

## ON TWO CLASSES OF REGULAR SEQUENCES

RI-XIANG CHEN

ABSTRACT. In this paper we get two new classes of regular sequences in the polynomial ring over the field of complex numbers.

**1. Introduction.** Throughout this paper,  $S = k[x_1, x_2, \dots, x_n]$  denotes the polynomial ring in  $n$  variables over a field  $k$ . Let  $f_1, f_2, \dots, f_n$  be some homogeneous polynomials in  $S$  of degrees  $a_1, a_2, \dots, a_n$  with  $1 \leq a_1 \leq a_2 \leq \dots \leq a_n$ . It is well known that the following statements are equivalent:

- (1)  $f_1, f_2, \dots, f_n$  is a regular sequence in  $S$ ;
- (2) The Koszul complex  $K.(f_1, f_2, \dots, f_n)$  is the minimal free resolution of  $S/(f_1, f_2, \dots, f_n)$  over  $S$ ;
- (3)  $(f_1, f_2, \dots, f_n)_N = S_N$  where  $N = 1 + \sum_{i=1}^n (a_i - 1)$ ;
- (4)  $\sqrt{(f_1, f_2, \dots, f_n)} = (x_1, x_2, \dots, x_n)$ , that is,  $\text{ht}(f_1, f_2, \dots, f_n) = n$ ;

and when the field  $k$  is algebraically closed, they are all equivalent to the following:

- (5)  $V(f_1, f_2, \dots, f_n) = \{(0, 0, \dots, 0)\}$ , that is,  $(0, 0, \dots, 0)$  is the only solution to the system of equations  $f_1 = 0, f_2 = 0, \dots, f_n = 0$ .

There are some other criterions for  $f_1, f_2, \dots, f_n$  being a regular sequence. Despite the many criterions, there remain some very difficult questions about regular sequences.

For example, in [2], Conca, et al., made the following conjecture:

**Conjecture 1.1.** [2]. In  $S = \mathbb{C}[x_1, x_2, x_3]$ , let  $p_a(3) = x_1^a + x_2^a + x_3^a$ ,  $p_b(3) = x_1^b + x_2^b + x_3^b$  and  $p_c(3) = x_1^c + x_2^c + x_3^c$ , where  $a, b, c \in \mathbb{N}^*$ ,

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$a < b < c$ ,  $\gcd(a, b, c) = 1$  and  $6 \mid abc$ , then  $p_a(3)$ ,  $p_b(3)$  and  $p_c(3)$  form a regular sequence in  $S$ .

Conca, et al., verified some special cases of this conjecture, and the conjecture remains mysterious. Another conjecture about regular sequences is the famous EGH conjecture, which is wide open except some special cases.

**Conjecture 1.2.** [4]. *If  $I \subset S = k[x_1, x_2, \dots, x_n]$  is a homogeneous ideal containing a regular sequence of homogeneous polynomials  $f_1, f_2, \dots, f_n$  of degrees  $2 \leq a_1 \leq a_2 \leq \dots \leq a_n$ , then there exists a homogeneous ideal in  $S$  containing  $x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n}$  with the same Hilbert function.*

These conjectures motivate us to further our study of regular sequences. It is well known that, generically,  $f_1, f_2, \dots, f_n$  form a regular sequence in  $S$  (see Proposition 3.1), but we do not have many specific examples of regular sequences. The most popular example is the sequence  $x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n}$ . Also, if under a monomial order  $<$ ,  $\text{in}_<(f_1) = x_1^{a_1}$ ,  $\text{in}_<(f_2) = x_2^{a_2}, \dots, \text{in}_<(f_n) = x_n^{a_n}$ , then  $f_1, f_2, \dots, f_n$  form a regular sequence in  $S$ . In addition, in the case of  $a_1 = a_2 = \dots = a_n = 1$ ,  $f_1, f_2, \dots, f_n$  is a regular sequence if and only if  $f_1, f_2, \dots, f_n$  are  $k$ -linearly independent, and, equivalently, the determinant of the  $n \times n$  coefficient matrix is nonzero.

It is easy to see that, if  $f_1 = g_1 h_1$  with  $\deg(g_1) \geq 1$  and  $\deg(h_1) \geq 1$ , then  $f_1, f_2, \dots, f_n$  is a regular sequence if and only if both  $g_1, f_2, \dots, f_n$  and  $h_1, f_2, \dots, f_n$  are regular sequences. So, in [1], Abedelfatah considered the following class of regular sequences. Let  $f_1, f_2, \dots, f_n$  be homogeneous polynomials which split into linear factors, then  $f_1, f_2, \dots, f_n$  is a regular sequence if and only if for  $i = 1, 2, \dots, n$  and for every linear factor  $l_i$  of  $f_i$ ,  $l_1, l_2, \dots, l_n$  is a regular sequence. Abedelfatah proved that the EGH conjecture holds for this class of regular sequences.

The goal of this paper is to get two new classes of regular sequences in the polynomial ring over the field of complex numbers.

In Section 2, we consider the power sum symmetric polynomial  $p_m(n) = x_1^m + x_2^m + \dots + x_n^m$  in  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$ . We prove that, if  $a, d \in \mathbb{N}^*$  with  $\gcd(a, d) = 1$  and  $n! \mid a(a+d) \cdots (a+(n-1)d)$ ,

then  $p_a(n), p_{a+d}(n), \dots, p_{a+(n-1)d}(n)$  is a regular sequence in  $S$  (see Theorem 2.4.)

