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EMBEDDING SUZUKI CURVES IN $\mathbb{P}^4$

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ABSTRACT. In this paper, we study the projective geometry of smooth models in $\mathbb{P}^4$ of Suzuki curves, employing the Weierstrass semigroup at the only singular point of the curves. In particular, we explicitly count the hypersurfaces of $\mathbb{P}^4$ containing the smooth projective model and provide a geometric characterization of those of small degree. We also prove that the characterization cannot be extended to higher-degree hypersurfaces of $\mathbb{P}^4$.

1. Introduction. Let $n \geq 2$ be an integer, and let q_0 and q be defined by $q_0 := 2^n$ and $q := 2q_0^2$. Let $\mathbb{F}_q$ denote the finite field with q elements, and fix any field $\mathbb{F}$ containing $\mathbb{F}_q$. For the rest of the paper, $\mathbb{F}$ will be the base field. Given an integer $r > 0$, we denote by $\mathbb{P}^r$ the r-dimensional projective space over $\mathbb{F}$. The projective plane $\mathbb{P}^2$ will be referred to as homogeneous coordinates $(x : y : z)$.

The *Suzuki curve* $S_n \subseteq \mathbb{P}^2$ associated to the integer n is defined over $\mathbb{F}$ by the following affine equation:

$$y^q - y = x^{q_0}(x^q - x)$$

(see **[9,** Example 5.24]). The curve is known to have only one point lying on the hyperplane at infinity $\{z = 0\}$, namely, $P_\infty := (0 : 1 : 0)$. The point P_∞, at which S_n has a cusp, is also the only singular point of the curve. The genus of S_n (i.e., by definition, the geometric genus of its normalization) is known to be $g_n := q_0(q - 1)$.

1.1. Main references on Suzuki curves. Suzuki curves are studied in depth throughout the book **[9]**. They are very interesting from a geometric viewpoint because of their optimality (Chapter 10) and their large group of automorphisms (Theorem 11.127 and, more generally,

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subsection 12.2). Relevant properties of the Suzuki group date back to [**8**]. A comprehensive view on Suzuki curves and their quotients is given in [**5**]. On the same topics, see also [**10**] and [**12,** Chapter V]. More recently, the p-torsion group scheme of Jacobians of Suzuki curves has been studied in [**3**]. Moreover, Eid and Duursma gave in [**2**] a complete set of five equations for the smooth model of a Suzuki curve in $\mathbb{P}^4$.

Interesting applications of Suzuki curves in coding theory have been successfully considered in [**1, 6, 11**], while also computing the Weierstrass semigroup associated to pairs of points of S_n ([**11,** Section III]).

1.2. Goal and layout of the paper. Let S_n be a Suzuki curve as defined above, and let $\pi : C_n \to S_n$ be its normalization. The normalization morphism, π, is known to be injective. In Section 2, we study linear systems of the form $|m\pi^{-1}(P_\infty)|$, $m \in \mathbb{Z}_{\geq 0}$. In particular, we give necessary and sufficient conditions for $|m\pi^{-1}(P_\infty)|$ to be very ample. The smallest integer m with this property is $q + 2q_0 + 1$. Moreover, the morphism induced by $|(q+2q_0+1)\pi^{-1}(P_\infty)|$ embeds C_n into $\mathbb{P}^4$. The curve obtained in this way, denoted by X_n, is a smooth model of S_n in $\mathbb{P}^4$. The goal of the paper is to study the projective geometry of X_n. More precisely, we are interested in explicitly counting the hypersurfaces of $\mathbb{P}^4$ containing X_n and describing those of small degree. Our main result is the following.

Theorem 1.1 (see Theorem 5.1 and Corollary 5.2)**.** *Let X_n be the curve defined above, and let $g_n = q_0(q-1)$ be its genus. The following facts hold.*

- (i) *There exists a unique degree two hypersurface $Q_n \subseteq \mathbb{P}^4$ containing X_n.*
- (ii) *Let $2 \leq t \leq q_0$ be an integer. The degree t hypersurfaces of $\mathbb{P}^4$ containing X_n are exactly those containing Q_n. Moreover, they form an $\mathbb{F}$-vector space of dimension $\binom{t+2}{4}$.*
- (iii) *The previous result is false in general for $t > q_0$. Indeed, there exist at least four linearly independent degree $q_0 + 1$ hypersurfaces of $\mathbb{P}^4$ containing X_n, and not containing Q_n.*

(iv) *Let $t \geq 2q_0+1$ be an integer. The degree t hypersurfaces of $\mathbb{P}^4$ containing X_n form an $\mathbb{F}$-vector space of dimension $\binom{t+4}{4} - t(q+2q_0+1) - 1 + g_n$.*

The theorem provides an interesting geometric characterization of the small-degree hypersurfaces of $\mathbb{P}^4$ containing X_n. It also proves that the characterization cannot be extended to higher-degree hypersurfaces.

Remark 1.2. Two linearly independent hypersurfaces of $\mathbb{P}^4$ containing X_n and not containing Q_n appear in [**3,** equations (3.2)].

Sections 3 and 4 are dedicated to preliminary results. In particular, in Section 3, we derive explicit formulas for the dimension of any Riemann-Roch space of the form $L(t(q+2q_0+1))$, $t \in \mathbb{Z}_{\geq 0}$. In Section 4, we consider some multiplication maps of geometric interest and study their properties. The computational results are interpreted from a geometric point of view in Section 5, leading to the main results of the paper.

Remark 1.3. The linear series $|(q+2q_0+1)\pi^{-1}(P_\infty)|$ here considered is of deep interest in the literature. Its properties can be used to characterize Suzuki curves in terms of the genus and the number of rational points (see [**9,** Theorem 10.102]).

