JOURNAL OF COMMUTATIVE ALGEBRA
Volume 7, Number 3, Fall 2015

FROBENIUS VECTORS, HILBERT SERIES AND GLUINGS OF AFFINE SEMIGROUPS

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ABSTRACT. Let S_1 and S_2 be two affine semigroups, and let S be the gluing of S_1 and S_2. Several invariants of S are related to those of S_1 and S_2; we review some of the most important properties preserved under gluings. The aim of this paper is to prove that this is the case for the Frobenius vector and the Hilbert series. Applications to complete intersection affine semigroups are also given.

1. On gluings of affine semigroups. In this section, we briefly summarize results on the gluing of affine semigroups. We also introduce concepts and notation used throughout the paper.

An *affine semigroup* S is a finitely generated submonoid of $\mathbb{Z}^m$ for some positive integer m. If $S \cap (-S) = 0$, that is to say, S is reduced, it can be shown that it has a unique minimal system of generators (see, for instance, [**25,** Chapter 3]). The cardinality of the minimal generating system of S is known as the *embedding dimension* of S. Recall that each reduced affine semigroup can be embedded into $\mathbb{N}^m$ for some m. In the following, we will assume that our affine semigroups are submonoids of $\mathbb{N}^m$.

Given an affine semigroup $S \subseteq \mathbb{N}^m$, denote by $\mathrm{G}(S)$ the group spanned by S, that is,

$$\mathrm{G}(S) = \big\{\mathbf{z} \in \mathbb{Z}^m \mid \mathbf{z} = \mathbf{a} - \mathbf{b}, \mathbf{a}, \mathbf{b} \in S\big\}.$$

2010 AMS *Mathematics subject classification.* Primary 20M14, Secondary 05A15, 11D07, 14M10.

Keywords and phrases. Affine semigroup, gluing, Frobenius vector, Hilbert series.

The first author is partially supported by project GDR CNRS 2945. The second author is supported by projects MTM2010-15595, MTM2014-55367-P, FQM-343, FQM-5849, Géanpyl (FR n°2963 du CNRS) and FEDER funds. The third author is partially supported by project MTM2012-36917-C03-01, National Plan I+D+I and by Junta de Extremadura (FEDER) funds.

Received by the editors on November 8, 2013.

DOI:10.1216/JCA-2015-7-3-317

Let A be the minimal generating system of S, and let $A = A_1 \cup A_2$ be a nontrivial partition of A. Let $S_i = \langle A_i \rangle$ (the monoid generated by A_i), $i \in \{1, 2\}$. Then $S = S_1 + S_2$. We say that S is the *gluing* of S_1 and S_2 by $\mathbf{d}$ if

- $\mathbf{d} \in S_1 \cap S_2$, and
- $\mathrm{G}(S_1) \cap \mathrm{G}(S_2) = \mathbf{d}\mathbb{Z}$.

We will denote this fact by $S = S_1 +_{\mathbf{d}} S_2$.

There are several properties that are preserved under gluings, and also some invariants of a gluing $S_1 +_{\mathbf{d}} S_2$ can be computed by knowing their values in S_1 and S_2. We summarize some of them next.

Assume that $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$. The monoid homomorphism $\varphi : \mathbb{N}^k \to S$ induced by $\mathbf{e}_i \mapsto \mathbf{a}_i$, $i \in \{1, \ldots, k\}$, is an epimorphism (where $\mathbf{e}_i$ is the ith row of the $k \times k$ identity matrix). Thus, S is isomorphic as a monoid to $\mathbb{N}^k / \ker \varphi$, where $\ker \varphi$ is the kernel congruence of φ, that is, the set of pairs $(\mathbf{a}, \mathbf{b}) \in \mathbb{N}^k \times \mathbb{N}^k$ with $\varphi(\mathbf{a}) = \varphi(\mathbf{b})$. A *presentation* of S is a system of generators of $\ker \varphi$. A *minimal presentation* is a presentation such that none of its proper subsets is a presentation. All minimal presentations have the same (finite) cardinality (see, for instance, [**25,** Corollary 9.5]). Suppose that $S = S_1 +_{\mathbf{d}} S_2$, where $S_i = \langle A_i \rangle$ for $i \in \{1, 2\}$, and that $A = A_1 \cup A_2$ is a nontrivial partition of A. We may assume, without loss of generality, that $A_1 = \{\mathbf{a}_1, \ldots, \mathbf{a}_l\}$ and $A_2 = \{\mathbf{a}_{l+1}, \ldots, \mathbf{a}_k\}$. According to [**22,** Theorem 1.4], if we know minimal presentations of S_1 and S_2, then we can construct a minimal presentation of S in the following way. Let ρ_i be a minimal presentation of S_i, $i \in \{1, 2\}$. Take $(\mathbf{a}, \mathbf{b}) \in \mathbb{N}^k \times \mathbb{N}^k$ with $\varphi(\mathbf{a}) = \varphi(\mathbf{b}) = \mathbf{d}$, the first l coordinates of $\mathbf{b}$ equal to zero and the last $k - l$ coordinates of $\mathbf{a}$ equal to zero. Then

$$\rho = \rho_1 \cup \rho_2 \cup \{(\mathbf{a}, \mathbf{b})\}$$

is a minimal presentation of S (actually, [**22,** Theorem 1.4] asserts that this characterizes that $S = S_1 +_{\mathbf{d}} S_2$).

For an affine semigroup S define $\mathrm{Betti}(S)$ as the set of $\mathbf{s} \in S$ for which there exists $\mathbf{a}, \mathbf{b} \in \varphi^{-1}(\mathbf{s})$ such that $(\mathbf{a}, \mathbf{b})$ belongs to a minimal presentation of S. Theorem 10 in [**14**] states that

$$\mathrm{Betti}(S_1 +_{\mathbf{d}} S_2) = \mathrm{Betti}(S_1) \cup \mathrm{Betti}(S_2) \cup \{\mathbf{d}\}.$$

Since several invariants such as the catenary degree and the maximum of the delta sets depend on the Betti elements of S (**[9, 8]**, respectively), the computation of these invariants for $S_1 +_{\mathbf{d}} S_2$ can be performed once we know their values for S_1, S_2 and $\mathbf{d}$ (see, for instance, **[7,** Corollary 4]).

Affine semigroups with a single Betti element can be characterized as a gluing of several copies of affine semigroups with empty minimal presentation (and thus isomorphic to $\mathbb{N}^t$ for some positive integer t) along this single Betti element (**[15]**).

We say that S is *uniquely presented* if, for every two minimal presentations σ and τ and every $(\mathbf{a}, \mathbf{b}) \in \sigma$, either $(\mathbf{a}, \mathbf{b}) \in \tau$ or $(\mathbf{b}, \mathbf{a}) \in \tau$, that is, there is a unique minimal presentation up to rearrangement of the pairs of the minimal presentation. It is known (**[14,** Theorem 12]) that $S_1 +_{\mathbf{d}} S_2$ is uniquely presented if and only if S_1 and S_2 are uniquely presented and $\pm(\mathbf{d} - \mathbf{a}) \notin S_1 +_{\mathbf{d}} S_2$ for every $\mathbf{a} \in \mathrm{Betti}(S_1) \cup \mathrm{Betti}(S_2)$.