In Section 3, we prove that, if  $f_1, f_2, \dots, f_n$  are homogeneous polynomials of degrees  $a_1, a_2, \dots, a_n$  in  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$  and for all  $i = 1, 2, \dots, n$ , the distance between  $f_i$  and  $x_i^{a_i}$  (see Definition 3.4) is less than 1, then  $f_1, f_2, \dots, f_n$  is a regular sequence in  $S$  (see Theorem 3.5.)

**2. Regular sequences of power sums.** Throughout this section,  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$ , and let  $p_m(n)$  denote the power sum symmetric polynomial  $x_1^m + x_2^m + \dots + x_n^m$  in  $S$ . Given positive integers  $a_1 < a_2 < \dots < a_n$ , let  $A$  be the set  $\{a_1, a_2, \dots, a_n\}$ . For simplicity, we will denote the sequence  $p_{a_1}(n), p_{a_2}(n), \dots, p_{a_n}(n)$  in  $S$  by  $p_A(n)$ .

The question is: when is  $p_A(n)$  a regular sequence in  $S$ ? To answer it, Conca, et al., made the following two observations.

**Lemma 2.1.** [2]. *Let  $a_1 < a_2 < \dots < a_n$  be positive integers and  $A = \{a_1, a_2, \dots, a_n\}$ . Set  $d = \gcd(a_1, a_2, \dots, a_n)$  and  $A' = \{a/d \mid a \in A\}$ . Then  $p_A(n)$  is a regular sequence in  $S$  if and only if  $p_{A'}(n)$  is a regular sequence in  $S$ .*

**Lemma 2.2.** [2]. *Let  $f_1, f_2, \dots, f_n$  be a regular sequence of homogeneous symmetric polynomials of degrees  $1 \leq a_1 \leq a_2 \leq \dots \leq a_n$  in  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$ . Then  $n!$  divides  $a_1 a_2 \dots a_n$ , that is,  $n! \mid a_1 a_2 \dots a_n$ .*

By Lemma 2.1, we can always assume that  $\gcd(a_1, a_2, \dots, a_n) = 1$ . And Lemma 2.2 implies that  $n! \mid a_1 a_2 \dots a_n$  is a necessary condition for  $p_A(n)$  to be a regular sequence in  $S$ . One may wonder if both  $\gcd(a_1, a_2, \dots, a_n) = 1$  and  $n! \mid a_1 a_2 \dots a_n$  would imply  $p_A(n)$  being a regular sequence. This is true for  $n = 2$ , which is proved in [2], and is also a special case of the following Theorem 2.4. The case of  $n = 3$  remains mysterious as Conjecture 1.1. However, for  $n \geq 4$ , the above two conditions are not sufficient for  $p_A(n)$  being a regular sequence, as is illustrated in the following example.

**Example 2.3.**  $\gcd(1, 3, 5, 24) = 1$  and  $4! \mid 1 \times 3 \times 5 \times 24$ , but  $p_1(4), p_3(4), p_5(4), p_{24}(4)$  is not a regular sequence in  $S = \mathbb{C}[x_1, x_2, x_3, x_4]$ , because  $(e^{\pi i/24}, -e^{\pi i/24}, 1, -1)$  is a nonzero solution to the system of

equations  $p_1(4) = 0$ ,  $p_3(4) = 0$ ,  $p_5(4) = 0$ ,  $p_{24}(4) = 0$ . Examples can be similarly constructed for  $n > 4$ .

In [2, Proposition 2.9], Conca, et al., proved that if  $a_1, a_2, \dots, a_n$  are consecutive positive integers, then  $p_A(n)$  is a regular sequence in  $S$ . To generalize this result, we have the following theorem.

**Theorem 2.4.** *Let  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$ ,  $n \geq 2$ ,  $a, d \in \mathbb{N}^*$  and  $A = \{a, a + d, a + 2d, \dots, a + (n - 1)d\}$ . Assume that  $\gcd(a, d) = 1$  and  $n! \mid a(a + d)(a + 2d) \cdots (a + (n - 1)d)$ . Then  $p_A(n)$  is a regular sequence in  $S$ .*

Note that the assumption  $\gcd(a, d) = 1$  is equivalent to  $\gcd(a, a + d, a + 2d, \dots, a + (n - 1)d) = 1$ . And, when  $d = 1$ , this theorem is [2, Proposition 2.9]. To prove Theorem 2.4, we need to prove the following two lemmas, which themselves are interesting.

**Lemma 2.5.** *Let  $n, a, d \in \mathbb{N}^*$ ,  $n \geq 2$  and  $\gcd(a, d) = 1$ . Then  $n! \mid a(a + d)(a + 2d) \cdots (a + (n - 1)d)$  if and only if  $\gcd(d, n!) = 1$ , that is, either  $d = 1$  or every prime factor of  $d$  is greater than  $n$ .*

*Proof. Only if.* Suppose, for contradiction, that there is a prime number  $p$  which divides  $\gcd(d, n!)$ . Then  $p \mid d$  and  $p \mid n!$ . Since  $n! \mid a(a + d)(a + 2d) \cdots (a + (n - 1)d)$ , it follows that  $p \mid a + id$  for some  $0 \leq i \leq n - 1$ , which implies that  $p$  is a prime factor of  $\gcd(d, a + id) = \gcd(d, a)$ . But  $\gcd(a, d) = 1$ , so we get a contradiction and, therefore,  $\gcd(d, n!) = 1$ .

