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QUASIDUALIZING MODULES

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ABSTRACT. We introduce and study "quasidualizing" modules. An artinian R-module T is *quasidualizing* if the homothety map $\widehat{R} \to \operatorname{Hom}_R(T,T)$ is an isomorphism and $\operatorname{Ext}^i_R(T,T) = 0$ for each integer $i > 0$. Quasidualizing modules are associated to semidualizing modules via Matlis duality. We investigate the associations via Matlis duality between subclasses of the Auslander class and Bass class and subclasses of derived T-reflexive modules.

Introduction. Let R be a commutative local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The $\mathfrak{m}$-adic completion of R is denoted $\widehat{R}$, the injective hull of k is $E = E_R(k)$, and the Matlis duality functor is $(-)^\vee = \operatorname{Hom}_R(-, E)$.

The motivation for this work comes from the study of semidualizing modules. Semidualizing modules were first introduced by Vasconcelos [**9**]. A finitely generated R-module C is *semidualizing* if the homothety map $R \to \operatorname{Hom}_R(C,C)$ is an isomorphism and $\operatorname{Ext}^i_R(C,C) = 0$ for each integer $i > 0$. For example, R is always a semidualizing R-module. Therefore, duality with respect to R is a special case of duality with respect to a semidualizing module, as is duality with respect to a dualizing R-module when R has one. On the other hand, Matlis duality is not covered in this way. The goal of this paper is to remedy this by introducing and studying the "quasidualizing" modules: An artinian R-module T is *quasidualizing* if the homothety map $\widehat{R} \to \operatorname{Hom}_R(T,T)$ is an isomorphism and $\operatorname{Ext}^i_R(T,T) = 0$ for each integer $i > 0$, see Definition 1.14. For example, E is always a quasidualizing module.

This paper is concerned with the properties of quasidualizing modules and how they compare with the properties of semidualizing modules. For instance, the next result gives a direct link between quasi-

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dualizing modules and semidualizing modules via Matlis duality; see Theorem 3.1.

Theorem A. *If R is complete, then the set of isomorphism classes of semidualizing R-modules is in bijection with the set of isomorphism classes of quasidualizing R-modules by Matlis duality.*

Following the literature on semidualizing modules, we use quasidualizing modules to define other classes of modules. For instance, given an R-module M, we consider the class $\mathcal{G}_M^{\text{full}}(R)$ of "derived M-reflexive R-modules" and their subclasses $\mathcal{G}_M^{\text{noeth}}(R)$ and $\mathcal{G}_M^{\text{artin}}(R)$ of noetherian modules and artinian modules, respectively. We also consider subclasses of the Auslander class $\mathcal{A}_M(R)$ and the Bass class $\mathcal{B}_M(R)$. See Section 1 for definitions. Some relations between these classes are listed in the next result which is proved in Section 3.

Theorem B. *Assume R is complete, and let T be a quasidualizing R-module. Then we have the following inverse equivalences and equalities*:

(i) $\mathcal{B}_{T^\vee}^{\text{noeth}}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}_T^{\text{artin}}(R) = \mathcal{A}_{T^\vee}^{\text{artin}}(R)$;

(ii) $\mathcal{B}_{T^\vee}^{\text{artin}}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}_T^{\text{noeth}}(R) = \mathcal{A}_{T^\vee}^{\text{noeth}}(R)$;

(iii) $\mathcal{B}_{T}^{\text{artin}}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}_{T^\vee}^{\text{noeth}}(R) = \mathcal{A}_{T}^{\text{noeth}}(R)$; *and*

(iv) $\mathcal{B}_{T}^{\text{noeth}}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}_{T^\vee}^{\text{artin}}(R) = \mathcal{A}_{T}^{\text{artin}}(R)$.

As a consequence of the previous result, we conclude that the classes $\mathcal{G}_{T^\vee}^{\text{noeth}}(R)$ and $\mathcal{G}_T^{\text{artin}}(R)$ are substantially different. For instance, as we observe next $\mathcal{G}_T^{\text{artin}}(R)$ satisfies the two-of-three condition, while the class $\mathcal{G}_{T^\vee}^{\text{noeth}}(R)$ does not; see Theorem 3.13.

