(G)^{s+1} : e_1 \cdots e_s)$ for an arbitrary s-fold product of edges (i.e., for $i \neq j$, $e_i = e_j$ is a possibility) in the spirit of [**1**]. Now, any s-fold product can be written as a product of s edges in various ways. In this section, we fix a presentation and work with respect to that. We first mention the following important result proved in [**1**], which says that these ideals are generated in degree 2 for any graph G.

Theorem 3.1. *For any graph G and for any s-fold product $e_1 \cdots e_s$ of edges in G (with the possibility of e_i being the same as e_j as an edge for $i \neq j$), the ideal*

$$(I(G)^{s+1} : e_1 \ldots e_s)$$

is generated by monomials of degree two.

As bipartite graphs have no odd cycles, the next result is from the definition of even-connectedness.

Proposition 3.2. *Let G be a bipartite graph and $s \geq 1$ an integer. Then, for every s-fold product $e_1 \cdots e_s$, $(I(G)^{s+1} : e_1 \cdots e_s)$ is a quadratic squarefree monomial ideal. Moreover, the graph G' associated to $(I(G)^{s+1} : e_1 \cdots e_s)$ is bipartite on the same vertex set and same bipartition as G.*

Proof. Note that, from Theorem 3.1, we know that $(I(G)^{s+1} : e_1 \cdots e_s)$ is a quadratic ideal. On the other hand, since G is a bipartite graph, from Theorem 2.2, we know that G contains no odd cycle. Therefore, for every vertex $v \in V(G)$, we can say that v is not even connected to itself with respect to $e_1 \cdots e_s$. Thus, $(I(G)^{s+1} : e_1 \cdots e_s)$ is squarefree. Then, a graph can be associated to it, namely, G'.

Now, we show that G' is also bipartite on $V(G)$ with the same bipartition. From Theorem 2.17, we have $G \subseteq G'$ and $V(G) = V(G')$; however, $E(G) \subseteq E(G')$. Let $V(G) = X \sqcup Y$ be the bipartition. We only need to show that X and Y are independent in G'.

Suppose that there is a $e = uv \in G'$ such that $u, v \in X$. Since G is bipartite $e \notin G$, using Theorem 2.17, we can say that u and v are even connected with respect to $e_1 \cdots e_s$. By the definition of even-connectedness there is a path $p_0 p_1 \cdots p_{2k+1}$ in G such that $p_0 = u$ and $p_{2k+1} = v$. Now, note that, since $u \in X$ and $p_0 p_1 \in G$ and G is bipartite, we can conclude that $p_1 \in Y$, since $p_1 p_2 \in G$ and $p_2 \in X$. By repeating this process, we can say $p_{2k} \in X$.

However, we know that $p_{2k} p_{2k+1} \in G$ and $p_{2k}, p_{2k+1} \in X$, which contradicts the fact that G is bipartite. Then X is independent in G'. Using the same method, we can show that Y is independent in G'. $\square$

The next corollary follows directly from Theorem 2.8 and [**1,** Lemmas 6.14, 6.15].

Corollary 3.3. *Let G be any graph and $e_1, \cdots, e_s$ some edges of G which are not necessarily distinct. If the minimal free resolution of $I(G)$ is linear, then*

$$(I(G)^{s+1} : e_1 \cdots e_s)$$

also has a linear minimal free resolution.

In order to prove the main result of this section, we need the next lemma.

Lemma 3.4. *Let G be a bipartite graph and $I = I(G)$ its edge ideal. Suppose that $e_1 \cdots e_s$ is an s-fold product of edges in G for a positive integer s. Then, we have*

$$(I^{s+1} : e_1 \cdots e_s) = \left(\left(I^2 : e_i\right)^s : \prod_{j \neq i} e_j \right)$$

for each $i \in \{1, 2, \ldots, s\}$.

Proof. Without loss of generality, we can suppose that $i = 1$.

Suppose that $V(G) = X \sqcup Y$ is a bipartition of $V(G)$, and assume that $uv \in (I^{s+1} : e_1 \cdots e_s)$. From Theorem 2.17, we have that $uv \in I$ or u and v are even connected with respect to $e_1 \cdots e_s$. Without loss of generality, we suppose that $u \in X$. If $uv \in I$, then the statement clearly follows. Let us assume that $uv \notin I$ and u and v are even connected with respect to $e_1 \cdots e_s$.