*If.* Assume  $\gcd(d, n!) = 1$ . Then there exist integers  $s, t$  such that  $sd + t(n!) = 1$ . Hence,

$$\begin{aligned} & s^n a(a + d)(a + 2d) \cdots (a + (n - 1)d) \\ & \equiv sa(sa + sd)(sa + 2sd) \cdots (sa + (n - 1)sd) \\ & \equiv sa(sa + 1 - t(n!))(sa + 2 - 2t(n!)) \cdots \\ & \quad (sa + (n - 1) - (n - 1)t(n!)) \\ & \equiv sa(sa + 1)(sa + 2) \cdots (sa + (n - 1)) \\ & \equiv 0 \pmod{n!}. \end{aligned}$$

Since  $\gcd(s, n!) = 1$ , it follows that  $n! \mid a(a+d)(a+2d)\cdots(a+(n-1)d)$ .  $\square$

In [5], Lam and Leung proved that if  $z_1, z_2, \dots, z_n \in \mathbb{C}$  are some  $d$ th roots of unity and  $z_1 + z_2 + \cdots + z_n = 0$ , then  $n$  must be a linear combination with non-negative integer coefficients of the prime factors of  $d$ . This is the main theorem of [5], and it was proved by group ring techniques. The next lemma can be viewed as a corollary to Lam and Leung's theorem. For the completeness of the paper, we also give an elementary proof of the lemma.

**Lemma 2.6.** *Let  $d, n \in \mathbb{N}^*$ ,  $d \geq 2$ ,  $n \geq 2$  and  $z_1, z_2, \dots, z_n \in \mathbb{C}$  be some  $d$ th roots of unity. If  $\gcd(d, n!) = 1$ , then  $z_1 + z_2 + \cdots + z_n \neq 0$ .*

*Proof.* Let  $d = q_1^{e_1} q_2^{e_2} \cdots q_s^{e_s}$  be the prime factorization of  $d$ , where  $q_1, q_2, \dots, q_s$  are distinct prime numbers and  $e_1, e_2, \dots, e_s \geq 1$ . Since  $\gcd(d, n!) = 1$ , it follows that  $q_1 > n, q_2 > n, \dots, q_s > n$ . We will prove this lemma by induction on  $s$ .

First, we consider the case  $s = 1$ . Suppose, for contradiction, that  $z_1 + z_2 + \cdots + z_n = 0$ . Let  $\omega = e^{2\pi i/d}$ . Without loss of generality, we can assume that  $z_1 = \omega^{b_1}, z_2 = \omega^{b_2}, \dots, z_n = \omega^{b_n}$  with  $0 = b_1 \leq b_2 \leq \cdots \leq b_n \leq d-1$ . Then

$$(b_2 - b_1) + (b_3 - b_2) + \cdots + (b_n - b_{n-1}) + (b_1 + d - b_n) = d.$$

Since  $q_1 > n$ , we have  $q_1^{e_1} > nq_1^{e_1-1}$ , so that  $d > n(d - \varphi(d))$ , where  $\varphi(d) = q_1^{e_1} - q_1^{e_1-1}$  is Euler's totient function. Hence, one of  $b_2 - b_1, b_3 - b_2, \dots, b_n - b_{n-1}$  and  $b_1 + d - b_n = d - b_n$  is greater than  $d - \varphi(d)$ .

Suppose  $d - b_n > d - \varphi(d)$ . Then  $b_n < \varphi(d)$ . Since  $z_1 + z_2 + \cdots + z_n = \omega^{b_1} + \omega^{b_2} + \cdots + \omega^{b_n} = 0$ , it follows that  $\omega$  is a root of the polynomial  $x^{b_1} + x^{b_2} + \cdots + x^{b_n}$ , which is of degree  $b_n < \varphi(d)$ . But this contradicts the well-known fact that the minimal polynomial of  $\omega$  over  $\mathbb{Q}$  has degree  $\varphi(d)$ .

Suppose  $b_i - b_{i-1} > d - \varphi(d)$  for some  $2 \leq i \leq n$ . Since  $z_1 + z_2 + \cdots + z_n = 0$ , it follows that

$$\frac{1}{z_i}(z_1 + z_2 + \cdots + z_n) = \frac{1}{z_i}(z_i + z_{i+1} + \cdots + z_n + z_1 + \cdots + z_{i-1})$$

$$= 1 + \omega^{b_{i+1}-b_i} + \dots + \omega^{b_n-b_i} + \omega^{b_1+d-b_i} + \dots \\ + \omega^{b_{i-1}+d-b_i} = 0.$$

Hence,  $\omega$  is a root of the polynomial  $1 + x^{b_{i+1}-b_i} + \dots + x^{b_n-b_i} + x^{d-b_i} + \dots + x^{b_{i-1}+d-b_i}$ , which is of degree  $d - (b_i - b_{i-1}) < d - (d - \varphi(d)) = \varphi(d)$ . But this also contradicts the fact that the minimal polynomial of  $\omega$  over  $\mathbb{Q}$  has degree  $\varphi(d)$ . So we have proved that  $z_1 + z_2 + \dots + z_n \neq 0$  for the case  $s = 1$ .