Theorem C. *Assume that R is complete, and let T be a quasidualizing R-module. Then $\mathcal{G}_T^{\mathrm{artin}}(R)$ satisfies the two-of-three condition, that is, given an exact sequence of R-module homomorphisms $0 \to L_1 \to L_2 \to L_3 \to 0$ if any two of the modules are in $\mathcal{G}_T^{\mathrm{artin}}(R)$, then so is the third.*

In Section 1 we provide some definitions and background material. Section 2 describes properties related to quasidualizing modules, and Section 3 describes relations between the different classes of modules using Matlis duality as well as Theorem C.

1. Background material.

Definition 1.1. We say that an R-module L is *Matlis reflexive* if the natural bidualitiy map $\delta_L^E : L \to L^{\vee\vee}$, given by $l \mapsto [\phi \mapsto \phi(l)]$ is an isomorphism.

Fact 1.2. Let L be an R-module. The natural biduality map δ_L is injective; see [**7,** Theorem 18.6(i)]. If L is Matlis reflexive, then $L^\vee$ is Matlis reflexive.

Fact 1.3. Assume R is complete, and let L be an R-module. If L is artinian, then $L^\vee$ is noetherian. If L is noetherian, then $L^\vee$ is artinian. Since R is complete, both artinian modules and noetherian modules are Matlis reflexive; see [**7,** Theorem 18.6(v)].

Lemma 1.4. *Let L and L' be R-modules such that L is Matlis reflexive. Then, for all $i \geqslant 0$, we have the isomorphisms*

$$\mathrm{Ext}_R^i(L', L) \cong \mathrm{Ext}_R^i(L^\vee, L^{'\vee}) \text{ and } \mathrm{Ext}_R^i(L', L^\vee) \cong \mathrm{Ext}_R^i(L, L^{p\vee}).$$

Proof. For the first isomorphism, since L is Matlis reflexive, by definition the map

$$\mathrm{Ext}_R^i(L', \delta_L) : \mathrm{Ext}_R^i(L', L) \to \mathrm{Ext}_R^i(L', \mathrm{Hom}_R(L^\vee, E))$$

is an isomorphism. A manifestation of Hom-tensor adjointness yields the following isomorphisms

$$\mathrm{Ext}_R^i(L', \mathrm{Hom}_R(L^\vee, E)) \xrightarrow{\cong} \mathrm{Ext}_R^i(L' \otimes_R L^\vee, E) \xrightarrow{\cong} \mathrm{Ext}_R^i(L^\vee, L'^\vee).$$

The composition of these maps provides us with the isomorphism $\operatorname{Ext}^i_R(L', L) \cong \operatorname{Ext}^i_R(L^\vee, L'^\vee)$.

For the second isomorphism, the fact that L is Matlis reflexive explains the second step in the following sequence $\operatorname{Ext}^i_R(L', L^\vee) \cong \operatorname{Ext}^i_R(L^{\vee\vee}, L'^\vee) \cong \operatorname{Ext}^i_R(L, L'^\vee)$. The first step follows from the first isomorphism since $L^\vee$ is Matlis reflexive. □

Fact 1.5. Assume R is complete, and let A and A' be artinian R-modules. Then $\operatorname{Hom}_R(A, A')$ is noetherian. This can be deduced using [**6,** Theorem 2.11].

Fact 1.6. Let L be an R-module. Then L is artinian over R if and only if it is artinian over $\widehat{R}$. See [**6,** Lemma 1.14] or [**2,** Remark 10.2.9].

Lemma 1.7. *Assume R is artinian, and let L be an R-module. Then the following are equivalent*:

(i) *L is noetherian over R*;
(ii) *L is finitely generated over R*; *and*
(iii) *L is artinian.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is standard; see [**1,** Propositions 6.2 and 6.5].

