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SERRE DIMENSION AND EULER CLASS GROUPS OF OVERRINGS OF POLYNOMIAL RINGS

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ABSTRACT. Let R be a commutative Noetherian ring of dimension d and

$$B = R[X_1, \dots, X_m, Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$$

a Laurent polynomial ring over R. If $A = B[Y, f^{-1}]$ for some $f \in R[Y]$, then we prove the following results:

(i) if f is a monic polynomial, then the Serre dimension of A is $\leq d$. The case $n = 0$ is due to Bhatwadekar, without the condition that f is a monic polynomial.

(ii) The pth Euler class group $E^p(A)$ of A, defined by Bhatwadekar and Sridharan, is trivial for $p \geq \max\{d+1, \dim A - p + 3\}$. The case $m = n = 0$ is due to Mandal and Parker.

1. Introduction. *In this paper, we will assume that all rings are commutative Noetherian of finite Krull dimension, all modules are finitely generated and all projective modules are of constant rank.* Throughout this paper, R will denote a ring of dimension d and B will denote the Laurent polynomial ring

$$R[X_1, \dots, X_m, Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$$

over R.

Let P be a projective R-module. An element $p \in P$ is said to be *unimodular* if there exist $\phi \in \text{Hom}(P, R)$ such that $\phi(p) = 1$. We write $\text{Um}(P)$ for the set of all unimodular elements of P. We say that the *Serre dimension* of R is $\leq t$ if every projective R-module of rank $\geq t+1$ has a unimodular element.

A classical result of Serre **[22]** is that the Serre dimension of R is $\leq d$. Quillen **[20]** and Suslin **[23]** proved Serre's conjecture that projective modules over polynomial rings $k[X_1, \dots, X_m]$ over a field k are free for all $m \geq 1$. In other words, the Serre dimension of $k[X_1, \dots, X_m]$ is 0.

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Plumstead [**19,** Theorem 2] generalized Serre's result by proving that the Serre dimension of $R[Y]$ is $\leq d$. Rao [**21,** Theorem 1.1] generalized Plumstead's result and proved that, if C is a birational overring of $R[Y]$, i.e.,

$$R[Y] \subset C \subset S^{-1}R[Y],$$

where S is the set of all nonzero divisors of $R[Y]$, then the Serre dimension of C is $\leq d$. As a consequence of Rao's result, we obtain the Serre dimension of $R[Y, f^{-1}] \leq d$ for any nonzero divisor $f \in R[Y]$.

Bhatwadekar and Roy [**6,** Theorem 3.1] generalized Plumstead's result to polynomial rings in many variables and proved that the Serre dimension of the polynomial ring $R[X_1, \ldots, X_m]$ is $\leq d$ for any $m \geq 1$. This result was generalized by Bhatwadekar, Lindel and Rao [**2,** Theorem 4.1] to the Laurent polynomial case. They proved that the Serre dimension of the Laurent polynomial ring

$$B := R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$$

is $\leq d$.

Bhatwadekar [**1,** Theorem 3.5] further generalized Bhatwadekar and Roy's result to polynomial extensions over a birational overring of $R[Y]$. More precisely, he proved that, if C is a birational overring of $R[Y]$, then the Serre dimension of

$$C[X_1, \ldots, X_m]$$

is $\leq d$. As a consequence of this result, we obtain that the Serre dimension of

$$R[X_1, \ldots, X_m, Y, f^{-1}]$$

is $\leq d$ for any nonzero divisor $f \in R[Y]$.

It is natural to ask whether an analogue of Bhatwadekar's result [**1**] is true for Laurent polynomial rings. More precisely, we can ask the following.

Question 1.1. *Let C be a birational overring of $R[Y]$. Is the Serre dimension of*

$$C[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] \leq d?$$

We answer this question when $C = R[Y, f^{-1}]$ with $f \in R[Y]$ a monic polynomial. Note that Lindel **[12]** gave another proof of **[2,** Theorem 4.1] mentioned above. Our proof closely follows Lindel's idea. Next, we state our result.

Theorem 1.2. *Let $A = B[Y, f^{-1}]$, where $f \in R[Y]$ is a monic polynomial. Then the Serre dimension of A is $\leq d$.*

Assume that $\dim R = d \geq 3$ and p is a positive integer such that $p \geq d - p + 3$. Then Bhatwadekar and Sridharan defined the pth Euler class group $E^p(R)$ of R which is an additive abelian group. We will not give the explicit definition of $E^p(R)$ (see **[5,** Section 4], for definition). Rather, we will describe the elements of $E^p(R)$, since this suffices for our purposes. Let I be an ideal of R of height p such that the R/I-module I/I^2 is generated by p elements. Let

$$\phi : (R/I)^p \twoheadrightarrow I/I^2$$

be a surjection, giving a set of p generators of R/I-module I/I^2. The surjection ϕ induces an element of the pth Euler class group $E^p(R)$, denoted by the pair (I, ϕ). Furthermore, it follows using the *moving lemma and addition principle*, that every element of $E^p(R)$ is a pair (I, ϕ) for some height p ideal I of R and some surjection $\phi : (R/I)^p \twoheadrightarrow I/I^2$. Bhatwadekar and Sridharan **[5,** Theorem 4.2] proved that there exists a surjection

$$\Phi : R^p \twoheadrightarrow I$$

which is a lift of ϕ, i.e., $\Phi \otimes A/I = \phi$, if and only if the associated element (I, ϕ) of the group $E^p(R)$ is the trivial element (identity element 0 of $E^p(R)$).

It is well known that a projective R-module of rank d need not, in general, have a unimodular element. The significance of Euler class group theory is demonstrated by the result of **[3]**, where it was proved that, for a rank d projective R-module P with trivial determinant, the precise obstruction for P to have a unimodular element lies in $E^d(R)$. More precisely, given a pair (P, χ), where

$$\chi : \wedge^d P \xrightarrow{\sim} R$$

is an isomorphism, an element $e(P,\chi)$ was associated with the Euler class group $E^d(R)$, and it was proved that P has a unimodular element if and only if $e(P,\chi)$ is the trivial element of $E^d(R)$. Such an obstruction theory is not known for projective R-modules of rank $d-1$ except for a special class of rings. When $R=S[Y]$ is a polynomial ring in one variable over a subring S of R, then Das [**7**] proved that, for a rank $d-1$ projective R-module Q with trivial determinant, the precise obstruction for Q to have a unimodular element lies in $E^{d-1}(R)$.

Let I be an ideal of $R[Y]$ containing a monic polynomial in the variable Y. Assume the $R[Y]/I$-module I/I^2 is generated by p elements, where $p \geq \dim(R[Y]/I)+2$. Mandal [**13,** Theorem 2.1] proved that any surjection

$$\phi : (R[Y]/I)^p \twoheadrightarrow I/I^2$$

can be lifted to a surjection

$$\Phi : R[Y]^p \twoheadrightarrow I.$$

Let $P=Q\oplus R$ be a projective R-module of rank p and

$$\psi : P[Y]/IP[Y] \twoheadrightarrow I/I^2$$

a surjection. Then Bhatwadekar et al. [**5,** Proposition 3.3] proved that ψ lifts to a surjection

$$\Psi : P[Y] \twoheadrightarrow I,$$

thus generalizing Mandal's result. If we further assume that height of I is p and $2p \geq \dim R[Y]+3$, then we can associate an element $(I,\phi) \in E^p(R[Y])$ to the surjection ϕ. Since Φ is a surjective lift of ϕ, by [**5,** Theorem 4.2], we obtain that (I,ϕ) is a trivial element of $E^p(R[Y])$.