It is well known that the cardinality of any minimal presentation of an affine semigroup is greater than or equal to its embedding dimension minus the dimension of the vector space spanned by the semigroup. An affine semigroup is a *complete intersection* affine semigroup if the cardinality of any of its minimal presentations attains this lower bound. It can be shown that an affine semigroup is a complete intersection if and only if it is either isomorphic to $\mathbb{N}^t$ for some positive integer t or it is the gluing of two complete intersection affine semigroups (**[12]**). This result generalizes **[23]**, which generalizes the classical result by Delorme for numerical semigroups (**[11]**; actually, the definition of gluing was inspired in that paper).

A *numerical semigroup* is a submonoid of $\mathbb{N}$ with finite complement in $\mathbb{N}$. It is easy to see that every numerical semigroup is finitely generated (see, for instance, **[26,** Chapter 1]), and thus every numerical semigroup is an affine semigroup. Let S be a numerical semigroup. The largest integer not belonging to S is known as its *Frobenius number*, $\mathrm{F}(S)$. By definition, $\mathrm{F}(S) + 1 + \mathbb{N} \subseteq S$. This is why the integer $\mathrm{F}(S) + 1$ is known as the *conductor* of S. Delorme **[11]** shows that the conductor of a numerical semigroup, that is, a gluing, say $S_1 +_d S_2$, can be computed in terms of the conductors of S_1, S_2 and d (we use d here instead of $\mathbf{d}$ because in this setting d is an integer). Thus, a formula for the Frobenius number of a numerical semigroup that is a gluing is easily derived (this idea is exploited in **[4]** to give a procedure for computing the set of all complete intersection numerical semigroups with given Frobenius number). One of the aims of this paper is to generalize this formula for affine semigroups.

Let S be a numerical semigroup. An element $g \in \mathbb{Z} \setminus S$ is a *pseudo-Frobenius number* if $g + (S \setminus \{0\}) \subseteq S$. In particular, $\mathrm{F}(S)$ is always a pseudo-Frobenius number. The cardinality of the set of pseudo-Frobenius numbers is known as the (Cohen-Macaulay) *type* of S, $\mathrm{t}(S)$. A numerical semigroup is *symmetric* if its type is one (there are plenty of characterizations of this property, see for instance, **[26,** Chapter 3]). Delorme in his above-mentioned paper **[11]** also proved that a numerical semigroup that is a gluing $S_1 +_d S_2$ is symmetric if and only if S_1 and S_2 are symmetric. Nari **[20,** Proposition 6.6] proved that, for a numerical semigroup of the form $S_1 +_d S_2$,

$$\mathrm{t}(S_1 +_d S_2) = \mathrm{t}(S_1)\,\mathrm{t}(S_2)$$

(actually the definition of gluing for numerical semigroups is slightly different and we have to divide S_1 and S_2 by their greatest common divisors in order to get S_1 and S_2 numerical semigroups; see the paragraph after Theorem 4.3). This formula can be seen as a generalization of the fact that the gluing of symmetric numerical semigroups is again symmetric, and it also shows that

- the gluing of pseudo-symmetric numerical semigroups (the only pseudo-Frobenius numbers are the Frobenius number and its half) cannot be pseudo-symmetric,
- the gluing of two nonsymmetric almost symmetric numerical semigroup is not almost symmetric (S is almost symmetric if the cardinality of $\mathbb{N} \setminus S$ equals $(\mathrm{F}(S) + \mathrm{t}(S))/2$).

Let S be an affine semigroup, and let $\mathbf{s} \in S \setminus \{0\}$. The *Apéry* set of $\mathbf{s}$ in S is the set

$$\mathrm{Ap}(S, \mathbf{s}) = \{\mathbf{x} \in S \mid \mathbf{x} - \mathbf{s} \notin S\}.$$

This set has, in general, infinitely many elements. If S is a numerical semigroup and $s \in S \setminus \{0\}$, then $\mathrm{Ap}(S, s)$ has exactly s elements (one for each congruent class modulo s). Let m be the least positive integer belonging to S, which is known as the *multiplicity* of S, and assume that S is minimally generated by $\{n_1, \ldots, n_k\}$, with $n_1 < \cdots < n_k$.

Clearly, $n_1 = m$ and $\mathrm{Ap}(S,m) \subseteq \{\sum_{i=2}^{k} a_i n_i \mid a_i \le \alpha_i, i \in \{2,\dots,k\}\}$, with $\alpha_i = \max\{k \in \mathbb{N} \mid k n_i \in \mathrm{Ap}(S,m)\}$. When the equality holds, we say that the Apéry set of S is α-rectangular. Theorem 2.3 in **[10]** shows that every numerical semigroup with α-rectangular Apéry set other than $\mathbb{N}$ can be constructed by gluing a numerical semigroup with the same property and a copy of $\mathbb{N}$.

For a given affine semigroup S and a field K, the *semigroup ring* $K[S]$ is defined as $K[S] = \bigoplus_{\mathbf{s}\in S} K t^{\mathbf{s}}$ with t an indeterminate. Addition is performed componentwise, and the product is calculated by using distributive law and $t^{\mathbf{s}} t^{\mathbf{s}'} = t^{\mathbf{s}+\mathbf{s}'}$ for all $\mathbf{s}, \mathbf{s}' \in S$. If S is a numerical semigroup, then $K[S]$ is a subring of $K[t]$. Recently (**[13]**), the following property has been shown to be preserved under gluing of numerical semigroups: for every relative I ideal of $K[S]$ generated by two monomials, $I \otimes_{K[S]} I^{-1}$ has nontrivial torsion. This partly solves a conjecture stated by Huneke and Wiegand (see **[13]** for details; also the restriction of being generated by just two elements can be removed if we take S_2 as a copy of $\mathbb{N}$).

If S is a numerical semigroup minimally generated by $\{n_1,\dots,n_k\}$, then $\mathfrak{m} = (t^{n_1},\dots,t^{n_k})$ is the unique maximal ideal of the power series ring $R = K[[t^{n_1},\dots,t^{n_k}]] = K[[S]]$. The Hilbert function of the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{n\in\mathbb{N}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is defined as $n \mapsto \dim_K(\mathfrak{m}^n/\mathfrak{m}^{n+1})$. In **[2]**, it is shown that, if the Hilbert functions of the associated graded rings of $K[[S_1]]$ and $K[[S_2]]$ are nondecreasing, then so is the Hilbert function of the associated graded ring of $K[[S_1 +_d S_2]]$ when the gluing is a "nice" gluing (see **[2,** Theorem 2.6] for details; this nice gluing has been also exploited in **[16]**).

Lastly, for $T = \langle an_1, an_2, an_3, an_4\rangle +_{ab} \langle b\rangle$, Barucci and Fröberg have been able to compute the Betti numbers of the free resolution of $K[T]$ in terms of that of $K[S]$, with $S = \langle n_1, n_2, n_3, n_4\rangle$ (**[5]**).

2. Gluings and cones. Given an affine semigroup $S \subseteq \mathbb{N}^m$, denote by $\mathrm{cone}(S)$ the cone spanned by S, that is,

$$\mathrm{cone}(S) = \left\{ q\,\mathbf{a} \mid q \in \mathbb{Q}_{\ge 0}, \mathbf{a} \in S \right\}.$$

Observe that $\mathrm{cone}(S)$ is pointed (the only subspace included in it is $\{0\}$), because S is reduced.

