JOURNAL OF COMMUTATIVE ALGEBRA
Volume 9, Number 1, Spring 2017

ULRICH IDEALS OF GORENSTEIN NUMERICAL SEMIGROUP RINGS WITH EMBEDDING DIMENSION 3

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ABSTRACT. The notion of Ulrich ideals was introduced by Goto et al. **[3]**. They developed an interesting theory on Ulrich ideals. In particular, they gave a characterization of Ulrich ideals of Gorenstein numerical semigroup rings that are generated by monomials. Using this result, in this paper, we investigate Ulrich ideals of Gorenstein numerical semigroup rings with embedding dimension 3 that are generated by monomials. In particular, we completely determine when such Ulrich ideals are existent in those rings.

1. Introduction. Throughout this paper, let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of nonnegative integers and integers, respectively. A *numerical semigroup* is a subset of $\mathbb{N}$ which is closed under addition, contains the zero element and whose complement in $\mathbb{N}$ is finite. Every numerical semigroup H is finitely generated and has the unique minimal system of generators $a_1, \ldots, a_r \in \mathbb{N}$, that is,

$$H = \langle a_1, \ldots, a_r \rangle := \{\lambda_1 a_1 + \cdots + \lambda_r a_r \mid \lambda_1, \ldots, \lambda_r \in \mathbb{N}\},$$

where $\gcd(a_1, \ldots, a_r) = 1$. The *Frobenius number* of H, denoted by $\mathrm{F}(H)$, is the maximal integer which does not belong to H. A numerical semigroup H is *symmetric* if, for any integers $x \in \mathbb{Z}$, either $x \in H$ or $\mathrm{F}(H) - x \in H$. Let k be a field and t be an indeterminate over k. The ring

$$k[[H]] := k[[t^{a_1}, \ldots, t^{a_r}]] \subset k[[t]]$$

is called the *semigroup ring* associated to $H = \langle a_1, \ldots, a_r \rangle$. A semigroup ring $k[[H]]$ is a one-dimensional Cohen-Macaulay local ring with

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2010 AMS *Mathematics subject classification.* Primary 13H10, 20M14, 20M25.

Keywords and phrases. Ulrich ideal, numerical semigroup, Gorenstein, symmetric, complete intersection, gluing.

Received by the editors on June 23, 2014, and in revised form on November 14, 2014.

DOI:10.1216/JCA-2017-9-1-143

the maximal ideal $\mathfrak{m} = (t^{a_1}, \ldots, t^{a_r})$. It is well known that $k[[H]]$ is Gorenstein if and only if H is symmetric.

Our purpose in this paper is to investigate Ulrich ideals of Gorenstein numerical semigroup rings that are generated by monomials. The notion of Ulrich ideals was introduced recently by Goto, et al. [**3**].

Definition 1.1 ([**3**])**.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R \geq 0$ and I an $\mathfrak{m}$-primary ideal of R. Suppose that I contains a parameter ideal $Q = (a_1, \ldots, a_d)$ of R as a minimal reduction. Then I is called an *Ulrich ideal* of R if the following two conditions are satisfied:

(1) $I^2 = QI$, and
(2) R/I-module I/I^2 is free.

By definition, any parameter ideal of R is Ulrich. For convenience, in this paper, we except parameter ideals whenever we refer to Ulrich ideals. We put $R = k[[H]]$, and let χ_R^g denote the set of Ulrich ideals of R which are generated by monomials in t. Then it is shown that χ_R^g is finite in [**3**] (but, the number of Ulrich ideals not generated by monomials depends on the field k).

When H is a numerical semigroup generated by 2-elements, the set χ_R^g is completely described in [**3**]. Therefore, in Section 3, we consider the case where H is symmetric and generated by 3-elements, and then we completely determine when χ_R^g is empty or not. We state the main theorem by using the next lemma.

Lemma 1.2 ([**4, 8**])**.** *Let $H = \langle a, b, c\rangle$ be a numerical semigroup generated by* 3*-elements. Then following assertions are equivalent.*

(1) *H is symmetric.*
(2) *Changing the order of a, b and c, if necessary, we can write $a = a'd$ and $b = b'd$, where $\gcd(a, b) = d > 1$ and $c \in \langle a', b'\rangle$, $c \neq a', b'$.*

In this case, we denote $H = \langle d\langle a', b'\rangle, c\rangle$.

The main result of this paper is the following.

Theorem 1.3 (Theorem 3.11)**.** *Let $H = \langle a, b, c\rangle$ be a symmetric numerical semigroup and assume that $H = \langle d\langle a', b'\rangle, c\rangle$. We put*

$R = k[[H]]$, $H_1 = \langle a', b' \rangle$ and $R_1 = k[[H_1]]$. Then the following assertions hold true.