For the implication (ii) $\Rightarrow$ (iii), assume that L is finitely generated over R. Then there exists an $n \in \mathbf{N}$ and a surjective map $R^n \xrightarrow{\phi} L$ so that we have $L \cong \operatorname{Im}(\phi) \cong R^n/\operatorname{Ker}(\phi)$. Since R is artinian, R^n is artinian. Thus, L is artinian because the quotient of an artinian module is artinian; see [**1,** Proposition 6.3].

For the implication (iii) $\Rightarrow$ (i), assume that L is artinian. Then there exists an $n \in \mathbf{N}$ such that $L \hookrightarrow E^n$; see [**3,** Theorem 3.4.3]. Since R is artinian, we have $R^\vee \cong E$ is noetherian over $\widehat{R}$ by Fact 1.3, where the isomorphism follows from [**7,** Theorem 18.6 (iv)]. Hence, we have that E^n is noetherian over $\widehat{R} = R$ since R is artinian. Since any submodule of a noetherian module is noetherian, we conclude that L is noetherian over R; see [**1,** Proposition 6.3]. □

Lemma 1.8. *Assume R is complete, and let A be an artinian R-module. Then there exists an injective resolution I of A such that,* *for each $i \geqslant 0$ we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbf{N}$. Furthermore, $I^\vee$ is a free resolution of $A^\vee$.*

Proof. Since A is artinian, we have the map $A \hookrightarrow E^{b_0}$ for some $b_0 \geqslant 1$; see [**3,** Theorem 3.4.3]. Because the finite direct sum of artinian modules is artinian, E^{b_0} is artinian, and we have $E^{b_0}/A \hookrightarrow E^{b_1}$ for some $b_1 \geqslant 0$. Recursively, we can construct an injective resolution of A such that, for each $i \geqslant 0$, we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbf{N}$.

Next we show that $I^\vee$ is a free resolution of $A^\vee$. The fact that $I_i \cong E^{b_i}$ explains the first step in the following sequence

$$I_i^\vee = \operatorname{Hom}_R(I_i, E) \cong \operatorname{Hom}_R(E^{b_i}, E) \cong \operatorname{Hom}_R(E, E)^{b_i} \cong \widehat{R}^{b_i} \cong R^{b_i}.$$

The second step is standard. The third step is from [**7,** Theorem 18.6(iv)], and the last step follows from the assumption that R is complete. The desired conclusion follows from the fact that $(-)^\vee$ is exact. □

Definition 1.9. Let L, L' and L'' be R-modules. The *Hom-evaluation* morphism

$$\theta_{LL'L''} : L \otimes_R \operatorname{Hom}_R(L', L'') \to \operatorname{Hom}_R(\operatorname{Hom}_R(L, L'), L")$$

is given by $a \otimes \phi \mapsto [\beta \mapsto \phi(\beta(a))]$.

Fact 1.10. The Hom-evaluation morphism $\theta_{LL'L''}$ is an isomorphism if the modules satisfy one of the following conditions:

(a) L is finitely generated and L'' is injective; or
(b) L is finitely generated and projective.

See [**5,** Lemma 1.6] and [**8,** Lemma 3.55].

Definition 1.11. An R-module C is *semidualizing* if it satisfies the following:

(i) C is finitely generated;
(ii) the homothety morphism $\chi_C^R : R \to \operatorname{Hom}_R(C, C)$, defined by $r \mapsto [c \mapsto rc]$, is an isomorphism; and
(iii) one has $\operatorname{Ext}_R^i(C, C) = 0$ for all $i > 0$.

Remark 1.12. Let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes of semidualizing R-modules.

Example 1.13. The ring R is always semidualizing.

Definition 1.14. An R-module T is *quasidualizing* if it satisfies the following:

(i) T is artinian;
(ii) the homothety morphism $\chi_T^{\widehat{R}} : \widehat{R} \to \operatorname{Hom}_R(T, T)$, defined by $r \mapsto [t \mapsto rt]$, is an isomorphism; and
(iii) one has $\operatorname{Ext}_R^i(T, T) = 0$ for all $i > 0$.

