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LATTICE-ORDERED ABELIAN GROUPS FINITELY GENERATED AS SEMIRINGS

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ABSTRACT. A lattice-ordered group (an ℓ-group) $G(\oplus, \vee, \wedge)$ can naturally be viewed as a semiring $G(\vee, \oplus)$. We give a full classification of (abelian) ℓ-groups which are finitely generated as semirings by first showing that each such ℓ-group has an order-unit so that we can use the results of Busaniche, Cabrer and Mundici [**8**]. Then, we carefully analyze their construction in our setting to obtain the classification in terms of certain ℓ-groups associated to rooted trees (Theorem 4.1).

This classification result has a number of interesting applications; for example, it implies a classification of finitely generated ideal-simple (commutative) semirings $S(+, \cdot)$ with idempotent addition and provides important information concerning the structure of general finitely generated ideal-simple (commutative) semirings, useful in obtaining further progress towards Conjecture 1.1 discussed in [**2, 15**].

1. Introduction. Lattice-ordered groups (or ℓ-groups for short) have long played an important role in algebra and related areas of mathematics. We briefly mention their relation to functional analysis and logic via correspondence with MV-algebras [**23, 24**], or the fact that the theory of factorization and divisibility on a Bézout domain yields an abelian ℓ-group. For this and further applications, see e.g., [**1, 13**]; the connections to Bézout domains were recently studied in detail by Yang [**27**].

Recently, there have been several interesting results concerning unital ℓ-groups. For example, Busaniche, Cabrer and Mundici [**8**] classified finitely generated unital (abelian) ℓ-groups G using the combinatorial

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notion of a stellar sequence, which is a sequence $|\Delta_0| \supset |\Delta_1| \supset \cdots$ of certain simplicial complexes in $[0,1]^n$. The idea is that each such G be of the form $G \simeq \mathcal{M}([0,1]^n)/I$, where $\mathcal{M}([0,1]^n)$ is the ℓ-group of all piecewise linear functions $f : [0,1]^n \to \mathbb{R}$ and I is the set of all functions f such that $f(|\Delta_i|) = 0$ for some i.

The aim of this paper is to explore and use the connections between semirings and ℓ-groups in the study of simple semirings, namely, an ℓ-group $G(\oplus, \vee, \wedge)$ is also a semiring $G(\vee, \oplus) = S(+, \cdot)$ such that the semiring addition $+$ is idempotent. By removing the idempotency condition, the notion of a parasemifield is obtained, i.e., a commutative semiring $S(+, \cdot)$ such that its multiplicative structure forms a group. (See the beginning of Sections 2 and 3 for precise definitions of the notions concerning ℓ-groups and semirings, respectively.)

In fact, it is easy to observe that there is a term-equivalence between lattice-ordered groups and additively idempotent parasemifields, i.e., satisfying $a + a = a$ for all a. In particular, this equivalence preserves finite generation in the sense that an ℓ-group is finitely generated if and only if it is finitely generated as a parasemifield. However, these are not equivalent to the property of being finitely generated as a semiring, which is stronger.

We shall assume all ℓ-groups and semirings to be automatically commutative, as we will be dealing only with these throughout the paper.

Our first result is Theorem 3.7 in which we show that every additively idempotent parasemifield, finitely generated as a semiring, is unital in the ℓ-group sense. Hence, it is natural to inquire whether we can identify those which are finitely generated as semirings among the unital ℓ-groups from the classification [**8**]. The answer is positive, although the proof is fairly involved and requires a careful discussion of the geometry of stellar sequences. The resulting Theorem 4.1 classifies all additively idempotent parasemifields which are finitely generated as semirings.

While this seems to be the first paper systematically to study semirings from the perspective of ℓ-groups and MV-algebras and to apply the strong classification results available therein to semirings, there is a long and fruitful tradition of proceeding in the other direction, namely, of attaching a semiring to an MV-algebra (note that MV-algebras are equivalent to unital ℓ-groups via the Mundici functor [**23**]). This was begun by Di Nola and Gerla [**10**], who defined an MV-semiring attached to an MV-algebra. Belluce and Di Nola [**4**] simplified it to an equivalent definition of MV-semirings. These two authors and Ferraioli [**5**] then established a categorical equivalence between MV-algebras and MV-semirings and used it to obtain a representation of MV-algebras as certain spaces of continuous functions via a corresponding representation of MV-semirings. The same authors [**6**] then very recently continued in the study of (prime) ideals of MV-semirings. We also note that there is a wealth of other interesting representation results for MV-algebras, see e.g., [**3, 7, 12**]. Finally, we remark that Belluce and Di Nola [**4**] have also established a connection to ring theory by studying a class of "Łukasiewicz rings," which are defined as those rings whose semirings of ideals form an MV-algebra.

Given the basic and fundamental nature of the notion of a semiring, it is not surprising that there is a wide variety of other applications of semirings and semifields, ranging from cryptography and other areas of computer science to dequantization, tropical mathematics and geometry; see, for example, [**14, 21, 22, 28**] for overviews of some of the applications and for further references.

Many parts of the structural theory of semirings and semifields mimic analogous results concerning rings and fields, see, e.g., [**14**]. However, much less is known overall; for instance, whereas simple commutative rings are merely fields and are very explicitly known, the analogous results for semirings are more subtle. First of all, one must distinguish between congruence-simple and ideal-simple semirings. Bashir, Hurt, Jančařík and Kepka [**2**] classified the congruence-simple ones and reduced the study of ideal-simple semirings to the study of parasemifields.

Together with their results, our Theorem 4.1 implies a full classification of additively idempotent finitely generated ideal-simple semirings. The structure of this classification follows [**2,** Theorem 11.2 and Section 12], but it is fairly technical, so we don't state the final result explicitly.

We have already mentioned that additively idempotent parasemifields are term-equivalent to ℓ-groups; the present Theorem 4.1 classifies those which are finitely generated semirings. A natural question to ask then is, "what is the structure of such parasemifields without the idempotency assumption?" Note that the corresponding result concerning rings is that, if a field is finitely generated as a ring, then it is finite.

There are no finite parasemifields and, in fact, we have the following conjecture:

Conjecture 1.1 ([**2, 15**]). *Every parasemifield which is finitely generated as a semiring is additively idempotent.*

Ježek, Kala and Kepka [**15**] proved this in the case of at most two generators by studying the geometry of semigroups $\mathcal{C}(S) \subset \mathbb{N}_0^2$ attached to parasemifields S. (For the definition and basic information on the semigroups $\mathcal{C}(S)$, see Section 3.) Since each parasemifield S has an additively idempotent factor $S/\sim$ such that the semigroup $\mathcal{C}(S)$ is equal to $\mathcal{C}(S/\sim)$, Theorem 4.1 may be used to obtain refined information on the structure of the semigroup $\mathcal{C}(S) \subset \mathbb{N}_0^m$, in general.

In a work in progress [**20**], the author and Korbelář use this to prove Conjecture 1.1. Our Theorem 4.1 then provides all parasemifields, finitely generated as a semiring, and hence, again using the results of [**2**], implies a complete classification of finitely generated ideal-simple semirings, see [**17**] for some details and Example 3.4 of the present paper.

There are various natural ways of extending and generalizing the classification of finitely generated unital ℓ-groups [**8**]. We only mention the cases of ℓ-groups which are not assumed to be unital, of finitely generated parasemifields, or even of non-commutative finitely generated parasemifields. To the author's knowledge, very little is known about any of these interesting problems.

