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# SIMPLE POLYOMINOES ARE PRIME

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ABSTRACT. In this paper, we show that the polyomino ideal of a simple polyomino coincides with the toric ideal of a weakly chordal bipartite graph, and hence, it has a quadratic Gröbner basis with respect to a suitable monomial order.

**Introduction.** Polyominoes are two-dimensional objects which originated in recreational mathematics and combinatorics. They have been widely discussed in connection with tiling problems of the plane. Typically, a polyomino is plane figure obtained by joining squares of equal sizes, which are known as *cells*. In connection with commutative algebra, polyominoes were first discussed in **[8]** by assigning each polyomino the ideal of its inner 2-minors or the *polyomino ideal*. The study of the ideal of $t$-minors of an $m \times n$ matrix is a classical subject in commutative algebra. The class of polyomino ideals widely generalizes the class of ideals of 2-minors of $m \times n$ matrix as well as the ideal of inner 2-minors attached to a two-sided ladder.

Let $\mathcal{P}$ be a polyomino and $K$ a field. We denote by $I_{\mathcal{P}}$ the polyomino ideal attached to $\mathcal{P}$, in a suitable polynomial ring over $K$. The residue class ring defined by $I_{\mathcal{P}}$ is denoted by $K[\mathcal{P}]$. It is natural to investigate the algebraic properties of $I_{\mathcal{P}}$ depending upon the shape of $\mathcal{P}$. In **[8]**, it was shown that, for a convex polyomino, the residue ring $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain. More generally, it was also shown that polyomino ideals attached to a row or column convex polyomino are also prime ideals. Later, in **[2]**, a classification of the convex polyominoes whose polyomino ideals are linearly related was given. For some special classes of polyominoes, the regularity of polyomino ideals is discussed in **[3]**.

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In [**8**], it was conjectured that a polyomino ideal attached to a simple polyomino is prime ideal. Roughly speaking, a *simple polyomino* is a polyomino without 'holes.' This conjecture was further studied in [**5**], where the authors introduced *balanced* polyominoes and proved that polyomino ideals attached to balanced polyominoes are prime. They expected that all simple polyominoes are balanced, which would then prove that simple polyominoes are prime. This question was further discussed in [**4**], where the authors proved that balanced and simple polyominoes are equivalent. Independent of the proofs given in [**4**], in this paper, we show, by using simpler arguments, that simple polyominoes are prime by identifying the attached residue class ring $K[\mathcal{P}]$ with the edge rings of weakly chordal graphs. This identification is profitable because it allows us to benefit from the result of [**6**] which states that the toric ideal of the edge ring of a weakly chordal bipartite graph has a quadratic Gröbner basis with respect to a suitable monomial order, which implies that $K[\mathcal{P}]$ is Koszul.

**1. Polyominoes and polyomino ideals.** First, we recall some definitions and notation from [**8**]. Given $a = (i, j)$ and $b = (k, l)$ in $\mathbb{N}^2$, we write $a \leq b$ if $i \leq k$ and $j \leq l$. The set

$$[a, b] = \{c \in \mathbb{N}^2 : a \leq c \leq b\}$$

is called an *interval*. If $i < k$ and $j < l$, then the elements $a$ and $b$ are called *diagonal* corners, and $(i, l)$ and $(k, j)$ are called *anti-diagonal* corners of $[a, b]$. An interval of the form $C = [a, a + (1, 1)]$ is called a *cell* (with left lower corner $a$). The elements (corners) $a, a + (0, 1), a + (1, 0), a + (1, 1)$ of $[a, a + (1, 1)]$ are called the *vertices* of $C$. The sets $\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}$ and $\{a + (0, 1), a + (1, 1)\}$ are called the *edges* of $C$. We denote the set of edges of $C$ by $E(C)$.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$. The vertex set of $\mathcal{P}$, denoted by $V(\mathcal{P})$ is given by

$$V(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} V(C).$$

The edge set of $\mathcal{P}$, denoted by $E(\mathcal{P})$ is given by

$$E(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} E(C).$$

Let $C$ and $D$ be two cells of $\mathcal{P}$. Then $C$ and $D$ are said to be *connected*, if there is a sequence of cells $\mathcal{C} : C = C_1, \ldots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of $C_i$ for $i = 1, \ldots, m-1$. If, in addition, $C_i \neq C_j$ for all $i \neq j$, then $\mathcal{C}$ is called a *path* (connecting $C$ and $D$). The collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected, see Figure 1.