Let

$$A = R[X_1,\dots,X_m]$$

be a polynomial ring over R and I an ideal of A of height $\geq d+1$. Let

$$p \geq \max\{\dim(A/I)+2, d+1\}$$

be an integer and

$$\phi : (A/I)^p \twoheadrightarrow I/I^2$$

a surjection. Since the height of $I > d$, by Suslin (2.5), there exists an automorphism Θ of A such that $\Theta(I)$ contains a monic polynomial in

X_m with coefficients from

$$R[X_1, \dots, X_{m-1}].$$

Therefore, replacing I by $\Theta(I)$, we may assume that I contains a monic polynomial in X_m. By the above-mentioned Mandal [**13,** Theorem 2.1], ϕ can be lifted to a surjection

$$\Phi : A^p \twoheadrightarrow I.$$

Therefore, if we further assume that

$$p \geq \max\{\dim A - p + 3, d + 1\},$$

then, by [**5,** Theorem 4.2], the associated element (I, ϕ) of $E^p(A)$ is trivial. Since any element of $E^p(A)$ is a pair (I, ϕ) for some height p ideal I of A, we get the pth Euler class group $E^p(A) = 0$. In particular,

$$E^{d+1}(R[Y]) = 0 \quad \text{for } d \geq 2.$$

This result is generalized by Mandal and Parker [**16,** Theorem 3.1] where

$$E^{d+1}(R[Y, f^{-1}]) = 0 \quad \text{for } d \geq 2 \text{ and } f \in R[Y]$$

is proved. We generalize Mandal and Parker's result as follows.

Theorem 1.3. *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$, and let p be an integer such that*

$$p \geq \max\{\dim A - p + 2, d + 1\}.$$

Let $P = Q \oplus R$ be a projective R-module of rank p and I a proper ideal of A of height $\geq d + 1$. *Assume there is a surjection*

$$\phi : P \otimes A/I(P \otimes A) \twoheadrightarrow I/I^2.$$

Then ϕ can be lifted to a surjection

$$\Phi : P \otimes A \twoheadrightarrow I.$$

As a consequence, taking P to be free, we obtain that any p generators of I/I^2 can be lifted to p generators of I.

The next result is a direct consequence of Theorem 1.3.

Corollary 1.4. *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$, and let p be an integer such that*

$$p \geq \max\{\dim A - p + 3, d + 1\}.$$

Then the pth Euler class group $E^p(A)$ of A is zero.

Let I be an ideal of $R[Y]$ containing a monic polynomial and P a projective R-module of rank p with $p \geq \dim(R[Y]/I) + 2$. Let

$$\phi : P[Y]/IP[Y] \twoheadrightarrow I/I^2$$

and

$$\delta : P \twoheadrightarrow I(0) := \{f(0) \mid f \in I\}$$

be two surjections such that $\phi(0) = \delta \otimes R/I(0)$. Then, Mandal **[14,** Theorem 2.1] proved that there exists a surjection $\Phi : P[Y] \twoheadrightarrow I$ such that

$$\Phi \otimes R[Y]/I = \phi$$

and

$$\Phi(0) = \delta,$$

thus answering a question of Nori, see **[14]**, on homotopy sections of projective modules, in case the ideal I contains a monic polynomial.

The above result of Mandal on the homotopy section was generalized by Kumar and Mandal **[10,** Theorem 1.2] to the Laurent polynomial case as follows. Let I be an ideal of $R[Y, Y^{-1}]$ containing a monic polynomial f in $R[Y]$ with $f(0) = 1$. Let P be a projective R-module of rank p with

$$p \geq \dim(R[Y, Y^{-1}]/I) + 2.$$

Let

$$\phi : P[Y, Y^{-1}]/IP[Y, Y^{-1}] \twoheadrightarrow I/I^2$$

and

$$\delta : P \twoheadrightarrow I(1) := \{g(Y = 1) | g \in I\}$$

be two surjections such that $\phi(1) = \delta \otimes R/I(1)$. Then, there exists a surjection

$$\Phi : P[Y, Y^{-1}] \twoheadrightarrow I$$

such that

$$\Phi \otimes R[Y, Y^{-1}]/I = \phi \quad \text{and} \quad \Phi(1) = \delta.$$

We prove the following analogue of Kumar and Mandal's result.

Theorem 1.5. *Let $A = B[Y, f^{-1}]$, where $f \in R[Y]$ is a monic polynomial with $f(1)$ a unit in R. Let I be an ideal of A of* height $\geq d+1$ *and P a projective B-module of* rank $\geq \max\{d+1, \dim(A/I)+2)\}$. *Let*

$$\phi : P[Y, f^{-1}]/IP[Y, f^{-1}] \twoheadrightarrow I/I^2$$

and

$$\delta : P \twoheadrightarrow I(1) \ (:= \{g(Y = 1) \mid g \in I\})$$

be two surjections such that $\delta \otimes I(1)/I(1)^2 = \phi \otimes A/(Y-1)$, where $I(1)$ is an ideal of B. Then there exists a surjection

$$\Psi : P[Y, f^{-1}] \twoheadrightarrow I$$

such that $\Psi \otimes A/I = \phi$ and $\Psi(1) = \delta$.

2. Preliminaries. In this section, we note some results for later use. For a ring A, $\operatorname{ht} I$ will denote the height of an ideal I of A. We begin by stating a result of Lindel [**12,** Lemma 1.1].

Proposition 2.1. *Let A be a ring, Q an A-module and $s \in A$ such that Q_s is a free A_s-module of rank r. Then there exist $p_1, \ldots, p_r \in Q$, $\phi_1, \ldots, \phi_r \in Q^*$ and $t \geq 1$ such that*:

(i) $0 :_A s'A = 0 :_A {s'}^2 A$, *where $s' = s^t$.*
(ii) *$s'Q \subset F$ and $s'Q^* \subset G$, where*

$$F = \sum_{i=1}^{r} Ap_i \subset Q$$

and

$$G = \sum_{i=1}^{r} A\phi_i \subset Q^*,$$

(iii) $(\phi_i(p_j))_{1 \leq i,j \leq r} = \text{diagonal}\,(s', \ldots, s')$. *We say F and G are s'-dual submodules of Q and Q^*, respectively.*

Definition 2.2.

(i) Let A be a ring, M an A-module and δ an A-endomorphism. We say that the maps

$$\xi : M \longrightarrow M \quad \text{and} \quad \xi^* : M^* \longrightarrow M^*$$

are δ-semilinear if ξ and ξ^* are group homomorphisms with respect to addition operation and

$$\xi(\alpha m) = \delta(\alpha)\xi(m), \qquad \xi^*(\alpha\phi) = \delta(\alpha)\xi^*(\phi)$$

for any $m \in M$, $\phi \in M^*$ and $\alpha \in A$.

(ii) Let I be an ideal of A and $s \in A$. An endomorphism $h : A \to A$ is called $s^t I$-analytic, $t \in \mathbb{N}$, if $h(s) = s$ and $h(a) - a \in s^t I$ with $0 :_A s^{t-1} = 0 :_A s^t$.

The next result is due to Lindel [**12,** Lemma 1.4].