Clearly, if A is finite and generates S, then

$$\mathrm{G}(S) = \left\{ \sum_{\mathbf{a}\in A} z_{\mathbf{a}}\mathbf{a} \,\middle|\, z_{\mathbf{a}} \in \mathbb{Z} \text{ for all } \mathbf{a} \right\}$$

and

$$\mathrm{cone}(S) = \left\{ \sum_{\mathbf{a}\in A} q_{\mathbf{a}}\mathbf{a} \,\middle|\, q_{\mathbf{a}} \in \mathbb{Q}_{\geq 0} \text{ for all } \mathbf{a} \right\}.$$

We will write $\mathrm{aff}(S)$ for the affine span of S, that is,

$$\mathrm{aff}(S) = G(S) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

As usual, we use the notation

$$\langle A \rangle = \left\{ \sum_{\mathbf{a}\in A} n_{\mathbf{a}}\mathbf{a} \,\middle|\, n_{\mathbf{a}} \in \mathbb{N} \text{ for all } \mathbf{a} \in A \right\}$$

(all sums are finite, that is, if A has infinitely many elements, all but a finite number of $z_{\mathbf{a}}$, $q_{\mathbf{a}}$ and $n_{\mathbf{a}}$ are zero).

Lemma 2.1. *Let* $\mathbf{r}_1, \dots, \mathbf{r}_k, \mathbf{r}_{k+1}$ *and* $\mathbf{x} \in \mathrm{cone}(\mathbb{N}^m) \setminus \{0\}$, *for some positive integers* m *and* k. *If* $\mathrm{cone}(\mathbf{r}_1, \dots, \mathbf{r}_k) = \mathrm{cone}(\mathbf{r}_1, \dots, \mathbf{r}_k, \mathbf{r}_{k+1})$, *then the following conditions are equivalent*:

(i) *There exist* $q_1, \dots q_k \in \mathbb{Q}_{>0}$ *such that* $\mathbf{x} = q_1\mathbf{r}_1 + \cdots + q_k\mathbf{r}_k$.

(ii) *There exist* $q'_1, \dots, q'_{k+1} \in \mathbb{Q}_{>0}$ *such that* $\mathbf{x} = q'_1\mathbf{r}_1 + \cdots + q'_k\mathbf{r}_k + q'_{k+1}\mathbf{r}_{k+1}$.

Proof. Observe that, from the hypothesis, $\mathbf{r}_{k+1} \in \mathrm{cone}(\mathbf{r}_1, \dots, \mathbf{r}_k)$, and thus there exists $t_1, \dots, t_k \in \mathbb{Q}_{\geq 0}$ such that $\mathbf{r}_{k+1} = t_1\mathbf{r}_1 + \cdots + t_k\mathbf{r}_k$. From this, it easily follows that (ii) implies (i).

Assume that there exist $q_1, \dots q_k \in \mathbb{Q}_{>0}$ such that $\mathbf{x} = q_1\mathbf{r}_1 + \cdots + q_k\mathbf{r}_k$. Let $N \in \mathbb{N}$ be such that, for all $i \in \{1, \dots, k\}$, $t_i/N < q_i$ (this is possible since $q_i > 0$ for all i). Take $q'_i = q_i - t_i/N$ (which is a positive rational number) for all $i \in \{1, \dots, k\}$, and $q'_{k+1} = 1/N$. Then, $q'_1\mathbf{r}_1 + \cdots + q'_k\mathbf{r}_k + q'_{k+1}\mathbf{r}_{k+1} = q_1\mathbf{r}_1 + \cdots + q_k\mathbf{r}_k - 1/N\mathbf{r}_{k+1} + 1/N\mathbf{r}_{k+1} = \mathbf{x}$. □

Given $\mathbf{r}_1, \ldots, \mathbf{r}_k \in \operatorname{cone}(\mathbb{N}^m) \setminus \{0\}$, we define the *relative interior* of $\operatorname{cone}(\mathbf{r}_1, \ldots, \mathbf{r}_k)$ by

$$\operatorname{relint}(\operatorname{cone}(\mathbf{r}_1, \ldots, \mathbf{r}_k)) = \left\{q_1\mathbf{r}_1 + \cdots + q_k\mathbf{r}_k \mid q_1, \ldots, q_k \in \mathbb{Q}_{>0}\right\}.$$

Observe that the relative interior of a cone, C, is the topological interior of C in its affine span, $\operatorname{aff}(\mathbf{r}_1, \ldots, \mathbf{r}_k)$, with the subspace topology.

For $A \subseteq \mathbb{N}^m$, we say that F is a *face* of $\operatorname{cone}(A)$ if $F \neq \emptyset$ and there exists $\mathbf{c} \in \mathbb{Q}^m \setminus \{0\}$ such that

- $F = \{\mathbf{x} \in \operatorname{cone}(A) \mid \mathbf{c} \cdot \mathbf{x} = 0\}$ and
- $\mathbf{c} \cdot \mathbf{y} \geq 0$ for all $\mathbf{y} \in \operatorname{cone}(A)$.

An element $\mathbf{a} \in A$ is an *extremal ray* of $\operatorname{cone}(A)$ if $\mathbb{Q}_{\geq 0}\mathbf{a}$ is a one-dimensional face of $\operatorname{cone}(A)$.

Now, according to Lemma 2.1, if A is the minimal system of generators of an affine semigroup $S \subseteq \mathbb{N}^m$, then we can say that $\mathbf{x} \in \operatorname{relint}(\operatorname{cone}(S))$ if and only if $\mathbf{x} \in \operatorname{relint}(\operatorname{cone}(A))$, even if A contains elements that are not extremal rays. We get also the following consequence.

Proposition 2.2. *Let A be a nonempty subset of $\mathbb{N}^m$, with m a positive integer. Assume that $A = A_1 \cup A_2$ is a nontrivial partition of A. Then* $\operatorname{relint}(\operatorname{cone}(A)) = \operatorname{relint}(\operatorname{cone}(A_1)) + \operatorname{relint}(\operatorname{cone}(A_2))$.

Proof. Obviously, if $\mathbf{x}_i \in \operatorname{relint}(\operatorname{cone}(A_i))$, $i \in \{1, 2\}$, then $\mathbf{x}_1 + \mathbf{x}_2 \in \operatorname{relint}(\operatorname{cone}(A))$. Now, consider $\mathbf{x} \in \operatorname{relint}(\operatorname{cone}(A))$. Without loss of generality, we may assume that $\mathbf{x} = \sum_{\mathbf{a} \in A} q_{\mathbf{a}}\mathbf{a}$ with $q_{\mathbf{a}} \in \mathbb{Q}_{>0}$. Thus, by taking $\mathbf{x}_i = \sum_{\mathbf{a} \in A_i} q_{\mathbf{a}}\mathbf{a}$, we are done. □

Notice that, if S is the gluing of S_1 and S_2 by $\mathbf{d}$, then

$$\mathbf{d} \notin \operatorname{relint}(\operatorname{cone}(S)) \text{ implies } \mathbf{d} \notin \operatorname{relint}(\operatorname{cone}(S_1)) \cap \operatorname{relint}(\operatorname{cone}(S_2)).$$

Otherwise, we may take $\mathbf{x}_i = (1/2)\mathbf{d}$, $i \in \{1, 2\}$.