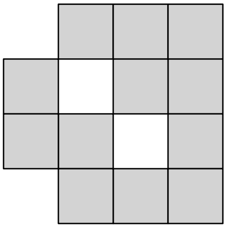

FIGURE 1. Polyomino.

Let $\mathcal{P}$ be a polyomino, and let $K$ be a field. We denote by $S$ the polynomial ring over $K$ with variables $x_{ij}$ with $(i,j) \in V(\mathcal{P})$. Following **[8]**, a 2-minor $x_{ij}x_{kl} - x_{il}x_{kj} \in S$ is called an *inner minor* of $\mathcal{P}$ if all the cells $[(r,s),(r+1,s+1)]$ with $i \leq r \leq k-1$ and $j \leq s \leq l-1$ belong to $\mathcal{P}$. In this case, the interval $[(i,j),(k,l)]$ is called an *inner interval* of $\mathcal{P}$. The ideal $I_{\mathcal{P}} \subset S$ generated by all inner minors of $\mathcal{P}$ is called the *polyomino ideal* of $\mathcal{P}$. We also set $K[\mathcal{P}] = S/I_{\mathcal{P}}$.

Let $\mathcal{P}$ be a polyomino. Following **[5]**, an interval $[a,b]$ with $a = (i,j)$ and $b = (k,l)$ is called a *horizontal edge interval* of $\mathcal{P}$ if $j = l$ and the sets $\{(r,j),(r+1,j\}$ for $r = i, \ldots, k-1$ are edges of cells of $\mathcal{P}$. If a horizontal edge interval of $\mathcal{P}$ is not strictly contained in any other horizontal edge interval of $\mathcal{P}$, then we call it a *maximal* horizontal edge interval. Vertical edge intervals and maximal vertical edge intervals of $\mathcal{P}$ are similarly defined.

Let $\{V_1, \ldots, V_m\}$ be the set of maximal vertical edge intervals and $\{H_1, \ldots, H_n\}$ the set of maximal horizontal edge intervals of $\mathcal{P}$. We denote by $G(\mathcal{P})$ the associated bipartite graph of $\mathcal{P}$ with vertex set

$$\{v_1, \ldots, v_m\} \bigsqcup \{h_1, \ldots, h_n\},$$

and the edge set is defined as:

$$E(G(\mathcal{P})) = \{\{v_i, h_j\} \mid V_i \cap H_j \in V(\mathcal{P})\}.$$

**Example 1.1.** Figure 2 shows a polyomino $\mathcal{P}$ with maximal vertical and maximal horizontal edge intervals labeled as $\{V_1, \dots, V_4\}$ and $\{H_1, \dots, H_4\}$, respectively, and Figure 3 shows the associated bipartite graph $G(\mathcal{P})$ of $\mathcal{P}$.

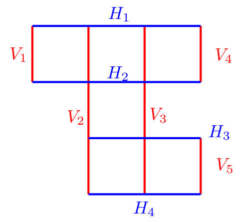


FIGURE 2. Maximal intervals of $\mathcal{P}$.

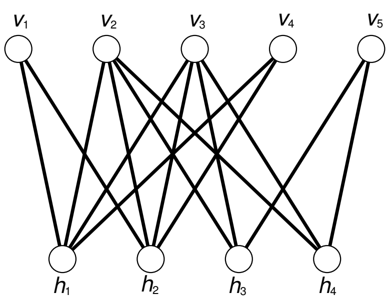


FIGURE 3. Associated bipartite graph of $\mathcal{P}$.