2. Geometry on the Weierstrass semigroup. Given a Suzuki curve S_n and an integer $m \geq 0$, we denote by $L(mP_\infty)$ the vector space of the rational functions on S_n whose pole order at P_∞ is at most m, i.e., the Riemann-Roch space associated to the divisor mP_∞ on S_n. We recall that the Weierstrass semigroup $H(P_\infty)$ associated to P_∞ is precisely the set of non-gaps at P_∞. In other words, $H(P_\infty)$ is the set of all the $m \in \mathbb{Z}_{\geq 0}$ such that there exists a rational function in $L(mP_\infty) \setminus L((m-1)P_\infty)$.

Remark 2.1. Since, for any $m \geq 0$, we have $0 \leq L((m+1)P_\infty) - L(mP_\infty) \leq 1$, by the definition of Weierstrass semigroup we clearly have $\dim_{\mathbb{F}} L(mP_\infty) = |\{s \in H(P_\infty) : s \leq m\}|$.

Lemma 2.2 (**[11]**, Lemma 3.1)**.** *Let $H(P_\infty)$ be the Weierstrass semigroup defined above. We have $H(P_\infty) = \langle q, q+q_0, q+2q_0, q+2q_0+1\rangle$.*

Notation 2.3. For any $(a,b,c,d) \in \mathbb{Z}^4_{\geq 0}$, we set

$$\|(a,b,c,d)\| := aq + b(q+q_0) + c(q+2q_0) + d(q+2q_0+1).$$

Notation 2.4. The normalization of S_n will always be denoted by C_n. It is well known (see for instance [**4,** Section 7.5]) that C_n is a smooth abstract curve which is birational to S_n. Since the normalization morphism $\pi : C_n \to S_n$ is injective, we will simply write P_∞ instead of $\pi^{-1}(P_\infty)$.

In the remainder of the section we focus on C_n curves and study linear systems of the form $|mP_\infty|$, providing a characterization of the very ample ones.

Lemma 2.5. *Let m be a positive integer. The linear system $|mP_\infty|$ is spanned by its global sections if and only if $m \in H(P_\infty)$.*

Proof. This is a well-known property of the one-point Weierstrass semigroup $H(P_\infty)$ (notice that C_n is smooth). □

Proposition 2.6. *Let m be a positive integer. The linear system $|mP_\infty|$ is very ample if and only if $m \in H(P_\infty)$ and $m-1 \in H(P_\infty)$.*

Proof. If $|mP_\infty|$ is very ample, then it is obviously spanned by its global sections. Hence, by Lemma 2.5, we get $m \in H(P_\infty)$. Denote by $\varphi_m : C_n \to \mathbb{P}^{r-1}$ the morphism induced by mP_∞. The linear system $|mP_\infty|$ is very ample if and only if φ_m is injective with non-zero differential at any point of C_n.

- ($\Rightarrow$) Assume that the linear system $|mP_\infty|$ is very ample. In particular, φ_m must have non-zero differential at P_∞. This implies the existence of a rational function $f \in L(mP_\infty)$ whose vanishing order at P_∞ is exactly one. Since $m \in H(P_\infty)$, this implies $m-1 \in H(P_\infty)$.
- ($\Leftarrow$) On the other hand, assume $m, m-1 \in H(P_\infty)$. We clearly have $m \geq q+2q_0+1$. As in Notation 2.4, let $\pi : C_n \to S_n$

denote the normalization morphism of S_n. Since $(x)_\infty = qP_\infty$ and $(y)_\infty = (q+q_0)P_\infty$ (see **[6,** Proposition 1.3]), we have $\{1,x,y\} \subseteq L(mP_\infty)$. Hence, the linear system $|mP_\infty|$ contains the linear system spanned by $\{1,x,y\}$, which induces the composition of π with the inclusion $S_n \hookrightarrow \mathbb{P}^2$. Since P_∞ is the only singular point of S_n, the morphism φ_m is injective with non-zero differential at any point of $C_n \setminus \{P_\infty\}$. Therefore, in order to prove that $|mP_\infty|$ is very ample, it is necessary and sufficient to show that $\dim_{\mathbb{F}} L((m-2)P_\infty) = \dim_{\mathbb{F}} L(mP_\infty)-2$. Since $m, m-1 \in H(P_\infty)$, this condition is clearly satisfied. $\square$

Remark 2.7. Proposition 2.6 shows that the smallest projective space in which C_n can be embedded by a one-point linear system $|mP_\infty|$ is $\mathbb{P}^4$.

3. Riemann-Roch spaces of Suzuki curves. In this section we provide an explicit formula for the dimension of any Riemann-Roch space of the form

$$L(t(q+2q_0+1)P_\infty), \quad t \in \mathbb{Z}_{\geq 0}.$$

Since the Weierstrass semigroup $H(P_\infty)$ is known (Proposition 2.2), the dimension of $L(mP_\infty)$ is also known, *in principle*, for any $m \geq 0$. On the other hand, deriving simple expressions from the semigroup's data is not completely trivial. Explicit formulas and their combination are key points in our approach. The main results of the Section are Propositions 3.4 and 3.8, whose proofs are split in some preliminary lemmas.

Lemma 3.1. *Let $(a,b,c,d) \in \mathbb{Z}^4_{\geq 0}$, and let $t \leq q_0 - 1$ be a positive integer. The following two facts are equivalent*:

(A) $\|(a,b,c,d)\| \leq t(q+2q_0+1)$,
(B) $a+b+c+d \leq t$.

Proof. Assume $a+b+c+d \leq t$. Then $\|(a,b,c,d)\| \leq (a+b+c+d)(q+2q_0+1) \leq t(q+2q_0+1)$. On the other hand, we have $q > q-q_0-1 = 2(q_0-1)q_0 + q_0 - 1 \geq 2tq_0 + t$. Hence, $(t+1)q > t(q+2q_0+1)$. If $a+b+c+d > t$, then $a+b+c+d \geq t+1$. As a consequence, we get $\|(a,b,c,d)\| \geq (a+b+c+d)q \geq (t+1)q > t(q+2q_0+1)$. $\square$

Lemma 3.2. *Let* $(a', b', c', d') \in \mathbb{Z}^4_{\geq 0}$. *Choose any integer* t *with* $1 \leq t \leq q_0 - 1$, *and assume* $\|(a', b', c', d')\| \leq t(q+2q_0+1)$. *There exists a unique four-tuple* $(a, b, c, d) \in \mathbb{Z}^4_{\geq 0}$ *with* $b \in \{0, 1\}$ *and* $\|(a, b, c, d)\| = \|(a', b', c', d')\|$.