Lemma 2.3. *Let A be a ring, I an ideal in A and M an A-module such that M_s is free for some $s \in A$. Let*

$$F = \sum_{i=1}^{r} Ap_i \subset M \quad \textit{and} \quad G = \sum_{i=1}^{r} A\phi_i \subset M^*$$

be two submodules as in Proposition 2.1. *Assume that an $s^{2t}I$-analytic endomorphism h of A is given. Then there exist h-semilinear maps*

$$\xi : M \longrightarrow M \quad \textit{and} \quad \xi^* : M^* \longrightarrow M^*$$

with the following properties:

(i) *$\xi(p) - p \in s^t IF$, $\xi^*(\phi) - \phi \in s^t IG$ and $\xi^*(\phi)\xi(p) = h(\phi(p))$ for all $p \in M$ and $\phi \in M^*$.*

(ii) *If N and N' are submodules of F and G, respectively, such that*

$$F \subset N \subset M \quad \textit{and} \quad G \subset N' \subset M^*,$$

then $\xi(N) = N$ and $\xi^*(N') = N'$.

The next result on fiber products is well known. For a reference, see [**15,** Proposition 2.2.1].

Proposition 2.4. *Let A be a ring, and let $f, g \in A$ be such that $fA + gA = A$. Let M and N be two A-modules. Suppose that*

$$\phi : M_f \longrightarrow N_f$$

is an A_f-homomorphism and

$$\psi : M_g \longrightarrow N_g$$

is an A_g-homomorphism such that $\phi_g = \psi_f$. Then,

(i) *there exists an A-homomorphism $\xi : M \to N$ such that $\xi_f = \phi$ and $\xi_g = \psi$.*
(ii) *If ϕ and ψ are surjective, then ξ is surjective.*

The following is implicit in Suslin's result [**24,** Lemma 6.2] and is known as Suslin's monic polynomial theorem.

Theorem 2.5. *Let I be an ideal of $R[X_1, \ldots, X_m]$ of* height $> d$. *Then there exist a positive integer N such that, for any integers $s_i > N$, if ϕ is the $R[X_m]$-automorphism of $R[X_1, \ldots, X_m]$ defined by $\phi(X_i) = X_i + X_m^{s_i}$ for $1 \leq i \leq m-1$, then $\phi(I)$ contains a monic polynomial in X_m with coefficients from $R[X_1, \ldots, X_{m-1}]$.*

The next result is implicit in Mandal's result [**13,** Lemma 2.3].

Lemma 2.6. *Let I be an ideal of B of* height $> d$ *and $n > 0$. Then there exists an $R[Y_n^{\pm 1}]$-automorphism Θ of B such that $\Theta(I)$ contains a monic polynomial in Y_n of the form $1 + Y_n h$ for some*

$$h \in R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_{n-1}^{\pm 1}, Y_n].$$

The next result is due to Bhatwadekar et al. [**2,** Theorem 4.1].

Theorem 2.7. *Let P be a projective B-module of* rank $> d$. *Then P has a unimodular element.*

The next result is due to Bhatwadekar et al. [**4,** Proposition 3.3].

Proposition 2.8. *Let I be an ideal of $R[X]$ containing a monic polynomial and $P = Q \oplus A$ a projective R-module of rank r, where $r \geq \dim(R[X]/I) + 2$. Let*

$$\phi : P[X] \twoheadrightarrow I/I^2$$

be a surjection. Then, ϕ can be lifted to a surjection

$$\Phi : P[X] \twoheadrightarrow I.$$

The next result is due to Dhorajia and Keshari [**8,** Theorem 3.12]. We will only state the necessary part here.

Theorem 2.9. *Let*

$$A = R[X_1, \ldots, X_m, Y_1, \ldots, Y_n, (f_1 \ldots f_n)^{-1}]$$

with $f_i \in R[Y_i]$ and P a projective A-module of $\operatorname{rank} r \geq d + 1$. *Then P is cancellative, i.e.,*

$$P \oplus A^t \tilde{\rightarrow} Q \oplus A^t$$

for some integer $t > 0$ implies $P \tilde{\rightarrow} Q$.

Definition 2.10. For an integer $n > 0$, a sequence of elements $a_1, \ldots, a_n$ in R is said to be a *regular sequence* of length n if a_i is a nonzero divisor in

$$R/(a_1, \ldots, a_{i-1}) \quad \text{for } i = 1, \ldots, n.$$

Let I be an ideal of R. We say I is *set theoretically* generated by n elements $f_1, \ldots, f_n \in R$ if

$$\sqrt{I} = \sqrt{(f_1, \ldots, f_n)}.$$

Assume that the height of I is n. Then I is said to be a *complete intersection* ideal if I is generated by a regular sequence of length n. Furthermore, I is said to be a *locally complete intersection* ideal if $I_{\mathfrak{p}}$ is a complete intersection ideal of height n for all prime ideals $\mathfrak{p}$ of R containing I. □

The next result is due to Mandal and Roy [**17,** Theorem 2.1]. See also [**13,** Theorem 6.2.2].

Theorem 2.11. *Let $J \subset I$ be two ideals of $R[X]$ such that I contains a monic polynomial. Assume that*

$$I = (f_1, \ldots, f_n) + I^2$$

and

$$J = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}.$$

Then J is generated by n elements. As a consequence, since $\sqrt{I} = \sqrt{J}$, I is set-theoretically generated by n elements.

The next result is due to Ferrand and Szpiro. For a proof, see [**18, 26**].

Theorem 2.12. *Let I be a locally complete intersection ideal of R of height $n \geq 2$ with $\dim(R/I) \leq 1$. Then there is a locally complete intersection ideal $J \subset R$ of height n such that*:

(i) $\sqrt{I} = \sqrt{J}$ *and*
(ii) J/J^2 *is a free R/J-module of* rank n.

The next result is easy to prove; hence, we omit the proof.

Lemma 2.13. *Let $f \in R[T] - R$. Then*

(i) *if I is a proper ideal of $R[T, f^{-1}]$, then*

$$\operatorname{ht} I = \operatorname{ht}(I \cap R[T]).$$

(ii) *If I is a proper ideal of $R[f, f^{-1}]$, then*

$$\operatorname{ht} I = \operatorname{ht}(I \cap R[f^{-1}]).$$

Lemma 2.14. *Let I be an ideal of $A = R[T, f^{-1}]$, where $f \in R[T] - R$. If*

$$J = I \cap R[f^{-1}],$$

then $\operatorname{ht} J = \operatorname{ht} I$.

Proof. Assume that I is a prime ideal. If we write $\mathfrak{a} = I \cap R$, then

$$\operatorname{ht} I = \operatorname{ht} IA_{\mathfrak{a}} \quad \text{and} \quad \operatorname{ht} J = \operatorname{ht} JR_{\mathfrak{a}}[f^{-1}].$$

Hence, we assume that $(R, \mathfrak{a})$ is a local ring. Furthermore, if $I = \mathfrak{a}A$ is an extended ideal, then

$$\operatorname{ht} I = \operatorname{ht} \mathfrak{a} = \operatorname{ht} J.$$

Hence, assume that $I \neq \mathfrak{a}A$. In this case $\operatorname{ht} I = \operatorname{ht} \mathfrak{a} + 1$. Since $R/\mathfrak{a}$ is a field, we obtain that

$$R/\mathfrak{a}[f, f^{-1}] \longrightarrow R/\mathfrak{a}[T, f^{-1}]$$

is an integral extension. Hence,

$$\operatorname{ht} I/\mathfrak{a} = \operatorname{ht} \widetilde{J}/\mathfrak{a},$$

where $\widetilde{J} = I \cap R[f, f^{-1}]$. Therefore,

$$\operatorname{ht} I = \operatorname{ht} \mathfrak{a} + 1 = \operatorname{ht} \widetilde{J} = \operatorname{ht} J,$$

by equation (2.13). The general case follows by noting that $\operatorname{ht} I = \operatorname{ht} \sqrt{I}$,

$$\sqrt{I} = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r,$$
$$\sqrt{J} = \mathcal{P}'_1 \cap \cdots \cap \mathcal{P}'_r,$$

where

$$\mathcal{P}'_i = \mathcal{P}_i \cap R[f^{-1}] \quad \text{and} \quad \operatorname{ht} \mathcal{P}_i = \operatorname{ht} \mathcal{P}'_i. \qquad \square$$