(1) *If d and c are odd, then*

$$\chi_R^g = \{(t^{d\alpha}, t^{d\beta}) \mid \alpha, \beta \in H_1 \text{ such that } (t^\alpha, t^\beta) \in \chi_{R_1}^g\}.$$

In particular, $\#\chi_R^g = \#\chi_{R_1}^g$. Hence, $\chi_R^g = \emptyset$ if a, b and c are odd.

(2) *If a', b' and d are odd and c is even, then*
  (a) *$\chi_R^g \neq \emptyset$ if and only if $H + \langle c/2 \rangle$ is symmetric.*
  (b) *if $\chi_R^g \neq \emptyset$, then*

$$\chi_R^g = \{(t^{(c/2)l}, t^{(c/2)d}) \mid \ l \text{ is even with } 1 < l < d\}.$$

  In particular, $\#\chi_R^g = (d-1)/2$.

(3) *If a', b' and c are odd, and c is even, then $\chi_R^g \neq \emptyset$. Furthermore, the number of χ_R^g does not depend on d.*

(4) *If a' or b' is even, then*

$$\chi_R^g \supset \{(t^{d\alpha}, t^{d\beta}) \mid \alpha, \beta \in H_1 \text{ such that } (t^\alpha, t^\beta) \in \chi_{R_1}^g\} \neq \emptyset.$$

In particular, $\chi_R^g \neq \emptyset$.

2. Preliminaries. We start this section by recalling some results on Ulrich ideals from **[3]**. The next theorem is very important for achieving our goal.

Theorem 2.1 (**[3]**). *Suppose that $R = k[[H]]$ is a Gorenstein numerical semigroup ring (equivalently, H is a symmetric numerical semigroup), and let I be an ideal of R. Then the following conditions are equivalent:*

(1) *$I \in \chi_R^g$.*

(2) *$I = (t^\alpha, t^\beta)$ $(\alpha, \beta \in H, \alpha < \beta)$ and if we put $x = \beta - \alpha$, the following conditions hold.*
  (i) *$x \notin H$, $2x \in H$.*
  (ii) *The numerical semigroup $S = H + \langle x \rangle$ is symmetric, and*
  (iii) *$\alpha = \min\{h \in H \mid h + x \in H\}$.*

In particular, we note that $\chi_R^g \neq \emptyset$ if and only if there is an integer $x \in \mathbb{Z}$ which satisfies conditions (i) and (ii) above.

Example 2.2.

(1) Let $H = \langle 4, 5\rangle = \{0, 4, 5, 8, 9, 10, 12 \rightarrow\}$. We can find the integers which satisfy condition (i):

$$x = 2,\ 6,\ 7,\ 11.$$

Among these integers, 2 and 6 merely satisfy condition (ii). Therefore, we have

$$\chi^g_{k[[H]]} = \{(t^8, t^{10}), (t^4, t^{10})\}$$

by condition (iii).

(2) If $H = \langle 3, 5\rangle$, then $\chi^g_{k[[H]]} = \emptyset$ since we can check that there are no integers which satisfy conditions (i) and (ii).

Actually, when H is generated by 2-elements, the set χ^g_R is completely described in **[3]**. In particular, the following assertion holds true.

Theorem 2.3 ([**3**])**.** *Let $H = \langle a, b\rangle$ be a numerical semigroup. Then the following conditions are equivalent.*

(1) $\chi^g_{k[[H]]} \neq \emptyset$.
(2) *a or b is even.*

We use the next lemma in Section 3.

Lemma 2.4 ([**3**])**.** *Under the notation in Definition* 1.1*, we suppose that $I^2 = QI$. Then*:

(1) $\mathrm{e}(I) \leq (\mu_R(I) - d + 1)\ell_R(R/I)$, *where* $\mathrm{e}(I)$, $\mu_R(I)$ *and* $\ell_R(R/I)$ *denote the multiplicity of I, the number of minimal generators of I, and the length of R/I, respectively.*
(2) *The following conditions for I are equivalent*:
  (i) *Equality holds in* (1)*.*
  (ii) *I is Ulrich.*
  (iii) *I/Q is a free R/I-module.*

The Apéry sets of a in H correspond to the k-basis of the ring $k[[H]]/(t^a)$, where $a \in H$.

Definition 2.5. Let H be a numerical semigroup, and take $0 \neq a \in H$. The *Apéry set* of a in H is

$$\mathrm{Ap}(H, a) = \{h \in H \mid h - a \notin H\}.$$

By definition, we see that $\mathrm{Ap}(H, a) = \{0 = w(0), w(1), \ldots, w(a-1)\}$, where $w(i) = \min\{h \in H \mid h \equiv i\}$ for each $1 \leq i \leq a - 1$. For more details, see [**7**].