Proposition 2.3. *Let A be a nonempty subset of $\mathbb{N}^m$, with m a positive integer. Assume that $A = A_1 \cup A_2$ is a nontrivial partition of A. Let F be a face of* $\operatorname{cone}(A)$. *Then every $\mathbf{x} \in F$ can be expressed as $\mathbf{x}_1 + \mathbf{x}_2$ with $\mathbf{x}_i$ in a face of* $\operatorname{cone}(A_i)$, *$i \in \{1, 2\}$.*

Proof. Let $\mathbf{x} \in F$. Then there exists $\mathbf{c} \in \mathbb{Q}^m \setminus \{0\}$ such that $\mathbf{c} \cdot \mathbf{x} = 0$ and $\mathbf{c} \cdot \mathbf{y} \geq 0$ for all $\mathbf{y} \in \operatorname{cone}(A)$. Notice that $\operatorname{cone}(A) = \operatorname{cone}(A_1)+\operatorname{cone}(A_2)$. Hence, there exists $\mathbf{x}_i \in \operatorname{cone}(A_i)$, $i \in \{1,2\}$, such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$. As $\operatorname{cone}(A_i) \subseteq \operatorname{cone}(A)$, $\mathbf{c} \cdot \mathbf{y}_i \geq 0$, for $i \in \{1,2\}$ and all $\mathbf{y}_i \in \operatorname{cone}(A_i)$. Hence, $0 = \mathbf{c}\cdot\mathbf{x} = \mathbf{c}\cdot\mathbf{x}_1+\mathbf{c}\cdot\mathbf{x}_2$ forces $\mathbf{c}\cdot\mathbf{x}_1 = \mathbf{c}\cdot\mathbf{x}_2 = 0$. We conclude that $\mathbf{x}_i$ is in the face $\{\mathbf{x} \in \mathbf{Q}^n \mid \mathbf{c} \cdot \mathbf{x} = 0\} \cap \operatorname{cone}(A_i)$ of $\operatorname{cone}(A_i)$, $i \in \{1,2\}$. □

We end this section by giving an affine-geometric characterization of gluings (Corollary 2.5). First we show how the cones in a gluing intersect.

Proposition 2.4. *Let S be an affine semigroup and $\mathbf{d} \in \mathbb{N}^n \setminus \{0\}$. If $S = S_1 +_{\mathbf{d}} S_2$, then*

$$\operatorname{cone}(S_1) \cap \operatorname{cone}(S_2) = \mathbf{d}\mathbb{Q}_{\geq 0}.$$

Proof. By definition, $\mathbf{d} \in S_1 \cap S_2$ and, clearly, $\mathbf{d}\mathbb{Q}_{\geq 0} \subseteq \operatorname{cone}(S_1) \cap \operatorname{cone}(S_2)$. If $\mathbf{d}' \in \operatorname{cone}(S_1) \cap \operatorname{cone}(S_2)$, then $\mathbf{d}' = (z_1/t_1)\mathbf{a}_1 = (z_2/t_2)\mathbf{a}_2$, with z_1, z_2, t_1, $t_2 \in \mathbb{N}$, and $\mathbf{a}_i \in S_i$, $i \in \{1,2\}$. Hence, $t_1, t_2\mathbf{d}' \in \mathrm{G}(S_1) \cap \mathrm{G}(S_2) = \mathbf{d}\mathbb{Z}$, that is, $\mathbf{d}' \in \mathbf{d}\mathbb{Q}_{\geq 0}$. □

The above result may also be obtained as a consequence of [**18,** Lemma 4.2].

Observe that the converse statement is not true, as the following simple example shows. Let S be semigroup generated by the columns of the matrix

$$A = \left(\begin{array}{ccc|ccc} 4 & 3 & 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 & 3 & 4 \end{array}\right),$$

and let S_1 and S_2 be the semigroups generated by the three first and the three last columns of A, respectively. In this case, $\mathbf{d} := (6,6)^\top \in S_1 \cap S_2$ and $\operatorname{cone}(S_1) \cap \operatorname{cone}(S_2) = \mathbf{d}\mathbb{Q}_{\geq 0}$. However, S_1 and S_2 cannot be glued by $\mathbf{d}$ because $\mathrm{G}(S_1) \cap \mathrm{G}(S_2)$ has rank 2; indeed, $3(2,2) = 2(3,3)$ and $(0,4) = -2(4,0) + 2(3,1) + (2,2)$.

Corollary 2.5. *Let S be an affine semigroup minimally generated by A. Let $A = A_1 \cup A_2$ be a nontrivial partition of A, and let $S_i = \langle A_i \rangle$, $i \in \{1,2\}$. Set $V = \operatorname{aff}(S_1) \cap \operatorname{aff}(S_2)$. Then, $S = S_1 +_{\mathbf{d}} S_2$ for some*

$\mathbf{d} \in \mathbb{N}^n \setminus \{0\}$, *if and only if* $V = \mathbf{d}\mathbb{Q}$ *and* $S \cap V = (S_1 \cap V) +_{\mathbf{d}} (S_2 \cap V)$ *for some* $\mathbf{d} \in \mathbb{N}^n \setminus \{0\}$.

Proof. If $S = S_1 +_{\mathbf{d}} S_2$ for some $\mathbf{d} \in \mathbb{N}^n \setminus \{0\}$, by an argument similar to that given in the proof of Proposition 2.4, we have that $V = \mathbf{d}\mathbb{Q}$. Now, since $\mathbf{d} \in (S_1 \cap V) \cap (S_2 \cap V)$ and $\mathrm{G}(S_1 \cap V) \cap \mathrm{G}(S_2 \cap V) = \mathrm{G}(S_1) \cap \mathrm{G}(S_2) = \mathbf{d}\mathbb{Z}$, we conclude that $S \cap V$ is the gluing of $S_1 \cap V$ and $S_2 \cap V$ by $\mathbf{d}$. Conversely, let $V = \mathbf{d}\mathbb{Q}$. Since $\mathrm{G}(S_1) \cap \mathrm{G}(S_2) = \mathrm{G}(S_1 \cap V) \cap \mathrm{G}(S_2 \cap V) = \mathbf{d}\mathbb{Z}$ and $\mathbf{d} \in (S_1 \cap V) \cap (S_2 \cap V) = S_1 \cap S_2$, because $\mathrm{G}(S_1) \cap \mathrm{G}(S_2) \subset V$, we are done. $\square$

Let S be the semigroup generated by the columns of the following matrix

$$A = \left(\begin{array}{ccc|ccc} 4 & 3 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right),$$

and let S_1 (S_2, respectively) be the semigroup generated by the three first (last, respectively) columns of A. Clearly, $V = \mathrm{aff}(S_1) \cap \mathrm{aff}(S_2) = (1,1,0)^\top\mathbb{Q}$. Now, since $S_1 \cap V \cong 2\mathbb{N}$, $S_2 \cap V \cong 3\mathbb{N}$ and $S \cap V \cong 2\mathbb{N} +_6 3\mathbb{N}$, in the light of the above corollary, we conclude that $S = S_1 +_{\mathbf{d}} S_2$, with $\mathbf{d} = (6,6,0)^\top$.

3. Gluings and Frobenius vectors.

Let S be an affine semigroup. We say that S has a *Frobenius vector* if there exists $\mathbf{f} \in \mathrm{G}(S) \setminus S$ such that

$$\mathbf{f} + \mathrm{relint}(\mathrm{cone}(S)) \cap \mathrm{G}(S) \subseteq S \setminus \{0\} \subseteq S.$$

Notice that $\mathbf{f} + (\mathrm{relint}(\mathrm{cone}(S)) \cap \mathrm{G}(S)) \subseteq S \setminus \{0\}$ is equivalent to $(\mathbf{f} + \mathrm{relint}(\mathrm{cone}(S))) \cap \mathrm{G}(S) \subseteq S \setminus \{0\}$, and thus we omit the parentheses in the above condition.

