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IDEAL CLASS GROUPS OF MONOID ALGEBRAS

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ABSTRACT. Let $A \subset B$ be an extension of commutative reduced rings and $M \subset N$ an extension of positive commutative cancellative torsion-free monoids. We prove that A is subintegrally closed in B and M is subintegrally closed in N if and only if the group of invertible A-submodules of B is isomorphic to the group of invertible $A[M]$-submodules of $B[N]$ Theorem 1.2 (b), (d). In the case $M = N$, we prove the same without the assumption that the ring extension is reduced Theorem 1.2 (c), (d).

1. Introduction. Throughout the paper, we assume that all rings are commutative with unity and all monoids are commutative cancellative torsion-free. For a ring extension $A \subset B$, the group of invertible A-submodule of B is denoted by $\mathcal{I}(A, B)$. This group has been extensively studied by Roberts and Singh [**6**]. Sadhu and Singh [**8,** Theorem 1.5] proved: *Let $A \subset B$ be an extension of rings and $\mathbb{Z}_+$ the monoid of positive integers. Then A is subintegrally closed in B if and only if $\mathcal{I}(A, B) \cong \mathcal{I}(A[\mathbb{Z}_+], B[\mathbb{Z}_+])$.*

Motivated by this result, we ask the following:

Question 1.1. *Let $A \subset B$ be an extension of rings and $M \subset N$ an extension of positive monoids. Are the following statements equivalent?*

(i) *A is subintegrally closed in B and M is subintegrally closed in N.*
(ii) *$A[M]$ is subintegrally closed in $B[N]$.*
(iii) *$\mathcal{I}(A, B)$ is isomorphic to $\mathcal{I}(A[M], B[N])$.*

It is always true that (ii) $\Rightarrow$ (i). If B is a reduced ring, then (i) $\Rightarrow$ (ii) is as well [**4,** Theorem 4.79].

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We answer Question 1.1 in the affirmative by proving the next result. Our proof uses Swan and Weibel's homotopy trick.

Theorem 1.2. *Let $A \subset B$ be an extension of rings and $M \subset N$ an extension of positive monoids.*

(a) *If $A[M]$ is subintegrally closed in $B[N]$ and N is affine, then $\mathcal{I}(A,B) \cong \mathcal{I}(A[M],B[N])$.*

(b) *If B is reduced, A is subintegrally closed in B and M is subintegrally closed in N, then $\mathcal{I}(A,B) \cong \mathcal{I}(A[M],B[N])$.*

(c) *If $M = N$, then the reduced condition on B is not needed, i.e., if A is subintegrally closed in B, then $\mathcal{I}(A,B) \cong \mathcal{I}(A[M],B[M])$.*

(d) (*converse of* (a), (b) *and* (c)). *If $\mathcal{I}(A,B) \cong \mathcal{I}(A[M],B[N])$, then*

(i) *A is subintegrally closed in B,*
(ii) *$A[M]$ is subintegrally closed in $B[N]$, and*
(iii) *B is reduced or $M = N$.*

The next result, which is immediate from (1.2), gives the exact conditions when (i) $\Rightarrow$ (ii) in Question 1.1.

Corollary 1.3. *Let $A \subset B$ be an extension of rings and $M \subset N$ an extension of positive monoids such that A is subintegrally closed in B and M is subintegrally closed in N.*

(i) *If B is reduced or $M = N$, then $A[M]$ is subintegrally closed in $B[N]$.*
(ii) *Conversely, if $A[M]$ is subintegrally closed in $B[N]$ and N is affine, then B is reduced or $M = N$.*

Let A be a seminormal ring with $\mathbb{Q} \subset A$ and M a positive seminormal monoid. Then [**4,** Theorem 8.42] proved that $\mathrm{Pic}(A) \cong \mathrm{Pic}(A[M])$. This result is due to Anderson [**2,** Theorem 1] in the case where $A[M]$ is an almost seminormal integral domain, see [**2,** Definition]. As an application of our result Theorem 1.2 (c), we deduce a special case of this result, see Remark 3.5.

Sadhu and Singh [**8,** Theorem 2.6] studied the relationship between the two groups $\mathcal{I}(A,B)$ and $\mathcal{I}(A[\mathbb{Z}_+],B[\mathbb{Z}_+])$, when A is not subintegrally closed in B. Using our Theorem 1.2, we generalize their result [**8,** Theorem 2.6] to the monoid algebra situation in a straightforward manner.