Let $S$ be the polynomial ring over field $K$ with variables $x_{ij}$ with $(i,j) \in V(\mathcal{P})$. Note that $|V_p \cap H_q| \leq 1$. If $V_p \cap H_q = \{(i,j)\}$, then we may write $x_{ij} = x_{V_p \cap H_q}$, when required. To each cycle $\mathcal{C} : v_{i_1}, h_{j_1}, v_{i_2}, h_{j_2}, \dots, v_{i_r}, h_{j_r}$ in $G(\mathcal{P})$, we associate a binomial in $S$ given by

$$f_{\mathcal{C}} = x_{V_{i_1} \cap H_{j_1}} \cdots x_{V_{i_r} \cap H_{j_r}} - x_{V_{i_2} \cap H_{j_1}} \cdots x_{V_{i_1} \cap H_{j_r}}.$$

We recall the definition of a cycle in $\mathcal{P}$ from [**5**]. A sequence of vertices $\mathcal{C}_\mathcal{P} = a_1, a_2, \ldots, a_m$ in $V(\mathcal{P})$ with $a_m = a_1$ and such that $a_i \neq a_j$ for all $1 \leq i < j \leq m-1$ is a called a *cycle* in $\mathcal{P}$ if the following conditions hold:

(i) $[a_i, a_{i+1}]$ is a horizontal or vertical edge interval of $\mathcal{P}$ for all $i = 1, \ldots, m-1$;
(ii) for $i = 1, \ldots, m$ we have: if $[a_i, a_{i+1}]$ is a horizontal edge interval of $\mathcal{P}$, then $[a_{i+1}, a_{i+2}]$ is a vertical edge interval of $\mathcal{P}$ and vice versa. Here, $a_{m+1} = a_2$.

We set $V(\mathcal{C}_\mathcal{P}) = \{a_1, \ldots, a_m\}$. Given a cycle $\mathcal{C}_\mathcal{P}$ in $\mathcal{P}$, we attach to $\mathcal{C}_\mathcal{P}$ the binomial

$$f_{\mathcal{C}_\mathcal{P}} = \prod_{i=1}^{(m-1)/2} x_{a_{2i-1}} - \prod_{i=1}^{(m-1)/2} x_{a_{2i}}.$$

Moreover, we call a cycle in $\mathcal{P}$ *primitive* if each maximal interval of $\mathcal{P}$ contains at most two vertices of $\mathcal{C}_\mathcal{P}$.

Note that, if $\mathcal{C} : v_{i_1}, h_{j_1}, v_{i_2}, h_{j_2}, \ldots, v_{i_r}, h_{j_r}$ defines a cycle in $G(\mathcal{P})$, then the sequence of vertices $\mathcal{C}_\mathcal{P} : V_{i_1} \cap H_{j_1}, V_{i_2} \cap H_{j_1}, V_{i_2} \cap H_{j_2}, \ldots, V_{i_r} \cap H_{j_r}, V_{i_1} \cap H_{j_r}$ is a primitive cycle in $\mathcal{P}$ and vice versa. Also, $f_C = f_{\mathcal{C}_\mathcal{P}}$.

We set

$$K[G(\mathcal{P})] = K[v_p h_q \mid \{p, q\} \in E(G(\mathcal{P}))] \subset T = K[v_1, \ldots, v_m, h_1, \ldots, h_n].$$

The subalgebra $K[G(\mathcal{P})]$ is called the *edge ring* of $G(\mathcal{P})$. Let $\varphi : S \to T$ be the surjective $K$-algebra homomorphism defined by $\varphi(x_{ij}) = v_p h_q$, where $\{(i,j)\} = V_p \cap H_q$. We denote the toric ideal of $K[G(\mathcal{P})]$ by $J_\mathcal{P}$. It is known from [**7**] that $J_\mathcal{P}$ is generated by the binomials associated with cycles in $G(\mathcal{P})$.