Lemma 2.15. *Let R be a ring of dimension d,*

$$B = R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}],$$
$$A = B[Y, f^{-1}],$$

where $f \in R[Y]$ is a nonconstant polynomial and I an ideal of A of height $> d$. *Then, there exists an integer $N > 0$ such that, for any set of integers $s_i, l_i > N$ and the $R[Y, f^{-1}]$-automorphism ϕ of A defined by*

$$\phi(X_i) = X_i + f^{-s_i} \quad \textit{and} \quad \phi(Y_i) = Y_i f^{l_i}$$

such that $\phi(I)$ contains a polynomial of the form $1 + fh$ for some $h \in B[Y]$.

Proof. We induct on n. Assume that $n = 0$. If $I_1 = I \cap B[f^{-1}]$, then by Lemma 2.14,

$$\operatorname{ht} I_1 = \operatorname{ht} I > d.$$

Applying Theorem 2.5 to the ring

$$B[f^{-1}] = R[X_1, \ldots, X_m, f^{-1}],$$

we can find a positive integer N_1 such that, for any integers $s_i > N_1$, if ϕ_1 is the $R[f^{-1}]$-automorphism of $B[f^{-1}]$ defined by

$$\phi_1(X_i) = X_i + f^{-s_i} \quad \text{for } 1 \le i \le m,$$

then $\phi_1(I_1)$ contains a monic polynomial, say F, of degree u, in the variable f^{-1} with coefficients from B. Since ϕ_1 naturally extends to an $R[Y, f^{-1}]$-automorphism of A, we obtain that $\phi_1(I)$ contains F, and hence, it contains $f^u F$ which is of the form $1 + fg$ for some $g \in B[Y]$.

Assume that $n > 0$. Define $L_{Y_n}(I)$ and $L_{Y_n{}^{-1}}(I)$ as the set of highest degree coefficients and lowest degree coefficients, respectively, of elements in I as a Laurent polynomial in the variable Y_n. It is easy to see that $L_{Y_n}(I)$ and $L_{Y_n{}^{-1}}(I)$ are ideals of $C[Y, f^{-1}]$, where

$$C = R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_{n-1}^{\pm 1}].$$

By [**13,** Lemma 3.1], we obtain that the height of the ideals $L_{Y_n}(I)$ and $L_{Y_n{}^{-1}}(I)$ is $\ge \operatorname{ht} I$.

If we write

$$L = L_{Y_n}(I) \cap L_{Y_n{}^{-1}}(I),$$

then L is an ideal of $C[Y, f^{-1}]$ of height $\ge \operatorname{ht} I > d$. Hence, by induction on n, there exists an integer N_2 such that, for any set of integers s_i, l_i all greater than N_2, if θ_1 is an $R[Y, f^{-1}]$-automorphism of $C[Y, f^{-1}]$ defined by

$$\theta_1(X_i) = X_i + f^{-s_i} \quad \text{and} \quad \theta_1(Y_j) = Y_j f^{l_j}$$

for $1 \le i \le m$ and $1 \le j \le n-1$, then $\theta_1(L)$ contains a polynomial

$$\widetilde{h} = 1 + fh' \quad \text{for some } h' \in C[Y].$$

We extend θ_1 to an $R[Y_n^{\pm 1}, Y, f^{-1}]$-automorphism of A. We can find a polynomial G in $\theta_1(I)$ of the form

$$G = \widetilde{h} + h_1 Y_n + \cdots + h_t Y_n^t \quad \text{for some } t \in \mathbb{N},$$

$h_i \in C[Y, f^{-1}]$ and $\widetilde{h}$ as above. We can choose an integer

$$N_3 = \max\{\text{power of } f^{-1} \text{ occuring in } G\}$$

such that, for any integer $l_n > N_3$, if θ_2 is an $C[Y, f^{-1}]$-automorphism of A defined by $\theta_2(Y_n) = Y_n f^{l_n}$, then

$$\theta_2(G) = 1 + fh \quad \text{for some } h \in B[Y].$$

We note that $\theta_2\theta_1$ is an $R[Y, f^{-1}]$-automorphism of A defined by

$$X_i \longmapsto X_i + f^{-s_i} \quad \text{and} \quad Y_j \longmapsto Y_j f^{l_j}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. Taking $N = \max\{N_2, N_3\}$ completes the proof. □

Proposition 2.16. *Let $A = B[Y, f^{-1}]$, where $f \in R[Y]$ is a monic polynomial and I an ideal of A of* height $> d$. *Then, there exists an integer $N > 0$ such that, for any set of integers t_i, s_i, l_i all greater than N, the $R[Y, f^{-1}]$-automorphism ϕ of A defined by*

$$\phi(X_i) = X_i + Y^{t_i} + f^{-s_i}$$

and

$$\phi(Y_i) = Y_i f^{l_i}$$

satisfies the following:

(i) *$\phi(I)$ contains a monic polynomial in Y with coefficients from B, and*
(ii) *$\phi(I)$ contains a polynomial of the form $1+fh$ for some $h \in B[Y]$.*

Proof. If $n = 0$, then

$$B = R[X_1, \ldots, X_m].$$

If $I_1 = I \cap B[f^{-1}]$, then by Lemma 2.14, $\operatorname{ht} I_1 = \operatorname{ht} I > d$. By Lemma 2.15, we can find a positive integer N_1 such that, for any integers $s_i > N_1$, if ϕ_1 is the $R[Y, f^{-1}]$-automorphism of $B[f^{-1}]$ defined by

$$\phi_1(X_i) = X_i + f^{-s_i} \quad \text{for } 1 \leq i \leq m.$$

Then $\phi_1(I_1)$ contains a polynomial of the form $1+fg$ for some $g \in B[Y]$.

If $I_2 = \phi_1(I) \cap B[Y]$, then by Lemma 2.13,

$$\operatorname{ht} I_2 = \operatorname{ht} I > d.$$

Applying Theorem 2.5 to the ring

$$B[Y] = R[X_1, \dots, X_m, Y],$$

we can find a positive integer N_2 such that, for any integers $t_i > N_2$, if ϕ_2 is the $R[Y]$-automorphism of $B[Y]$ defined by

$$\phi_2(X_i) = X_i + Y^{t_i} \quad \text{for } 1 \le i \le m,$$

then $\phi_2(I_2)$ contains a monic polynomial, say G, in the variable Y with coefficients from B. Since ϕ_2 naturally extends to an $R[Y, f^{-1}]$-automorphism of A, we obtain that $\phi_2\phi_1(I)$ contains

(i) a monic polynomial G in the variable Y with coefficients from B, and
(ii) an element $1 + fh$, where $h = \phi_2(g) \in B[Y]$.

Note that $\phi_2\phi_1$ is an $R[Y, f^{-1}]$-automorphism of A defined by

$$X_i \longmapsto X_i + Y^{t_i} + f^{-s_i}.$$

This proves the result in case $n = 0$ by taking $N = \max\{N_1, N_2\}$.