From the definition of even connection there is a path $P : u = x_0 y_1 x_1 \cdots x_k y_{k+1} = v$ in G such that for each $i \in \{1, \ldots, k\}$,

$$y_i x_i = e_j \quad \text{for some } j \in \{1, 2, \ldots, s\}.$$

We set $G' = G((I^2 : e_1))$. Since $G \subset G'$, if there is no $i \in \{1, \ldots, k\}$ such that $y_i x_i = e_1$, then the path P is an even connection with respect to $e_2 \cdots e_s$ in G'. Therefore, from Theorem 2.17, the result follows. Thus, we assume there are $\alpha_1, \ldots, \alpha_\ell \in \{1, \ldots, k\}$ such that

$$\alpha_1 < \alpha_2 < \cdots < \alpha_\ell \quad \text{and} \quad y_{\alpha_t} x_{\alpha_t} = e_1 \text{ for every } t.$$

For each t, since $y_{\alpha_t} x_{\alpha_t} = e_1$, $x_{\alpha_t - 1}$ and $y_{\alpha_t + 1}$ are even connected with respect to e_1, $x_{\alpha_t - 1} y_{\alpha_t + 1} \in G'$. In particular, we have $x_{\alpha_1 - 1}$ and $y_{\alpha_\ell + 1}$ are even connected with respect to e_1. Then, we have the following even connection with respect to $e_2 \ldots e_s$ in G'

$$P' : u = x_0 y_1 x_1 \cdots x_{\alpha_1 - 1} y_{\alpha_\ell + 1} x_{\alpha_\ell + 1} \cdots y_{k+1} = v,$$

and thus, $uv \in ((I^2 : e_1)^s : e_2 \cdots e_s)$. Conditions (ii), (iii) and (iv) of even connectedness follow, as P is an even connection in G.

In order to show the converse, suppose that $uv \in ((I^2 : e_1)^s : e_2 \cdots e_s)$. Then, from Theorem 2.17, either $uv \in (I^2 : e_1)$ or u, v

are even connected with respect to $e_2 \cdots e_s$ in G'. If $uv \in (I^2 : e_1)$, the statement is evident (since then, clearly, $uv \in (I^{s+1} : e_1 \cdots e_s)$). Thus, we assume that $uv \notin (I^2 : e_1)$. From the definition of even connectedness there is a path $P : u = x_0 y_1 x_1 \cdots y_k x_k y_{k+1} = v$ in G' such that for each $i \in \{1, \dots, k\}$,

$$y_i x_i = e_j \quad \text{for some } j \in \{2, \dots, s\}.$$

If, for each i, $x_i y_{i+1} \in G$, then P is an even connection in G and, from Theorem 2.17, the claim is evident. Therefore, suppose that there exists an i such that $x_i y_{i+1} \in G' \setminus G$. From Theorem 2.17, x_i and y_{i+1} are even connected with respect to e_1. Thus, by the definition, there is a path $x_i e_1 y_{i+1}$ in G. Let $e_1 = yx$. Therefore, if we replace $x_i y_{i+1}$ by $x_i e_1 y_{i+1}$ in P, we have the following path in G:

$$P' : u = x_0 y_1 x_1 \dots x_i y x y_{i+1} \dots y_{k+1} = v.$$

If we have only one copy of e_1 in P', then clearly, P' is an even connection with respect to $e_1 \cdots e_s$ in G, and the theorem will follow.

Otherwise, assume that there exist i and j such that $i < j$ and $|j - i|$ maximum such that

$$x_i y x y_{i+1}, x_j y x y_{j+1} \in P'.$$

Then, P' can be reduced to the following path in G:

$$P'' : u = x_0 y_1 x_1 \dots x_i y x y_{j+1} \dots y_{k+1} = v.$$

We observe that this is an even connection with respect to $e_1 \cdots e_s$. Conditions (i), (ii) and (iv) in the definition are satisfied as P is an even connection in G', and condition (iii) follows from the fact that P'' has only one copy of e_1, by construction. This proves the converse. $\square$

We are now ready to compute $\operatorname{reg}(I(G)^s)$ for a bipartite graph G in which we have $\operatorname{reg}(G) = 3$. By using induction on s and from Theorem 2.14, we need to investigate $\operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s)$ for an arbitrary s-fold product $e_1 \cdots e_s$. In the next theorem, we will show that $\operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) \leq 3$.