Now we consider the case  $s \geq 2$ . Suppose, for contradiction, that  $z_1 + z_2 + \dots + z_n = 0$ . Let  $d_1 = q_1^{e_1}$  and  $d_2 = q_2^{e_2} \dots q_s^{e_s}$ . Then  $d = d_1 d_2$  and  $\gcd(d_1, d_2) = 1$ . Let  $\omega = e^{2\pi i/d_1}$ ,  $\varepsilon = e^{2\pi i/d_2}$  and  $\eta = e^{2\pi i/d}$ . Then it is well known that  $\mathbb{Q}(\eta) = \mathbb{Q}(\omega)(\varepsilon) = \mathbb{Q}(\varepsilon)(\omega)$ . Hence, without loss of generality, we can assume that  $z_1 = \omega^{b_1} \varepsilon^{c_1}$ ,  $z_2 = \omega^{b_2} \varepsilon^{c_2}$ ,  $\dots$ ,  $z_n = \omega^{b_n} \varepsilon^{c_n}$  with  $0 = b_1 \leq b_2 \leq \dots \leq b_n \leq d_1 - 1$  and  $0 = c_1 \leq c_2 \leq \dots \leq c_n \leq d_2 - 1$ . Then

$$(b_2 - b_1) + (b_3 - b_2) + \dots + (b_n - b_{n-1}) + (b_1 + d_1 - b_n) = d_1.$$

By the proof of case  $s = 1$ , it is easy to see that we can assume  $d_1 - b_n \geq \max\{b_i - b_{i-1} \mid i = 2, \dots, n\}$ . Since  $q_1 > n$ , we have  $q_1^{e_1} > n q_1^{e_1-1}$ , so that  $d_1 > n(d_1 - \varphi(d_1))$ , which implies  $d_1 - b_n > d_1 - \varphi(d_1)$ , that is,  $b_n < \varphi(d_1)$ .

Let  $j = \min\{i = 1, \dots, n \mid b_i = b_n\} \geq 1$ . Then  $\varepsilon^{c_j}, \varepsilon^{c_{j+1}}, \dots, \varepsilon^{c_n}$  are some  $d_2$ th roots of unity. Since  $\gcd(d, n!) = 1$ , it follows that  $\gcd(d_2, (n - j + 1)!) = 1$ . Thus, by the induction hypothesis, it is easy to see that  $\varepsilon^{c_j} + \varepsilon^{c_{j+1}} + \dots + \varepsilon^{c_n} \neq 0$ . Therefore,  $\omega$  is a root of the polynomial  $\varepsilon^{c_1} x^{b_1} + \varepsilon^{c_2} x^{b_2} + \dots + \varepsilon^{c_{j-1}} x^{b_{j-1}} + (\varepsilon^{c_j} + \varepsilon^{c_{j+1}} + \dots + \varepsilon^{c_n}) x^{b_n} \in \mathbb{Q}(\varepsilon)[x]$ , which is of degree  $b_n < \varphi(d_1)$ . However, since  $\varphi(d) = \varphi(d_1)\varphi(d_2)$  and

$$\varphi(d) = [\mathbb{Q}(\eta) : \mathbb{Q}] = [\mathbb{Q}(\varepsilon)(\omega) : \mathbb{Q}(\varepsilon)][\mathbb{Q}(\varepsilon) : \mathbb{Q}] = [\mathbb{Q}(\varepsilon)(\omega) : \mathbb{Q}(\varepsilon)]\varphi(d_2),$$

it follows that  $[\mathbb{Q}(\varepsilon)(\omega) : \mathbb{Q}(\varepsilon)] = \varphi(d_1)$ , so that the minimal polynomial of  $\omega$  over  $\mathbb{Q}(\varepsilon)$  has degree  $\varphi(d_1)$ . So we have a contradiction, and therefore,  $z_1 + z_2 + \dots + z_n \neq 0$ .  $\square$

*Proof of Theorem 2.4.* Suppose, for contradiction, that  $p_A(n)$  is not a regular sequence in  $S$ . Then the polynomial system associated to  $p_A(n)$  has a nonzero solution  $(z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ . Since the polynomials  $p_A(n)$  are homogeneous and symmetric, without the loss

of generality, we can assume that  $z_1 = 1$  and that  $z_1^d, z_2^d, \dots, z_t^d$  are all the distinct values among  $z_1^d, z_2^d, \dots, z_n^d$ . Note that  $1 \leq t \leq n$ .

If  $t = 1$  and  $d = 1$ , then  $z_1 = z_2 = \dots = z_n = 1$ . Hence,  $z_1^a + z_2^a + \dots + z_n^a = n \neq 0$ , which contradicts  $(z_1, z_2, \dots, z_n)$  being a root of  $p_a(n)$ .

If  $t = 1$  and  $d \geq 2$ , then  $z_1^d = z_2^d = \dots = z_n^d = 1$ . Hence,  $z_1^a, z_2^a, \dots, z_n^a$  are some  $d$ th roots of unity. By Lemmas 2.5 and 2.6, we have that  $z_1^a + z_2^a + \dots + z_n^a \neq 0$ , which also contradicts  $(z_1, z_2, \dots, z_n)$  being a root of  $p_a(n)$ .

If  $t \geq 2$ , then, for every  $1 \leq i \leq t$ , we set

$$A_i = \{j = 1, 2, \dots, n \mid z_j^d = z_i^d\} \quad \text{and} \quad w_i = \sum_{j \in A_i} z_j^a.$$

Since  $(z_1, z_2, \dots, z_n)$  is a solution of the polynomial system associated to  $p_A(n)$ , we have the following system of equations:

$$\begin{cases} w_1 + w_2 + \dots + w_t = 0 \\ w_1 z_1^d + w_2 z_2^d + \dots + w_t z_t^d = 0 \\ \dots \dots \dots \\ w_1 z_1^{(n-1)d} + w_2 z_2^{(n-1)d} + \dots + w_t z_t^{(n-1)d} = 0. \end{cases}$$