As for the contents of this paper, Section 2 reviews the definitions and basic facts on ℓ-groups, including the statement of the classification of finitely generated unital ones (and the required notions concerning simplicial and abstract complexes). Then, in Section 3, we briefly review some preliminaries on semirings and parasemifields and prove that, if an additively idempotent parasemifield is finitely generated as a semiring, then it is unital. For the sake of completeness we outline the proofs of some classical results concerning semirings that we need.

In Section 4 we then give the classification of additively idempotent parasemifields, finitely generated as semirings.

2. ℓ-groups and complexes. In this section, we briefly review some basics about ℓ-groups and simplicial complexes that we will need, including the classification of Busaniche, Cabrer and Mundici [**8**]. Our outline is quite terse, but we at least also provide a brief (very) informal overview at the end of this section. For a more detailed treatment we refer the reader to [**8**]. Also see [**9**], where rational polyhedra are used in the study of projective unital ℓ-groups. For more general background information on ℓ-groups, see, for example, [**1, 13**].

A *lattice-ordered abelian group* (*ℓ-group* for short) $G(+,-,0,\vee,\wedge)$ is an algebraic structure such that $G(+,-,0)$ is an abelian group, $G(\vee,\wedge)$ is a lattice and $a+(b\vee c)=(a+b)\vee(a+c)$ for all $a,b,c\in G$.

An *order-unit* $u\in G$ is an element such that, for each $g\in G$, there exists an $n\in\mathbb{N}$ so that $nu\geq g$, i.e., $nu\vee g=nu$. A *unital ℓ-group* (G,u) is an ℓ-group with an order-unit u. A *unital ℓ-homomorphism* is a homomorphism of ℓ-groups which maps one order-unit to the other one. An *ℓ-ideal* is the kernel of a unital ℓ-homomorphism; any ℓ-ideal I then determines the factor-homomorphism $G\to G/I$.

We now review the classification of [**8**]. Denote by $\mathcal{M}([0,1]^n)$ the set of piecewise linear continuous functions $f:[0,1]^n\to\mathbb{R}$ such that each piece has integral coefficients (and the number of pieces is finite). $\mathcal{M}([0,1]^n)$ is a group under pointwise addition of functions, and we can define $(f\vee g)(x)=\max(f(x),g(x))$ and $(f\wedge g)(x)=\min(f(x),g(x))$ for $f,g\in\mathcal{M}([0,1]^n)$. This makes $\mathcal{M}([0,1]^n)$ an ℓ-group with the constant function 1 being an order-unit. Note that $\mathcal{M}([0,1]^n)$ is (finitely) generated (as an ℓ-group) by the constant function 1 and projections on the ith coordinate $\pi_i:[0,1]^n\to\mathbb{R}$ (but it is not finitely generated as a semiring, as we shall see in Corollary 4.6). Also, for $D\subset[0,1]^n$, we define $\mathcal{M}(D)$ as the ℓ-group whose elements are restrictions $f|D$ of functions $f\in\mathcal{M}([0,1]^n)$ to D. Thus, $\mathcal{M}(D)$ is a factor of $\mathcal{M}([0,1]^n)$.

The classification then states that each finitely generated unital ℓ-group is of the form $\mathcal{M}([0,1]^n)/I$ for an explicitly defined ℓ-ideal I (and provides a criterion for when two ideals give the same ℓ-group). The ideal I comes from a stellar sequence $\mathcal{W}$ of simplicial complexes as follows: from $\mathcal{W}$, we construct a sequence $\mathcal{P}_0\supset\mathcal{P}_1\supset\mathcal{P}_2\supset\cdots$

of polyhedra in $[0,1]^n$ and define $I = \{f \in \mathcal{M}([0,1]^n) \mid f(P_i) = 0$ for some $i\}$. In order to give more details we first need to give some definitions concerning (abstract) simplicial complexes, following **[8]**.

We assume the reader is familiar with the usual notion of a (simplicial) complex in $\mathbb{R}^n$. Let us just note that a *simplex* is the convex hull of a finite set of points, a *k-simplex* is a simplex of dimension k, a *complex* $\mathcal{K}$ is a finite set of simplices such that, if T_1, T_2 are simplices with $\dim T_1 = \dim T_2 - 1$, $T_1 \subset \partial T_2$, and $T_2 \in \mathcal{K}$, then also $T_1 \in \mathcal{K}$ (where, by ∂T, we denote the boundary of T). The *support* $|\mathcal{K}|$ of a complex $\mathcal{K}$ is the union of all simplices in $\mathcal{K}$. Throughout this paper, we shall often identify a complex with its support. A simplex $\operatorname{conv}(v_0, \ldots, v_k)$ is *rational* if all of the coordinates of all vertices v_i are rational. A complex is *rational* if all its simplices are rational. For more background information on simplicial complexes, see for example, **[11]**.

Definition 2.1 ([**8,** page 262])**.** A (finite) *abstract simplicial complex* is a pair $H = (\mathcal{V}, \Sigma)$, where $\mathcal{V}$ is a non-empty finite set of vertices of H and Σ is a collection of subsets of $\mathcal{V}$ whose union is $\mathcal{V}$ with the property that every subset of an element of Σ is again an element of Σ. Given $\{v, w\} \in \Sigma$ and $a \notin \mathcal{V}$, we define the *binary subdivision* $(\{v, w\}, a)$ of H as the abstract simplicial complex $(\{v, w\}, a)H$ obtained by adding a to the vertex set and replacing every set $\{v, w, u_1, \ldots, u_t\} \in \Sigma$ by the two sets $\{v, a, u_1, \ldots, u_t\}$ and $\{a, w, u_1, \ldots, u_t\}$ and all their subsets.

A *weighted abstract simplicial complex* is a triple $W = (\mathcal{V}, \Sigma, \omega)$ where $(\mathcal{V}, \Sigma)$ is an abstract simplicial complex and ω is a map of $\mathcal{V}$ into $\mathbb{N}$. For $\{v, w\} \in \Sigma$ and $a \notin \mathcal{V}$, the *binary subdivision* $(\{v, w\}, a)W$ is the abstract simplicial complex $(\{v, w\}, a)(\mathcal{V}, \Sigma)$ equipped with the weight function $\widetilde{\omega} : \mathcal{V} \cup \{a\} \to \mathbb{N}$ given by $\widetilde{\omega}(a) = \widetilde{\omega}(v) + \widetilde{\omega}(w)$ and $\widetilde{\omega}(u) = \omega(u)$ for all $u \in \mathcal{V}$.

Definition 2.2 ([**8,** page 264])**.** Let $W = (\mathcal{V}, \Sigma, \omega)$ and W' be two weighted abstract simplicial complexes. A map $\mathfrak{b} : W \to W'$ is a *stellar transformation* if it is either a deletion of a maximal set of Σ or a binary subdivision or the identity map.

A sequence $\mathcal{W} = (W_0, W_1, W_2, \ldots)$ of weighted abstract simplicial complexes is *stellar* if W_{i+1} is obtained from W_i by a stellar transformation.

Definition 2.3 ([**8**], page 263)**.** Now, let $W = (\mathcal{V}, \Sigma, \omega)$ be an abstract simplicial complex with the set of vertices $V = \{v_1, \ldots, v_n\}$. Choose the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, and let Δ_W be the complex whose vertices are $v'_1 = e_1/\omega(v_1), \ldots, v'_n = e_1/\omega(v_n)$ and whose k-dimensional simplices are given by $\mathrm{conv}(v'_{i(0)}, \ldots, v'_{i(k)}) \in \Delta_W$ if and only if $\{v_{i(0)}, \ldots, v_{i(k)}\} \in \Sigma$.