3. The case of $H = \langle a, b, c\rangle$. In this section, we consider the case where $H = \langle a, b, c\rangle$ is symmetric. We provide some lemmas to prove our main theorem.

Definition 3.1 ([**7**]). For two numerical semigroups $H_1 = \langle a_1, \ldots, a_m\rangle$ and $H_2 = \langle b_1, \ldots, b_n\rangle$, we define a *gluing* of H_1 and H_2 as follows:

$$\langle d_1H_1, d_2H_2\rangle := \langle d_1a_1, \ldots, d_1a_m, d_2b_1, \ldots, d_2b_n\rangle\,,$$

where $d_1 \in H_2\backslash\{b_1, \ldots, b_n\}, d_2 \in H_1\backslash\{a_1, \ldots, a_m\}$ and $\gcd(d_1, d_2) = 1$.

By the constructions of gluings, we have the following result.

Proposition 3.2. *Let $H = \langle d_1H_1, d_2H_2\rangle$ be a gluing of two numerical semigroups H_1 and H_2. Then the ring $k[[H]]$ is a $k[[H_1]]$, (respectively, $k[[H_2]]$)-free module of rank d_1 (respectively, d_2).*

Proof. By the k-algebra map $\phi : k[[H_1]] \to k[[H]]$, where $t^a \mapsto t^{d_1a}$ for all $a \in H_1$, we can regard $k[[H]]$ as a $k[[H_1]]$-module. From [**7**, Theorem 9.2], we have

$$\begin{aligned}\mathrm{Ap}(H, d_1d_2) &= \{d_1h_1 + d_2h_2 \mid h_1 \in \mathrm{Ap}(H_1, d_2), h_2 \in \mathrm{Ap}(H_2, d_1)\}\\ &= \bigcup_{h_2\in\mathrm{Ap}(H_2,d_1)} (d_2h_2 + d_1\,\mathrm{Ap}(H_1, d_2)) \text{ (disjoint union).}\end{aligned}$$

This implies that

$$H = \bigcup_{h_2 \in \mathrm{Ap}(H_2, d_1)} (d_2 h_2 + d_1 H_1) \text{ (disjoint union).}$$

Hence, we get the isomorphism $k[[H]] \cong k[[H_1]]^{\oplus d_1}$ as a $k[[H_1]]$-module. By exactly the same argument, we see that $k[[H]] \cong k[[H_2]]^{\oplus d_2}$ as a $k[[H_2]]$-module. $\square$

We say that a numerical semigroup H is a *complete intersection* if its semigroup ring $k[[H]]$ is a complete intersection.

Theorem 3.3 ([**2, 7**]). *The following assertions hold true.*

(1) *Let $H = \langle d_1 H_1, d_2 H_2 \rangle$ be a gluing of two numerical semigroups H_1 and H_2. Then H is symmetric (respectively, a complete intersection) if and only if H_1 and H_2 are symmetric (respectively, complete intersections).*
(2) *A numerical semigroup other than $\mathbb{N}$ is a complete intersection if and only if it is a gluing of two complete intersection numerical semigroups.*

Remark 3.4. When a numerical semigroup H is generated by 3-elements, H is symmetric if and only if H is a complete intersection, see [**4**]. Therefore, Lemma 1.2 is a special case of Theorem 3.3 (2) since $H = \langle a, b, c \rangle = \langle d \langle a', b' \rangle, c \rangle$ is a gluing of $\langle a', b' \rangle$ and $\langle 1 \rangle = \mathbb{N}$.

The following is one of the key lemmas in this section. We remark that, if $k[[H]]$ is not Gorenstein, then Ulrich ideals of $k[[H]]$ need not be 2-generated, see [**3**]. Hence, we can state (1) in the next lemma as follows since we do not assume Gorensteiness of R or R_i in the proof.

Lemma 3.5. *Let $H = \langle d_1 H_1, d_2 H_2 \rangle$ be a gluing of two numerical semigroups $H_1 = \langle a_1, \ldots, a_m \rangle$ and $H_2 = \langle b_1, \ldots, b_n \rangle$. We put $R = k[[H]]$ and $R_i = k[[H_i]]$ for $i = 1, 2$. The following assertions hold true for $i = 1, 2$.*

(1) *If $(t^{\alpha_1}, t^{\alpha_2}, \ldots, t^{\alpha_r}) \in \chi^g_{R_i}$, then $(t^{d_i \alpha_1}, t^{d_i \alpha_2}, \ldots, t^{d_i \alpha_r}) \in \chi^g_R$.*
(2) *Suppose that H_1 and H_2 is symmetric (equivalently, H is symmetric by Theorem 3.3). Then if $(t^\gamma, t^\delta) \in \chi^g_R$ and d_i divides*

$x := \delta - \gamma > 0$, then there exists two integers $\alpha, \beta \in H_i$ with $x/d_i = \beta - \alpha > 0$ such that $(t^{\alpha}, t^{\beta}) \in \chi^g_{R_i}$.