Theorem 1.4. *Let $A \subset B$ be an extension of rings, and let ${}^+\!A$ denote the subintegral closure of A in B. Assume that M is a positive monoid. Then,*

(i) *the diagram:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{I}(A,{}^+\!A) & \longrightarrow & \mathcal{I}(A,B) & \xrightarrow{\phi(A,{}^+\!A,B)} & \mathcal{I}({}^+\!A,B) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \theta(A,{}^+\!A)} & & \downarrow{\scriptstyle \theta(A,B)} & & \simeq\downarrow{\scriptstyle \theta({}^+\!A,B)} & & \\
1 & \longrightarrow & \mathcal{I}(A[M],{}^+\!A[M]) & \longrightarrow & \mathcal{I}(A[M],B[M]) & \longrightarrow & \mathcal{I}({}^+\!A[M],B[M]) & \longrightarrow & 1
\end{array}$$

is commutative with exact rows.

(ii) *If $\mathbb{Q} \subset A$, then $\mathcal{I}(A[M],{}^+\!A[M]) \cong \mathbb{Z}[M] \otimes_{\mathbb{Z}} \mathcal{I}(A,{}^+\!A)$.*

2. Preliminaries.

Definition 2.1.

(i) Let $A \subset B$ be an extension of rings. The extension $A \subset B$ is called *elementary subintegral* if $B = A[b]$ for some b with $b^2, b^3 \in A$. If B is a union of subrings obtained from A by a finite succession of elementary subintegral extensions, then the extension $A \subset B$ is called *subintegral*. The *subintegral closure* of A in B, denoted by ${}_B^{+}A$, is the largest subintegral extension of A in B. We say A is *subintegrally closed* in B if ${}_B^{+}A = A$. A ring A is called *seminormal* if it is reduced and subintegrally closed in $PQF(R) := \prod_{\mathfrak{p}} QF(R/\mathfrak{p})$, where $\mathfrak{p}$ runs through the minimal prime ideals of R and $QF(R/\mathfrak{p})$ is the quotient field of $R/\mathfrak{p}$, see [**4,** page 154].

(ii) Let $A \subset B$ and $A' \subset B'$ be two ring extensions. A morphism ϕ between the pairs $(A,B) \to (A',B')$ is a ring homomorphism $\phi : B \to B'$ with $\phi(A) \subset A'$. For a ring extension $A \subset B$, if $\mathcal{I}(A,B)$ denotes the multiplicative group of invertible A-submodules of B, then $\mathcal{I}$ is a functor from the category of ring extensions to the category of abelian groups. Let $\mathcal{I}(\phi)$ denote the group homomorphism which is induced by the morphism ϕ of a ring extension. If $B \subset B'$ and $A \subset A'$, then the inclusion map $i : B \to B'$ defines a morphism of pairs $(A, B) \to (A', B')$. We will denote $\mathcal{I}(i)$ by $\theta(A, B)$. For basic facts pertaining to ring extensions and the functor $\mathcal{I}$, we refer the reader to [**6**].

(iii) Let $M \subset N$ be an extension of monoids. The extension $M \subset N$ is called *elementary subintegral* if $N = M \cup xM$ for some x with $x^2, x^3 \in M$. If N is a union of submonoids which are obtained from M by a finite succession of elementary subintegral extensions, then the extension $M \subset N$ is called *subintegral.* The *subintegral closure* of M in N, denoted by ${}_N{}^+M$, is the largest subintegral extension of M in N. We say M is *subintegrally closed* in N if ${}_N{}^+M = M$. Let $\phi(M)$ denote the group of fractions of the monoid M. We say M is *seminormal* if it is subintegrally closed in $\phi(M)$.

(iv) For a monoid M, let $U(M)$ denote the group of units of M. If $U(M)$ is a trivial group, then M is called *positive.* If M is finitely generated, then M is called *affine.*

For basic definitions and facts pertaining to monoids and monoid algebras, we refer the reader to [**4,** Chapters 2, 4]).

Notation 2.2. For a ring A, Pic(A) denotes the Picard group of A, $U(A)$ denotes the multiplicative group of units of A and nil(A) denotes the nil radical of A.

Now, we give some results for later use.

The next result, which follows with repeated applications of [**8,** Corollary 1.6], is due to Sadhu and Singh.