Proof. To prove the existence, write $b' = 2\beta + B$, with $\beta \geq 0$ and $B \in \{0, 1\}$, and set $a := a' + \beta$, $b := B$, $c := c' + \beta$ and $d := d'$. The following uniqueness argument is inspired by [**3,** Proposition 3.7]. Assume that there exist $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in \mathbb{Z}^4_{\geq 0}$ such that:

(A) $b_1, b_2 \in \{0, 1\}$,
(B) $\|(a_1, b_1, c_1, d_1)\| = \|(a_2, b_2, c_2, d_2)\|$,
(C) $\|(a_1, b_1, c_1, d_1)\|, \|(a_2, b_2, c_2, d_2)\| \leq t(q + 2q_0 + 1)$.

As in the proof of Lemma 3.1, we have $(t + 1)q > t(q + 2q_0 + 1)$. Condition (iii) implies, in particular, $c_1, c_2, d_1, d_2 \leq t \leq q_0 - 1$. Condition (ii) is equivalent to
(1)
$$(a_1-a_2)q+(b_1-b_2)(q+q_0)+(c_1-c_2)(q+2q_0)+(d_1-d_2)(q+2q_0+1) = 0.$$

Reducing modulo q_0, we have $d_1 - d_2 \equiv 0 \mod q_0$. Since $-q_0 + 1 \leq d_1, d_2 \leq q_0 - 1$, we deduce $d_1 = d_2$. Hence, equation (1) becomes

$$(a_1 - a_2)q + (b_1 - b_2)(q + q_0) + (c_1 - c_2)(q + 2q_0) = 0. \tag{2}$$

Reducing modulo $2q_0$, we obtain $(b_1 - b_2)q_0 \equiv 0 \mod 2q_0$. Since $b_1, b_2 \in \{0, 1\}$, one gets $b_1 = b_2$. By substitution into equation (2), we may write

$$(a_1 - a_2)q + (c_1 - c_2)(q + 2q_0) = 0. \tag{3}$$

Reducing modulo q, we get $(c_1 - c_2)2q_0 \equiv 0 \mod q$. Since $q = 2q_0^2$ and $c_1, c_2 \leq q_0 - 1$, we conclude $c_1 = c_2$. Clearly $a_1 = a_2$ at this point. $\square$

The following lemma summarizes some trivial facts which we need later on in the paper. A proof can easily be obtained by induction.

Lemma 3.3. *Let* h *be a positive integer. The following formulas hold.*

(A) $\sum_{i=0}^{h} i = h(h + 1)/2$.
(B) $\sum_{i=0}^{h} i^2 = h^3/3 + h^2/2 + h/6$.

(C) *Let $\mathcal{T}_h$ be the set of all the three-tuple $(a,b,c) \in \mathbb{Z}^3_{\geq 0}$ satisfying $a+b+c=h$. We have $|\mathcal{T}_h| = (h+1)(h+2)/2$.*

Proposition 3.4. *Let t be a non-negative integer, and let $g_n = q_0(q-1)$ be the genus of the Suzuki curve S_n (see Section 1). The dimension of the one point Riemann-Roch space $L(t(q+2q_0+1)P_\infty)$ is given by the following formulas*:

$$\dim_{\mathbb{F}} L(t(q+2q_0+1)P_\infty)$$
$$= \begin{cases} 4t+1 & \text{if } t=0 \text{ or } t=1, \\ \binom{t+4}{4} - \binom{t+2}{4} & \text{if } 2 \leq t \leq q_0-1, \\ t(q+2q_0+1)+1-g_n+\binom{2q_0-t+2}{4}-\binom{2q_0-t}{4} & \text{if } q_0 \leq t \leq 2q_0-4, \\ t(q+2q_0+1)+6-g_n & \text{if } t=2q_0-3, \\ t(q+2q_0+1)+2-g_n & \text{if } t=2q_0-2, \\ t(q+2q_0+1)+1-g_n & \text{if } t \geq 2q_0-1. \end{cases}$$

Proof. We recall (Remark 2.1) that $\dim_{\mathbb{F}} L(t(q+2q_0+1))$ is exactly the cardinality of the set $H_t(P_\infty) = \{s \in H(P_\infty) : s \leq t(q+2q_0+1)\}$. The proof is divided into five steps.

(A) If $t \in \{0,1\}$, the dimension is easily computed by hand (Lemma 2.2).

(B) Assume $2 \leq t \leq q_0 - 1$. Combining Lemmas 3.1 and 3.2, we see that, for any $t \in \{2, \ldots, q_0-1\}$, the cardinality of $H_t(P_\infty)$ may be computed as

$$|H_t(P_\infty)| = |\{(a,b,c,d) \in \mathbb{Z}^4_{\geq 0} : b \in \{0,1\} \text{ and } a+b+c+d \leq t\}|.$$

Hence, following the notation of Lemma 3.3, we have

$$\begin{aligned} |H_t(P_\infty)| &= \sum_{h=0}^{t} |\{(a,b,c,d) \in \mathbb{Z}^4_{\geq 0} : b \in \{0,1\} \\ &\qquad\qquad \text{and } a+b+c+d = h\}| \\ &= \sum_{h=0}^{t} |\mathcal{T}_h| + \sum_{h=1}^{t} |\mathcal{T}_{h-1}| = |\mathcal{T}_t| + 2\sum_{h=0}^{t-1} |\mathcal{T}_h| \\ &= (t+1)(t+2)/2 + \sum_{h=0}^{t-1} h^2 + 3h + 2 \end{aligned}$$

$$= (2t^3 + 9t^2 + 13t + 6)/6$$
$$= \binom{t+4}{4} - \binom{t+2}{4},$$

which is the expected formula.