(3) $\#\chi^g_{R_i} \leq \#\chi^g_R$.

Proof.

(1) We note that $R \cong R_1^{\oplus d_1}$ by Proposition 3.2. Therefore, by Lemma 2.4, if $I = (t^{\alpha_1}, t^{\alpha_2}, \ldots, t^{\alpha_r}) \in \chi^g_{R_1}$, then

$$\mathrm{e}(IR) = d_1 \,\mathrm{e}(I) = d_1 \mu_{R_1}(I) \ell_{R_1}(R_1/I) = \mu_R(IR)\ell_R(R/IR),$$

where $IR = (t^{d_i\alpha_1}, t^{d_i\alpha_2}, \ldots, t^{d_i\alpha_r})$. Hence $IR \in \chi^g_R$ by Lemma 2.4.

(2) It suffices to prove that $x/d_1 \notin H_1, 2x/d_1 \in H_1$ and $H_1 + \langle x/d_1 \rangle$ is symmetric by Theorem 2.1. It is clear that $x/d_1 \notin H_1$ and $2x/d_1 \in H_1$ since $x \notin H$ and $2x \in H$ by Theorem 2.1. We use Theorem 3.3 to see that $H_1 + \langle x/d_1 \rangle$ is symmetric: since $H + \langle x \rangle = \langle d_1(H_1 + \langle x/d_1 \rangle), d_2 H_2 \rangle$ is symmetric, so is $H_1 + \langle x/d_1 \rangle$.

(3) This is clear by (1). $\square$

By Theorem 2.3 and Lemma 3.5, when $H = \langle d \langle a', b' \rangle, c \rangle$, we see that $\chi^g_{k[[H]]} \neq \emptyset$ whenever a' or b' is even.

Example 3.6.

(1) Let $H_1 = \langle 4, 5 \rangle$ and $H_2 = \mathbb{N}$. We know that

$$\chi^g_{R_1} = \{(t^8, t^{10}), (t^4, t^{10})\}$$

and $\chi^g_{R_2} = \emptyset$ (see Example 2.2). Let $H = \langle 3H_1, 13H_2 \rangle = \langle 12, 13, 15 \rangle$ be a gluing of H_1 and H_2. By Theorem 2.1, we can check that

$$\chi^g_R = \{(t^{24}, t^{30}), (t^{12}, t^{30})\}.$$

In that case, there is a one-to-one correspondence between the sets $\chi^g_{R_1}$ and χ^g_R. In other words, all Ulrich ideals of R are extensions from those of R_1 (this example illustrates Theorem 3.11 (1) since 3 and 13 are odd).

(2) Let H_1 and H_2 be as above, and let $H = \langle 3H_1, 16H_2 \rangle = \langle 12, 15, 16 \rangle$ be a gluing of H_1 and H_2. Then we see that

$$\chi^g_R = \{(t^{24}, t^{30}), (t^{12}, t^{30}), (t^{16}, t^{24}), (t^{16}, t^{30})\}.$$

In that case, the ideals (t^{16}, t^{24}) and (t^{16}, t^{30}) are not extensions from those of R_1.

Now, let $H = \langle a_1, a_2, a_3, a_4 \rangle$ be a numerical semigroup. We define the positive integer $\alpha_1 > 0$ as follows:

$$\alpha_1 = \min\{n > 0 \mid na_1 \in \langle a_2, a_3, a_4 \rangle\}.$$

We also define $\alpha_2, \alpha_3, \alpha_4$ in the same way. Then Bresinsky completely characterized the defining ideal $\mathfrak{p}$ of $k[[H]] \cong k[[X_1, X_2, X_3, X_4]]/\mathfrak{p}$ when H is symmetric but not a complete intersection.