Lemma 2.3. *Let $A \subset B$ be an extension of rings. Then A is subintegrally closed in B if and only if $A[\mathbb{Z}_+^r]$ is subintegrally closed in $B[\mathbb{Z}_+^r]$ for any integer $r > 0$.*

The next result is obtained [**4,** Theorem 4.79] by observing that $\mathrm{sn}_B(A)$ (the seminormalization of A in B) and is the same as ${}_B{}^+A$ (the subintegral closure of A in B) in our notation.

Lemma 2.4. *Let $A \subset B$ be an extension of reduced rings and $M \subset N$ an extension monoid. Then ${}_B{}^+A[N \cap \mathrm{sn}(M)]$ is the subintegral closure of $A[M]$ in $B[N]$, where $\mathrm{sn}(M)$ is the seminormalization (subintegral closure) of M in $\phi(M)$.*

3. Main results. The next result is motivated from ([**1**], Lemma 5.7).

Lemma 3.1. *Let $R = R_0 \oplus R_1 \oplus \cdots$ and $S = S_0 \oplus S_1 \oplus \cdots$ be two positively graded rings with $R \subset S$ and $R_i \subset S_i$ for all $i \geq 0$. If the canonical map $\theta(R,S) : \mathcal{I}(R,S) \to \mathcal{I}(R[X],S[X])$ is an isomorphism, then the canonical map $\theta(R_0,S_0) : \mathcal{I}(R_0,S_0) \to \mathcal{I}(R,S)$ is an isomorphism.*

Proof. This result uses Swan and Weibel's homotopy trick. Let $j : (R_0,S_0) \to (R,S)$ be the inclusion map and $\pi : (R,S) \to (R_0,S_0)$ the canonical surjection defined as $\pi(s_0 + s_1 + \cdots + s_r) = s_0$, where $s_o + s_1 + \cdots + s_r \in S$. Then $\pi j = Id_{(R_0,S_0)}$. Applying the functor $\mathcal{I}$, we get that $\mathcal{I}(\pi)\theta(R_0,S_0) = \mathrm{Id}_{\mathcal{I}(R_0,S_0)}$, where $\theta(R_0,S_0) = \mathcal{I}(j)$. Hence, the canonical map $\theta(R_0,S_0)$ is injective. So we must prove that $\theta(R_0,S_0)$ is surjective.

Let $e_0, e_1 : (R[X],S[X]) \to (R,S)$ be two evaluation maps defined as $X \to 0$, $X \to 1$, respectively, and i the inclusion map from $(R,S) \to (R[X],S[X])$. Then, we obtain that $e_0 i = e_1 i$. Let $w : (R,S) \to (R[X],S[X])$ be a map defined as $w(s) = s_0 + s_1 X + \cdots + s_r X^r$, where $s = s_0 + s_1 + \cdots + s_r \in S$. It is easy to see that w is a ring homomorphism from $S \to S[X]$, and moreover, w is a morphism of ring extensions, i.e., $w(R) \subset R[X]$. It is easy to see that $e_0 w = j\pi \cdots (a)$.

Since $e_0 i = e_1 i = \mathrm{Id}_{(R,S)}$, we obtain that $\mathcal{I}(e_0)\theta(R,S) = \mathcal{I}(e_1)\theta(R, S) = \mathrm{Id}_{\mathcal{I}(R,S)}$ (recall that $\theta(R,S) = \mathcal{I}(i)$). Therefore, $\mathcal{I}(e_0)$ and $\mathcal{I}(e_1)$ are inverses of the canonical isomorphism $\theta(R,S)$. Hence, $\mathcal{I}(e_0) = \mathcal{I}(e_1)$. By (a), we have $\mathcal{I}(e_0)\mathcal{I}(w) = \theta(R_0,S_0)\mathcal{I}(\pi)$. Hence, $\mathcal{I}(e_1)\mathcal{I}(w) = \theta(R_0,S_0)\mathcal{I}(\pi)$. Note that $\mathcal{I}(e_1)\mathcal{I}(w) = \mathrm{Id}_{\mathcal{I}(R,S)} = \theta(R_0,S_0)\mathcal{I}(\pi)$. Therefore, we obtain that $\theta(R_0,S_0)$ is surjective. This completes the proof. □

The next result is [**4,** Theorem 4.79] when the ring extension is reduced. We use the same arguments as in [**4,** Theorem 4.42, 4.79] to prove the following result. For an alternate proof of the following result, see Remark 3.3.