**2. Simple polyominoes are prime.** Let $\mathcal{P}$ be a polyomino, and let $[a, b]$ an interval with the property that $\mathcal{P} \subset [a, b]$. According to [**8**], a polyomino $\mathcal{P}$ is called *simple* if, for any cell $C$ not belonging to $\mathcal{P}$, there exists a path $C = C_1, C_2, \ldots, C_m = D$ with $C_i \notin \mathcal{P}$ for $i = 1, \ldots, m$ and such that $D$ is not a cell of $[a, b]$. For example, the polyomino illustrated in Figure 1 is not simple, but that in Figure 4 is simple. It is conjectured in [**8**] that $I_\mathcal{P}$ is a prime ideal if $\mathcal{P}$ is simple.

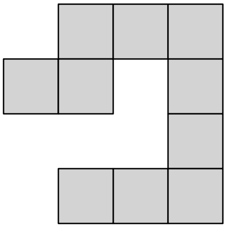

FIGURE 4. Simple polyomino.

We recall from graph theory that a graph is called *weakly chordal* if every cycle of length greater than 4 has a chord. In order to prove following lemma, we define some terms. We say that a cycle $\mathcal{C}_{\mathcal{P}}$ : $a_1, a_2, \ldots, a_m$ in $\mathcal{P}$ with $a_m = a_1$ has a self crossing if there exist indices $i$ and $j$ such that $a_i, a_{i+1} \in V_k$ and $a_j, a_{j+1} \in H_l$ and $a_i, a_{i+1}, a_j, a_{j+1}$ are all distinct and $V_k \cap H_l \neq \emptyset$. In this situation, if $\mathcal{C}$ is the associated cycle in $G(\mathcal{P})$, then it also shows that $\{v_k, h_l\} \in E(G(\mathcal{P}))$ which gives us a chord in $\mathcal{C}$.

Let $\mathcal{C}_{\mathcal{P}} : a_1, a_2, \ldots, a_r$ be a cycle in $\mathcal{P}$ which does not have any self crossing. Then we call the area bounded by edge intervals $[a_i, a_{i+1}]$ and $[a_r, a_1]$ for $i \in \{1, r-1\}$, the *interior* of $\mathcal{C}_{\mathcal{P}}$. Moreover, we call a cell $C$ is an *interior* cell of $\mathcal{C}_{\mathcal{P}}$ if $C$ belongs to the interior of $\mathcal{C}_{\mathcal{P}}$.

**Lemma 2.1.** *Let $\mathcal{P}$ be a simple polyomino. Then the graph $G(\mathcal{P})$ is weakly chordal.*

*Proof.* Let $\mathcal{C}$ be a cycle of $G(\mathcal{P})$ of length $2n$ with $n \geq 3$, and let $\mathcal{C}_{\mathcal{P}}$ be the associated primitive cycle in $\mathcal{P}$. We may assume that $\mathcal{C}_{\mathcal{P}}$ does not have any 1. Otherwise, by following the definition of self crossing, we know that $\mathcal{C}$ has a chord.

Let

$$\mathcal{C} : v_{i_1}, h_{j_1}, v_{i_2}, h_{j_2}, \ldots, v_{i_r}, h_{j_r}$$

and

$$\mathcal{C}_{\mathcal{P}} : V_{i_1} \cap H_{j_1}, V_{i_2} \cap H_{j_1}, V_{i_2} \cap H_{j_2}, \ldots, V_{i_r} \cap H_{j_r}, V_{i_1} \cap H_{j_r}.$$

We may write $a_1 = V_{i_1} \cap H_{j_1}$, $a_2 = V_{i_2} \cap H_{j_1}$, $a_3 = V_{i_2} \cap H_{j_2}, \ldots, a_{2r-1} = V_{i_r} \cap H_{j_r}$, $a_{2r} = V_{i_1} \cap H_{j_r}$. Also, we may assume that $a_1$ and $a_2$ belong to the same maximal horizontal edge interval. Then, $a_{2r}$ and $a_1$ belong to the same maximal vertical edge interval.