Then, Δ_W is a complex, $|\Delta_W| \subset [0,1]^n$, and we have a map $\iota : \mathcal{V} \to |\Delta_W|$ given by $\iota(v_i) = v'_i$, the so-called *canonical realization* of W.

Definition 2.4 ([**8**], pages 256-257)**.** Let $\mathcal{K}$ be a complex and $p \in |\mathcal{K}| \subset \mathbb{R}^n$ a point in $\mathcal{K}$. The *blow-up* $\mathcal{K}_{(p)}$ of $\mathcal{K}$ at p is the complex obtained by replacing each simplex $T \in \mathcal{K}$ that contains p by the set of all simplices of the form $\mathrm{conv}(F \cup \{p\})$, where F is any face of T not containing p.

For a rational 1-simplex $E = \mathrm{conv}(v, w) \in \mathbb{R}^n$ we define the *Farey mediant* of E as the rational point

$$u = \frac{\mathrm{den}(v)v + \mathrm{den}(w)w}{\mathrm{den}(v) + \mathrm{den}(w)} \in E,$$

where $\mathrm{den}(v)$ denotes the least common denominator of the coordinates of vector v.

If E belongs to a rational complex $\mathcal{K}$ and v is the Farey mediant of E, the (binary) *Farey blow-up* is $\mathcal{K}_{(v)}$.

Remark 1 ([**8,** Lemma 4.4]). Note that if $\mathcal{W} = (W_0, W_1, W_2, \ldots)$ is a stellar sequence of weighted abstract simplicial complexes and $\iota_0 : \mathcal{V}_0 \to |\Delta_0|$ the canonical realization, we can naturally extend this to attach a complex $\Delta_i = \Delta_{W_i}$ to each W_i:

Let $\mathfrak{b}_0 : W_0 \to W_1$ be the given stellar transformation. We define Δ_1 as follows: If $\mathfrak{b}_1$ deletes a maximal set $M \in \Sigma$, we delete the corresponding maximal simplex from Δ_0. If $\mathfrak{b}_1$ is a binary subdivision $(\{a,b\}, c)W_0$ at some $E = \{a,b\} \in \Sigma$, let e be the Farey mediant of the 1-simplex $\mathrm{conv}(\iota_0(E))$. Then, Δ_1 is the Farey blow-up of Δ_0 at e. If $\mathfrak{b}_1$ does not do anything, we also keep Δ_0 unchanged.

In all cases, we accordingly modify ι_0 to obtain a realization $\iota_1 : \mathcal{V}_1 \to |\Delta_1|$. Then, we can continue by considering $\mathfrak{b}_1 : W_1 \to W_2$, and so on.

Eventually, we obtain a sequence of complexes corresponding to $[0,1]^n \supset |\Delta_0| \supset |\Delta_1| \supset \cdots$.

Definition 2.5 ([**8,** Lemma 2.3])**.** Given a sequence $\mathcal{P} = (P_1 \supset P_2 \supset \cdots)$ of subsets of $[0,1]^n$, define an ℓ-ideal $I = I(\mathcal{P})$ of $\mathcal{M}([0,1]^n)$ by $I(\mathcal{P}) = \{f \in \mathcal{M}([0,1]^n) | f(P_i) = 0 \text{ for some } i\}$. This gives an ℓ-group $\mathcal{M}([0,1]^n)/I(\mathcal{P})$.

Theorem 2.6 ([**8,** Theorem 5.1])**.** *For every finitely generated unital ℓ-group (G,u), there is a stellar sequence $\mathcal{W} = (W_0, W_1, W_2, \ldots)$ such that $(G,u) \simeq \mathcal{G}(\mathcal{W})$, where $\mathcal{G}(\mathcal{W}) = \mathcal{M}([0,1]^n)/I$ for I the ideal corresponding to the sequence $[0,1]^n \supset |\Delta_0| \supset |\Delta_1| \supset \cdots$ defined using $\mathcal{W}$ as in Remark 1.*

All of this is not nearly as complicated as it sounds: we start with suitable complex Δ_0 and then modify it in infinitely many steps. In each step, we either

- delete a maximal simplex from the previous complex, or
- suitably divide a one-dimensional simplex E into two (and then we have to correspondingly divide all the simplices containing E), or
- do nothing.

This produces a sequence $[0,1]^n \supset |\Delta_0| \supset |\Delta_1| \supset \cdots$, and we define $G = \mathcal{M}([0,1]^n)/I$, where I is the set of all functions f such that $f(|\Delta_i|) = 0$ for some i. Every finitely generated unital ℓ-group is obtained in this way.

3. Existence of order-unit. Let us now review the connection between ℓ-groups and semirings.

By a (commutative) *semiring*, we shall mean a non-empty set S equipped with two associative and commutative operations (addition and multiplication) where the multiplication distributes over the addition from both sides. We shall be dealing with commutative semirings only, so we merely call them semirings. Note that our definition of a semiring is slightly more general than that used in the context of MV-semirings, see e.g., [**5**], since we do not require a semiring to contain a zero or a one.

A non-trivial semiring S is a *parasemifield* if the multiplication defines a non-trivial group. A non-trivial semiring S is a *semifield* if there is an element $0 \in S$ such that $0 \cdot S = 0$ and such that the set $S \backslash \{0\}$ is a group (for the semiring multiplication).

A semiring is *additively idempotent* if $x + x = x$ for all $x \in S$.

As already mentioned in the introduction, there is a well known term-equivalence (and, hence, a categorical isomorphism) between additively idempotent parasemifields and ℓ-groups. We shall use this to switch between the languages of parasemifields and ℓ-groups, sometimes without explicitly mentioning it.

Proposition 3.1 ([**25, 26**])**.** *There is a term-equivalence between additively idempotent parasemifields and ℓ-groups.*

Proof. Let $S(+, \cdot, ^{-1}, 1)$ be an additively idempotent parasemifield, and define $a \vee b = a + b$, $a \wedge b = (a^{-1} + b^{-1})^{-1}$. Then, $S(\cdot, ^{-1}, 1, \vee, \wedge)$ is an ℓ-group. Conversely, if $S(\cdot, ^{-1}, 1, \vee, \wedge)$ is an ℓ-group (written multiplicatively), then $S(+, \cdot, ^{-1}, 1)$ is an additively idempotent parasemifield, where $a + b = a \vee b$. We see that every basic operation on an ℓ-group is a term operation on an additively idempotent parasemifield, and vice versa. This implies that these two classes of algebras have the same clones of operations, i.e., they are term-equivalent. $\square$

We define a (pre-)ordering $\leq$ on a semiring S by $a \leq b$ if and only if $a = b$ or there exists a $c \in S$ such that $a + c = b$. Note that it is preserved by addition and multiplication in S. Also, this is the same ordering as that on the corresponding ℓ-group.

Note that if S is a parasemifield, then the ordering $\leq$ on S is antisymmetric:

Proposition 3.2 ([**14,** Proposition 20.37])**.** *Let S be a parasemifield. For all $a, b, c \in S$, we have*:

(a) *if $a + b + c = a$, then $a + b = a$.*
(b) *If $a \leq b \leq a$, then $a = b$.*

Proof.

(a) Let $a+b+c=a$. Multiply both sides by $a^{-2}b$, and then add $a^{-1}c$. We get $a^{-1}b+a^{-2}b^2+a^{-2}bc+a^{-1}c = a^{-1}b+a^{-1}c$, and thus, $(a^{-1}b+a^{-1}c)(a^{-1}b+1) = (a^{-1}b+a^{-1}c)$. Dividing by $a^{-1}b+a^{-1}c$, we obtain $a^{-1}b+1=1$ as desired.