Lemma 3.2. *Let $A \subset B$ be an extension of rings and M an affine monoid. Assume that A is subintegrally closed in B. Then, $A[M]$ is subintegrally closed in $B[M]$.*

Proof. It is easy to see that $A[\phi(M)] \cap B[M] = A[M]$. Hence, it is enough to prove that $A[\phi(M)]$ is subintegrally closed in $B[\phi(M)]$. Since M is affine, $\phi(M) \cong \mathbb{Z}^r$ for some integer $r > 0$. Hence, we must prove that $A[\mathbb{Z}^r]$ is subintegrally closed in $B[\mathbb{Z}^r]$. Since subintegrality commutes with localization, see [**4,** Theorem 4.75d], we only have to prove that $A[\mathbb{Z}^r_+]$ is subintegrally closed in $B[\mathbb{Z}^r_+]$. This is indeed the case because of (2.3). □

3.1. Proof of Theorem 1.2.

Proof.

(a) Since N is positive affine, N has a positive grading by [**4,** Proposition 2.17 f]. Since M is a submonoid of N, it has a positive grading induced from N. Therefore, both $A[M]$ and $B[N]$ have positive gradings. Hence, we can write

$$A[M] = A_0 \oplus A_1 \oplus \cdots$$

and

$$B[N] = B_0 \oplus B_1 \oplus \cdots$$

with $A_0 = A$ and $B_0 = B$. We define $R := A[M]$, $S := B[N]$ and $R_0 := A$, $S_0 := B$. By the hypothesis, R is subintegrally closed in S; hence, by ([**8,** Theorem 1.5], $\mathcal{I}(R,S) \cong \mathcal{I}(R[X], S[X])$. Therefore, by Lemma 3.1, we obtain that $\mathcal{I}(A,B) \cong \mathcal{I}(A[M], B[N])$.

(b) First, assume that N is affine. Since B is reduced, by (2.4), the subintegral closure of $A[M]$ in $B[N]$ is ${}_B{}^+A[N \cap \operatorname{sn}(M)]$. Note that $\operatorname{sn}(M) ={}_{\phi(M)}{}^+M$ in our notation. It is easy to see that ${}_N{}^+M = N \cap_{\phi(M)}{}^+M$. By hypothesis, ${}_B{}^+A = A$ and ${}_N{}^+M = M$. Hence, $A[M]$ is subintegrally closed in $B[N]$. Therefore, by (a), we obtain that $\mathcal{I}(A,B) \cong \mathcal{I}(A[M], B[N])$.

Now we will discuss the case when N is not affine. Let $\Lambda := \{N_i : i \in I\}$ be the set of all affine submonoids of N. Then Λ forms a directed set by defining $N_i \leq N_j$ if N_i is a submonoid of N_j. Let $M_i := M \cap N_i$, where $N_i \in \Lambda$. Since M is subintegrally closed in N, it is easy to see that M_i is subintegrally closed in N_i. Then,

$$N = \cup_{N_i \in \Lambda} N_i \quad \text{and} \quad M = \cup M_i.$$

If $N_i \leq N_j$, then there exists a morphism of ring extension $\phi_{ij} : (A[M_i], B[N_i]) \to (A[M_j], B[N_j])$ induced from the inclusion map $B[N_i] \to B[N_j]$. Hence, $(\{(A[M_i], B[N_i])\}_{N_i \in \Lambda}, \{\phi_{ij}\}_{N_i \leq N_j})$ forms a directed system in the category of ring extensions. Then the direct limit of this system is $((A[M], B[N]), \{\phi_i\})$, where $\phi_i : (A[M_i], B[N_i]) \to (A[M], B[N])$, i.e., $\varinjlim_\Lambda (A[M_i], B[N_i]) = (A[M], B[N])$.

Similarly, as in the above paragraph, one may see that

$$(\mathcal{I}(A[M_i], B[N_i])_{N_i \in \Lambda}), \{\mathcal{I}(\phi_{ij})\}_{N_i \leq N_j})$$

forms a directed system in the category of abelian groups.