Theorem 3.5. *Let G be a bipartite connected graph and $s \geq 1$ an integer. If $\operatorname{reg}(I(G)) = 3$, then $\operatorname{reg}((I(G)^{s+1} : e_1 \cdots e_s)) \leq 3$ for every s-fold product $e_1 \cdots e_s$.*

Proof. Let $e_1 \cdots e_s$ be an s-fold product. The proof is by induction on s. Let $s = 1$. From Proposition 3.2, $(I(G)^2 : e)$ is a quadratic squarefree monomial ideal, and its associated graph G' is bipartite with the same vertex set and bipartition as G. Also, note that, if G'^c contains no induced cycle of length ≥ 4, then from Theorem 2.8 and Proposition 2.18, we can say that $(I(G)^2 : e)$ has a linear resolution, and thus,

$$\mathrm{reg}((I(G)^2 : e)) = 2.$$

Then, we may assume that G'^c has an induced cycle of length ≥ 4. From Proposition 2.13, we only need to show that there is no cycle of length ≥ 6 in G'^{bc}.

Let $V(G) = X \sqcup Y$ be a bipartition of G, and assume that there is an induced cycle C_{2n} in G'^{bc}, $n \geq 3$, on the vertex set:

$$V(C) = \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}.$$

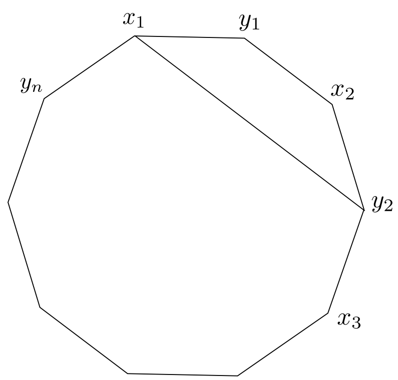

FIGURE 1.

Let $C_{2n} : x_1y_1, x_2y_2, \dots, x_ny_n$. Since $G \subseteq G'$, C_{2n} is also a cycle in G^{bc}. Since $\mathrm{reg}(I(G)) = 3$, and the length of $C_{2n} \geq 6$ from Proposition 2.13, C_{2n} must contain a chord. We can assume that this chord divides C_{2n} into two cycles of smaller length. If one of these cycles is C_4, we stop; otherwise, since its length is ≥ 6, it must have a chord (Proposition 2.13). We obtain that chord. Again, this chord divides C_{2n} into two cycles of smaller length. If one of these cycles is C_4, we stop; otherwise, we continue finding chords to finish with C_4 in G^{bc}. Without loss of generality, we can assume that $C_4 = x_1, y_1, x_2, y_2$.

First note that, by applying Theorem 2.17 since $x_1y_2 \in G' \backslash G$, we can conclude that x_1y_2 is even connected with respect to $e = xy$. Then, by the definition of even connectedness, there is a path x_1yxy_2 in G. Using the proof by contradiction, we will show that the cycle C_{2n} contains an induced cycle of length ≥ 6 in G^{bc}.

We first prove the next useful statements

(I) $x_2y, x_3y \notin G$. If $x_2y \in G$ (or $x_3y \in G$), then, since $xy_2 \in G$, we have the even connection x_2yxy_2 (or x_3yxy_2) with respect to e in G. Thus, from Theorem 2.17, $x_2y_2 \in G'$ (or $x_3y_2 \in G'$), a contradiction.

(II) $x_3y_1, x_2y_3 \in G$. Suppose that $x_3y_1 \in G^{bc}$ (or $x_2y_3 \in G^{bc}$). Then, from Theorem 2.17, x_3 and y_1 (or x_2 and y_3) are even connected with respect to e, and then, from the definition, $xy_1 \in G$ (or $x_2y \in G$). Since $xy_2 \in G$ and $x_1y \in G$, we have the even connections xy_1x_1y (or xy_2x_2y) in G. Then, by applying Theorem 2.17, we can conclude x_1y_1 (or $x_2y_2 \in G'$), a contradiction.

We now settle our claim for $n = 3$.