(C) Since the genus of S_n is $g_n = q_0(q-1)$, we compute $2g_n - 2 = 2(q_0-1)(q+2q_0+1)$. Hence, for $t \geq 2q_0 - 2$, the dimension of $L(t(q+2q_0+1))$ is given by a trivial application of the Riemann-Roch theorem and the fact that $\dim_{\mathbb{F}} L(0) = 1$.

(D) Now assume $q_0 \leq t \leq 2q_0 - 4$, and set $D_t := t(q+2q_0+1)P_\infty$. A canonical divisor on S_n is $K = (2g_n - 2)P_\infty \sim 2(q_0-1)(q+2q_0+1)P_\infty$. See also **[3]** for details. We have a linear equivalence of divisors

$$K - D_t \sim (2q_0 - 2 - t)(q + 2q_0 + 1)P_\infty.$$

Since $2 \leq 2q_0 - 2 - t \leq q_0 - 1$, thanks to step (B), we are able to explicitly compute $\dim_{\mathbb{F}} L(K - D_t)$ and obtain $\dim_{\mathbb{F}} L(D_t)$ by applying the Riemann-Roch theorem as follows:

$$\dim_{\mathbb{F}} L(D_t) = t(q+2q_0+1)+1-g_n+\binom{2q_0-t+2}{4}-\binom{2q_0-t}{4}.$$

(E) Finally, assume $t = 2q_0 - 3$, and set $D := (2q_0-3)(q+2q_0+1)P_\infty$. We have a linear equivalence $K - D \sim (q+2q_0+1)P_\infty$ and so, by step (A), the dimension of $L(D)$ is again computed by the Riemann-Roch theorem. □

We conclude this section providing an explicit monomial basis of any Riemann-Roch space $L(mP_\infty)$, $m \geq 0$. The following preliminary result generalizes Lemma 3.2.

Lemma 3.5. *Let $(a', b', c', d') \in \mathbb{Z}^4_{\geq 0}$. There exists a unique $(a, b, c, d) \in \mathbb{Z}^4_{\geq 0}$ which satisfies the following properties:*

$$0 \leq b \leq 1, \qquad 0 \leq c \leq q_0 - 1,$$
$$0 \leq d \leq q_0 - 1, \qquad \|(a,b,c,d)\| = \|(a',b',c',d')\|.$$

Proof. To prove the uniqueness, we may apply the same argument as Lemma 3.2, which uses only our hypothesis on b, c and d. Let us prove the existence. Write $d' = \delta q_0 + D$ with $0 \leq D \leq q_0 - 1$ and set $(a_1, b_1, c_1, d_1) := (a' + \delta q_0, b' + \delta, c', D)$. Write $b_1 = 2\beta + B$ with $0 \le B \le 1$, and set $(a_2, b_2, c_2, d_2) := (a_1 + \beta, B, c_1 + \beta, D)$. Write $c_2 = \gamma q_0 + C$ with $0 \le C \le q_0 - 1$, and define

$$(a, b, c, d) := (a_2 + \gamma q_0 + \gamma, b_2, C, d_2) = (a' + \delta q_0 + \gamma q_0 + \beta + \gamma, B, C, D).$$

It is easily checked that (a, b, c, d) has the expected properties. $\square$

Definition 3.6. Following **[6]** and **[11]**, we define the rational functions $v := y^{2q_0} + x^{2q_0+1}$ and $w := y^{2q_0}x + v^{2q_0}$. The pole divisors of x, y, v, w are computed in **[6,** Proposition 1.3], as follows:

$$(x)_\infty = qP_\infty, \qquad (y)_\infty = (q + q_0)P_\infty,$$
$$(v)_\infty = (q + 2q_0)P_\infty, \qquad (w)_\infty = (q + 2q_0 + 1)P_\infty.$$

Remark 3.7. From the pole divisors given in the previous definition we see that, for any $(a, b, c, d) \in \mathbb{Z}^4_{\ge 0}$, the pole order of $x^a y^b v^c w^d$ at P_∞ is exactly $\|(a, b, c, d)\|$.

Proposition 3.8. *Let $m \ge 0$ be an integer. A basis of the Riemann-Roch space $L(mP_\infty)$ is given by all the rational functions $x^a y^b v^c w^d$ such that*:

$$a, b, c, d \in \mathbb{Z}_{\ge 0}, \quad 0 \le b \le 1, \ 0 \le c, \ d \le q_0 - 1, \ \|(a, b, c, d)\| \le m.$$

Proof. By Lemma 3.5, such rational functions have different pole orders at P_∞. In particular, they are linearly independent. By Remark 3.7, they all belong to $L(mP_\infty)$. Finally, by definition of $H(P_\infty)$ and Lemma 3.5, their number is $\dim_{\mathbb{F}} L(mP_\infty)$. The result follows. $\square$

4. Multiplication maps and their geometry. Let S_n be the Suzuki curve defined in Section 1, and let $\pi : C_n \to S_n$ be its normalization (see Notation 2.4). By Proposition 2.6, the linear system $|(q + 2q_0 + 1)P_\infty|$ defines an embedding $\varphi_{q+2q_0+1} : C_n \to \mathbb{P}^4$. We set $X_n := \varphi_{q+2q_0+1}(C_n)$, a smooth curve of degree $q + 2q_0 + 1$ in $\mathbb{P}^4$.

Definition 4.1. Given non-negative integers a, b and t, we will denote by $\mu(a,b)$ and $\mu_t(a)$, respectively, the multiplication maps

$$\mu(a,b) : L(aP_\infty) \otimes L(bP_\infty) \longrightarrow L((a+b)P_\infty),$$
$$\mu_t(a) : L(aP_\infty)^{\otimes t} \longrightarrow L(taP_\infty).$$

Since, in the function field defined by S_n, multiplication is commutative, each of the maps $\mu_t(a)$ induces a multiplication map $\sigma_t(a) : S^t(L(aP_\infty)) \to L(taP_\infty)$, where $S^t(L(aP_\infty))$ denotes the tth power of the symmetric tensor product.