Assume that $n > 0$, and use induction on n. Defining $L_{Y_n}(I)$, $L_{Y_n^{-1}}(I)$ and L as in Lemma 2.15, we obtain that L is an ideal of $C[Y, f^{-1}]$ of height $\ge \operatorname{ht} I > d$. Hence, by induction on n, there exists an integer N_3 such that, for any set of integers t_i, s_i, l_i all greater than N_3, if θ_1 is an $R[Y, f^{-1}]$-automorphism of $C[Y, f^{-1}]$ defined by

$$\theta_1(X_i) = X_i + Y^{t_i} + f^{-s_i} \quad \text{and} \quad \theta_1(Y_j) = Y_j f^{l_j}$$

for $1 \le i \le m$ and $1 \le j \le n-1$, then $\theta_1(L)$ contains

(a) a monic polynomial, say $\widetilde{g}$, in Y with coefficients from C, and
(b) a polynomial $\widetilde{h}$ of the form $1 + fh'$ for some $h' \in C[Y]$.

We extend θ_1 to an $R[Y_n^{\pm 1}, Y, f^{-1}]$-automorphism of A. We can find polynomials F and G in $\theta_1(I)$ of the form

$$F = \widetilde{g} Y_n^s + g_{n-1} Y_n^{s-1} + \cdots + g_0$$

and

$$G = \widetilde{h} + h_1 Y_n + \cdots + h_t Y_n^t,$$

for some $s, t \in \mathbb{N}$, $g_i, h_i \in C[Y, f^{-1}]$ and $\widetilde{g}, \widetilde{h}$ as in (a) and (b). We can choose an integer

$$N_4 = \max\{\text{power of } f^{-1} \text{ occuring in } G \text{ and degrees of } \widetilde{g},\, g_i \text{ in } Y\}$$

such that, for any integer $l_n > N_4$, if θ_2 is an $C[Y, f^{-1}]$-automorphism of A defined by $\theta_2(Y_n) = Y_n f^{l_n}$, then

(i) $Y_n^{-s}\theta_2(F)$ is a monic polynomial in Y with coefficients from $C[Y_n^{\pm 1}] = B$ (here we are using f to be monic), and
(ii) $\theta_2(G) = 1 + fh$ for some $h \in B[Y]$.

We note that $\theta_2\theta_1$ is an $R[Y, f^{-1}]$-automorphism of A defined by

$$X_i \longmapsto X_i + Y^{t_i} + f^{-s_i}$$

and

$$Y_j \longmapsto Y_j f^{l_j}$$

for $1 \le i \le m$ and $1 \le j \le n$. Taking $N = \max\{N_3, N_4\}$ completes the proof. $\square$

3. Main theorems. In this section, we prove the results stated in the introduction.

Theorem 3.1. *Let R be a ring of dimension d,*

$$B = R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}]$$

and $A = B[Y, f^{-1}]$, where $f \in R[Y]$ is a monic polynomial. Then the Serre dimension of A is $\le d$.

Proof. Without loss of generality, we may assume that R is reduced. If $m = 0$, then replacing A by $A[X_1]$, we will assume that $m > 0$. Let

P be a projective A-module of rank $r > d = \dim R$. We need to show that P has a unimodular element. If S denotes the set of all nonzero divisors of R, then $S^{-1}R$ is a zero-dimensional ring. Therefore, by [**8,** Lemma 3.9], we can find some $s \in S$ such that P_s is a free A_s-module of rank r. By Proposition 2.1, there exist an integer $t > 0$,

$$p_1, \dots, p_r \in P \quad \text{and} \quad \phi_1, \dots, \phi_r \in P^*$$

such that the submodules

$$F = \sum_{i=1}^{r} Ap_i \quad \text{of } P \quad \text{and} \quad G = \sum_{i=1}^{r} A\phi_i \quad \text{of } P^*$$

satisfy:

$$s^t P \subset F, \qquad s^t P^* \subset G$$

and the matrix

$$(\phi_i(p_j)) = \operatorname{diag}(s^t, \dots, s^t).$$

Submodules F and G are called s^t*-dual submodules* of P and P^*, respectively. Replacing s by s^t, we assume that F and G satisfy

$$sP \subset F, \qquad sP^* \subset G$$

and

$$(\phi_i(p_j)) = \operatorname{diag}(s, \dots, s).$$

Since

$$A/(s(Y-1)) = \widetilde{R}[X_1, \dots, X_m, Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$$

is a Laurent polynomial ring over a d dimensional ring $\widetilde{R} := R[Y, f^{-1}]/(s(Y-1))$, by Theorem 2.7, $P/(s(Y-1))$ has a unimodular element. Let $p \in P$ be such that its image $\overline{p}$ in $P/s(Y-1)P$ is a unimodular element.

Let us write $\phi_i(p) = a_i \in A$ for $1 \le i \le r$ and define $b := (1-Y)\prod_{i=1}^{m} X_i \prod_{j=1}^{n} Y_j$. Then sb is a nonzero divisor in A. We can find an integer $l > \deg(a_1)$ such that $a_1' := a_1 + s^2 b^l$ is a nonzero divisor in A, where $\deg(a_1)$ is the total degree of a_1 as a polynomial in $X_1, \dots, X_m$ with coefficients from $R[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}, Y, f^{-1}]$. Hence height of the ideal $a_1'A$ is ≥ 1.

Since $\overline{p}$ is a unimodular element in $P/s(Y-1)P$ and $\phi_1, \dots, \phi_r$ is a basis of the free module P_s^*, we get that

$$(a_1, a_2, \dots, a_r, s^2(Y-1)) \in \mathrm{Um}_{r+1}(A_s).$$

Since

$$a_1' \in a_1 + s^2(Y-1)A,$$

we obtain

$$(a_1', a_2, \dots, a_r, s^2(Y-1)) \in \mathrm{Um}_{r+1}(A_s).$$

Hence, by the prime avoidance argument, we can choose $c_2, \dots, c_r$ in A such that if

$$a_i' = a_i + s^2(Y-1)c_i \quad \text{for } 2 \le i \le r,$$

then height of the ideal

$$(a_1', \dots, a_r')A_{s(Y-1)}$$

is $\ge r$. Let $l' > 2\widetilde{d}$ be an integer, where $\widetilde{d}$ is the maximum of total degrees of $a_1', \dots, a_r'$ as a polynomial in $X_1, \dots, X_m$. If we write

$$a_r'' := a_r' + s^2(Y-1)(a_1')^{l'},$$

then the degree of a_r'' as a polynomial in $X_1, \dots, X_m$ is $e' := mll'$.

Let

$$q = c_2p_2 + \cdots + c_{r-1}p_{r-1} + (c_r + (a_1')^{l'})p_r.$$

Then

$$\widetilde{p} := p + sb^lp_1 + s(Y-1)q$$

is also a lift of $\overline{p}$. Furthermore, we have $\phi_i(\widetilde{p}) = a_i'$ for $1 \le i \le r-1$ and $\phi_r(\widetilde{p}) = a_r''$. Hence, replacing p by $\widetilde{p}$, we see that height of the ideal

$$O_P(p)A_{s(Y-1)} = (a_1', \dots, a_{r-1}', a_r'')A_{s(Y-1)}$$

is $\ge r$.