Theorem 3.7 ([**1**]). *Let $H = \langle a_1, a_2, a_3, a_4 \rangle$ be a numerical semigroup and $\mathfrak{p}$ its defining ideal. Then H is symmetric, which is not a complete intersection if and only if after changing order of a_1, a_2, a_3 and a_4, if necessary,*

$$\begin{aligned}\mathfrak{p} = (f_1 = X_1^{\alpha_1} - X_3^{\alpha_{13}} X_4^{\alpha_{14}}, f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}} X_4^{\alpha_{24}},& \\ f_3 = X_3^{\alpha_3} - X_1^{\alpha_{31}} X_2^{\alpha_{32}}, f_4 = X_4^{\alpha_4} - X_2^{\alpha_{42}} X_3^{\alpha_{43}},& \\ f_5 = X_3^{\alpha_{43}} X_1^{\alpha_{21}} - X_2^{\alpha_{32}} X_4^{\alpha_{14}}),&\end{aligned}$$

where each f_i is unique up to isomorphism for each i and $0 < \alpha_{ij} < \alpha_j$ for each i, j.

The following is important for our goal.

Theorem 3.8. *Let $H = \langle a, b, c \rangle$ be a symmetric numerical semigroup. If $H + \langle x \rangle$ is symmetric for an integer $x \in \mathbb{Z}$ such that $x \notin H$ and $2x \in H$, then $H + \langle x \rangle$ is a complete intersection.*

Proof. When $S = H + \langle x \rangle$ is generated by at most 3-elements, S is a complete intersection if and only if it is symmetric. Thus, we may assume that S is generated by 4-elements.

We assume that $H + \langle x \rangle$ is symmetric but not a complete intersection. By Lemma 1.2, we may assume that $H = \langle d \langle a', b' \rangle, c \rangle$. In Theorem 3.7, we may also assume that $x = a_4$ without loss of generality. Then $\alpha_4 = 2$, $\alpha_{14} = \alpha_{24} = 1$ and $\alpha_{34} = 0$ by our assumption and Theorem 3.7.

Next, we show that we may put $c = a_3$. Otherwise, we may put $a = a_1$, $b = a_3$ and $c = a_2$ (note the situation is symmetric in the order of a and b). Then, since $\alpha_1 a = \alpha_{13} b + x$, we see that d divides x. Therefore, we can write $S = \langle d \langle a', b', x' \rangle, c \rangle$, where $x = x'd$. Since S is symmetric which is not a complete intersection, so is $H_1 = \langle a', b', x' \rangle$ by Theorem 3.3, which is a contradiction since H_1 is generated by 3-elements. Thus, we put $a = a_1$, $b = a_2$ and $c = a_3$. But then, we again see that d divides x since we have $\alpha_2 b = \alpha_{21} a + x$, a contradiction. This completes the proof. $\square$

Remark 3.9. In Theorem 3.8, the condition, $2x \in H$, is essential. For example, let $H = \langle 13, 16, 20 \rangle$, which is symmetric, and let $x = 22$. Then $x, 2x \notin H$, but $3x \in H$. We can check that $H + \langle x \rangle = \langle 13, 16, 20, 22 \rangle$ is symmetric but not a complete intersection.

Using Theorem 3.8, we can prove the next lemma.

Lemma 3.10. *Let $H = \langle a, b, c \rangle = \langle d \langle a', b' \rangle, c \rangle$ be a symmetric numerical semigroup. Suppose that $S = H + \langle x \rangle$ is symmetric for an integer $x \in \mathbb{Z}$ such that $x \notin H$ and $2x \in H$. We write $2x = \lambda_1 a + \lambda_2 b + \lambda_3 c$, where $\lambda_1, \lambda_2, \lambda_3 \geq 0$. It follows that if $\lambda_3 > 0$, and $\lambda_1 > 0$ or $\lambda_2 > 0$, then a or b is even, and c is even.*

Proof. We consider each case individually. First we consider the case where S is generated by 3-elements.

(i) Assume that $c \in \langle a, b, x \rangle$. Then we can write $c = \mu_1 a + \mu_2 b + \mu_3 x$, where $\mu_1, \mu_2 \geq 0$, $\mu_3 > 0$. Therefore, we can write $2x = (\lambda_1 + \lambda_3 \mu_1) a + (\lambda_2 + \lambda_3 \mu_2) b + \lambda_3 \mu_3 x$. In this equality, we know that $\lambda_3 \mu_3 = 1$ or 2 since $\lambda_3 > 0$ and $\mu_3 > 0$. If $\lambda_3 \mu_3 = 1$, then $x \in H$, which is a contradiction. Thus, we get $\lambda_3 \mu_3 = 2$, but this is impossible, since $\lambda_1 > 0$ or $\lambda_2 > 0$. Hence, this case does not occur.