Rewriting the first  $t$  equations in the matrix form, we get

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1^d & z_2^d & \dots & z_t^d \\ \vdots & \vdots & & \vdots \\ z_1^{(t-1)d} & z_2^{(t-1)d} & \dots & z_t^{(t-1)d} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_t \end{pmatrix} = 0.$$

By Lemmas 2.5 and 2.6, it is easy to see that  $w_1 \neq 0$ , so that  $(w_1, w_2, \dots, w_t)$  is a nonzero vector. Thus, we have that

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1^d & z_2^d & \dots & z_t^d \\ \vdots & \vdots & & \vdots \\ z_1^{(t-1)d} & z_2^{(t-1)d} & \dots & z_t^{(t-1)d} \end{vmatrix} = 0,$$

which implies

$$\prod_{1 \leq i < j \leq t} (z_j^d - z_i^d) = 0.$$

This contradicts the assumption that  $z_1^d, z_2^d, \dots, z_t^d$  are all distinct. Therefore,  $p_A(n)$  is a regular sequence in  $S$ .  $\square$

**Remark 2.7.** In [2], Conca, et al., verified some special cases of Conjecture 1.1. Now Theorem 2.4 proves some new cases of this conjecture. For example, if  $A = \{1, 8, 15\}$  or  $A = \{2, 7, 12\}$ , then  $p_A(3)$  is a regular sequence in  $S = \mathbb{C}[x_1, x_2, x_3]$ . However, Conjecture 1.1 still remains mysterious. For example, if  $A = \{2, 5, 9\}$ , then it is hard to see if  $p_A(3)$  is a regular sequence. Also, when  $A = \{1, 6, m\}$ , the proof for  $p_A(3)$  being a regular sequence took almost eight pages in [2], so one might wonder if there is an easier way to prove it.

**3. Regular sequences close to  $x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n}$ .** Let  $S = k[x_1, x_2, \dots, x_n]$  be the polynomial ring in  $n$  variables over an infinite field  $k$ . Let  $f_1, f_2, \dots, f_n$  be a sequence of homogeneous polynomials of degrees  $1 \leq a_1 \leq a_2 \leq \dots \leq a_n$  in  $S$ . Then it is well-known that, generically,  $f_1, f_2, \dots, f_n$  is a regular sequence in  $S$ . In the next proposition, we will give this basic fact an elementary proof, which will also be useful when proving Theorem 3.5.

**Proposition 3.1.** *Let  $S = k[x_1, x_2, \dots, x_n]$  with  $k$  being any field. Given  $1 \leq a_1 \leq a_2 \leq \dots \leq a_n$ , let  $W = S_{a_1} \oplus S_{a_2} \oplus \dots \oplus S_{a_n}$  be the affine  $k$ -space of dimension  $m$ , where  $m = \sum_{i=1}^n \binom{n+a_i-1}{a_i}$ . Let*

$$\Sigma = \{(f_1, f_2, \dots, f_n) \in W \mid f_1, f_2, \dots, f_n \text{ is a regular sequence in } S\}.$$

*Then  $\Sigma$  is a non-empty Zariski-open subset of  $W$ . Furthermore, if  $k$  is an infinite field, then  $\Sigma$  is a dense open subset of  $W$ , and then, generically, a sequence of homogeneous polynomials  $f_1, f_2, \dots, f_n$  of degrees  $a_1, a_2, \dots, a_n$  is a regular sequence in  $S$ .*

*Proof.* For any  $d \geq 1$ , the set of all monomials in  $S_d$  ordered in the lexicographic order form a basis of the  $k$ -vector space  $S_d$ . And  $S_0 = k$  has basis  $\{1\}$ . These bases give rise to a basis of the  $k$ -vector space  $W = S_{a_1} \oplus S_{a_2} \oplus \dots \oplus S_{a_n}$ . We call these bases the monomial bases of these  $k$ -vector spaces. So, under the monomial basis,  $W$  is isomorphic to  $k^m$ .

Let  $N = \sum_{i=1}^n (a_i - 1)$ . Then the  $n$ -tuple  $(f_1, f_2, \dots, f_n)$  is in  $\Sigma$  if and only if for the ideal  $(f_1, f_2, \dots, f_n) \subset S$ , we have that

$(f_1, f_2, \dots, f_n)_N = S_N$ , or, equivalently, given any  $h \in S_N$ , there exist  $g_1 \in S_{N-a_1}, g_2 \in S_{N-a_2}, \dots, g_n \in S_{N-a_n}$  such that

$$(3.1) \quad f_1 g_1 + f_2 g_2 + \dots + f_n g_n = h.$$

Under the monomial basis, let  $\lambda_1, \lambda_2, \dots, \lambda_m$  be the coordinates of the  $n$ -tuple  $(f_1, f_2, \dots, f_n)$  in  $W$ , let  $\mu_1, \mu_2, \dots, \mu_p$  be the coordinates of  $h$  in  $S_N$  where  $p = \binom{n+N-1}{N}$ , and let  $z_1, z_2, \dots, z_q$  be the coordinates of the  $n$ -tuple  $(g_1, g_2, \dots, g_n)$  in  $S_{N-a_1} \oplus S_{N-a_2} \oplus \dots \oplus S_{N-a_n}$  where  $q = \sum_{i=1}^n \binom{n+N-a_i-1}{N-a_i}$ . Then equation (3.1) is equivalent to the following matrix equation:

$$(3.2) \quad A \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_q \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix},$$