Since $\overline{p}$ is a unimodular element in $P/s(Y-1)P$ and $p \in P$ is a lift of $\overline{p}$, we get

$$O_P(p) + s(Y-1)A = A.$$

Furthermore, height of the ideal $O_P(p)A_{s(Y-1)}$ is $\ge r$. Therefore, we get that the height of the ideal $O_P(p)$ is $\ge r$. By Proposition 2.16, there exists an integer $N > 0$ such that, for any integers t', s', l'' all greater than N, if Θ is the $R[Y, f^{-1}]$-automorphism of A defined by

$$\Theta(X_i) = X_i + Y^{t'} + f^{-s'} \quad \text{and} \quad \Theta(Y_j) = Y_j f^{l''}$$

for $1 \le i \le m$ and $1 \le j \le n$, then the following hold:

(a) $\Theta(O_P(p))$ contains a monic polynomial in Y with coefficients from B.
(b) $\Theta(O_P(p))$ contains a polynomial $g \in B[Y]$ of the form $1 + fh$ for some $h \in B[Y]$.

Furthermore, if we choose s' and l'' in the automorphism Θ such that $s' > nl/(ml-1)l''$, then with $e := (ms' - nl'')ll'$, the following hold:

(c) $f^e\Theta(a_i') \in B[Y]$ for $1 \le i \le r-1$.
(d) $f^e\Theta(a_r'') \in s^{2l'+2}\prod_1^n Y_i^{ll'} + fB[Y]$.

Parts (a) and (b) follow from Proposition 2.16. For (c), recall that $l' >$ is the maximum of total degrees of $a_1', \dots, a_r'$; hence, we must only ensure $e > l's'$. This is indeed the case because of our choice of s'. Part (d) is a direct consequence of the choice of e and s'.

Replacing A by $\Theta(A)$, we assume that:

(a′) $O_P(p)$ contains a monic polynomial in Y with coefficients from B.
(b′) $O_P(p)$ contains a polynomial $g \in B[Y]$ of the form $1 + fh$ for some $h \in B[Y]$.
(c′) $f^e a_i' \in B[Y]$ for $1 \le i \le r-1$.
(d′) $f^e a_r'' \in s^{2l'+2}\prod_1^n Y_i^{ll'} + fB[Y]$.

We have $g = 1 + fh \in O_P(p)$ for some $h \in B[Y]$, hence

$$A = B[Y] + gY. \qquad (*)$$

In order to see $(*)$, let $a \in A$. Then $a = b/f^t$ for some $b \in B[Y]$. Hence, $f^t a = b$, giving $(g-1)^t a = bh^t$ which implies $a = b' + ga'$, where $a' \in A$, $b' \in B[Y]$.

Since $A = O_P(p) + s(Y-1)A$, using equation $(*)$, we obtain

$$A = O_P(p) + s(Y-1)B[Y].$$

Therefore,

$$B[Y] = A \cap B[Y] = O_P(p) \cap B[Y] + s(Y-1)B[Y].$$

Using (a′), [**11,** page 100, Lemma 1.1] and

$$B[Y] = O_P(p) \cap B[Y] + sB[Y],$$

we get $B = (O_P(p) \cap B) + sB$. Hence, we now obtain:

(i) $O_P(p)$ contains an element $1 + b's$ for some $b' \in B$.
(ii) $O_P(p)$ contains an element $1 + s(Y-1)a$ for some $a \in B[Y]$.

Let ψ_1' and ψ_2' in P^* be such that

$$\psi_1'(p) = 1 + b's \quad \text{and} \quad \psi_2'(p) = 1 + s(Y-1)a.$$

We can choose an integer $l_0 > 0$ such that $f^{l_0}\psi_j'(p_i) \in B[Y]$ for $j = 1, 2$ and $1 \le i \le r$. Write $\phi_{r+j} = f^{l_0}\psi_j' \in P^*$ for $j = 1, 2$ and $p_{r+1} = f^e p \in P$.

Consider the $B[Y]$-modules

$$M := \sum_{i=1}^{r+1} B[Y]p_i \quad \text{and} \quad H := \sum_{i=1}^{r+2} B[Y]\phi_i.$$

We have $\phi_i(p_j) \in B[Y]$ for $1 \le i \le r+2$ and $1 \le j \le r+1$. Furthermore, we have

$$M \otimes_{B[Y]} A \subset P \quad \text{and} \quad H \otimes_{B[Y]} A \subset P^*.$$

Since $sP \subset F$, we obtain

$$sp_{r+1} = \sum_{i=1}^{r} b_i p_i$$

for some $b_i \in A$, and hence, $\phi_i(sp_{r+1}) = sb_i$ for $1 \le i \le r$. Since s is a nonzero divisor in A, we get $b_i = \phi_i(p_{r+1}) \in B[Y]$. Therefore,

$$sp_{r+1} = \sum_{1}^{r} \phi_i(p_{r+1})p_i.$$

Hence, if we write

$$F' := \sum_{i=1}^{r} B[Y]p_i,$$

then we have $sp_{r+1} \in F'$. Similarly, if we write

$$G' := \sum_{i=1}^{r} B[Y]\phi_i,$$

then we obtain $s\phi_{r+j} \in G'$ for $j = 1, 2$. Therefore, M_s and H_s are free modules over $B_s[Y]$ with $M_s = F'_s$ and $H_s = G'_s$. Furthermore, F' and G' are s-dual submodules of M and M^*, respectively, i.e., $sM \subset F'$, $sH \subset G'$ and the matrix

$$(\phi_i(p_j)) = \operatorname{diag}(s, \ldots, s).$$

Let us define a B-algebra endomorphism

$$\delta : B[Y] \longrightarrow B[Y]$$

by

$$\delta \mid B = \mathrm{id}$$

and

$$\delta(Y) = 1 + (Y - 1)(1 - b'^2 s^2) = Y + s^2 b'^2 (1 - Y),$$

where $b' \in B$ was chosen earlier as

$$\phi_{r+1}(p) = f^{l_0}(1 + b's).$$

Since $\delta(Y^t) - Y^t \in s^2 B[Y]$ for all integers $t \geq 0$, we obtain that

$$\delta(\alpha) - \alpha \in s^2 B[Y]$$

for any $\alpha \in B[Y]$. Such an endomorphism δ of $B[Y]$ is called s^2-*analytic*, see [**12,** page 304]. Recall that

$$M = \sum_{1}^{r+1} B[Y]p_i$$

is a $B[Y]$-module.

Applying Lemma 2.3 to the above data, we obtain δ-semilinear maps

$$\xi : M \longrightarrow M \quad \text{and} \quad \xi^* : M^* \longrightarrow M^*$$

such that

$$\xi^*(\phi)(\xi(p)) = \delta(\phi(p))$$

for any $\phi \in M^*$ and $p \in M$. Furthermore, $\xi^*(H) \subset H$. Therefore,

$$A \otimes_{B[Y]} \xi^*(H) \subset P^*.$$

We have the following:

(i′) $\phi_r(p_{r+1}) = \phi_r(f^e p) = s^{2l'+2} \prod_1^n Y_i^{ll'} + f\widetilde{b}$ for some $\widetilde{b} \in B[Y]$, using (d′).
(ii′) $\phi_{r+1}(p_{r+1}) = f^{l_0}\psi_1'(f^e p) = f^{l_0+e}(1 + b's)$ for $b' \in B$, using (i).
(iii′) $\phi_{r+2}(p_{r+1}) = f^{l_0}\psi_2'(f^e p) = f^{l_0+e}(1 + s(Y-1)a)$ for some $a \in B[Y]$, using (ii).