(b) Write $b=a+x$ and $a=a+x+y$. By part (a), $a=a+x=b$. □

Definition 3.3. An additively idempotent parasemifield S is *order-unital* if there exists an element $u \in S$ such that, for each $s \in S$, there is an $n \in \mathbb{N}$ so that $u^n s + 1 = 1$.

Note that this definition is equivalent to the corresponding definition of a unital ℓ-group. For, if $v \in S$ is an order-unit in the ℓ-group sense, we have that, for each $s \in S$, there is some $n \in \mathbb{N}$ so that $v^n \geq s$. Now choose $u = v^{-1}$. Then, $1 \geq u^n s$, and so $1 = u^n s + t$ for some $t \in S$. Now, $1 + u^n s = (u^n s + t) + u^n s = u^n s + t = 1$. Conversely, if u is an element from Definition 3.3, then $v = u^{-1}$ will be an order-unit in the ℓ-group sense.

As usual, $\mathbb{N}$ and $\mathbb{Q}^+$ denote the semirings of positive integers and rational numbers, respectively; $\mathbb{N}_0$ is the semiring of non-negative integers.

While not as many classes of semirings have been studied as in the case of ℓ-groups or MV-algebras, we mention some examples of parasemifields and simple semirings in order to give our presentation a more concrete flavor. A basic example of additively idempotent parasemifields is given by a totally ordered group G (written multiplicatively), where we define the semiring addition $a+b := \max(a,b)$, obtaining so-called "tropical semirings" or "max-plus algebras". Standard examples are $\mathbb{R}(\max,+)$ and $\mathbb{Z}(\max,+)$. Note that the parasemifields defined in Definition 4.2 are a generalization of the latter case.

Example 3.4. Simple semirings were considered in detail by Bashir, Hurt, Jančařík and Kepka [**2**]. The study of ideal-simple ones reduces to the case of parasemifields (not necessarily additively idempotent); a basic example of such a construction is the following. For a parasemifield $P(+,\cdot,^{-1},1)$, consider the disjoint union $S := P \cup \{0\}$ and extend the operations by setting $x + 0 = x$ and $x \cdot 0 = 0$. Then, the semifield S is an ideal-simple semiring.

Congruence-simple semirings are essentially completely classified, with the exception of a rather mysterious class of subsemirings S of positive real numbers $\mathbb{R}^+(+, \cdot)$. We refer the interested reader to [**2**] for details and note only that the author and Korbelář [**19**] have provided examples of congruence-simple subsemirings of $\mathbb{Q}^+$ defined using p-adic valuations, such as $S = \{x \in \mathbb{Q}^+ \mid 2^{-\mathrm{v}_p(x)} < x\}$ (here $\mathrm{v}_p(x)$ is the additive p-adic valuation of x).

For more background information on semirings, see e.g., [**14**].

We will need further basic properties of (finitely generated) parasemifields.

In the rest of this section, let S be a parasemifield m-generated as a semiring. That means that there is a surjective semiring homomorphism $\varphi : \mathbb{N}[x_1, \dots, x_m] \to S$ (where x_i are indeterminates). For $a = (a_1, \dots, a_m) \in \mathbb{N}_0^m$, we use the notation $x^a = x_1^{a_1} \cdots x_m^{a_m}$.

Let A be the prime subparasemifield of S, i.e., the smallest (possibly trivial) parasemifield contained in S.

Let Q be the subsemiring of elements which are smaller than some element of A, i.e., $Q = \{s \in S \mid \text{there exists } q \in A : s \leq q\}$. Let $\mathcal{C} = \mathcal{C}(S) = \{a \in \mathbb{N}_0^m \mid \varphi(x^a) \in Q\}$ be the corresponding semigroup (or a cone) in N_0^m.

The structure of Q and $\mathcal{C}$ carries much information about S. For example, in [**15**], it was used to show that every parasemifield, two generated as a semiring, is additively idempotent.

Proposition 3.5.

(a) *If S is additively idempotent, then $A = \{1\}$; otherwise, $A \simeq \mathbb{Q}^+$.*
(b) *For $q_1, q_2 \in S$ we have $q_1 + q_2 \in Q$ if and only if $q_1, q_2 \in Q$. For $q \in S$, $n \in \mathbb{N}$ we have $q^n \in Q$ if and only if $q \in Q$.*
(c) *$\mathcal{C}$ is a pure subsemigroup of $\mathbb{N}_0^m$, i.e., it is closed under addition and, for $n \in \mathbb{N}$ and $a \in \mathbb{N}_0^m$, we have $na \in \mathcal{C}$ if and only if $a \in \mathcal{C}$.*

Proof. This a summary of various statements in [**15, 18**]. We merely sketch the proofs.

(a) $\mathbb{Q}^+$ is the free 0-generated parasemifield. Therefore, A is a factor of $\mathbb{Q}^+$. Now, it suffices only to note that $\mathbb{Q}^+$ is congruence simple.
(b) $q_1 + q_2 \in Q$ means that $q_1 + q_2 \le s$ for some $s \in A$. Therefore, $q_i \le q_1 + q_2 \le s$ and $q_i \in Q$ $(i = 1, 2)$.
(c) Follows directly from (b). □

We shall use the structure of $\mathcal{C}$ to show Theorem 3.7. In particular, we will need the next proposition which essentially states that there is an element c which is "inside" of the cone $\mathcal{C}$.

Proposition 3.6. *There exists $c \in \mathcal{C}$ such that:*

(a) *$c + e_i \in \mathcal{C}$ for each $i = 1, \ldots, m$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_0^m$ is the vector having 1 at the ith position and 0 elsewhere.*
(b) *$nc + a \in \mathcal{C}$ for each $a = (a_i) \in \mathbb{N}_0^m$, where $n = a_1 + \cdots + a_m$.*

Proof.

(a) Take $f = 1 + x_1 + \cdots + x_m \in \mathbb{N}[x_1, \ldots, x_m]$. Since S is a parasemifield, there is a $g = \sum_j a_j x^{c^{(j)}}$ (where $a_j \in \mathbb{N}$) such that $\varphi(g)$ is the inverse of $\varphi(f)$ in S, i.e., $\varphi(fg) = 1$. Thus, $\varphi(fg) \in Q$, and since

$$fg = \sum_j a_j (x^{c^{(j)}} + x^{c^{(j)}+e_1} + \cdots + x^{c^{(j)}+e_m}),$$

by Proposition 3.5 (b), each of the monomials $x^{c^{(j)}}, x^{c^{(j)}+e_1}, \ldots, x^{c^{(j)}+e_m}$ lies in Q, and thus, $c^{(j)}, c^{(j)} + e_1, \ldots, c^{(j)} + e_m$ all lie in $\mathcal{C}$. Hence, we can just choose $c = c^{(j)}$ for any j.

(b) $a = a_1 e_1 + \cdots + a_m e_m$, and thus, $nc + a = a_1(c + e_1) + \cdots + a_m(c + e_m) \in \mathcal{C}$. □

We are now ready to prove the main result of this section:

Theorem 3.7. *Let S be an additively idempotent parasemifield, finitely generated as a semiring. Then, S is order-unital.*

Proof. Choose $u = \varphi(x^c)$ with $c \in \mathcal{C}$ chosen by Proposition 3.6. We want to show that $u^n s + 1 = 1$ for each $s \in S$ and some $n \in \mathbb{N}$. Clearly, it suffices to show it for $s = \varphi(x^a)$, $a \in \mathbb{N}_0^m$ (each element of S is a finite sum of elements of this form).