We are going to prove that, if S_1 and S_2 have Frobenius vectors, then so does $S = S_1 +_{\mathbf{d}} S_2$.

Theorem 3.1. *Let S be an affine semigroup. Assume that $S = S_1 +_{\mathbf{d}} S_2$. If S_1 and S_2 have Frobenius vectors, so does S. Moreover, if $\mathbf{f}_1$ and $\mathbf{f}_2$ are, respectively, Frobenius vectors of S_1 and S_2, then*

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{d}$$

is a Frobenius vector of S.

Proof. Let $G_1 = \mathrm{G}(S_1)$, $G_2 = \mathrm{G}(S_2)$ and $G = \mathrm{G}(S)$. Clearly $G = G_1 + G_2$, since $S = S_1 + S_2$.

We start by proving that $\mathbf{f} \in G \setminus S$. As $\mathbf{f}_1 \in G_1$, $\mathbf{f}_2 \in G_2$ and $\mathbf{d} \in G_1 \cap G_2$, we have $\mathbf{f} \in G$. Assume that $\mathbf{f} \in S$. Then there exist $\mathbf{s}_1 \in S_1$ and $\mathbf{s}_2 \in S_2$ such that $\mathbf{f} = \mathbf{s}_1 + \mathbf{s}_2$. Then $\mathbf{f}_1 + \mathbf{d} - \mathbf{s}_1 = \mathbf{s}_2 - \mathbf{f}_2 \in G_1 \cap G_2 = \mathbf{d}\mathbb{Z}$. So, we can find $k \in \mathbb{Z}$ such that $\mathbf{f}_1 + \mathbf{d} - \mathbf{s}_1 = \mathbf{s}_2 - \mathbf{f}_2 = k\mathbf{d}$. If $k \leq 0$, then $\mathbf{f}_2 = \mathbf{s}_2 - k\mathbf{d} \in S_2$, a contradiction. If $k > 0$, then $\mathbf{f}_1 = \mathbf{s}_1 + (k-1)\mathbf{d} \in S_1$, which is also impossible, and this proves that $\mathbf{f} \notin S$.

In order to simplify the notation, set $C_1 = \mathrm{relint}(\mathrm{cone}(S_1))$, $C_2 = \mathrm{relint}(\mathrm{cone}(S_2))$ and $C = \mathrm{relint}(\mathrm{cone}(S))$. Now, let us prove that, for all $\mathbf{x} \in C \cap G$, we have that $\mathbf{f} + \mathbf{x} \in S$. Since $\mathbf{f} + \mathbf{x} \in G$, there must be $\mathbf{g}_1 \in G_1$ and $\mathbf{g}_2 \in G_2$ such that $\mathbf{f} + \mathbf{x} = \mathbf{g}_1 + \mathbf{g}_2$. In light of Proposition 2.2, there exists $\mathbf{x}_1 \in C_1$ and $\mathbf{x}_2 \in C_2$ such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$. Then $\mathbf{f} + \mathbf{x} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{d} + \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{g}_1 + \mathbf{g}_2$. Let $t \in \mathbb{Z}_{>0}$ be such that $\mathbf{s}_1 = t\mathbf{x}_1 \in S_1$ and $\mathbf{s}_2 = t\mathbf{x}_2 \in S_2$. This yields $t\mathbf{f}_1 + t\mathbf{d} + \mathbf{s}_1 - t\mathbf{g}_1 = t\mathbf{g}_2 - t\mathbf{f}_2 - \mathbf{s}_2 = k\mathbf{d}$ for some integer k. Assume that $k \leq 0$. Then $t\mathbf{f}_1 + \mathbf{s}_1 + (t-k)\mathbf{d} = t\mathbf{g}_1$, and thus $\mathbf{f}_1 + (\mathbf{x}_1 + (t-k)/t\mathbf{d}) = \mathbf{g}_1$. Observe that $\mathbf{x}_1 + (t-k)/t\mathbf{d} \in C_1$, which implies that $\mathbf{g}_1 \in S_1$ because $\mathbf{f}_1$ is a Frobenius vector for S_1.

Let n be the maximum nonnegative integer such that $\mathbf{g}_1 - nd \in S_1$. Hence, $\mathbf{g}_1 - (n+1)\mathbf{d} = \mathbf{f}_1 + \mathbf{x}_1 + (t - k/t)\mathbf{d} - (n+1)\mathbf{d} \notin S_1$, and consequently $tn + k > 0$, since otherwise $(t - k/t) - (n+1) \geq 0$, which leads to $\mathbf{x}_1 + (t - k/t)\mathbf{d} - (n+1)\mathbf{d} \in C_1$, yielding $\mathbf{g}_1 - (n+1)\mathbf{d} \in S_1$, a contradiction. Now, $t\mathbf{g}_2 - t\mathbf{f}_2 - \mathbf{s}_2 + tn\mathbf{d} = (tn+k)\mathbf{d}$, which means that $\mathbf{g}_2 + n\mathbf{d} = \mathbf{f}_2 + \mathbf{x}_2 + (tn + k/t)\mathbf{d}$. As $\mathbf{x}_2 + (tn + k/t)\mathbf{d} \in C_2$ and $\mathbf{f}_2$ is a Frobenius vector for S_2, we deduce that $\mathbf{g}_2 + n\mathbf{d} \in S_2$. Finally, $\mathbf{f} + \mathbf{x} = \mathbf{g}_1 + \mathbf{g}_2 = (\mathbf{g}_1 - n\mathbf{d}) + (\mathbf{g}_2 + n\mathbf{d}) \in S_1 + S_2 = S$.

If $k \geq 0$, then $t\mathbf{f}_2 + \mathbf{s}_2 + t\mathbf{d} - t\mathbf{g}_2 = t\mathbf{g}_1 - t\mathbf{f}_2 - \mathbf{s}_1 = -k\mathbf{d}$, and we repeat the above argument by swapping $\mathbf{g}_1$ and $\mathbf{g}_2$. □

If A is a set of positive integers, and $S = \langle A \rangle$, then $T = S/\gcd(A)$ is a numerical semigroup, and $\mathrm{F}(T) = \max(\mathbb{N} \setminus T)$. It follows easily that $\mathrm{F}(S) = \gcd(A)\,\mathrm{F}(T)$. Recall that the conductor of T is defined as the Frobenius number of T plus one. Hence, Theorem 3.1 generalizes the well-known formula for the gluing of two submonoids of $\mathbb{N}$ (**[11,** Proposition 10 (i)]).