Using the relation

$$\xi^*(\phi)(\xi(p_{r+1})) = \delta(\phi(p_{r+1})),$$

we see that the δ-images of elements in (i′)–(iii′) belong to $O_M(\xi(p_{r+1}))$. Furthermore, using

$$A \otimes_{B[Y]} \xi(M) \subset P$$

and

$$A \otimes_{B[Y]} \xi^*(H) \subset P^*,$$

we see that δ-images of the above three elements belong to

$$O_P(1 \otimes \xi(p_{r+1})).$$

We will show that $1 \otimes \xi(p_{r+1})$ is a unimodular element of P by showing that the δ-images of the above three elements generate the unit ideal. Suppose not. Then there exists a maximal ideal $\mathfrak{m}$ containing elements

(i″)

$$\delta\Big(s^{2l'+2} \prod_1^n Y_i^{ll'} + f\widetilde{b}\Big) = s^{2l'+2} \prod_1^n Y_i^{ll'} + \delta(f)\delta(\widetilde{b}),$$

(ii″) $\delta(f^{l_0+e}(1 + b's)) = \delta(f)^{l_0+e}(1 + b's)$, and
(iii″) $\delta(f^{l_0+e}(1 + s(Y-1)a)) = \delta(f)^{l_0+e}(1 + s\delta(Y-1)\delta(a))$.

Assume that $\delta(f) \in \mathfrak{m}$. Then, by (i″), we obtain that $s \in \mathfrak{m}$ as the Y_i's are units in A. Since $\delta(f) - f \in (s^2)$, we get $f \in \mathfrak{m}$. This is a contradiction, since f is a unit in A. In the other case, assume

$\delta(f) \notin \mathfrak{m}$. Then

$$1 + s\delta(Y-1)\delta(a) \in \mathfrak{m} \quad \text{and} \quad 1 + b's \in \mathfrak{m}.$$

Since

$$\delta(Y-1) = (Y-1)(1 - b'^2 s^2) \in (1 + b's)A,$$

we obtain $\delta(Y-1) \in \mathfrak{m}$. This shows that $1 \in \mathfrak{m}$, a contradiction. Therefore, we get that $1 \otimes \xi(p_{r+1})$ is a unimodular element. This completes the proof. $\square$

Theorem 3.2. *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$ and p be an integer such that*

$$p \geq \max\{\dim A - p + 2, d + 1\}.$$

Let $P = Q \oplus R$ be a projective R-module of rank p and I a proper ideal of A of height $\geq$ *$d+1$. Assume that there is a surjection $\phi : P \otimes A/I(P \otimes A) \twoheadrightarrow I/I^2$. Then ϕ can be lifted to a surjection $\Phi : P \otimes A \twoheadrightarrow I$. As a consequence, taking P to be free, we get that any p generators of I/I^2 can be lifted to p generators of I.*

Proof. We assume that $n \geq 1$ because we can always use more variables as well as retraction. If

$$C := R[X_1, \ldots, X_m, Y_1^{\pm 1}, \ldots, Y_{n-1}^{\pm 1}],$$

then

$$A = C[Y_n^{\pm 1}, Y, f^{-1}] \quad \text{with } f \in R[Y].$$

We are given a surjection

$$\phi : P \otimes (A/I) \twoheadrightarrow I/I^2,$$

where $P = Q \oplus R$. We want to show that ϕ can be lifted to a surjection

$$\Phi : P \otimes A \twoheadrightarrow I.$$

Let

$$\Phi_1 : P \otimes A \longrightarrow I$$

be a lift of ϕ. We can find an integer $k > 0$ such that $f^k \Phi_1$ maps $P \otimes C[Y_n^{\pm 1}, Y]$ into $J := I \cap C[Y_n^{\pm 1}, Y]$. Now, $f^k \Phi_1$ induces a map

$$\psi : P \otimes (C[Y_n^{\pm 1}, Y]/J) \longrightarrow J/J^2.$$

Note that

$$\psi_f = f^k\phi : P \otimes (A/I) \twoheadrightarrow (J/J^2)_f$$

is a surjection.

Using the height of $I > d$ and applying Lemma 2.15, we obtain an $R[Y, f^{-1}]$-automorphism Θ of A such that $\Theta(I)$ contains $1+fh$ for some $h \in C[Y_n^{\pm 1}, Y]$. Replacing A by $\Theta(A)$ and I by $\Theta(I)$, we can assume that $1 + fh \in I$. Since $1 + fh \in J$, we obtain that $(J/J^2)_{1+fh}$ is the zero module. Hence, ψ_{1+fh} is a surjection. Applying Proposition 2.4, we get that

$$\psi : P \otimes (C[Y_n^{\pm 1}, Y]/J) \longrightarrow J/J^2$$

is a surjection. If ψ has a surjective lift

$$\Psi : P \otimes C[Y_n^{\pm 1}, Y] \twoheadrightarrow J,$$

then

$$f^{-k}\Psi_f : P \otimes A \twoheadrightarrow I$$

will be our required surjective lift of ϕ. Therefore, it is enough to show that ψ has a surjective lift from $P \otimes C[Y_n^{\pm 1}, Y]$ onto J.

Note that $C[Y_n^{\pm 1}, Y] = B[Y]$ is a Laurent polynomial ring over R and J is an ideal of $C[Y_n^{\pm 1}, Y]$ of height $> d = \dim R$. By Lemma 2.6, there exists an $R[Y_n^{\pm 1}]$-automorphism Θ of $C[Y_n^{\pm 1}, Y]$ such that $\Theta(J)$ contains a monic polynomial in Y_n of the form $1 + Y_n h'$ for some $h' \in C[Y, Y_n]$. Replacing J by $\Theta(J)$, we can assume that J contains a monic polynomial $1 + Y_n h'$ in the variable Y_n.

Lift ψ to a map

$$\Psi_1 : P \otimes C[Y, Y_n^{\pm 1}] \longrightarrow J.$$

If we set $K := J \cap C[Y, Y_n]$, then $Y_n^l \Psi_1$ will map $P \otimes C[Y, Y_n]$ into K for some integer $l > 0$. Now $Y_n^l \Psi_1$ will induce a map

$$\delta : P \otimes (C[Y, Y_n]/K) \longrightarrow K/K^2$$

such that $\delta_{Y_n} = Y_n^l \psi$ is a surjection from $P \otimes (C[Y, Y_n^{\pm 1}]/J)$ onto J/J^2. Since K contains a monic polynomial $1 + Y_n h'$ in Y_n, we get $(K/K^2)_{1+Y_n h'} = 0$. Applying Proposition (2.4), we obtain that

$$\delta : P \otimes (C[Y, Y_n]/K) \twoheadrightarrow K/K^2$$

is a surjection. Applying Proposition 2.8, we have that δ can be lifted to a surjection

$$\Delta : P \otimes C[Y, Y_n] \twoheadrightarrow K.$$

Therefore, $Y_n^{-l}\Delta$ is a surjective lift of ψ. This completes the proof. $\square$

Theorem 3.3. *Let $A = B[Y, f^{-1}]$, where $f \in R[Y]$ is a monic polynomial with $f(1)$ a unit in R. Let I be an ideal of A of* height $\geq d+1$ *and P a projective B-module of* rank $\geq \max\{d+1, \dim(A/I)+2)\}$. *Let*

$$\phi : P[Y, f^{-1}]/IP[Y, f^{-1}] \twoheadrightarrow I/I^2$$

and

$$\delta : P \twoheadrightarrow I(1)\ (:= \{g(Y = 1) | g \in I\})$$

be two surjections such that $\delta = \phi \otimes A/(Y-1)$, where $I(1)$ is an ideal of B. Then there exists a surjection

$$\Psi : P[Y, f^{-1}] \twoheadrightarrow I$$

such that $\Psi \otimes A/I = \phi$ and $\Psi(1) = \delta$.