where the  $p \times q$  matrix  $A$  is determined by equation (3.1) and the entries of  $A$  are either some  $\lambda_i$  or zero. Thus,  $f_1, f_2, \dots, f_n$  is a regular sequence if and only if, given any vector  $(\mu_1, \mu_2, \dots, \mu_p)^T$ , there exists a vector  $(z_1, z_2, \dots, z_q)^T$  satisfying equation (3.2), or equivalently,  $\text{rank}(A) = p$ . So, the  $n$ -tuple  $(f_1, f_2, \dots, f_n)$  is in  $\Sigma$  if and only if the  $m$ -tuple  $(\lambda_1, \lambda_2, \dots, \lambda_m)$  is in  $k^m - V(I_p(A))$ . Since  $k^m - V(I_p(A))$  is a Zariski-open subset of  $k^m$ , it follows that  $\Sigma$  is a Zariski-open subset of  $W$ . Note that  $x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n}$  is a regular sequence in  $S$ . Therefore,  $\Sigma$  is a non-empty Zariski-open subset of  $W$ .

If the field  $k$  is infinite, then the affine space  $W$  is irreducible, so that the non-empty Zariski-open subset  $\Sigma$  is dense in  $W$ , which implies that, generically, a sequence of homogeneous polynomials  $f_1, f_2, \dots, f_n$  of degrees  $a_1, a_2, \dots, a_n$  is a regular sequence in  $S$ .  $\square$

To illustrate equation (3.2) in the above proof of Proposition 3.1, we have the following example.

**Example 3.2.** Let  $S = k[x_1, x_2]$  and  $a_1 = a_2 = 2$ . Then the numbers defined in the above proof are  $m = 6$ ,  $N = 3$  and  $p = q = 4$ . Let

$$\begin{aligned} f_1 &= \lambda_1 x_1^2 + \lambda_2 x_1 x_2 + \lambda_3 x_2^2, & g_1 &= z_1 x_1 + z_2 x_2, \\ f_2 &= \lambda_4 x_1^2 + \lambda_5 x_1 x_2 + \lambda_6 x_2^2, & g_2 &= z_3 x_1 + z_4 x_2, \end{aligned}$$

$$h = \mu_1 x_1^3 + \mu_2 x_1^2 x_2 + \mu_3 x_1 x_2^2 + \mu_4 x_2^3.$$

Then  $f_1 g_1 + f_2 g_2 = h$  is equivalent to the following matrix equation:

$$A \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & 0 \\ \lambda_2 & \lambda_1 & \lambda_5 & \lambda_4 \\ \lambda_3 & \lambda_2 & \lambda_6 & \lambda_5 \\ 0 & \lambda_3 & 0 & \lambda_6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix}.$$

Since  $z_1$  in  $g_1$  can only be multiplied by  $f_1$ , it follows that only the coefficients  $\lambda_1, \lambda_2, \lambda_3$  of  $f_1$  can appear in the first column of the matrix  $A$  and each coefficient of  $f_1$  appears only once in the first column. The other columns of  $A$  have similar properties. Also, one can see that  $f_1, f_2$  is a regular sequence in  $S$  if and only if  $\det(A) \neq 0$ .

**Remark 3.3.** Illustrated by the above example, we have that, in general, the matrix  $A$  in the proof of Proposition 3.1 has the following property: for any column of  $A$ , there exists some  $f_i$  such that the entries of the column are either the coefficients of  $f_i$  or zero, and each coefficient of  $f_i$  appears only once in that column. This observation about the matrix  $A$  will be used in the proof of Theorem 3.5, which is the main result of this section.

If  $k = \mathbb{C}$ , then by [3, Theorem 2.3, Chapter 3], we see that  $f_1, f_2, \dots, f_n$  is a regular sequence in  $S$  if and only if the resultant  $\text{Res}(f_1, f_2, \dots, f_n)$  is nonzero. If  $n = 2$  or  $a_1 = a_2 = \dots = a_n = 1$ , then  $\text{Res}(f_1, f_2, \dots, f_n) = \det(A)$ . For other cases, the matrix  $A$  is not even a square matrix. And, although we have that  $V(I_p(A)) = V(\text{Res}(f_1, f_2, \dots, f_n))$ , it is not clear if  $\text{Res}(f_1, f_2, \dots, f_n)$  can always be expressed as a single determinant.

In order to state Theorem 3.5, we need the following definition. In the rest of this section, we will assume  $k = \mathbb{C}$ .

**Definition 3.4.** Let  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$  and  $f, g$  be two homogeneous polynomials in  $S$  of degree  $d$ . Suppose

$$\begin{aligned} f &= \lambda_1 x_1^d + \lambda_2 x_1^{d-1} x_2 + \dots + \lambda_r x_n^d, \\ g &= \nu_1 x_1^d + \nu_2 x_1^{d-1} x_2 + \dots + \nu_r x_n^d, \end{aligned}$$

where  $r = \binom{n+d-1}{d}$ . Then we define

$$d(f, g) = |\lambda_1 - \nu_1| + |\lambda_2 - \nu_2| + \cdots + |\lambda_r - \nu_r|,$$

and we call it *the distance between  $f$  and  $g$* .