By Proposition 3.6 (b), we can choose n large enough so that $nc + a \in \mathcal{C}$. Thus, $u^n s \in Q$. Since S is additively idempotent, $A = \{1\}$, and this means that $u^n s \leq 1$. Therefore, $1 \leq u^n s + 1 \leq 1$, and so by Proposition 3.2, $u^n s + 1 = 1$ and S is unital. □

Note that the order-unit we have just constructed is in no way unique. However, we shall see that the resulting classification Theorem 4.1 is independent of the choice of the order-unit.

4. The classification. By Theorem 3.7, we know that every additively idempotent parasemifield, finitely generated as a semiring, is order-unital. Considered as an ℓ-group via Proposition 3.1, we see that it is one of the ℓ-groups classified in [**8**]. In this section, we use these results to classify all such parasemifields, namely, we show the following Theorem 4.1. Its proof is somewhat long and will end near the conclusion of this article.

Theorem 4.1. *Let S be an additively idempotent parasemifield, finitely generated as a semiring. Then, S is a* (*finite*) *product of parasemifields of the form $G(T_i, v_i)$, where (T_i, v_i) are rooted trees and $G(T_i, v_i)$ are associated additively idempotent parasemifields* (*or, equivalently, ℓ-groups*), *defined in Definition* 4.2.

Two such products

$$\prod_{i=1}^{k} G(T_i, v_i) \quad \text{and} \quad \prod_{j=1}^{k'} G(T'_j, v'_j)$$

are isomorphic parasemifields if and only if $k = k'$, and there is some permutation σ of $\{1, \dots, k\}$ such that, for all i, we have $(T_i, v_i) \simeq (T'_{\sigma(i)}, v'_{\sigma(i)})$ as rooted trees.

Theorem 4.1 may be viewed as occurring in the category of additively idempotent parasemifields; in particular, it gives an equivalence between the subcategory of finitely generated objects and the subcategory consisting of finite products $\prod_{i=1}^{k} G(T_i, v_i)$. Equivalently, by Proposition 3.1, we may also view it in the isomorphic category of ℓ-groups: it feels slightly more natural to define $G(T, v)$ there, in the language of ℓ-groups.

First, let us briefly introduce some notions related to rooted trees.

Note that a *rooted tree* (T, v) is a (finite, non-oriented) connected graph T containing no cycles and having a specified vertex, the *root* v. By an *initial segment* T' of a rooted tree (T, v), we shall mean a (possibly empty) subtree such that, if $w \in T'$, then all the vertices on the (unique) path in T from v to w lie in T'. If T' is a non-empty initial segment of a rooted tree (T, v), the set of *next vertices* $N(T')$ is the set of all vertices $w \in T \backslash T'$ such that there is a $t \in T'$ and an edge (w, t) in T. If T' is empty, we set $N(T') = \{v\}$. For a vertex w, define a tree $T_w \subset T$ consisting of exactly all vertices $u \in T$ such that the (unique) path from u to the root v passes through w.

We are now ready to define $G(T, v)$.

Definition 4.2. Let T be a tree with root v. Define an ℓ-group $G(T, v)$ as follows: first, attach a copy of the group of integers $\mathbb{Z} = \mathbb{Z}_w$ to each vertex w of T. Then, $G(T, v)$ as an additive group is merely the direct product of these groups $\mathbb{Z}_w$. We shall denote elements of $G(T, v)$ as tuples (g_w) with $g_w \in \mathbb{Z}_w$.

Next, take tuples (g_w) and (h_w), and define $(g_w) \vee (h_w) = (k_w)$ and $(g_w) \wedge (h_w) = (m_w)$ as follows. Let T' be the largest initial segment of T such that $g_w = h_w$ for all $w \in T'$. For $w \in T'$, set $k_w = m_w = g_w (= h_w)$. Now take $w \in N(T')$. Then, $g_w \neq h_w$, and, without loss of generality, assume that $g_w > h_w$. Then, define $k_u = g_u$ and $m_u = h_u$ for all $u \in T_w$.

It is straightforward to check that $G(T, v)$ is indeed an abelian lattice ordered group; note that the lattice operations essentially come from some lexicographical ordering on $G(T, v)$ with respect to the structure of the tree.

We note that the construction of $G(T, v)$ is closely related to the Hahn embedding: the tree T is a chain if and only if the ℓ-group G is linearly ordered. In this case, the group $G(T, v)$ is exactly the group $\mathbb{Z}^n$ equipped with the lexicographic ordering, where n is the number of vertices of T.

We will need a few properties of the construction of [**8**] and of piecewise linear convex functions, especially in relation to being finitely generated as a semiring. They are detailed in the following three lemmas.

Lemma 4.3. *Let* $\mathcal{W} = (W_0, W_1, \ldots)$ *be the stellar sequence corresponding to the* ℓ*-group* $G = \mathcal{M}([0,1]^n)/I$. *Let* $[0,1]^n \supset D_0 \supset D_1 \supset D_2 \supset \cdots$ *be the corresponding sequence of complexes, let* $D = \bigcap D_i$, *and consider the* ℓ*-group* $\mathcal{M}(D)$ *of restrictions of functions in* $\mathcal{M}([0,1]^n)$ *to* D.

Then, there is a surjection

$$G = \mathcal{M}([0,1]^n)/I \longrightarrow \mathcal{M}(D).$$

Proof. Let $\mathrm{res} : \mathcal{M}([0,1]^n) \to \mathcal{M}(D)$ be the restriction map, and let $\pi : \mathcal{M}([0,1]^n) \to \mathcal{M}([0,1]^n)/I$ be the projection. By the definition of I, if $\pi(f) = \pi(g)$ then $\mathrm{res}(f) = \mathrm{res}(g)$, and thus, res factors through π, i.e.,

$$\mathrm{res} : \mathcal{M}([0,1]^n) \longrightarrow \mathcal{M}([0,1]^n)/I \longrightarrow \mathcal{M}(D).$$

Let

$$r : \mathcal{M}([0,1]^n)/I \longrightarrow \mathcal{M}(D)$$

be the corresponding map. Since res is a surjective homomorphism by definition, r is surjective as well. $\square$

Lemma 4.4. *Let* $A \subset [0,1]^n$ *be a simplex and* $f, g \in \mathcal{W}(A)$ *convex functions. Then,* $\max(f,g)$ *and* $f+g$ *are also convex.*

Proof. A function h is convex if the set $G(h)$ of all points above its graph is convex (in $A \times \mathbb{R}$), i.e., if the line segment between any two points in $G(h)$ lies in $G(h)$. Let $X, Y \in G(\max(f,g))$, and denote the line segment between these points as XY. Since f and g are both convex, $XY \in G(f)$ and $XY \in G(g)$. However, then $XY \in G(f) \cap G(g) = G(\max(f,g))$.