(ii) Assume that $a \in \langle b, c, x \rangle$ or $b \in \langle a, c, x \rangle$. Since a and b are interchangeable, it suffices to consider $a \in \langle b, c, x \rangle$. Then, we can put $a = \mu_1 b + \mu_2 c + \mu_3 x$, where $\mu_1, \mu_2 \geq 0, \mu_3 > 0$. Thus, we have $2x = (\lambda_1 \mu_1 + \lambda_2) b + (\lambda_1 \mu_2 + \lambda_3) c + \lambda_1 \mu_3 x$. Since $\lambda_3 > 0$, we see that $\lambda_1 \mu_3 = 0$ or 1. If $\lambda_1 \mu_3 = 1$, then $x \in H$, which is a contradiction, and hence, $\lambda_1 \mu_3 = 0$. Then, since $\mu_3 > 0$ and

$\lambda_1 = 0$, we have $\lambda_2 > 0$ by our assumption. By Lemma 1.2, we have $S = \langle 2\langle b'', c''\rangle, x\rangle$, where $b = 2b'', c = 2c''$. This yields that b and c are even.

Next, we consider the case where $S = \langle a, b, c, x\rangle$ is generated by 4-elements. Then by Lemma 3.8, we know that S is a complete intersection. Thus, S is a gluing of two complete intersection numerical semigroups H_1 and H_2 by Theorem 3.3. In our situation, we can determine H_1 and H_2 (see **[2]** or **[7,** Chapter 8]). In particular, if $2x = \lambda_1 a + \lambda_3 c$ with $\lambda_1, \lambda_3 > 0$, then we can take $H_1 = \langle a/d_1, c/d_1, x/d_1\rangle$, where $d_1 = \gcd(a, c, x) > 1$, and if $2x = \lambda_1 a + \lambda_2 b$ with $\lambda_1, \lambda_2 > 0$, then $H_1 = \langle a/d_1, b/d_1, x/d_1\rangle$, where $d_1 = \gcd(a, b, x) > 1$.

(i) If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $S = \langle 2\langle a/2, b/2, c/2\rangle, x\rangle$. This contradicts $\gcd(a, b, c) = 1$.

(ii) When $\lambda_1 > 0$ and $\lambda_2 = 0$, we see that $S = \langle d_1\langle a/d_1, c/d_1, x/d_1\rangle, b\rangle$. Furthermore, $\langle a/d_1, c/d_1, x/d_1\rangle = \langle 2\langle a/2d_1, c/2d_1\rangle, x/d_1\rangle$. Hence, a and c are even.

(iii) When $\lambda_1 = 0$ and $\lambda_2 > 0$, we see that b and c are even in the same manner as in (ii).

The proof is complete. $\square$

Now we give the proof of our main theorem.

Theorem 3.11. *Let $H = \langle a, b, c\rangle$ be a symmetric numerical semigroup, and assume that $H = \langle d\langle a', b'\rangle, c\rangle$. We set $R = k[[H]]$, $H_1 = \langle a', b'\rangle$ and $R_1 = k[[H_1]]$. Then the following assertions hold true.*

(1) *If d and c are odd, then*

$$\chi_R^g = \{(t^{d\alpha}, t^{d\beta}) \mid \alpha, \beta \in H_1 \text{ such that } (t^\alpha, t^\beta) \in \chi_{R_1}^g\}.$$

In particular, $\#\chi_R^g = \#\chi_{R_1}^g$. Hence, $\chi_R^g = \emptyset$ if a, b and c are odd.

(2) *If a', b' and d are odd, and c is even, then*

(i) *$\chi_R^g \neq \emptyset$ if and only if $H + \langle c/2\rangle$ is symmetric.*

(ii) *if $\chi_R^g \neq \emptyset$, then*

$$\chi_R^g = \{(t^{(c/2)l}, t^{(c/2)d}) \mid \ l \text{ is even with } 1 < l < d\}.$$

In particular, $\#\chi_R^g = (d-1)/2$.

(3) *If a', b' and c are odd, and d is even, then $H+\langle da'/2\rangle$ or $H+\langle db'/2\rangle$ is symmetric. In particular, $\chi_R^g \neq \emptyset$. Furthermore, the number of χ_R^g does not depend on d.*

(4) *If a' or b' is even, then*

$$\chi_R^g \supset \{(t^{d\alpha}, t^{d\beta}) \mid \alpha, \beta \in H_1 \text{ such that } (t^{\alpha}, t^{\beta}) \in \chi_{R_1}^g\} \neq \emptyset.$$

In particular, $\chi_R^g \neq \emptyset$.

Remark 3.12. When a' or b' is even, and c or d is even, there are cases where χ_R^g is lifted from $\chi_{R_1}^g$ and also the cases where some element of χ_R^g is not lifted (see Example 3.13 (4)).

Proof.