Given  $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ , let  $W = S_{a_1} \oplus S_{a_2} \oplus \cdots \oplus S_{a_n}$ . For any two  $n$ -tuples  $(f_1, f_2, \dots, f_n)$  and  $(g_1, g_2, \dots, g_n)$  in  $W$ , we can define the distance between them as follows:

$$d((f_1, f_2, \dots, f_n), (g_1, g_2, \dots, g_n)) = d(f_1, g_1) + d(f_2, g_2) + \cdots + d(f_n, g_n).$$

This definition makes  $W$  into a metric space, whose induced topology is larger than the Zariski topology on  $W$ . So, by Proposition 3.1, the set  $\Sigma$  is a non-empty open subset in the metric space  $W$ . Therefore, given any regular sequence  $f_1, f_2, \dots, f_n$  of homogeneous polynomials with degrees  $a_1, a_2, \dots, a_n$  in  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$ , there exists  $\varepsilon > 0$  such that, for any homogeneous polynomials  $g_1, g_2, \dots, g_n \in S$  of degrees  $a_1, a_2, \dots, a_n$ , if  $d(g_1, f_1) < \varepsilon$ ,  $d(g_2, f_2) < \varepsilon, \dots, d(g_n, f_n) < \varepsilon$ , then  $g_1, g_2, \dots, g_n$  is also a regular sequence in  $S$ .

The question is: if  $f_1 = x_1^{a_1}, f_2 = x_2^{a_2}, \dots, f_n = x_n^{a_n}$ , what is the maximal value for  $\varepsilon$ ? The answer is  $\varepsilon = 1$ , as is shown in the following theorem.

**Theorem 3.5.** *Let  $f_1, f_2, \dots, f_n$  be a sequence of homogeneous polynomials of degrees  $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$  in  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$ . If  $d(f_i, x_i^{a_i}) < 1$  for  $i = 1, 2, \dots, n$ , then  $f_1, f_2, \dots, f_n$  is a regular sequence in  $S$ .*

To prove this theorem, we need the following lemma.

**Lemma 3.6.** *Let  $A = (a_{ij})_{n \times n} \in M_n(\mathbb{C})$ . If, for every  $j = 1, 2, \dots, n$ , we have*

$$|a_{jj}| > |a_{1j}| + \cdots + |a_{j-1,j}| + |a_{j+1,j}| + \cdots + |a_{nj}|,$$

*then  $\det(A) \neq 0$ .*

*Proof.* Suppose, for contradiction, that  $\det(A) = 0$ . Let  $\alpha_i = (a_{i1}, a_{i2}, \dots, a_{in})$  for  $i = 1, 2, \dots, n$ . Then  $\alpha_1, \alpha_2, \dots, \alpha_n$  are linearly

dependent. Hence, there exist  $l_1, l_2, \dots, l_n \in \mathbb{C}$ , which are not all zero, such that

$$l_1\alpha_1 + l_2\alpha_2 + \dots + l_n\alpha_n = 0.$$

Let  $|l_j| = \max\{|l_i| \mid i = 1, 2, \dots, n\} > 0$ . Then

$$\alpha_j = -\frac{l_1}{l_j}\alpha_1 - \dots - \frac{l_{j-1}}{l_j}\alpha_{j-1} - \frac{l_{j+1}}{l_j}\alpha_{j+1} - \frac{l_n}{l_j}\alpha_n.$$

Hence,

$$a_{jj} = -\frac{l_1}{l_j}a_{1j} - \dots - \frac{l_{j-1}}{l_j}a_{j-1,j} - \frac{l_{j+1}}{l_j}a_{j+1,j} - \frac{l_n}{l_j}a_{nj}.$$

Thus,

$$\begin{aligned} |a_{jj}| &\leq \left| \frac{l_1}{l_j} \right| |a_{1j}| + \dots + \left| \frac{l_{j-1}}{l_j} \right| |a_{j-1,j}| \\ &\quad + \left| \frac{l_{j+1}}{l_j} \right| |a_{j+1,j}| + \left| \frac{l_n}{l_j} \right| |a_{nj}| \\ &\leq |a_{1j}| + \dots + |a_{j-1,j}| + |a_{j+1,j}| + |a_{nj}|, \end{aligned}$$

which contradicts the assumption. Therefore,  $\det(A) \neq 0$ .  $\square$

*Proof of Theorem 3.5.* For every  $i = 1, 2, \dots, n$ , let  $\nu_i$  be the coefficient of  $x_i^{\alpha_i}$  in  $f_i$  and  $c_i$  the sum of the absolute value of the other coefficients in  $f_i$ . Since  $d(f_i, x_i^{\alpha_i}) < 1$ , it follows that  $c_i + |\nu_i - 1| < 1$ , which implies that  $c_i < 1 - |\nu_i - 1| \leq |1 + (\nu_i - 1)| = |\nu_i|$ .

Let  $A$  be the  $p \times q$  matrix defined in the proof of Proposition 3.1. Since  $x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n}$  is a regular sequence in  $S$ , it follows that there is a  $p \times p$  submatrix  $B$  of  $A$ , which is invertible when evaluated at the  $n$ -tuple  $(x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n})$ . Let  $B(x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n})$  denote the matrix  $B$  evaluated at  $(x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n})$ . By Remark 3.3, it is easy to see that the column vectors of  $B(x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n})$  are of the form  $(0, \dots, 0, 1, 0, \dots, 0)^T$ . Since  $B(x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n})$  is invertible, by rearranging the column vectors of  $B$ , we can assume that  $B(x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n})$  is the identity matrix.