Remark 4.2. This section is rather technical. More precisely, we study the surjectivity of the multiplication maps $\sigma_t(q+2q_0+1)$, $t \geq 1$, introduced in Definition 4.1. Interesting geometric applications will be shown later in the paper. The main results of this section are Proposition 4.7 and its consequences (Corollary 4.8). The proof of the previously cited proposition is split in Lemmas 4.3, 4.4, 4.5 and 4.6.

Lemma 4.3. *Let α and β be non negative integers such that $\alpha+\beta \leq q_0 - 1$. The multiplication map $\mu(\alpha(q+2q_0+1), \beta(q+2q_0+1))$ of Definition* 4.1 *is surjective.*

Proof. Since α and β play interchangeable roles and the case $\alpha = 0$ is trivial, we may assume $\beta \geq \alpha > 0$. Keep in mind Proposition 3.8 and consider a basis element, $x^a y^b v^c w^d$, of the Riemann-Roch space $L((\alpha+\beta)(q+2q_0+1)P_\infty)$. We clearly have

$$aq + b(q+q_0) + c(q+2q_0) + d(q+2q_0+1) \leq (\alpha+\beta)(q+2q_0+1). \tag{4}$$

Since $\alpha+\beta \leq q_0 - 1$, we get $\alpha+\beta \leq q_0 - 1 < 2q_0^2/(2q_0+1) = q/(2q_0+1)$. As a consequence, $(\alpha+\beta)(2q_0+1) < q$, i.e., $q(\alpha+\beta+1) > (\alpha+\beta)(q+2q_0+1)$. By inequality (4), we have, in particular, $(a+b+c+d)q \leq (\alpha+\beta)(q+2q_0+1) < q(\alpha+\beta+1)$. Dividing by q, one obtains $a+b+c+d < \alpha+\beta+1$ and so $a+b+c+d \leq \alpha+\beta$. Now we write $(a,b,c,d) = (a_1,b_1,c_1,d_1)+(a_2,b_2,c_2,d_2)$ with $a_1+b_1+c_1+d_1 \leq \alpha$ and $a_2+b_2+c_2+d_2 \leq \beta$. It follows that $\|(a_1,b_1,c_1,d_1)\| \leq \alpha(q+2q_0+1)$

and $\|(a_2, b_2, c_2, d_2)\| \leq \beta(q + 2q_0 + 1)$, and so

$$x^a y^b v^c w^d = \mu(\alpha(q + 2q_0 + 1), \beta(q + 2q_0 + 1))$$
$$(x^{a_1} y^{b_1} v^{c_1} w^{d_1} \otimes x^{a_2} y^{b_2} v^{c_2} w^{d_2}).$$

In other words, a generic basis element $x^a y^b v^c w^d \in L((\alpha+\beta)(q+2q_0+1)P_\infty))$ is in the image of $\mu(\alpha(q+2q_0+1), \beta(q+2q_0+1))$, as claimed. $\square$

Lemma 4.4. *Let $t \geq 1$ be an integer, and let $(a, b, c, d) \in \mathbb{Z}^4_{\geq 0}$ with $a + b + c + d \leq t$. There exist four 4-tuples $\{(a_i, b_i, c_i, d_i)\}_{i=1}^t \subseteq \{0, 1\}^4$ such that $a_i + b_i + c_i + d_i \leq 1$ for all $i = 1, \ldots, t$ and*

$$(a, b, c, d) = \sum_{i=1}^{t} (a_i, b_i, c_i, d_i).$$

Proof. We use induction on t. If $t = 1$, then we take $(a_1, b_1, c_1, d_1) := (a, b, c, d)$. Now assume $a + b + c + d \leq t + 1$. If $a + b + c + d \leq t$, then, by inductive hypothesis, we write $(a, b, c, d) = \sum_{i=1}^t (a_i, b_i, c_i, d_i)$ with $a_i, b_i, c_i, d_i \in \{0, 1\}$ and $a_i + b_i + c_i + d_i \leq 1$ for all $i = 1, \ldots, t$. Define the 4-tuple $(a_{t+1}, b_{t+1}, c_{t+1}, d_{t+1}) := (0, 0, 0, 0)$ and obtain $(a, b, c, d) = \sum_{i=1}^{t+1} (a_i, b_i, c_i, d_i)$. On the other hand, if $a+b+c+d = t+1$ then one among a, b, c, d must be positive. Assume, without loss of generality, $a > 0$. Then, by induction, $(a - 1, b, c, d) = \sum_{i=1}^t (a_i, b_i, c_i, d_i)$ with $a_i, b_i, c_i, d_i \in \{0, 1\}$ and $a_i + b_i + c_i + d_i \leq 1$ for $i = 1, \ldots, t$. By setting $(a_{t+1}, b_{t+1}, c_{t+1}, d_{t+1}) := (1, 0, 0, 0)$ we have $(a, b, c, d) = \sum_{i=1}^{t+1} (a_i, b_i, c_i, d_i)$, and the lemma is proved. $\square$

The following lemma is well known and easy to prove.

Lemma 4.5. *Let m be a positive integer. Let $\{f_1, \ldots, f_h\} \subseteq L(mP_\infty)$ be a set of rational functions. Assume that, for any $s \in H(P_\infty)$, with $s \leq m$, there exists a $1 \leq j_m \leq h$ such that $(f_{j_m})_\infty = s$. Then $\{f_1, \ldots, f_h\}$ is a generating set of $L(mP_\infty)$.*

Lemma 4.6. *Let $t \geq 2q_0 + 1$ be an integer. For any $s \in H(P_\infty)$, with $s \leq t(q + 2q_0 + 1)$, there exists a four-tuple $(a, b, c, d) \in \mathbb{Z}^4_{\geq 0}$ such that $\|(a, b, c, d)\| = s$ and $a + b + c + d \leq t$.*

Proof. The argument is divided into two steps.