(1) We assume that d and c are odd. Then the statement implies that all Ulrich ideals of R are extensions from those of R_1. Therefore, it suffices to prove that, if $H+\langle x\rangle$ is symmetric for an integer x such that $x \notin H$ and $2x \in H$, then d divides x by Theorem 2.1 and Lemma 3.5.

Since $2x \in H$, we put $2x = \lambda_1 a + \lambda_2 b + \lambda_3 c$, where $\lambda_1, \lambda_2, \lambda_3 \geq 0$. If $\lambda_3 = 0$, then d divides x since a and b are divided by d, and hence we are done. Therefore, we assume that $\lambda_3 > 0$. If $\lambda_1 = \lambda_2 = 0$, then $x = \lambda_3 c/2 \in H$, a contradiction. Hence, we have $\lambda_1 > 0$ or $\lambda_2 > 0$. But then, c is even by Lemma 3.10, which contradicts our assumption. Hence, we must have $\lambda_3 = 0$. The last statement of (1) follows from Theorem 2.3 and the first statement of (1).

(2) Next we assume that a', b' and d are odd (equivalently, a and b are odd), and c is even. Then we note that $\chi_{R_1}^g = \emptyset$ by Theorem 2.3. It is clear that, if $H + \langle c/2\rangle$ is symmetric, then $\chi_R^g \neq \emptyset$ by Theorem 2.1. Conversely, if $\chi_R^g \neq \emptyset$, then there exists an integer x such that $x \notin H, 2x \in H$ and $H+\langle x\rangle$ is symmetric by Theorem 2.1. We claim that the set of such integers is equal to $\{\lambda c/2 \mid \lambda \text{ is odd with } 1 \leq \lambda \leq d-1\}$. Then the statements of (2) follow from this.

We put $2x = \lambda_1 a + \lambda_2 b + \lambda_3 c$, where $\lambda_1, \lambda_2, \lambda_3 \geq 0$. If $\lambda_3 = 0$, then d divides x, and hence, $\chi_{R_1}^g \neq \emptyset$ by Lemma 3.5, a contradiction. Therefore, we have $\lambda_3 > 0$ and $\lambda_1 = \lambda_2 = 0$ since, if $\lambda_1 > 0$ or $\lambda_2 > 0$, then a or b is even by Lemma 3.10, a contradiction. By this discussion, we can write $x = \lambda c/2$, where $\lambda = \lambda_3$ is odd. We need to show that $H + \langle x\rangle$ is symmetric for any $1 \leq \lambda \leq d - 1$. Since

$H + \langle x \rangle = \langle a, b, c, \lambda c/2 \rangle$ is symmetric, it is a complete intersection by Theorem 3.8. Hence, we can write

$$H + \langle x \rangle = \left\langle d \langle a', b' \rangle, \frac{c}{2} \langle 2, \lambda \rangle \right\rangle.$$

by Theorem 3.3 since we know that both $\langle a', b' \rangle$ and $\langle 2, \lambda \rangle$ are complete intersections. Then it is easily seen that $d \in \langle 2, \lambda \rangle \setminus \{2, \lambda\}$ if and only if $1 \leq \lambda \leq d - 1$. This completes the proof.

(3) Assume that a', b' and c are odd, and d is even. Since $c \in \langle a', b' \rangle$, we put $c = \lambda_1 a' + \lambda_2 b'$, where $\lambda_1, \lambda_2 \geq 0$. We know that either λ_1 or λ_2 is odd and the other is even since c is even. If λ_1 is odd and λ_2 is even (respectively, λ_1 is even and λ_2 is odd), then $H + \langle da'/2 \rangle = \langle d/2 \langle a', 2b' \rangle, c \rangle$ (respectively, $H + \langle db'/2 \rangle = \langle d/2 \langle 2a', b' \rangle, c \rangle$ is symmetric. Hence, $\chi_R^g \neq \emptyset$ by Theorem 2.1.

We see that the number of χ_R^g does not depend on d as follows. Let $H' = \langle 2 \langle a', b' \rangle, c \rangle$ and $H_m = \langle d \langle a', b' \rangle, c \rangle$, where $d = 2m$, $m > 1$. We show that $\#\chi^g_{k[[H']]} = \#\chi^g_{k[[H_m]]}$ for any m. Assume that $H' + \langle x \rangle$ is symmetric for $x \in \mathbb{Z}$ such that $x \notin H'$ and $2x \in H'$. Then it is easily seen that $mx \notin H_m$, $2mx \in H_m$ and $H + \langle mx \rangle$ is symmetric, which implies that $\#\chi^g_{k[[H']]} \leq \#\chi^g_{k[[H_m]]}$ by Theorem 2.1.