Lemma 3.2. *Let S be an affine semigroup minimally generated by A. If A is a set of linearly independent elements, then $\mathbf{f} = -\sum_{\mathbf{a}\in A}\mathbf{a}$ is a Frobenius vector for S.*

Proof. Let $\mathbf{x} \in \operatorname{relint}(\operatorname{cone}(S)) \cap G(S)$. Then $\mathbf{x} = \sum_{\mathbf{a}\in A} q_{\mathbf{a}}\mathbf{a} = \sum_{\mathbf{a}\in A} z_{\mathbf{a}}\mathbf{a}$, with $q_{\mathbf{a}} \in \mathbb{Q}_{>0}$ and $z_{\mathbf{a}} \in \mathbb{Z}$ for all $\mathbf{a}$. Since the elements in A are linearly independent, this forces $z_{\mathbf{a}} = q_{\mathbf{a}}$ for all $\mathbf{a}$; in particular, $z_{\mathbf{a}} - 1 \geq 0$ for all $\mathbf{a}$. Hence $\mathbf{f} + \mathbf{x} = \sum_{\mathbf{a}\in A}(z_{\mathbf{a}} - 1)\mathbf{a} \in S$. □

Since every complete intersection affine semigroup has either no relations (free in the categorical sense, that is, its minimal set of generators is a set of linearly independent vectors) or it is the gluing of two affine semigroups ([**12**]), we get the following result.

Theorem 3.3. *Let S be a complete intersection affine semigroup. Then S has a Frobenius vector.*

Remark 3.4. Let $S = S_1 +_{\mathbf{d}} S_2$ be the gluing of S_1 and S_2 by $\mathbf{d}$, and assume that $S_2 = \langle \mathbf{v} \rangle$. Hence, $\mathbf{d} = \theta\mathbf{v}$ for some $\theta \in \mathbb{N}$. Clearly $-\mathbf{v}$ is a Frobenius vector for S_2 (Lemma 3.2), and, if S_1 has a Frobenius vector $\mathbf{f}_1$, then the formula of Theorem 3.1 implies that $\mathbf{f} = \mathbf{f}_1 - \mathbf{v} + \theta\mathbf{v} = \mathbf{f}_1 + (\theta - 1)\mathbf{v}$ is a Frobenius vector of S. More generally let $\mathbf{v}_1, \ldots, \mathbf{v}_e$ be a set of $\mathbb{Q}$ linearly independent vectors of $\mathbb{N}^e$. Let $S_0 = \langle \mathbf{v}_1, \ldots, \mathbf{v}_e \rangle$, and let $\mathbf{v}_{e+1}, \ldots, \mathbf{v}_{e+h}$ be a set of vectors of $\mathbb{N}^e \cap \operatorname{cone}(\mathbf{v}_1, \ldots, \mathbf{v}_e)$. Set $S_i = \langle \mathbf{v}_1, \ldots, \mathbf{v}_{e+i} \rangle$ for all $1 \leq i \leq h$, and assume that $S_i = S_{i-1} +_{\theta_i \mathbf{v}_i} \langle \mathbf{v}_i \rangle$ (such semigroups are called free semigroups). A Frobenius vector $\mathbf{f}_0$ of S_0 being $\mathbf{f}_0 = -\sum_{k=1}^{e} \mathbf{v}_k$ (Lemma 3.2), it follows that

$$\mathbf{f}_i = \sum_{j=1}^{i} (\theta_j - 1)\mathbf{v}_j - \sum_{k=1}^{e} \mathbf{v}_k \tag{3.1}$$

is a Frobenius vector of S_i. This formula has also been proved by the first author in [**3**], who gave the following uniqueness condition: this Frobenius vector $\mathbf{f}$ is minimal with respect to the order induced by $\operatorname{cone}(S)$, that is, for every other Frobenius vector $\mathbf{f}'$ of S, $\mathbf{f}' \in \mathbf{f} + \operatorname{cone}(S)$.

We recall that a reduced affine semigroup S is said to be *simplicial* if there are linearly independent elements $\mathbf{a}_1, \ldots, \mathbf{a}_n \in S$ such that

$\text{cone}(S) = \text{cone}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$. Under this hypothesis, conditions for the existence and conditions for uniqueness of a Frobenius vector of S are given in [**1**].

Formula (3.1) is a special case of the following general formula for a Frobenius vector of a complete intersection affine semigroup.

Remark 3.5. Recall that, according to [**12**], any complete intersection affine semigroup is either generated by a set of linearly independent vectors or it is a gluing of two complete intersection numerical semigroups. Thus, repeating this argument recursively, if S is a complete intersection affine semigroup A, then there exists a partition $A_1 \cup \cdots \cup A_t = A$ such that A_i are sets of linearly independent vectors and

$$S = S_1 +_{\mathbf{d}_1} S_2 +_{\mathbf{d}_2} \cdots +_{\mathbf{d}_{t-1}} S_t,$$

with $S_i = \langle A_i \rangle$. From Theorem 3.1 and Lemma 3.2, it follows that

$$\sum_{i=1}^{t-1} \mathbf{d}_i - \sum_{\mathbf{a} \in A} \mathbf{a} \tag{3.2}$$

is a Frobenius vector for S.

Next, we show that this Frobenius vector is unique in the sense defined above.

Proposition 3.6. *Let S be a complete intersection affine semigroup, and let $\mathbf{f}$ be defined as in* (3.2)*. Then, for every face F of* $\text{cone}(S)$, $(\mathbf{f} + F) \cap S$ *is empty.*

Proof. Since either S is free or the gluing of two complete intersection affine semigroups S_1 and S_2, we proceed by induction. If S is free, then Lemma 3.2 asserts that $\mathbf{f} = -\sum_{\mathbf{a} \in A} \mathbf{a}$, with A the minimal generating set of S. Clearly, in this case, the assertion is true.

Now assume that $S = S_1 +_{\mathbf{d}} S_2$ for some $\mathbf{d} \in S_1 \cap S_2$. From Theorem 3.1, $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{d}$, where $\mathbf{f}_i$, $i \in \{1, 2\}$, is also defined by (3.2). By the induction hypothesis, for every face F_i of $\text{cone}(S_i)$, $i \in \{1, 2\}$, $(\mathbf{f}_i + F_i) \cap S_i = \emptyset$.

Assume, to the contrary, that there exists $\mathbf{x} \in F$ such that $\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{d} + \mathbf{x} \in S$. According to Proposition 2.3, there exists $\mathbf{x}_i \in F_i$, $i \in \{1, 2\}$, such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, for some face F_i of cone(S_i). Hence, there are $\mathbf{s}_1 \in S_1$ and $\mathbf{s}_2 \in S_2$ such that $\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{d} + \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{s}_1 + \mathbf{s}_2$. Then $\mathbf{f}_1 + \mathbf{x}_1 - \mathbf{s}_1 = \mathbf{s}_2 - \mathbf{f}_2 - \mathbf{d} - \mathbf{x}_2 = k\mathbf{d}$ for some integer k. As by the induction hypothesis $\mathbf{f}_1 + \mathbf{x}_1 \notin S_1$, we deduce $k < 0$. Therefore, $\mathbf{f}_2 + \mathbf{x}_2 = \mathbf{s}_2 - (k + 1)\mathbf{d}$. But $\mathbf{f}_2 + \mathbf{x}_2 \notin S_2$, which forces $k + 1 > 0$, or equivalently, $k \geq 0$. But this is in contradiction with $k < 0$. $\square$

Theorem 3.7. *Let S be a complete intersection, and let $\mathbf{f}$ be as in* (3.2). *Assume that $\mathbf{f}'$ is another Frobenius vector of S. Then $\mathbf{f}' \in \mathbf{f} + \mathrm{cone}(S)$.*

Proof. Write $\mathbf{f} = \mathbf{a} - \mathbf{b}$ and $\mathbf{f}' = \mathbf{a}' - \mathbf{b}'$ with $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}' \in S$, and let $\mathbf{c} \in \mathrm{relint}(\mathrm{cone}(S))$. Then $\mathbf{x} = \mathbf{f} + \mathbf{b} + \mathbf{a}' + \mathbf{c} = \mathbf{f}' + \mathbf{b}' + \mathbf{a} + \mathbf{c} \in (\mathbf{f} + \mathrm{relint}(\mathrm{cone}(S))) \cap (\mathbf{f}' + \mathrm{relint}(\mathrm{cone}(S)))$.