(A) Assume $s \leq tq$, and take any 4-tuple (a,b,c,d) such that $\|(a,b,c,d)\| = s$. Such a 4-tuple exists because $s \in H(P_\infty)$. If $a+b+c+d \geq t+1$, then we clearly have the contradiction $s = \|(a,b,c,d)\| \geq (t+1)q > s$. Hence, $a+b+c+d \leq t$, and we are done.

(B) Now assume $s > tq$. Write $s = \alpha(q+2q_0+1) - \beta$ with $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ and $0 \leq \beta \leq q+2q_0$. Since $s \leq t(q+2q_0+1)$, we have $0 \leq \alpha \leq t$. Set:

$$\begin{aligned} e_1 &:= \left\lfloor \frac{\beta}{2q_0+1} \right\rfloor, \\ e_2 &:= \left\lfloor \frac{\beta - e_1(2q_0+1)}{q_0+1} \right\rfloor, \\ e_3 &:= \beta - e_1(2q_0+1) - e_2(q_0+1). \end{aligned}$$

Notice that $e_2 \in \{0,1\}$ and $e_3 \leq q_0$. Since $\beta \leq q+2q_0 = (2q_0+1)q_0$, we get $e_1 \leq q_0$, and the equality holds if and only if $\beta = q+2q_0$. In this case, we have $e_2 = \lfloor q_0/(q_0+1) \rfloor = 0$. Hence, in any case, $e_1+e_2+e_3 \leq 2q_0$. Notice also that

$$s > tq \geq (2q_0+1)q > (2q_0-1)(q+2q_0+1).$$

Since $s = \alpha(q+2q_0+1) - \beta$ with $0 \leq \beta \leq q+2q_0$, we deduce $\alpha \geq 2q_0 \geq e_1+e_2+e_3$. Hence, $e_1+e_2+e_3 \leq \alpha \leq t$, and we may take $a := e_1$, $b := e_2$, $c := e_3$ and $d := \alpha - e_1 - e_2 - e_3$ to conclude the proof. □

Proposition 4.7. *Let t be a positive integer, and let $\sigma_t(q+2q_0+1)$ be as in Definition* 4.1*.*

(1) *If $1 \leq t \leq q_0$, then $\sigma_t(q+2q_0+1)$ is surjective.*
(2) *If $t \geq 2q_0+1$, then $\sigma_t(q+2q_0+1)$ is surjective.*

Proof. Let us divide the proof into three steps.

(A) Here we assume $1 \leq t \leq q_0-1$. The image of the map $\sigma_t(q+2q_0+1)$ and the image of the map $\mu_t(q+2q_0+1)$ coincide. Moreover, the image of $\mu_t(q+2q_0+1)$ contains the image of $\mu(q+2q_0+1, (t-1)(q+2q_0+1))$. Since $t = 1+(t-1) \leq q_0-1$, by Lemma 4.3, the map $\mu(q+2q_0+1, (t-1)(q+2q_0+1))$ is surjective, and we are done.

(B) Assume $t \geq 2q_0+1$. We recall that $L(q+2q_0+1)$ has $\{1, x, y, v, w\}$ as a basis. Moreover, $1, x, y, v, w$ have the following pole divisors:

$$(1)_\infty = 0, \qquad (x)_\infty = q, \qquad (y)_\infty = q + q_0,$$
$$(v)_\infty = q + 2q_0, \qquad (w)_\infty = q + 2q_0 + 1.$$

Take any $s \in H(P_\infty)$ with $s \leq t(q + 2q_0 + 1)$. By Lemma 4.6, there exists a 4-tuple $(a, b, c, d) \in \mathbb{Z}^4_{\geq 0}$ such that $\|(a, b, c, d)\| = s$ and $a+b+c+d \leq t$. Thanks to Lemma 4.4, we write $(a, b, c, d) = \sum_{i=1}^t (a_i, b_i, c_i, d_i)$, with $a_i, b_i, c_i, d_i \in \{0, 1\}$ and $a_i+b_i+c_i+d_i \leq 1$ for any $i \in \{1, \ldots, t\}$. Hence, for any $i \in \{1, \ldots, t\}$, we have $x^{a_i} y^{b_i} v^{c_i} w^{d_i} \in L((q + 2q_0 + 1)P_\infty)$. Moreover,

$$\sigma_t(q + 2q_0 + 1)\left(\bigotimes_{i=1}^{t} x^{a_i} y^{b_i} v^{c_i} w^{d_i}\right) = x^a y^b v^c w^d$$

is a rational function in the image of σ_t whose pole divisor is exactly sP_∞. Notice that s is arbitrary in $H(P_\infty)$ with $s \leq t(q + 2q_0 + 1)$. Hence, by Lemma 4.5, the image of $\sigma_t(q + 2q_0 + 1)$ spans the vector space $L(t(q + 2q_0 + 1)P_\infty)$, i.e., $\sigma_t(q + 2q_0 + 1)$ is surjective.

(C) Let us consider the case $t = q_0$. By Lemma 4.5, it is enough to prove that, for any $s \in H(P_\infty)$, with $s \leq q_0(q + 2q_0 + 1)$, there exists a rational function, say f, in the image of $\sigma_{q_0}(q + 2q_0 + 1)$ with the property $(f)_\infty = sP_\infty$. Write $s = \|(a, b, c, d)\|$ for a certain $(a, b, c, d) \in \mathbb{Z}^4_{\geq 0}$.

(C.1) If $s \leq (q_0 - 1)(q + 2q_0 + 1)$ then, by Lemma 3.1, we conclude $a + b + c + d \leq q_0 - 1 < q_0$. In this case we are done, as in step (B).