For $f+g$, choose $X = (x_1, x_2), Y = (y_1, y_2) \in G(f+g)$ (x_1, y_1 are n-tuples in A and $x_2, y_2 \in \mathbb{R}$). Then, there are

$$X' = (x_1, x'), \qquad Y' = (y_1, y') \in G(f)$$

and

$$X'' = (x_1, x''), \qquad Y'' = (y_1, y'') \in G(g)$$

such that $x_2 = x' + x''$ and $y_2 = y' + y''$. If we now take points $X_0 = (a, b) \in G(f)$ and $Y_0 = (a, c) \in G(g)$ on the line segments $X'Y'$ and $X''Y''$, respectively, then the point $(a, b+c)$ is on the line segment XY and lies in $G(f+g)$ (and each point of the line segment XY is of this form). □

Lemma 4.5. *Let $a_1, a_2, \ldots$ be a sequence of points in D such that $\lim a_i = a \in [0,1]^n$. Then, $\mathcal{M}(\{a_1, a_2, \ldots\})$ and $\mathcal{M}(D)$ are not finitely generated semirings.*

Proof. Assume that there are functions $f_1, \ldots, f_k \in \mathcal{M}([0,1]^n)$ whose restrictions generate $\mathcal{M}(\{a_1, a_2, \ldots\})$ as a semiring. Since each f_i is piecewise linear, we can find a simplex A such that each f_i is linear on A and infinitely many of the a_i lie in A. Denote the set of all such a_i as B. Using the (surjective) restriction map

$$\mathcal{M}(\{a_1, a_2, \ldots\}) \longrightarrow \mathcal{M}(B),$$

we see that the functions $f_1, \ldots, f_k$ generate $\mathcal{M}(B)$ as well.

Now consider the subset M of $\mathcal{M}(A)$ semiring-generated by $f_1, \ldots, f_k$. Since each linear function is convex, each function in M is convex by Lemma 4.4. However, there are clearly functions in $\mathcal{M}(B)$ which are not restrictions of convex functions on A, a contradiction.

The restriction map is a surjection from $\mathcal{M}(D)$ onto $\mathcal{M}(\{a_1, a_2, \ldots\})$. Thus, neither is $\mathcal{M}(D)$ finitely generated. □

Note that the same proof shows the following corollary:

Corollary 4.6. *Let $A \subset [0,1]^n$ be a simplex of dimension ≥ 1. Then, $\mathcal{W}(A)$ is not a finitely generated semiring.*

We are now ready to start discussing the structure of additively idempotent parasemifields. We will first show that our parasemifield S is a direct product of finitely many parasemifields corresponding to germs of functions at certain points.

Definition 4.7. Let p be a point in $[0,1]^n$, and let $\mathcal{P} = ([0,1]^n \supset P_0 \supset P_1 \supset \ldots)$ be a sequence of complexes such that $\bigcap P_i = \{p\}$. Then, we define the $\mathcal{P}$-*germ of functions* at p as $\mathcal{M}_{\mathcal{P}}(p) = \mathcal{M}([0,1]^n)/I$, where I is the ideal corresponding to the sequence $\mathcal{P}$, i.e., I consists of functions $f \in \mathcal{M}([0,1]^n)$ such that $f(P_i) = 0$ for some i.

The germ of functions at a point p is exactly what it should intuitively be: it is the set of all functions viewed locally at p "in the directions given by $\mathcal{P}$."

Proposition 4.8. *Let S be an additively idempotent parasemifield, finitely generated as a semiring. View S as a (unital) ℓ-group, and let $\mathcal{W} = (W_0, W_1, \ldots)$ be the corresponding stellar sequence, $\mathcal{D} = ([0,1]^n \supset D_0 \supset D_1 \supset \cdots)$ the corresponding sequence of complexes, and I the defining ideal. Then*

$$D = D(\mathcal{W}) = \bigcap D_i = \{d_1, \ldots, d_k\}$$

is finite and $S = \mathcal{M}([0,1]^n)/I$ is isomorphic to the direct product of $S_i = \mathcal{M}_{\mathcal{D}^i}(d_i)$, where $\mathcal{D}^i = ([0,1]^n \supset D_0^i \supset D_1^i \supset \cdots)$ with $D_j^i := D_j \cap C^i$ for some fixed simplex C^i containing an open neighborhood of the given point d_i.

Remark 4.9. The formulation of Proposition 4.8 is fairly technical, but the idea is simple. The intersection D is finite, and the parasemifield S will decompose as a direct product of parasemifields S_i, each of which corresponds to a germ of functions at a point $d_i \in D$.

Note that, strictly speaking, the local sequences of complexes $\mathcal{D}^i$ we are using do not come from a stellar sequence. This is merely a technicality, though: we can modify the stellar sequence $\mathcal{W}$ by first deleting all the simplices outside of C^i (using suitable subdivisions) and only then continuing with the stellar transformations which created $\mathcal{W}$. This produces a stellar sequence $\mathcal{W}^i$ whose corresponding sequence of complexes is $\mathcal{D}^{i'} = ([0,1]^n \supset D_1' \supset D_1' \supset \cdots \supset D_k' \supset D_1^i \supset D_2^i \supset \cdots)$, which differs from $\mathcal{D}^i$ only in finitely many complexes, and thus, produces the same germ of functions (as defined in Definition 4.7 above).

Proof of Proposition 4.8. Assume that D is not finite. Since $D \subset [0,1]^n$, we see that D has a cumulation point. Thus, by Lemma 4.5 it follows that $\mathcal{M}(D)$ is not finitely generated. By Lemma 4.3, $\mathcal{M}([0,1]^n)/I = S$ surjects onto $\mathcal{M}(D)$, and thus, neither is S a finitely generated semiring.

Therefore, $D = \{d_1, \dots, d_k\}$ is finite, and we can find suitable disjoint simplices C^i containing open neighborhoods of the points d_i and define $\mathcal{D}^i$ and $S_i = \mathcal{M}_{\mathcal{D}^i}(d_i)$ as in the statement of the proposition. Then, the restriction map gives a surjection $r : S \to \prod S_i$ similarly as in Lemma 4.3.

In order to show that r is injective, assume that $r(f) = 0$ for some $f \in \mathcal{M}([0,1]^n)$, i.e., there is a j such that $f(D^i_j) = 0$ for all i. We want to show that $\pi(f) = 0$. Since an open neighborhood of $D = \bigcap D_i$ is contained in $\bigcup_i D^i_j$, we see that there is a k such that $D_k \subset \bigcup_i D^i_j$. Thus, $f(D_k) = 0$, which means that $f \in I$ and $\pi(f) = 0$. □

Therefore, to finish the classification, we only need to describe the structure of the germs $\mathcal{M}_{\mathcal{D}}(d)$. This will be given in terms of ℓ-groups $G(T, v)$ associated to rooted trees, defined in Definition 4.2.

Proposition 4.10. *Let $\mathcal{W} = (W_0, W_1, \dots)$ be a stellar sequence, $\mathcal{D} = ([0,1]^n \supset D_0 \supset D_1 \supset D_2 \supset \cdots)$ the corresponding sequence of complexes and $D = \bigcap D_i$. Assume that $D = \{d\}$ has one element. Then, the corresponding ℓ-group of germs of functions $G = \mathcal{M}([0,1]^n)/I = \mathcal{M}_{\mathcal{D}}(d)$ is either not finitely generated as a semiring or is isomorphic to an ℓ-group $G(T, v)$ associated to a (finite) rooted tree (T, v).*

Proof. Assume that $\mathcal{M}_{\mathcal{D}}(d)$ is finitely generated as a semiring.

In order to prove the proposition we shall modify the sequence $\mathcal{D}$ in several steps while preserving the ℓ-group $\mathcal{M}_{\mathcal{D}}(d)$. The fairly long proof is divided into five steps:

A. *Simplices containing d.* First form a new sequence of complexes $\mathcal{D}^1 = (D^1_0 \supset D^1_1 \supset \cdots)$, where D^1_i is obtained from D_i by recursively removing all maximal simplices which do not contain d. Note that the simplices without d play no role in determining the germ of local functions, and so $\mathcal{M}_{\mathcal{D}^1}(d) = \mathcal{M}_{\mathcal{D}}(d)$. Also, note that $\mathcal{D}^1$ is still obtained from a stellar sequence (taking into account the potential need for modifications as in Remark 4.9; we shall not mention this further).