Let  $B(f_1, f_2, \dots, f_n)$  be the matrix  $B$  evaluated at the  $n$ -tuple  $(f_1, f_2, \dots, f_n)$ . For every  $i = 1, 2, \dots, p$ , let  $\eta_i$  be the  $i$ th column vector of  $B(f_1, f_2, \dots, f_n)$ . Then, by Remark 3.3, it is easy to see that the  $i$ th entry in  $\eta_i$  is  $\nu_j$  for some  $1 \leq j \leq n$ , all the other

coefficients of  $f_j$  appear as the entries of  $\eta_i$  exactly once, and the remaining entries of  $\eta_i$  are zeros. Since  $c_j < |\nu_j|$ , it follows that the matrix  $B(f_1, f_2, \dots, f_n)$  satisfies the assumptions of Lemma 3.6, and then  $\det(B(f_1, f_2, \dots, f_n)) \neq 0$ . So the matrix  $A$  evaluated at the  $n$ -tuple  $(f_1, f_2, \dots, f_n)$  has rank  $p$ . Therefore,  $f_1, f_2, \dots, f_n$  is a regular sequence in  $S$ .  $\square$

**Remark 3.7.** It is easy to see that in  $S = \mathbb{C}[x_1, x_2]$ , if  $d(f_1, x_1^{a_1}) = 1$  and  $d(f_2, x_2^{a_2}) = 1$ , then  $f_1, f_2$  may not be a regular sequence in  $S$ . For example,  $f_1 = x^2 + xy$ ,  $f_2 = xy^2 + y^3$  is not a regular sequence in  $S$ .

The regular sequences obtained in Theorem 3.5 are very different from the regular sequences obtained in Theorem 2.4. It would be interesting to know if the EGH conjecture holds for these two classes of regular sequences. Besides, the above two classes of regular sequences are in  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$ . If  $S = \mathbb{Z}/2\mathbb{Z}[x_1, x_2, \dots, x_n]$ , then the sequence in Theorem 2.4 is not a regular sequence because the polynomial system has a non-zero solution  $(1, 1, 0, \dots, 0)$ . It would be interesting to find a new class of regular sequences in  $S = \mathbb{Z}/2\mathbb{Z}[x_1, x_2, \dots, x_n]$ .

We end this paper with the following proposition. We put it in this paper because its proof is similar to the proof of Proposition 3.1.

**Proposition 3.8.** *Let  $S = \mathbb{C}[x_1, x_2, \dots, x_n]$ . Let  $f_1, f_2, \dots, f_t$  be  $\mathbb{C}$ -linearly independent polynomials in  $S_i$  with  $i \geq 1$ . Then there exists  $\varepsilon > 0$  such that, for any homogeneous polynomials  $g_1, g_2, \dots, g_t$  in  $S_i$  satisfying  $d(g_1, f_1) < \varepsilon$ ,  $d(g_2, f_2) < \varepsilon, \dots, d(g_t, f_t) < \varepsilon$ , we have that  $g_1, g_2, \dots, g_t$  are  $\mathbb{C}$ -linearly independent and*

$$\dim_{\mathbb{C}}(f_1, f_2, \dots, f_t)_{i+1} \leq \dim_{\mathbb{C}}(g_1, g_2, \dots, g_t)_{i+1}.$$

*Proof.* It is easy to see that  $g_1, g_2, \dots, g_t$  are  $\mathbb{C}$ -linearly independent when  $\varepsilon$  is sufficiently small. Let

$$\begin{aligned} \Sigma &= \{(g_1, g_2, \dots, g_t) \in (S_i)^t \mid \dim_{\mathbb{C}}(g_1, g_2, \dots, g_t)_{i+1} \\ &\geq \dim_{\mathbb{C}}(f_1, f_2, \dots, f_t)_{i+1}\}. \end{aligned}$$

Since the  $t$ -tuple  $(f_1, f_2, \dots, f_t)$  is in  $\Sigma$ , it follows that  $\Sigma$  is non-empty, so that it suffices to prove that  $\sigma$  is an open subset of the affine  $\mathbb{C}$ -space

$(S_i)^t$ . Note that under the monomial basis as defined in the proof of Proposition 3.1,  $(S_i)^t$  is isomorphic to  $\mathbb{C}^N$  where  $N = t \binom{n+i-1}{i}$ .

For any  $t$ -tuple  $(g_1, g_2, \dots, g_t) \in (S_i)^t$ ,  $(g_1, g_2, \dots, g_t)_{i+1}$  is the  $\mathbb{C}$ -vector space spanned by the  $nt$  polynomials  $x_1g_1, x_2g_1, \dots, x_n g_t$ . Let  $B$  be the  $nt \times \binom{n+i}{i+1}$  matrix with each row being the coordinates of some  $x_l g_j$  under the monomial basis of  $S_{i+1}$ , where  $l = 1, 2, \dots, n$  and  $j = 1, 2, \dots, t$ . If the coordinates of  $t$ -tuple  $(g_1, g_2, \dots, g_t)$  under the monomial basis of  $(S_i)^t$  is  $(\lambda_1, \lambda_2, \dots, \lambda_N)$ , then the entries of  $B$  are either some  $\lambda_i$  or zero. Let  $p = \dim_{\mathbb{C}}(f_1, f_2, \dots, f_t)_{i+1}$ , then it is easy to see that the  $t$ -tuple  $(g_1, g_2, \dots, g_t)$  is in  $\Sigma$  if and only if the  $N$ -tuple  $(\lambda_1, \lambda_2, \dots, \lambda_N)$  is in  $\mathbb{C}^N - V(I_p(B))$ . Since  $\mathbb{C}^N - V(I_p(B))$  is an open subset of  $\mathbb{C}^N$ , it follows that  $\Sigma$  is an open subset of  $(S_i)^t$ .  $\square$

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DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY OF SCIENCE AND TECHNOLOGY, NANJING, JIANGSU, 210094, P.R.CHINA

Email address: rc429@cornell.edu