Assume that $\mathbf{f}' \notin \mathbf{f} + \mathrm{cone}(S)$. Then the segment joining $\mathbf{f}'$ and $\mathbf{x}$ cuts some face of $\mathbf{f} + \mathrm{cone}(S)$. Denote by $\mathbf{f} + F$ this face, and let $\mathbf{f} + \mathbf{y}$ be this intersection point ($\mathbf{y} \in F$ and F is a face of cone(S)). There exists a positive integer k such that $k\mathbf{y}$ is in S, and thus $\mathbf{f} + k\mathbf{y} \in \mathrm{G}(S) \cap (\mathbf{f} + F)$. Notice that $\mathbf{f} + \mathbf{y} = \mathbf{f}' + \mathbf{y}'$ for some $\mathbf{y}' \in \mathrm{relint}(\mathrm{cone}(S))$. As $\mathbf{y} \in F$, $(k - 1)\mathbf{y} \in \mathrm{cone}(S)$, and consequently $\mathbf{f} + k\mathbf{y} = \mathbf{f}' + (\mathbf{y}' + (k - 1)\mathbf{y}) \in \mathbf{f}' + \mathrm{relint}(\mathrm{cone}(S))$. Hence, $\mathbf{f} + k\mathbf{y} \in (\mathbf{f}' + \mathrm{relint}(\mathrm{cone}(S))) \cap \mathrm{G}(S) \subseteq S$, in contradiction with Proposition 3.6. $\square$

4. Gluings and Hilbert series. The *Hilbert series* of S is the Hilbert series associated to $K[S]$: $\mathrm{H}(S, \mathbf{x}) = \sum_{\mathbf{s} \in S} \mathbf{x}^{\mathbf{s}}$, where for $\mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{N}^m$, $\mathbf{x}^{\mathbf{s}} = x_1^{s_1} \cdots x_m^{s_m}$. This map is sometimes known in the literature as a generating function of S, and it has been shown to be of the form $g(S, \mathbf{x}) / \prod_{\mathbf{a} \in A}(1 - \mathbf{x}^{\mathbf{a}})$, with A the minimal generating set of S (see **[6,** subsection 7.3]).

The next lemma is a straightforward generalization of (4) in **[21]**.

Lemma 4.1. *Let S be an affine semigroup, and let $\mathbf{m} \in S \setminus \{0\}$. Then*

$$\mathrm{H}(S, x) = \frac{1}{1 - x^{\mathbf{m}}} \sum_{\mathbf{w} \in \mathrm{Ap}(S, \mathbf{m})} x^{\mathbf{w}}. \qquad (4.1)$$

Proof. It follows directly from the definition of $\mathrm{Ap}(S,\mathbf{m})$, that for every $\mathbf{s}\in S$, there exist unique $k\in\mathbb{N}$ and $\mathbf{w}\in\mathrm{Ap}(S,\mathbf{m})$ such that $\mathbf{s}=k\mathbf{m}+\mathbf{w}$. Hence,

$$\mathrm{H}(S,\mathbf{x})=\sum_{\substack{k\in\mathbb{N}\\ \mathbf{w}\in\mathrm{Ap}(S,\mathbf{m})}}\mathbf{x}^{k\mathbf{m}+\mathbf{w}}=\sum_{k\in\mathbb{N}}(\mathbf{x}^{\mathbf{m}})^{\mathbf{k}}\sum_{\mathbf{w}\in\mathrm{Ap}(S,\mathbf{m})}x^{\mathbf{w}}.$$

The proof follows by taking into account that $\sum_{k\in\mathbb{N}}(\mathbf{x}^{\mathbf{m}})^{\mathbf{k}}=1/(1-\mathbf{x}^{\mathbf{m}})$. □

The following result can also be understood as a generalization of (4) in [**21**], since for simplicial affine semigroups that are Cohen-Macaulay, the set $\bigcap_{i=1}^m\mathrm{Ap}(S,\mathbf{v}_i)$, with $\mathbf{v}_1,\ldots,\mathbf{v}_m$ a set of extremal rays of S, plays a similar role to the Apéry set of an element in a numerical semigroup (compare [**24,** Theorem 1.5] and [**26,** Lemma 2.6]).

Proposition 4.2. *Let S be a simplicial affine semigroup with extremal rays $\mathbf{v}_1,\ldots,\mathbf{v}_m$. Then $\mathrm{H}(S,\mathbf{x})=P(\mathbf{x})/\prod_{i=1}^m(1-\mathbf{x}^{\mathbf{v}_i})$, with $P(\mathbf{x})$ a polynomial.*

Proof. Let $Ap=\bigcap_{i=1}^m\mathrm{Ap}(S,\mathbf{v}_i)$. In view of [**24,** Section 1], this set is finite. Moreover, from [**24,** Theorem 1.5], we know that every element $\mathbf{s}$ in S can be expressed uniquely as $\mathbf{s}=\sum_{i=1}^m a_i\mathbf{v}_i+\mathbf{w}$ with $a_1,\ldots,a_d\in\mathbb{N}$ and $\mathbf{w}\in Ap$. Arguing as in Lemma 4.1,

$$\mathrm{H}(S,\mathbf{x})=\sum_{\mathbf{s}\in S}\mathbf{x}^{\mathbf{s}}=\frac{\sum_{\mathbf{w}\in Ap}\mathbf{x}^{\mathbf{w}}}{\prod_{i=1}^m(1-x^{\mathbf{v}_i})},$$

which concludes the proof. □

Theorem 4.3. *Let S, S_1 and S_2 be affine semigroups, and let $\mathbf{d}\in S$. Assume that $S=S_1+_{\mathbf{d}}S_2$. Then*

$$\mathrm{H}(S_1+_{\mathbf{d}}S_2,\mathbf{x})=(1-\mathbf{x}^{d})\,\mathrm{H}(S_1,\mathbf{x})\,\mathrm{H}(S_2,\mathbf{x}).$$

Proof. From (4.1),

$$\mathrm{H}(S,\mathbf{x})=\frac{1}{1-\mathbf{x}^{\mathbf{d}}}\sum_{\mathbf{w}\in\mathrm{Ap}(S,\mathbf{d})}\mathbf{x}^{\mathbf{w}}.$$

From [**22,** Theorem 1.4], the mapping

$$\mathrm{Ap}(S_1,\mathbf{d}) \times \mathrm{Ap}(S_2,\mathbf{d}) \longrightarrow \mathrm{Ap}(S,\mathbf{d}), \quad (x,y) \mapsto x+y, \tag{4.2}$$

is a bijection, and thus, $\mathrm{Ap}(S,\mathbf{d}) = \mathrm{Ap}(S_1,\mathbf{d}) + \mathrm{Ap}(S_2,\mathbf{d})$. Hence,

$$\begin{aligned}\sum_{\mathbf{w}\in \mathrm{Ap}(S,\mathbf{d})} x^{\mathbf{w}} &= \sum_{\mathbf{w}_1\in \mathrm{Ap}(S_1,\mathbf{d})} \sum_{\mathbf{w}_2\in \mathrm{Ap}(S_2,\mathbf{d})} \mathbf{x}^{\mathbf{w}_1+\mathbf{w}_2} \\ &= \Bigg(\sum_{\mathbf{w}_1\in \mathrm{Ap}(S_1,\mathbf{d})} \mathbf{x}^{\mathbf{w}_1}\Bigg)\Bigg(\sum_{\mathbf{w}_2\in \mathrm{Ap}(S_2,\mathbf{d})} \mathbf{x}^{\mathbf{w}_2}\Bigg).\end{aligned}$$