Proof. Without loss of generality, we assume that $f \in R[Y] - R$. Let $\Phi_1 : P[Y, f^{-1}] \to I$ be any lift of ϕ. Then, $\Phi_1(1) = \delta$ modulo $I(1)^2$. Hence, $\Phi_1(1) - \delta \in I(1)^2 \operatorname{Hom}(P, B)$. Set

$$\Phi_1(1) - \delta = f_1(1)g_1(1)\alpha_1 + \cdots + f_r(1)g_r(1)\alpha_r$$

for some $f_i, g_i \in I$ and $\alpha_i \in \operatorname{Hom}(P, B)$. If we write

$$\Phi_2 := \Phi_1 - (f_1 g_1 \widetilde{\alpha}_1 + \cdots + f_r g_r \widetilde{\alpha}_r),$$

where $\widetilde{\alpha}_i = \alpha_i \otimes id : P \otimes_B A \to A$, then

$$\Phi_2 : P[Y, f^{-1}] \longrightarrow I$$

is also a lift of ϕ with $\Phi_2(1) = \delta$.

Let $J := I \cap B[Y]$. Then there exists $k > 0$ such that $f^k \Phi_2$ maps $P[Y]$ into J. Now $f^k \Phi_2$ induces a map

$$\psi : P[Y]/JP[Y] \longrightarrow J/J^2.$$

Note that

$$\psi_f = f^k\phi : P[Y, f^{-1}]/IP[Y, f^{-1}] \twoheadrightarrow (J/J^2)_f$$

is a surjection.

Since $\operatorname{ht} I > d$, by Proposition 2.16, applying an $R[Y, f^{-1}]$-automorphism of A, we may assume that I contains

(i) a monic polynomial g in Y with coefficients from B, and
(ii) an element $1 + fh$ for some $h \in B[Y]$.

Since $1 + fh \in J$, we obtain $(J/J^2)_{1+fh} = 0$. Therefore, ψ_{1+fh} is the zero map. By equation (2.4), we get that

$$\psi : P[Y]/JP[Y] \twoheadrightarrow J/J^2$$

is a surjection. Furthermore,

$$f(1)^k\delta : P \twoheadrightarrow J(1)$$

is a surjection with $\psi(1) = f(1)^k\delta \otimes B/J(1)$. Since the rank of $P \geq \dim B[Y]/J + 2$ holds and J contains monic polynomial g, using **[14,** Theorem 2.1], there exists a surjection $\Psi : P[Y] \twoheadrightarrow J$ which is a lift of ψ and $\Psi(1) = f(1)^k\delta$. Therefore,

$$\Phi = f^{-k}\Psi_f : P[Y, f^{-1}] \twoheadrightarrow I$$

is a surjection which is a lift of $f^{-k}\psi = \phi$ with $\Phi(1) = \delta$. This completes the proof. □

4. Applications. Let M be a finitely generated R-module. If we write $\mu(M)$ for the minimum number of generators of M as an R-module, then Föster **[9]** and Swan **[25]** proved that

$$\mu(M) \leq \max\{\mu(M_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R), M_{\mathfrak{p}} \neq 0\}.$$

In particular, if P is a projective R-module of rank r, then $\mu(P) \leq r+d$. As a consequence of our result (1.2), we prove the next theorem.

Theorem 4.1. *Let $A = B[Y, f^{-1}]$ for some monic polynomial $f \in R[Y]$ and P be a projective A-module of rank r. Then $\mu(P) \leq r + d$.*

Proof. Assume that P is generated by s elements, where $s > r + d$. Then we will show that P is also generated by $s - 1$ elements. By Förster and Swan, we have $s \leq \dim A + r = d + m + n + 1 + r$. Let

$\phi : A^s \twoheadrightarrow P$ be a surjection. If Q is the kernel of ϕ, then the rank of Q is $s - r > d$. Hence, by Theorem 1.2, Q has a unimodular element, say $q \in \mathrm{Um}(Q)$. Since $A^s \tilde{\to} P \oplus Q$, we obtain $q \in \mathrm{Um}(A^s)$. Since $\phi(q) = 0$, ϕ induces a surjection

$$\overline{\phi} : A^s/qA \twoheadrightarrow P.$$

Since $s - 1 > d$, by Theorem 2.9, A^{s-1} is cancelative. Hence,

$$A^s/qA \tilde{\to} A^{s-1}.$$

Therefore, P is generated by $s - 1$ elements. This completes the proof. □

Proposition 4.2. *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$. Let $J \subset I$ be two ideals of A such that*

$$I = (f_1, \ldots, f_n) + I^2$$

and

$$J = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}.$$

Assume that I contains

(i) *a monic polynomial $F \in C[Y]$ in the variable Y, and*

(ii) *an element of the form $1 + fh$ for some $h \in C[Y]$.*

Then J is generated by n elements. As a consequence, I is set-theoretically generated by n elements.

Proof. Replacing f_i by $f^N f_i$ for an integer $N > 0$, we may assume that $f_i \in B[Y]$ for all i. Let $K = I \cap B[Y]$ be an ideal of $B[Y]$. Let

$$\phi : (B[Y]/K)^n \longrightarrow K/K^2$$

be the map defined by $e_i \mapsto \overline{f}_i$. Then ϕ_f is surjective and ϕ_{1+fh} is a zero map since $1+fh \in K$. Hence, by Proposition 2.4, ϕ is a surjection. Therefore, we obtain

$$K = (f_1, \ldots, f_n) + K^2.$$

If

$$L := (f_1, \ldots, f_{n-1}) + K^{(n-1)!},$$

then $L_f = J$. Since K contains a monic polynomial F, using Theorem 2.11, we get that L is generated by n elements. Therefore, J is generated by n elements. □

Theorem 4.3. *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$. Let $J \subset I$ be two ideals of A such that*

$$I = (f_1, \dots, f_n) + I^2$$

and

$$J = (f_1, \dots, f_{n-1}) + I^{(n-1)!}.$$

Assume that the height of $I > d$. Then J is generated by n elements. In particular, I is set-theoretically generated by n elements.

Proof. Applying an automorphism as in Lemma 2.15, we may assume that I contains an element $1 + fh$ for some $h \in B[Y]$. Now, as in Proposition 4.2, replacing f_i by $f^N f_i$, we may assume that $f_i \in B[Y]$. Then, if $K = I \cap B[Y]$, then

$$K = (f_1, \dots, f_n) + K^2$$

as in Proposition 4.2. Since the height of $K > d$, using an automorphism of $B[Y]$, we may assume that K contains a monic polynomial in Y. Now, if

$$L = (f_1, \dots, f_{n-1}) + K^{(n-1)!},$$

then, by Theorem 2.11, L is generated by n elements. Hence, $J = K_f$ is generated by n elements. □

Theorem 4.4. *Let $A = B[Y, f^{-1}]$ for some $f \in R[Y]$ with the further condition that $m + n \geq 1$. Let $I \subset A$ be a locally complete intersection ideal of height*

$$r \geq \max\{\dim A - 1, \dim A - r + 2\}$$

with $\dim A/I \leq 1$. Then, I is set-theoretically generated by r elements.

Proof. By Ferrand and Szpiro's Theorem 2.12, there is a locally complete intersection ideal J of height r such that (i) $\sqrt{J} = \sqrt{I}$ and (ii) J/J^2 is a free A/J-module of rank r. Since $m + n \geq 1$, we get $r \geq d + 1$. By Theorem 1.3, the r generators of free module J/J^2 can be lifted to r generators of J. Hence, I is set-theoretically generated by r elements. □

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