First, we show that every interior cell of $\mathcal{C}_{\mathcal{P}}$ belongs to $\mathcal{P}$. Suppose that we have an interior cell $C$ of $\mathcal{C}_{\mathcal{P}}$ which does not belong to $\mathcal{P}$. Let $\mathcal{J}$ be any interval such that $\mathcal{P} \subset \mathcal{J}$. Then, by using the definition of simple polyomino, we obtain a path of cells $C = C_1, C_2, \ldots, C_t$ with $C_i \notin P$, $i = 1, \ldots t$ and $C_t$ a boundary cell in $\mathcal{J}$. This shows that

$$V(C_1) \cup V(C_2) \cup \ldots \cup V(C_t)$$

intersects at least one of $[a_i, a_{i+1}]$ for $i \in \{1, \ldots, r-1\}$ or $[a_r, a_1]$, which is not possible since $\mathcal{C}_{\mathcal{P}}$ is a cycle in $\mathcal{P}$. Hence, $C \in \mathcal{P}$. This shows that an interval in the interior of $\mathcal{C}_{\mathcal{P}}$ is an inner interval of $\mathcal{P}$.

Let $\mathcal{I}$ be the maximal inner interval of $\mathcal{C}_{\mathcal{P}}$ to which $a_1$ and $a_2$ belong, and let $b$ and $c$ be the corner vertices of $\mathcal{I}$. We may assume that $a_1$ and $c$ are the diagonal corners and $a_2$ and $b$ are the anti-diagonal corners of $\mathcal{I}$. If $b, c \in V(\mathcal{C}_{\mathcal{P}})$, then primitivity of $\mathcal{C}$ implies that $\mathcal{C}$ is a cycle of length 4. We may assume that $b \notin V(\mathcal{C}_{\mathcal{P}})$. Let $H'$ be the maximal horizontal edge interval which contains $b$ and $c$. The maximality of $\mathcal{I}$ implies that $H' \cap V(\mathcal{C}_{\mathcal{P}}) \neq \emptyset$. For example, see Figure 5. Therefore, $\{v_{i_1}, h'\}$ is a chord in $\mathcal{C}$, as desired. □

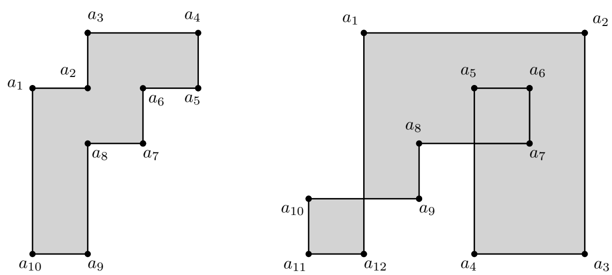


FIGURE 5. Self crossings.

**Theorem 2.2.** *Let $\mathcal{P}$ be a simple polyomino. Then, $I_{\mathcal{P}} = J_{\mathcal{P}}$.*

*Proof.* First we show that $I_{\mathcal{P}} \subset J_{\mathcal{P}}$. Let $f = x_{ij}x_{kl} - x_{il}x_{kj} \in I_{\mathcal{P}}$. Then, maximal vertical edge intervals $V_p$ and $V_q$ and maximal

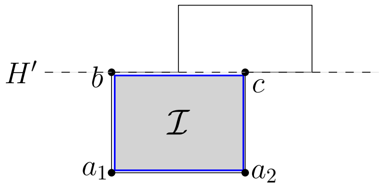


FIGURE 6. When $b, c \in V(\mathcal{C}_{\mathcal{P}})$.

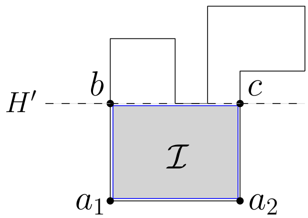


FIGURE 7. When $b \notin V(\mathcal{C}_{\mathcal{P}})$.

horizontal edge intervals $H_m$ and $H_n$ of $\mathcal{P}$ exist such that $(i,j),(i,l) \in V_p$, $(k,j),(k,l) \in V_q$ and $(i,j),(k,j) \in H_m$, $(i,l),(k,l) \in H_n$. This gives that

$$\phi(x_{ij}x_{kl}) = v_p h_m h_n v_q = \phi(x_{il}x_{kj}),$$

which implies that $f \in J_P$.