As

$$\mathrm{H}(S_1,\mathbf{x}) = \frac{1}{1-\mathbf{x}^{\mathbf{d}}} \sum_{\mathbf{w}_1\in \mathrm{Ap}(S_1,\mathbf{d})} x^{\mathbf{w}_1},$$

$$\mathrm{H}(S_2,\mathbf{x}) = \frac{1}{1-\mathbf{x}^{\mathbf{d}}} \sum_{\mathbf{w}_2\in \mathrm{Ap}(S_2,\mathbf{d})} \mathbf{x}^{\mathbf{w}_2},$$

we get

$$\mathrm{H}(S,\mathbf{x}) = (1-\mathbf{x}^{\mathbf{d}})\,\mathrm{H}(S_1,\mathbf{x})\,\mathrm{H}(S_2,\mathbf{x}). \qquad \square$$

If S is a numerical semigroup ($\gcd(S) = 1$), and it is a gluing of M_1 and M_2, then $S_1 = M_1/d_1$ and $S_2 = M_2/d_2$ are also numerical semigroups, with $d_i = \gcd(M_i)$, $i \in \{1,2\}$. Hence, $S = d_1S_1 +_{d_1d_2} d_2S_2$ and $\mathrm{lcm}\,(d_1,d_2) = d_1d_2$. We say in this setting that S is a gluing of S_1 and S_2 at d_1d_2.

From the definition of the Hilbert series associated to a submonoid M of N, it follows easily that, if $k \mid \gcd(M)$, then

$$\mathrm{H}(M/k, x^k) = \mathrm{H}(M,x). \tag{4.3}$$

We get the following corollary.

Corollary 4.4. *Let S be a numerical semigroup. Assume that $S = d_1S_1 +_{d_1d_2} d_2S_2$ is a gluing of the numerical semigroups S_1 and S_2. Then*

$$\mathrm{H}(S,x) = (1-x^{d_1d_2})\,\mathrm{H}(S_1,x^{d_1})\,\mathrm{H}(S_2,x^{d_2}).$$

Example 4.5. Let $S = \langle a, b\rangle$ with a and b coprime positive integers. Then $S = a\mathbb{N} +_{ab} b\mathbb{N}$. Then, by Corollary 4.4,

$$\mathrm{H}(\langle a, b\rangle, x) = (1 - x^{ab})\,\mathrm{H}(\mathbb{N}, x^a)\,\mathrm{H}(\mathbb{N}, x^b) = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)}.$$

If we do this computation by using the formula

$$\mathrm{H}(\langle a, b\rangle, x) = \frac{1}{1 - x^a} \sum_{w\in \mathrm{Ap}(\langle a,b\rangle,a)} x^w,$$

we obtain

$$\mathrm{H}(\langle a, b\rangle, x) = \frac{1}{1 - x^a} \sum_{k=0}^{a-1} x^{kb} = \frac{1}{1 - x^a}\,\frac{1 - x^{ab}}{1 - x^b}.$$

Observe that this is a particular case of [**21,** Proposition 2] (see also [**19,** Theorem 4] for a relationship with inclusion-exclusion polynomials).

This idea can be generalized to any complete intersection affine semigroup. The base setting is the following.

Lemma 4.6. *Let $A \subseteq \mathbb{N}^m$ be a set of linearly independent vectors. Then*

$$\mathrm{H}(\langle A\rangle, \mathbf{x}) = \frac{1}{\prod_{\mathbf{a}\in A}(1 - \mathbf{x}^{\mathbf{a}})}.$$

Proof. Assume that $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$, and write $S = \langle A\rangle$. Notice that the map $\mathbb{N}^k \to S$, $(n_1, \ldots, n_k) \mapsto \sum_{i=1}^{k} n_i\mathbf{a}_i$ is a monoid isomorphism. Hence,

$$\sum_{\mathbf{s}\in S} x^s = \sum_{n_1\in\mathbb{N},\ldots,n_k\in\mathbb{N}} \mathbf{x}^{n_1\mathbf{a}_1+\cdots+n_k\mathbf{a}_k} = \prod_{i=1}^{k} \sum_{n\in\mathbb{N}} (x^{\mathbf{a}_i})^n,$$

and the proof follows easily. □

Proposition 4.7. *Let S be a free affine semigroup. Assume that*

$$S = (\cdots(\langle \mathbf{v}_1, \ldots, \mathbf{v}_e\rangle +_{\theta_{e+1}\mathbf{v}_{e+1}} \langle \mathbf{v}_{e+1}\rangle) +_{\theta_{e+2}\mathbf{v}_{e+2}} \cdots) +_{\theta_{e+h}\mathbf{v}_{e+h}} \langle \mathbf{v}_{e+h}\rangle.$$

Then,

$$\mathrm{H}(S,\mathbf{x}) = \frac{\prod_{i=1}^{h}(1-\mathbf{x}^{\theta_{e+i}\mathbf{v}_{e+i}})}{\prod_{i=1}^{e+h}(1-\mathbf{x}^{\mathbf{v}_i})}.$$

This is indeed a particular case of the following theorem.

Theorem 4.8. *Let S be a complete intersection affine semigroup minimally generated by A. Let $\mathbf{d}_1,\dots,\mathbf{d}_{t-1}$ be as in Remark* 3.5*,*

$$\mathrm{H}(S,\mathbf{x}) = \frac{\prod_{i=1}^{t-1}(1-\mathbf{x}^{\mathbf{d}_i})}{\prod_{\mathbf{a}\in A}(1-\mathbf{x}^{\mathbf{a}})}.$$

If S is a complete intersection, then any minimal system of generators $\{f_1,\dots,f_{t-1}\}$ of the toric ideal associated to S forms a regular sequence. With the same notation as in our Remark 3.5, $\mathrm{Betti}\,(S) = \{d_1,\dots,d_{t-1}\}$. So, we may assume that $\deg_S(f_i) = d_i$, $i\in\{1,\dots,t-1\}$ (repetitions may occur). Now, Theorem 4.8 also follows from [**17,** Exercise 5].

Remark 4.9. Observe that, if we subtract the degree of the numerator and denominator of the formula given in Theorem 4.8, we obtain formula (3.2).

Example 4.10. Let $S=\langle 4,5,6\rangle = \langle 4,6\rangle +_{10} 5\mathbb{N} = (4\mathbb{N} +_{12} 6\mathbb{N}) +_{10} 5\mathbb{N}$. Then

$$\mathrm{H}(\langle 4,5,6\rangle, x) = \frac{(1-x^{10})(1-x^{12})}{(1-x^4)(1-x^5)(1-x^6)}.$$

The Frobenius number of S is $10+12-(4+5+6)=7$.

Acknowledgments. Part of this research was performed while the second author visited the Université d'Angers, and he wants to thank the Département de Mathématiques of this university for its kind hospitality.

The authors would also like to thank the referees for their comments and suggestions.

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