(C.2) Assume $s > (q_0 - 1)(q + 2q_0 + 1)$. If $a + b + c + d \leq q_0$ then, as above, the result is proved. Hence, we will assume $a+b+c+d \geq q_0+1$ for the rest of the proof. If $a+b+c+d \geq q_0 + 2$, then $\|(a, b, c, d)\| \geq (q_0 + 2)q > q_0(q + 2q_0 + 1)$. As a consequence, we have $a + b + c + d \leq q_0 + 1$, and so $a+b+c+d = q_0+1$. Assume $a \leq q_0-1$. Then $b+c+d \geq 2$, and so $\|(a, b, c, d)\| \geq (q_0 + 1)q + 2q_0 > q_0(q + 2q_0 + 1)$, a contradiction. It follows that $a \in \{q_0, q_0 + 1\}$, and we can study the two cases separately.

- If $a = q_0 + 1$, then clearly $b = c = d = 0$, and so x^{q_0+1} is a rational function with the expected pole divisor. Working modulo the equation of S_n, we have $x^{q_0+1} = v^{q_0} - y$ (see also [**3,** equations (3.2)]). Since v^{q_0} and y trivially belong to the image of $\sigma_{q_0}(q + 2q_0 + 1)$, x^{q_0+1} also belongs to such an image.

- If $a = q_0$ and $(c, d) \neq (0, 0)$, then $\|(a, b, c, d)\| \geq q_0 q + (q + 2q_0) > q_0(q + 2q_0 + 1)$, a contradiction. It follows that $c = d = 0$ and $b = 1$. Notice that $x^{q_0} y$ is a rational function with the expected pole divisor. Moreover, $x^q y = w^{q_0} - v$ (again [**3,** equations (3.2)]), and so we conclude as in the previous step. □

Corollary 4.8. *Let t be an integer. If $1 \leq t \leq q_0$ or $t \geq 2q_0 + 1$, then the restriction map of cohomology groups $\rho_t : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \to H^0(X_n, \mathcal{O}_{X_n}(t))$ is surjective.*

Proof. Since the embedding $\varphi_{q+2q_0+1} : C_n \to X_n$ is induced by the linear system $|(q + 2q_0 + 1)P_\infty|$, the pull-back bundle of $\mathcal{O}_{X_n}(1)$ through φ_{q+2q_0+1} is that associated to the linear system $|(q + 2q_0 + 1)P_\infty|$. By Proposition 4.7, the restriction map $S^t(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))) \to H^0(X_n, \mathcal{O}_{X_n}(t))$ is surjective. The result follows. □

5. The smooth model of a Suzuki curve in $\mathbb{P}^4$. Here we study the geometric properties of the smooth model $X_n \subseteq \mathbb{P}^4$ of a Suzuki curve S_n. We apply the computational results derived in the previous parts of the paper in order to count the hypersurfaces of $\mathbb{P}^4$ containing X_n (Theorem 5.1). Moreover, we provide an explicit geometric characterization of those of small degree (Corollary 5.2).

Theorem 5.1. *Let t be a positive integer, and let $\mathcal{K}(t, X_n)$ denote the $\mathbb{F}$-vector space of all the degree t hypersurfaces of $\mathbb{P}^4$ containing X_n. Let $\kappa(t, X_n)$ be the dimension of $\mathcal{K}(t, X_n)$. The following formulas hold*:

$$\kappa(t, X_n) = \begin{cases} \binom{t+2}{4} & \text{if } 2 \leq t \leq q_0, \\ \binom{t+4}{4} - t(q + 2q_0 + 1) - 1 + g_n & \text{if } t \geq 2q_0 + 1. \end{cases}$$

Proof. The vector space $\mathcal{K}(t, X_n)$, whose dimension is in question, is exactly the kernel of the restriction map $\rho_t : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \to H^0(X_n, \mathcal{O}_{X_n}(t))$. If $2 \leq t \leq q_0$ or $t \geq 2q_0 + 1$, then ρ_t is surjective by Corollary 4.8. It follows that

$$\begin{aligned}\kappa(t, X_n) &= h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) - h^0(X_n, \mathcal{O}_{X_n}(t)) \\ &= h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) - \dim_{\mathbb{F}} L(t(q + 2q_0 + 1)P_\infty).\end{aligned}$$

Now it suffices to apply the formulas given in Proposition 3.4. Notice that, for the case $t = q_0$, we also use the identity

$$\binom{t+4}{4} - \binom{t+2}{4} = 2t^2 + 2t + 1 + \binom{t+2}{4} - \binom{t}{4}. \qquad \square$$

Theorem 5.1 allows us to geometrically characterize all the small-degree hypersurfaces of $\mathbb{P}^4$ containing the smooth model X_n of a Suzuki curve.

Corollary 5.2. *Let X_n be the smooth projective model of the Suzuki curve S_n in $\mathbb{P}^4$, embedded by the linear system $|(q + 2q_0 + 1)P_\infty|$. The following facts hold.*

(1) *There exists a unique degree two hypersurface $Q_n \subseteq \mathbb{P}^4$ containing X_n.*
(2) *Let $2 \le t \le q_0$ be an integer. The degree t hypersurfaces of $\mathbb{P}^4$ containing X_n are exactly those containing Q_n. Moreover, they form an $\mathbb{F}$-vector space of dimension $\binom{t+2}{4}$.*
(3) *There exist at least four linearly independent degree $q_0 + 1$ hypersurfaces of $\mathbb{P}^4$ containing X_n and not containing Q_n.*

Proof. Theorem 5.1 immediately gives the existence of a unique degree two hypersurface of $\mathbb{P}^4$, say Q_n, containing X_n. Fix any integer $t \ge 2$. Since $h^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t-2)) = 0$, the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(t-2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(t) \longrightarrow \mathcal{O}_{Q_n}(t) \longrightarrow 0$$

induces the following exact sequence of cohomology groups:

$$\begin{aligned}0 \longrightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t-2)) \longrightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \\ \longrightarrow H^0(Q_n, \mathcal{O}_{Q_n}(t)) \longrightarrow 0.\end{aligned}$$

So we have $h^0(Q_n, \mathcal{O}_{Q_n}(t)) = \binom{t+4}{4} - \binom{t+2}{4}$. The restriction map $\rho_t : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \to H^0(X_n, \mathcal{O}_{X_n}(t))$ factors through the restriction map $\rho'_t : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \to H^0(Q_n, \mathcal{O}_{Q_n}(t))$. More precisely, the following diagram commutes (we recall that Q_n contains X_n)

$$\begin{tikzcd} H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \arrow[r, "\rho'_t"] \arrow[dr, "\rho_t"'] & H^0(Q_n, \mathcal{O}_{Q_n}(t)) \arrow[d, "\rho''_t"] \\ & H^0(X_n, \mathcal{O}_{X_n}(t)). \end{tikzcd}$$

Since ρ'_t is surjective, we clearly have $\mathrm{Im}\,(\rho_t) = \mathrm{Im}\,(\rho''_t)$. Let us divide the rest of the proof into two steps.