We want to prove that

$$\varinjlim_\Lambda (\mathcal{I}(A[M_i], B[N_i])) \cong \mathcal{I}(\varinjlim_\Lambda (A[M_i], B[N_i])) = \mathcal{I}(A[M], B[N]).$$

For each N_i, we have a map

$$\mathcal{I}(j) : \mathcal{I}(A[M_i], B[N_i]) \longrightarrow \mathcal{I}(A[M], B[N])$$

induced by the inclusion map $j : B[N_i] \to B[N]$. Hence, by the universal property of the direct limit, there exists a map

$$\phi : \varinjlim_\Lambda (\mathcal{I}(A[M_i], B[N_i])) \longrightarrow \mathcal{I}(A[M], B[N]).$$

We claim that ϕ is an isomorphism. For surjectivity, let $I \in \mathcal{I}(A[M], B[N])$. Hence, there exists $N_k \in \Lambda$ such that $I \in \mathcal{I}(A[M_k], B[N_k])$. Taking the image of I inside $\varinjlim_\Lambda \mathcal{I}(A[M_i], B[N_i])$, we obtain that ϕ is surjective. Since the natural inclusion $j : B \to B[N_i]$ induces an isomorphism $\mathcal{I}(j) : \mathcal{I}(A, B) \cong \mathcal{I}(A[M_i], B[N_i])$ for each N_i, we obtain that $\mathcal{I}(A, B) \cong \varinjlim_\Lambda \mathcal{I}(A[M_i], B[N_i])$. Now, it is easy to see that ϕ is injective. Therefore, we obtain that $\mathcal{I}(A, B) \cong \mathcal{I}(A[M], B[N])$.

(c) As in (b), we can assume that $M = N$ is affine. Then, by (3.2), $A[M]$ is subintegrally closed in $B[M]$. Hence, as in the proof of (b), we obtain that $\mathcal{I}(A, B) \cong \mathcal{I}(A[M], B[M])$.

(d) (i) In order to prove that A is subintegrally closed in B, let $b \in B$ with $b^2, b^3 \in A$. Let $m \in M$. Let $I := (b^2, 1 - bm)$ and $J := (b^2, 1 + bm)$ be two $A[M]$-submodules of $B[N]$. Note that $IJ \subset A[M]$ and $(1 - bm)(1 + bm)(1 + b^2m^2) = 1 - b^4m^4 \in IJ$. Hence, $1 = b^4m^4 + 1 - b^4m^4 \in IJ$, i.e., $IJ = A[M]$. Therefore, $I \in \mathcal{I}(A[M], B[N])$. Let π be the natural surjection from $B[N] \to B$ sending $N \to 0$. Then $\mathcal{I}(\pi)(I) = A$. By hypothesis, $\mathcal{I}(\pi)$ is an isomorphism; hence, $I = A[M]$. Therefore, $b \in A$. Hence, A is subintegrally closed in B.

(ii) Let $g \in B[N]$ be such that $g^2, g^3 \in A[M]$. Let $I := (g^2, 1+g+g^2)$ and $J := (g^2, 1 - g + g^2)$ be two $A[M]$-submodules of $B[N]$. Then

$$\begin{aligned}(1 + g + g^2)(1 - g + g^2) &= (1 + g^2 + g^4) \in IJ \\ &\Longrightarrow 1 + g^2 \in IJ \Longrightarrow 1 \\ &= g^4 + (1 + g^2)(1 - g^2) \in IJ.\end{aligned}$$

Note that $IJ \subset A[M]$, hence, $IJ = A[M]$. Therefore, $I \in \mathcal{I}(A[M], B[N])$. Let $\pi(g) = b \in B$ (π as defined in (i)). Then $\mathcal{I}(\pi)(I) = (b^2, 1 - b + b^2)$. Since $g^2, g^3 \in A[M]$, we obtain that $b^2, b^3 \in A$. However, A is subintegrally closed in B by (i). Hence, we obtain that $b \in A$. Therefore, $\mathcal{I}(\pi)(I), \mathcal{I}(\pi)(J)$ are contained in A and $\mathcal{I}(\pi)(I) = A \Rightarrow I = A[M]$. Hence, $g \in A[M]$. This proves that $A[M]$ is subintegrally closed in $B[N]$.

(iii) Note the commutative diagram

$$\begin{array}{ccc} \mathcal{I}(A, B) & \xrightarrow{\phi_1} & \mathcal{I}(A/\operatorname{nil}(A), B/\operatorname{nil}(B)) \\ \Big\downarrow{\scriptstyle \phi_2} & & \Big\downarrow{\scriptstyle \phi_3} \\ \mathcal{I}(A[M], B[N]) & \xrightarrow[\phi_4]{} & \mathcal{I}\left(\frac{A[M]}{\operatorname{nil}(A)[M]}, \frac{B[N]}{\operatorname{nil}(B)[N]}\right), \end{array}$$