B. *Stable subspaces.* By a *stable line* in $\mathcal{D}^1$, we shall mean a line ℓ passing through d such that $\ell \cap D_i^1$ is a line segment (and not just the point d) for each i. This means that, while the stellar transformations which give $\mathcal{D}^1$ may (and will) subdivide the one-dimensional simplex which gives a line segment lying on ℓ, they will never delete this simplex.

We point out that stable lines give non-trivial elements in $\mathcal{M}_{\mathcal{D}}(d)$: the germ of linear functions on ℓ will lie in $\mathcal{M}_{\mathcal{D}}(d)$. A linear function on a line is determined by its slope (and value at the point d) and, since the functions we are considering are restrictions of linear functions with integral coefficients, the set of possible slopes is $\mathbb{Z}$. Thus, to each stable line ℓ corresponds a copy of $\mathbb{Z} \subset \mathcal{M}_{\mathcal{D}}(d)$.

Similarly, for $k \geq 1$, we can define a *stable k-subspace* in $\mathcal{D}^1$ as a k-dimensional (affine) space L containing d such that $L \cap D_i^1$ has dimension k for each i. (A stable 1-subspace is just a stable line.)

By the definition of stable subspaces it follows that, if a simplex in D_i^1 intersects every stable line only in the point d (and thus the same is true for the intersection with any stable subspace), then it does not contribute to $\mathcal{M}_{\mathcal{D}}(d)$. Therefore, we can form D_i^2 and $\mathcal{D}^2$ by omitting all such simplices with no non-trivial intersection with a stable line. Then, $\mathcal{M}_{\mathcal{D}^2}(d) = \mathcal{M}_{\mathcal{D}}(d)$.

C. *Simplices defined using the generators.* By an open simplex, we shall mean a point, or the interior of a k-simplex for $k \geq 1$.

Denote the (semiring) generators of $S = \mathcal{M}_{\mathcal{D}}(d)$ by $f_1, \dots, f_k$ (as usual, we identify a function $f \in \mathcal{M}([0,1]^n)$ with its image in $\mathcal{M}_{\mathcal{D}}(d)$). Each of these functions is piecewise linear, and thus, there is a finite set $\mathcal{P}_0$ of open simplices which cover $[0,1]^n$ such that the restriction of each f_j to any $P \in \mathcal{P}_0$ is linear. In fact, we can modify $\mathcal{P}_0$ to obtain the following lemma.

Lemma 4.11. *There is a finite set $\mathcal{P}$ of open simplices such that*

(i) *elements of $\mathcal{P}$ are pairwise disjoint,*

(ii) *the restriction $f_j|P$ is linear for all j and all $P \in \mathcal{P}$,*

(iii) $\dim P \cap D_i^2 = \dim P$ *for all i and all $P \in \mathcal{P}$,*

(iv) $P \cap \ell = \emptyset$ *for all stable lines* ℓ *and all* $P \in \mathcal{P}$ *with* $\dim P > 1$,

(v) *for each* e *and each* $P \in \mathcal{P}$ *with* $\dim P > e$*, there is exactly one* $Q \in \mathcal{P}$ *such that* $\dim Q = e$ *and* $Q \subset \overline{P}$ ($\overline{P}$ *denotes the closure in* $\mathbb{R}^n$),

(vi) $\mathcal{M}_{\mathcal{D}^3}(d) = \mathcal{M}_{\mathcal{D}}(d)$*, where* $\mathcal{D}^3 = \mathcal{D}^2 \cap U = (D_0^2 \cap U \supset D_1^2 \cap U \supset \cdots)$ *and* $U = \bigcup_{P \in \mathcal{P}} P$.

Note that the simplices $P \in \mathcal{P}$ from the lemma can be viewed as a refinement of the notion of stable subspaces.

We shall recursively modify $\mathcal{P}_0$ in several steps while making sure that $\mathcal{M}_{\mathcal{D}^2 \cap U}(d)$ remains unchanged and equal to $S = \mathcal{M}_{\mathcal{D}}(d)$ (this is clearly true at the beginning as $\bigcup_{P \in \mathcal{P}_0} P = [0,1]^n$).

In order to begin, let $\mathcal{P} = \mathcal{P}_0$. Now recursively repeat the following set of modifications:

1. Only those simplices $P \in \mathcal{P}$ which have non-empty intersection with infinitely many (and hence all) of the D_i^2 are relevant for determining S. Hence, we can delete all other P from $\mathcal{P}$. Continue to Step 2.

2. If there is a $P \in \mathcal{P}$ and finitely many open simplices $S_1, \ldots, S_a \subset P$ such that $\dim S_i < \dim P$ and $P \backslash (S_1 \cup \cdots \cup S_a)$ has non-empty intersection with all D_i^2, then replace P by $S_1, \ldots, S_a$ in $\mathcal{P}$. Return to Step 1 if $\mathcal{P}$ has been modified, else continue to Step 3.

Note that $\mathcal{P}$ has finitely many elements at any time and dimensions of elements of $\mathcal{P}$ are decreasing; thus, this step will occur only finitely many times. Also, note that, after finishing Steps 1 and 2, $\mathcal{P}$ contains only open simplices P with $\dim(P \cap D_i^2) = \dim P$ for all i.

3. Assume that $d \neq P \in \mathcal{P}$ has non-empty intersection with infinitely many stable lines. Arguing as in the proof of Lemma 4.5, we see that S is then not finitely generated as a semiring, a contradiction: namely, any function f which is linear along finitely many of these lines (and suitably defined at the other lines) will be non-trivial in S. By considering the set of slopes of f along these lines, it is easy to construct a function $f \in S$ which will not be convex on P. However, this contradicts Lemma 4.4 as all the semiring generators are linear on P. Therefore, every $d \neq P \in \mathcal{P}$ has non-empty intersection with only finitely many stable lines.

Suppose that $\dim P > 1$, ℓ is a stable line and $\ell \cap P \neq \emptyset$. Then, choose a $(\dim P - 1)$-dimensional hypersurface H containing ℓ and subdivide P along this hypersurface, i.e., $P = (P \cap H) \cup (P \backslash H)$ and $P \backslash H$ has two connected components, P_1 and P_2. We can choose H so that $P \cap H, P_1$, and P_2 are all open simplices; in $\mathcal{P}$, then replace P by $P \cap H$, P_1 and P_2.

After performing this finitely many times (since for each P there are only finitely many stable lines), we arrive at $\mathcal{P}$ satisfying property (iv). Return to Step 1 if $\mathcal{P}$ has been modified, else continue to Step 4.

4. Assume that there are $P, Q, R \in \mathcal{P}$ such that $Q, R \subset \overline{P}$ and $Q \not\subset \overline{R}$ and $R \not\subset \overline{Q}$. Take such a P of the smallest dimension. Since Q and R are disjoint, we can again subdivide P by a $(\dim P - 1)$-dimensional hypersurface H as above so that ($Q \subset \overline{P_1}$ and $R \subset \overline{P_2}$) or ($Q \subset \overline{P_1}$ and $R \subset \overline{P \cap H}$) or ($R \subset \overline{P_1}$ and $Q \subset \overline{P \cap H}$) (and replace P in $\mathcal{P}$ by $P \cap H$, P_1 and P_2).