(A) Assume $2 \leq t \leq q_0$. We proved in Corollary 4.8 that the restriction map ρ_t is surjective. So ρ''_t is surjective as well. On the other hand, as in the proof of Theorem 5.1, $h^0(X_n, \mathcal{O}_{X_n}(t)) = \binom{t+4}{4} - \binom{t+2}{4}$. Hence, ρ''_t is bijective. It follows that $\ker(\rho_t) = \ker(\rho'_t)$, i.e., a degree t hypersurface of $\mathbb{P}^4$ contains X_n if and only if it is a union of Q_n and a degree $t-2$ hypersurface of $\mathbb{P}^4$. By Theorem 5.1, the degree t hypersurfaces of $\mathbb{P}^4$ containing X_n form a vector space of dimension $\binom{t+2}{4}$.

(B) Assume $t = q_0+1$. Proposition 3.4 and straightforward brute force computations allow us to write the dimension of the vector space $L(t(q+2q_0+1)P_\infty)$ as

$$\begin{aligned} \dim_{\mathbb{F}} L(t(q+2q_0+1)P_\infty) &= \binom{q_0+5}{4} - \binom{q_0+3}{4} - 4 \\ &= \binom{t+4}{4} - \binom{t+2}{4} - 4. \end{aligned}$$

Since X_n is obtained by embedding C_n through the linear system $|(q+2q_0+1)P_\infty|$, we have also $h^0(X_n, \mathcal{O}_{X_n}(t)) = \binom{t+4}{4} - \binom{4+2}{4} - 4$. Since ρ'_t is surjective, $\dim_{\mathbb{F}} \ker(\rho'_t) = \binom{t+2}{4}$. As a consequence, we deduce the following inequality:

$$\begin{aligned} \dim_{\mathbb{F}} \ker(\rho_t) &\geq h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t-2)) - h^0(X_n, \mathcal{O}_{X_n}(t)) \\ &= \binom{t+4}{4} - \left[\binom{t+4}{4} - \binom{t+2}{4} - 4\right] \\ &= \binom{t+2}{4} + 4 \end{aligned}$$

$$= \dim_{\mathbb{F}} \ker(\rho_t') + 4.$$

Since $\ker(\rho_t') \subseteq \ker(\rho_t)$, there must exist at least four linearly independent hypersurfaces of $\mathbb{P}^4$ vanishing on X_n and not vanishing on Q_n, as claimed. $\square$

Example 5.3. By Proposition 3.8, a basis of the Riemann-Roch space $L((q+2q_0+1)P_\infty)$ is given by $\{1, x, y, v, w\}$. Taking homogeneous coordinates $(x_1 : x_2 : x_3 : x_4 : x_5)$ in $\mathbb{P}^4$, we assume without loss of generality that X_n is the embedding of C_n defined by the following relations:

$$x_1/x_5 = x, \qquad x_2/x_5 = y, \qquad x_3/x_5 = v, \qquad x_4/x_5 = w.$$

It is easily checked that the degree two hypersurface $Q_n \subseteq \mathbb{P}^4$ defined by the affine equation $x_2^2 = x_1x_3 + x_4$ contains X_n. By Corollary 5.2, Q_n is the unique degree two hypersurface of $\mathbb{P}^4$ containing X_n (its equation is defined up to a scalar multiplication). The equations of two linearly independent degree q_0+1 hypersurfaces of $\mathbb{P}^4$ and not containing Q_n appeared in the proof of Proposition 4.7, step (C.2):

$$x^{q_0+1} = v^{q_0} - y, \qquad x^q y = w^{q_0} - v.$$

As pointed out in Section 1, we find the same equations in [**3,** equations (3.2)].

Remark 5.4. Lemma 2.6 provides an explicit characterization of all the very ample linear systems of the form $|mP_\infty|$. We studied in detail the case $m = q + 2q_0 + 1$, which provides the 'smallest' possible embedding of C_n. Other very ample linear systems can be considered, obtaining projective models of Suzuki curves in higher-dimensional projective spaces. We notice that the smallest $m > q + 2q_0 + 1$ such that $|mP_\infty|$ is very ample is $2q + 2q_0 + 1$. Moreover, $|(2q + 2q_0 + 1)P_\infty|$ embeds C_n into $\mathbb{P}^9$. A systematic study of higher-degree embeddings seems to be difficult.

Conclusions. In this paper, we constructed projective smooth models of a plane Suzuki curve S_n through linear systems of the form $|mP_\infty|$, where P_∞ is the only singular point of any S_n. Computational results on the Weierstrass semigroup at P_∞ were applied in order to study in depth the smallest possible embedding $X_n \subseteq \mathbb{P}^4$ from a geometric point of view. In particular, the small-degree hypersurfaces of $\mathbb{P}^4$ containing X_n are characterized by Corollary 5.2, proving also that the result cannot be extended to higher-degree hypersurfaces. On the other hand, high-degree hypersurfaces of $\mathbb{P}^4$ containing X_n are explicitly counted. In order to derive such geometric results, here we solve some one-point Riemann-Roch problems in the range which is not trivially covered by the homonymous theorem, providing closed formulas.

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