Conversely, we assume that $H_m + \langle y \rangle$ is symmetric for $y \in \mathbb{Z}$ such that $y \notin H_m$ and $2y \in H_m$. By Lemma 3.10, we can write as $2y = \lambda_1(2ma') + \lambda_2(2mb')$, where $\lambda_1, \lambda_2 \geq 0$. Therefore, m divides y, and we put $x = y/m$. Then we see that $x \notin H'$, $2x \in H'$ and $H' + \langle x \rangle$ is symmetric. This yields $\#\chi^g_{k[[H']]} \geq \#\chi^g_{k[[H_m]]}$ by Theorem 2.1.

(4) This follows immediately from Theorem 2.3 and Lemma 3.5. □

Example 3.13.

(1) By Theorem 3.11 (1), if all of a, b and c are odd, then $\chi_R^g = \emptyset$, but the converse is not true. For example, let $H = \langle 8, 9, 15 \rangle = \langle 3 \langle 3, 5 \rangle, 8 \rangle$. Then we can check that $\chi_R^g = \emptyset$, which also illustrates Theorem 3.11 (2).

(2) Let $H_m = \langle 2m \langle 3, 7 \rangle, 23 \rangle$, where $m \geq 1$. Then, for any m,

$$\chi^g_{k[[H_m]]} = \{(t^{20m}, t^{23m}), (t^{14m}, t^{23m}), (t^{6m}, t^{23m})\}.$$

(3) In the case of Theorem 3.11 (3), the number of χ_R^g may not be described by using a', b' or c. For example, we let $H = \langle 6, 10, c\rangle = \langle 2\langle 3,5\rangle, c\rangle$. Then:
 - if $c = 11$, $\chi_R^g = \{(t^6, t^{11})\}$.
 - if $c = 13$, $\chi_R^g = \{(t^{10}, t^{13})\}$.
 - if $c = 15$, $\chi_R^g = \{(t^{12}, t^{15}), (t^{10}, t^{15}), (t^6, t^{15})\}$.
 - if $c = 17$, $\chi_R^g = \{(t^{12}, t^{17}), (t^6, t^{17})\}$.
 - if $c = 19$, $\chi_R^g = \{(t^{16}, t^{19}), (t^{10}, t^{19}), (t^6, t^{19})\}$.

(4) We give examples in the case where a' or b' is even. If $H = \langle 3\langle 3,4\rangle, 10\rangle$, then

$$\chi_R^g = \{(t^{12}, t^{18})\}.$$

In that case, χ_R^g consists of the extension from $\chi_{R_1}^g$, and hence the equality holds true in Theorem 3.11 (4). If $H = \langle 3\langle 2,3\rangle, 4\rangle$, however, then

$$\chi_R^g = \{(t^4, t^6), (t^6, t^9), (t^4, t^9)\}.$$

Then the ideal (t^6, t^9) is the only extension from $\chi_{R_1}^g$ (see also Example 3.6).

4. Some remarks. We conclude the paper by giving some remarks. We say that a numerical semigroup H is generated by an *arithmetic sequence* if it is in the form of

$$H = \langle a, a+d, \ldots, a+nd\rangle,$$

where $a, d > 0, n \geq 2$ and $\gcd(a, d) = 1$. In that case, the following result is shown in **[6]**.

Theorem 4.1 (**[6]**). *Let $H = \langle a, a+d, \ldots, a+nd\rangle$ be a symmetric numerical semigroup generated by an arithmetic sequence. Then $\chi_{k[[H]]}^g \neq \emptyset$ if and only if $n = 2$.*

It is known that, when $H = \langle a, a+d, \ldots, a+nd\rangle$ is symmetric, it is a complete intersection if and only if $n = 2$, see **[5]**. Therefore, we may expect that if H is a symmetric numerical semigroup but not a complete intersection, then $\chi_{k[[H]]}^g = \emptyset$. However, unfortunately, there are counterexamples:

Example 4.2. A numerical semigroup $H = \langle 10, 12, 13, 14, 15\rangle$ is symmetric but not a complete intersection. However, $H + \langle 5\rangle = \langle 5, 12, 13, 14\rangle$ is symmetric, and hence, $\chi^g_{k[[H]]} \neq \emptyset$. In general, $H_m = \langle 2m, 2m+2, 2m+3, \ldots, 3m\rangle$ is symmetric but not a complete intersection if $m \geq 5$. Then we can check that $H + \langle m\rangle$ is symmetric. Therefore, $\chi^g_{k[[H_m]]} \neq \emptyset$.

Acknowledgments. The author would like to thank Professor Keiichi Watanabe for useful conversations.

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