where ϕ_i are natural maps for all i. By (i), we obtain that A is subintegrally closed in B; hence, by [**8,** Lemma 1.2], $\operatorname{nil}(B) \subset A$. Hence, $\operatorname{nil}(B) = \operatorname{nil}(A)$. Therefore, by [**6,** Proposition 2.6], we get that ϕ_1 is an isomorphism. Since N is a cancellative torsion-free monoid, by [**4,** Theorem 4.19], $\operatorname{nil}(B[N]) = \operatorname{nil}(B)[N]$. By (c), we obtain that ϕ_3 is an isomorphism. Hence, ϕ_2 is an isomorphism if and only if ϕ_4 is an isomorphism. By [**6,** Proposition 2.7], ϕ_4 is an isomorphism if and only if $(1 + \mathrm{nil}(B)[N])/(1 + \mathrm{nil}(A)[M])$ is a trivial group. Since $\mathrm{nil}(B) = \mathrm{nil}(A)$, this is equivalent to $\mathrm{nil}(B) = 0$, i.e., B is reduced, or $M = N$. □

Remark 3.3. If A is subintegrally closed in B and $M = N$, then we obtain $\mathcal{I}(A, B) \cong \mathcal{I}(A[M], B[M])$ from the arguments, as in (1.2) (d) (iii), without using Lemma 3.2. Hence, using Theorem 1.2 (d) (ii), we obtain that $A[M]$ is subintegrally closed in $B[M]$. This gives an alternate proof of Lemma 3.2 without the hypothesis that M is affine.

Corollary 3.4. *Let $A \subset B$ be an extension of reduced rings such that A is subintegrally closed in B. Then,*

$$\mathcal{I}(A, B) \cong \mathcal{I}(A[X_1, \ldots, X_m], B[X_1, \ldots, X_m, Y_1, \ldots, Y_n]).$$

Proof. Observe that the submonoid generated by $(X_1, \ldots, X_m)$ is subintegrally closed in the monoid generated by $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$. Hence, we obtain the result using Theorem 1.2 (b). □

In the next remark, we give an application of the result Theorem 1.2 (c).

Remark 3.5 (cf., [**8,** Remark 1.8])**.** Let A be a seminormal ring which is Noetherian or an integral domain. Let M be a positive seminormal monoid. Let K be the total quotient ring of A. Then, K is a finite product of fields; hence, $\mathrm{Pic}(K)$ is a trivial group. By [**3,** Corollary 2], $\mathrm{Pic}(K[M])$ is a trivial group. By [**4,** Proposition 4.20], $U(K) = U(K[M])$ and $U(A) = U(A[M])$. Now, using the same arguments as in [**8,** Remark 1.8], one can easily deduce that $\mathrm{Pic}(A) \cong \mathrm{Pic}(A[M])$ from (1.2).

3.2. Proof of Theorem (1.4).

(i) Following the arguments of [**8,** Theorem 2.6], we observe that we have only to prove that the maps $\phi(A, {}^+A, B)$ and $\phi(A[M], {}^+A[M], B[M])$ are surjective. Since ${}^+A$ is subintegrally closed in B, $\theta({}^+A, B)$ is surjective by (1.2) (c). Therefore, we need only show that $\phi(A, {}^+A, B)$ is surjective. However, this follows from [**7,** Proposition 3.1] by taking $C = {}^+A$.

(ii) If $A \subset B$ is a subintegral extension of $\mathbb{Q}$-algebras, then a natural isomorphism $\xi_{B/A} : B/A \to \mathcal{I}(A, B)$ is defined in [**6**]. As in [**6,** Lemma 5.3], this yields a commutative diagram

$$\begin{array}{ccc} {}^+A/A & \xrightarrow{\ \xi_{{}^+A/A}\ } & \mathcal{I}(A, {}^+A) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \theta(A, {}^+A)} \\ {}^+A[M]/A[M] & \xrightarrow[\xi]{} & \mathcal{I}(A[M], {}^+A[M]), \end{array}$$

where $\xi := \xi_{{}^+A[M]/A[M]}$. Both $\xi_{{}^+A/A}$ and ξ are isomorphisms by ([**6**], main Theorem 5.6 and [**5,** Theorem 2.3]). Now $\mathcal{I}(A[M], {}^+A[M]) \cong {}^+A[M]/A[M] \cong \mathbb{Z}[M] \otimes_{\mathbb{Z}} {}^+A/A \cong \mathbb{Z}[M] \otimes_{\mathbb{Z}} \mathcal{I}(A, {}^+A)$. $\square$

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