Next, we show that $J_{\mathcal{P}} \subset I_{\mathcal{P}}$. It is known from **[6, 7]** that the toric ideal of the weakly chordal bipartite graph is minimally generated by quadratic binomials associated with cycles of length 4. It suffices to show that $f_{\mathcal{C}} \in I_P$, where $\mathcal{C}$ is a cycle of length 4 in $G(\mathcal{P})$.

Let $\mathcal{I}$ be an interval such that $\mathcal{P} \subset \mathcal{I}$. Let $\mathcal{C} : h_1, v_1, h_2, v_2$. Then, $\mathcal{C}_{\mathcal{P}} : a_{11} = H_1 \cap V_1$, $a_{21} = H_2 \cap V_1$, $a_{22} = H_2 \cap V_2$ and $a_{12} = H_1 \cap V_2$ is the associated cycle in $\mathcal{P}$ which also determines an interval in $\mathcal{I}$. Let $a_{11}$ and $a_{22}$ be the diagonal corners of this interval. We must show that $[a_{11}, a_{22}]$ is an inner interval in $\mathcal{P}$. Assume that $[a_{11}, a_{22}]$ is not an inner interval of $\mathcal{P}$, that is, a cell $C \in [a_{11}, a_{22}]$ exists which does not belong to $\mathcal{P}$. Using the fact that $\mathcal{P}$ is a simple polyomino, we obtain a path of cells $C = C_1, C_2, \ldots, C_r$ with $C_i \notin \mathcal{P}$, $i = 1, \ldots, r$ and $C_r$ a cell in $\mathcal{I}$. Then, $V(C_1 \cup \ldots \cup C_r)$ intersects at least one of the maximal intervals

$H_1, H_2, V_1, V_2$, say $H_1$, which contradicts the fact that $H_1$ is an interval in $\mathcal{P}$. Hence, $[a_{11}, a_{22}]$ is an inner interval of $\mathcal{P}$ and $f_{\mathcal{C}} \in I_{\mathcal{P}}$. □

**Corollary 2.3.** *Let $\mathcal{P}$ be a simple polyomino. Then $K[\mathcal{P}]$ is Koszul and a normal Cohen-Macaulay domain.*

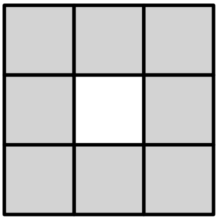

FIGURE 8. Polyomino with prime polyomino ideal.

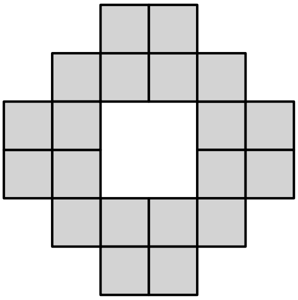

FIGURE 9. Polyomino with "one hole."

*Proof.* From **[6]**, we know that $J_{\mathcal{P}} = I_{\mathcal{P}}$ has a squarefree quadratic Gröbner basis with respect to a suitable monomial order. Hence, $K[\mathcal{P}]$ is Koszul. By a theorem of Sturmfels **[9]**, we obtain that $K[\mathcal{P}]$ is normal and then, following a theorem of Hochster **[1,** Theorem 6.3.5], we obtain that $K[\mathcal{P}]$ is Cohen-Macaulay. □

A polyomino ideal may be prime even if the polyomino is not simple. The polyomino ideal attached to the polyomino in Figure 8 is prime.

However, the polyomino ideal attached to the polyomino attached in Figure 9 is not prime. It would be interesting to find a complete characterization of polyominoes whose attached polyomino ideals are prime, but it is not easy. However, as a first step, it is already an interesting question to classify polyominoes with only "one hole" such that their associated polyomino ideal is prime.

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