Note that $y \neq y_1, y_3$ (if $y = y_1$, then x_1y is an edge in G as x_1y_1 is, and, if $y = y_3$, then x_2y is an edge in G as x_2y_3 is an edge in G by assumption which forces x_2y_2 to be an edge in G'; both lead to a contradiction). Therefore, we can consider the 6-cycle $x_1y_1x_2yx_3y_3$ in G^{bc}. From (I) and (II), we know this cycle has no chords in G^{bc}; thus, it is an induced cycle in G^{bc} of length 6, which contradicts Proposition 2.13 and the fact that $\operatorname{reg}(G) = 3$.

We now assume that $n > 3$. We show the following statements

- For each $i \geq 2$, if $xy_i \in G$, then we have

$$x_{i+1}y_j \in G \qquad \text{for each } j \notin \{i, i+1\} \tag{3.1}$$

and

$$x_iy_j \in G \qquad \text{for each } j \notin \{i, i-1\}.$$

Proof. Assume that, for some j we have $x_{i+1}y_j \in G^{bc}$ (or $x_iy_j \in G^{bc}$). Then, we have an even connection $x_{i+1}yxy_j$ (or x_iyxy_j), that is, x_{i+1} (or x_i) is connected to y in G. Also, since $xy_i \in G$, these even connections can be converted to the even connection $x_{i+1}yxy_i$ (or x_iyxy_i), which, from Theorem 2.17, means that $x_{i+1}y_i$ or x_iy_i belong to G', a contradiction. $\square$

- For each $i \notin \{1, n\}$,

$$x_iy_n \notin G^{bc}. \tag{3.2}$$

Proof. Suppose that, for some i, $x_iy_n \in G^{bc}$. Since $x_iy_n \in G'$, using Theorem 2.17, we have a path x_iyxy_n in G. However, since $x_1y \in G$, we can conclude that x_1 and y_n are even connected with respect to e. Thus, from Theorem 2.17, we have $x_1y_n \in G'$, a contradiction. $\square$

We proceed by showing the following

$$xy_\ell, x_1y_\ell \in G \quad \text{for all } 3 \le \ell \le n-1. \tag{3.3}$$

We prove this by using induction on ℓ.

First, we assume that $\ell = 3$. If $x_1y_3 \in G^{bc}$, since $y \neq y_1, y_3$ (if $y = y_1$, then x_1y is an edge in G as x_1y_1 is, and, if $y = y_3$, then x_2y is an edge in G as x_2y_3 is an edge in G by assumption which forces x_2y_2 to be an edge in G'; both lead to a contradiction) and from (I) and (II), we can consider the induced 6-cycle $x_1y_1x_2yx_3y_3$ in G^{bc}, which contradicts the fact that $\operatorname{reg}(G) = 3$. Then, $x_1y_3 \notin G^{bc}$.

If $xy_3 \in G^{bc}$, then, since $x \neq x_1, x_3$ (otherwise, since $xy_2 \in G$, we have $x_1y_2 \in G$ or $x_3y_2 \in G$) we can consider the 6-cycle $x_1y_2x_3y_3xy_n$ in G^{bc}. From the facts that $x_1y_3 \notin G^{bc}$ and (3.2), we know that this cycle contains no chords, contradicting the fact that $\operatorname{reg}(G) = 3$.

We now suppose, for each $3 \le \ell < n$, that our claim is true. We show that $xy_{\ell+1}, x_1y_{\ell+1} \in G$. Suppose that $x_1y_{\ell+1} \in G^{bc}$. Note that, from the induction hypothesis and (3.4) and (3.2), we have

$$\begin{aligned} &x_ty_j \notin G^{bc} \quad \text{for } 3 \le t \le \ell < n \text{ and } j \notin \{t, t+1\} \\ &x_1y_t \notin G^{bc} \qquad\qquad\quad \text{for } 3 \le t \le \ell. \end{aligned}$$

Then the cycle $C_{2\ell} : x_1y_2x_3y_3 \cdots x_{\ell+1}y_{\ell+1}$ is an induced cycle of length ≥ 6 in G^{bc}, which contradicts the fact that $\operatorname{reg}(G) = 3$. Then, we have

$$x_1y_t \notin G^{bc} \quad \text{for } 3 \leq t \leq \ell+1. \tag{3.4}$$

We show that $xy_{\ell+1} \in G$. Suppose that $xy_{\ell+1} \in G^{bc}$.