After performing this finitely many times (since for each P such a situation can occur only finitely many times), we arrive at $\mathcal{P}$ satisfying property (v): we have just ensured the uniqueness of such a Q; its existence easily follows from the fact that $\dim P \cap D_i^2 = \dim P$.

Return to Step 1 if $\mathcal{P}$ has been modified, else we are done.

Note that the entire algorithm terminates after finitely many steps and that (i)–(vi) are satisfied at the end, completing the proof of Lemma 4.11. $\square$

D. *Construction of the tree T.* Now we can easily construct a rooted tree (T, v) attached to the sequence $\mathcal{M}_{D^3}(d)$ obtained using Lemma 4.11: Associate a vertex v_P to each $P \in \mathcal{P}$; there will be an edge connecting vertices v_P and v_Q if and only if ($P \subset \overline{Q}$ and $\dim P = \dim Q - 1$) or ($Q \subset \overline{P}$ and $\dim Q = \dim P - 1$). The vertex v_d is the root v.

By Lemma 4.11 we see that (T, v) is a (connected) rooted tree.

E. *Description of $M_{\mathcal{D}}(d)$.* The germ of a function f in $\mathcal{M}_{\mathcal{D}^3}(d) = M_{\mathcal{D}}(d)$ can have any value at d, which gives the $\mathbb{Z}_v$ at the root v of the tree T.

Given $f \in \mathcal{M}_{\mathcal{D}^3}(d)$, choose a small ball B containing d so that the restriction of f to $B \cap r$ is linear for all rays $r \ni d$. Since we are considering only the germ of functions at d, f is uniquely determined by $f|B$ as an element of $\mathcal{M}_{\mathcal{D}^3}(d) = M_{\mathcal{D}}(d)$.

Take a 1-simplex $P \in \mathcal{P}$. The value of f at the endpoint d of $\overline{P}$ has already been selected, and so the restriction $f|(P \cap B)$ (which is linear by Lemma 4.11 (ii)) is uniquely determined by its value at any point $p \in P \cap B$. The choice of this value gives the $\mathbb{Z}_{v_P}$ at the vertex v_P of the tree T.

After having dealt with all the 1-simplices, take a 2-simplex $P \in \mathcal{P}$. There is a unique 1-simplex $Q \in \mathcal{P}$, $Q \subset \overline{P}$; $f|(Q \cap B)$ has already been determined, and thus, the restriction $f|(P \cap B)$ is uniquely determined by its value at any point $p \in P$. The choice of this value gives the $\mathbb{Z}_{v_P}$ at the vertex v_P of the tree T.

We can continue in this way, successively dealing with simplices of larger and larger dimensions, until we have covered the entire tree T and uniquely determined the function $f|B$ as an element of $\mathcal{M}_{\mathcal{D}^3}(d) = M_{\mathcal{D}}(d)$.

Now, it is straightforward to check that the ℓ-group $M_{\mathcal{D}}(d)$ is exactly $G(T, v)$. $\square$

Together with Proposition 4.8, this finishes the proof of Theorem 4.1, except for the uniqueness part. This follows from the proof and from the uniqueness statement of [**8,** Corollary 5.4]. However, for the sake of completeness and to make sure that our classification is indeed independent of the choice of order-unit in Theorem 3.7, let us give (a sketch of) a direct proof.

Assume that

$$G = \prod_{i=1}^{k} G(T_i, v_i)$$

is one of our ℓ-groups from Theorem 4.1, abstractly given as an ℓ-group $(G, +, -, 0, \vee, \wedge)$, i.e., without specifying the corresponding rooted tree structure (or the order-unit). In order to make the notation more uniform, we consider the disjoint union F of the rooted trees (T_i, v_i) as a "rooted forest" $\mathcal{F} = (F, v_1, v_2, \ldots, v_k)$.

We shall show how to reconstruct this rooted forest $\mathcal{F}$ from G, which will then imply the uniqueness statement of the theorem.

First, we introduce some notation for the "standard" basis" of $\prod_{i=1}^k G(T_i, v_i)$. In Definition 4.2, we have attached a copy $\mathbb{Z}_w$ of the additive group of integers to each vertex w. Denote by $b(w)$ the element of G which corresponds to $1 \in \mathbb{Z}_w$, i.e., $b(w) = (g_v)_{v \in F}$ is the tuple with $g_w = 1$ and $g_v = 0$ if $v \neq w$. Note that, by definition, we have $b(w) > 0$. We shall say that an element $g \in G$ is *infinitesimally smaller* than $h \in G$ if $ng < h$ for all $n \in \mathbb{Z}$ and denote this by $g \ll h$. We say that an element $g \in G$ is *infinitesimal* if $g \ll h$ for some $h \in G$.

Now we try to identify the basis elements $b(v)$ corresponding to roots $v = v_i$. Define $B_0 = \{g_1, \ldots, g_m\}$ as a maximal set of elements which satisfy all of the following properties for all pairs $i \neq j$:

- $g_i > 0$,
- g_i is not infinitesimal,
- g_i is not a sum of positive non-infinitesimal elements,
- $g_i \vee g_j = g_i + g_j$.

Considering the elements g_i as linear combinations of the basis $b(w)$, it is easy to see that $k = m$ and that there are infinitesimal elements $h_i \ll g_i$ such that $\{g_1 + h_1, \ldots, g_k + h_k\} = \{b(v_1), \ldots, b(v_k)\}$. After permuting the indices, if necessary, we can assume that $g_i + h_i = b(v_i)$. Hence, up to the infinitesimal elements h_i, we see that B_0 is the set of basis elements corresponding to the roots v_i.

Now, for $i = 1, \ldots, k$ define $G_i = \{g \in G \mid g \ll g_i\}$. This is an ℓ-subgroup of G isomorphic to the ℓ-group $G(\mathcal{F}_i)$ attached to a rooted forest $\mathcal{F}_i$, obtained by removing the root v_i from the tree T_i and designating the vertices $v \in N(v_i)$, i.e., those that are connected to v_i by an edge in T_i, as the roots of the trees in the forest $\mathcal{F}_i$.

We can now proceed in the same way with each G_i and define a set B_i of elements that correspond to basis elements $b(v)$, $v \in N(v_i)$ (again up to elements that are infinitesimally smaller).

Proceeding by induction in this fashion, we eventually define an element g_w for each $w \in F$ so that the set $\{g_w \mid w \in F\}$ with the ordering $\ll$ is isomorphic to the rooted forest $\mathcal{F}$ (viewed as an ordered set whose maximal elements are the roots). Note that we have intrinsically defined the ordered set $\{g_w \mid w \in F\}$, without referring to the forest $\mathcal{F}$ (or the chosen order-unit).

Assume now that

$$\prod_{i=1}^{k} G(T_i, v_i) \quad \text{and} \quad \prod_{j=1}^{k'} G(T'_j, v'_j)$$

are isomorphic ℓ-groups. As above, we can attach to them rooted forests $\mathcal{F}$ and $\mathcal{F}'$, respectively, which then must be isomorphic rooted forests. This proves the uniqueness statement of Theorem 4.1. □

We note that, as a group, each $G(T_i, v_i)$ is merely $\mathbb{Z}^{n_i}$ for some n_i, and thus, we obtain the following corollary to Theorem 4.1.

Corollary 4.12. *If an additively idempotent parasemifield is finitely generated as a semiring $S(+, \cdot)$, then it is finitely generated as a group $S(\cdot) \simeq \mathbb{Z}^n$.*

We are not aware of any more direct or elementary proof of this surprising fact. It would certainly be very interesting to obtain one.

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