Note that $x \notin \{x_1, \ldots, x_{\ell+1}\}$. Otherwise, using the induction hypothesis, we have that $xy_t \in G$ for each $t \in \{1, 2, \ldots, \ell\}$. We can say that $x_{j+1}y_j \in G$ for some $j \in \{1, 2 \ldots, \ell\}$ which is a contradiction.

Then, we can consider the $2\ell+1$-cycle $x_1y_2x_3y_3 \cdots y_{\ell+1}xy_n$. By applying the induction hypothesis, 3.1), (3.2) and (3.4), we can conclude this cycle has no chords, contradicting the fact that $\operatorname{reg}(G) = 3$. This settles (3.3).

Then, by using (3.1), (3.2) and (3.3), we can find the $2(n-1)$ induced cycle $x_1y_2x_3y_3 \cdots x_ny_n$ in G^{bc}, contradicting Proposition 2.13 and the fact that $\operatorname{reg}(G) = 3$.

Now suppose that $s > 1$ and our claim holds for each $t < s$. Then, by Lemma 3.4 and induction, we have $\operatorname{reg}(G') \leq 3$.

If $\operatorname{reg}(G') = 2$, then, from Corollary 3.3 and Lemma 3.4,

$$\operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) = \operatorname{reg}(I(G')^s : e_2 \cdots e_s) = 2.$$

Also, if $\operatorname{reg}(G') = 3$, from the induction hypothesis and Lemma 3.4, we have

$$\operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) = \operatorname{reg}(I(G')^s : e_2 \cdots e_s) \leq 3. \qquad \square$$

Theorem 3.6. *Let G be a bipartite connected graph with edge ideal $I(G)$. If $\operatorname{reg}(G) = 3$, then, for all $s \geq 1$, $\operatorname{reg}(I(G)^s) = 2s+1$.*

Proof. The proof is by induction on s. The case $s = 1$ is true by assumption. Suppose that $\operatorname{reg}(I(G)^s) = 2s+1$. We will show that $\operatorname{reg}(I(G)^{s+1}) = 2s+3$.

Note that, from Theorem 3.5 and Theorem 2.14, we have

$$\operatorname{reg}(I(G))^{s+1} \leq 2s+3.$$

On the other hand, since $\operatorname{reg}(I(G))^{s+1} \geq 2s+2$, if $\operatorname{reg}(I(G))^{s+1} < 2s+3$, then we have $\operatorname{reg}(I(G))^{s+1} = 2s+2$, or, in other words, $\operatorname{reg}(I(G))^{s+1}$ has a linear minimal free resolution.

By applying Theorem 2.10, we can conclude that $I(G)$ has a linear presentation, and since G is bipartite from Proposition 2.18, it has a linear minimal free resolution. Thus, $\operatorname{reg}(G) = 2$, a contradiction. $\square$

Concluding Remark 3.7. In recent work, Beyarslan, Há and Trung answered Question 1.1 for forests and cycles in [**3**]. Francisco, Há and Van Tuyl ([**11**]) showed that, if any power of an edge ideal has linear minimal free resolution, then the complement of the corresponding graph has no induced four cycles, which is equivalent to having a linear presentation ([**14**]). In light of these, and based on the Macaulay 2 calculations by Francisco, the following question was asked by Nevo and Peeva, which is the base case of [**14,** Open Problem 1.11 (2)].

Question 3.8. *Let $I(G)$ be the edge ideal of a graph G that does not have any induced four cycle in its complement. If* $\operatorname{reg}(I(G)) \leq 3$, *then is it true that, for all $s \geq 2$, $I(G)^s$ has linear minimal free resolution?*

One important fact regarding bipartite graphs is that the complement of a bipartite graph cannot have any induced cycle of length > 4. In light of Fröberg's theorem and this fact, it can be said that, for the bipartite graphs, a linear presentation implies a linear resolution, in which case, the $\operatorname{reg}((I(G)) \leq 3$ condition becomes redundant. However, Theorem 3.5 provides some evidence that the $\operatorname{reg}(I(G)) \leq 3$ condition is interesting in its own right. While powers of edge ideals with regularity 3 may not have linear resolutions in general, they may very well have regularity which shows a nice pattern.

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