Remark 1.15. The homothety morphism $\chi_T^{\widehat{R}}$ is well defined since T is artinian implying by Fact 1.6 that T is an $\widehat{R}$-module.

Remark 1.16. Let $\mathfrak{Q}_0(R)$ denote the set of isomorphism classes of quasidualizing modules.

Example 1.17. The injective hull of the residue field E is always quasidualizing. See [**3,** Theorem 3.4.1] and [**7,** Theorem 18.6(iv)] for conditions (i) and (ii) of Definition 1.14. Since E is injective by definition, we have $\operatorname{Ext}_R^i(E, E) = 0$ for all $i > 0$ satisfying the last condition.

Definition 1.18. Let M be an R-module. Then an R-module L is *derived M-reflexive* if:

(i) the natural biduality map $\delta_L^M : L \to \operatorname{Hom}_R(\operatorname{Hom}_R(L, M), M)$ defined by $l \mapsto [\phi \mapsto \phi(l)]$ is an isomorphism; and
(ii) one has $\operatorname{Ext}_R^i(L, M) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(L, M), M)$ for all $i > 0$.

We write $\mathcal{G}_M^{\text{full}}(R)$ to denote the class of all derived M-reflexive R-modules, $\mathcal{G}_M^{\text{mr}}(R)$ to denote the class of all Matlis reflexive derived M-reflexive R-modules, $\mathcal{G}_M^{\text{artin}}(R)$ to denote the class of all artinian derived M-reflexive R-modules, and $\mathcal{G}_M^{\text{noeth}}(R)$ to denote the class of all noetherian derived M-reflexive R-modules.

Remark 1.19. When $M = C$ is a semidualizing R-module, the class $\mathcal{G}_M^{\text{noeth}}(R)$ is the class of *totally C-reflexive* R-modules, sometimes denoted $\mathcal{G}_C(R)$.

Definition 1.20. Let L and L' be R-modules. We say that L is in the *Bass class* $\mathcal{B}_{L'}(R)$ with respect to L' if it satisfies the following:

(i) the natural evaluation homomorphism $\xi_L^{L'} : \operatorname{Hom}_R(L', L) \otimes_R L' \to L$, defined by $\phi \otimes l \mapsto \phi(l)$, is an isomorphism; and
(ii) one has $\operatorname{Ext}_R^i(L', L) = 0 = \operatorname{Tor}_i^R(L', \operatorname{Hom}_R(L', L))$ for all $i > 0$.

We write $\mathcal{B}_{L'}^{\text{mr}}(R)$ to denote the class of all Matlis reflexive R-modules in the Bass class with respect to L'. We write $\mathcal{B}_{L'}^{\text{artin}}(R)$ to denote the class of all artinian R-modules in the Bass class with respect to L', and $\mathcal{B}_{L'}^{\text{noeth}}(R)$ to denote the class of all noetherian R-modules in the Bass class with respect to L'.

Definition 1.21. Let L and L' be R-modules. We say that L is in the *Auslander class* $\mathcal{A}_{L'}(R)$ with respect to L' if it satisfies the following:

(i) the natural homomorphism $\gamma_L^{L'} : L \to \operatorname{Hom}_R(L', L' \otimes_R L)$, which is defined by $l \mapsto [l' \mapsto l' \otimes l]$, is an isomorphism; and
(ii) one has $\operatorname{Tor}_i^R(L', L) = 0 = \operatorname{Ext}_R^i(L', L' \otimes_R L)$ for all $i > 0$.

We write $\mathcal{A}_{L'}^{\text{mr}}(R)$ to denote the class of all Matlis reflexive R-modules in the Auslander class with respect to L'. We write $\mathcal{A}_{L'}^{\text{artin}}(R)$ to denote the class of all artinian R-modules in the Auslander class with respect to L', and $\mathcal{A}_{L'}^{\text{noeth}}(R)$ to denote the class of all noetherian R-modules in the Auslander class with respect to L'.

2. Quasidualizing Modules. We begin with a few preliminary results pertaining to quasidualizing modules.

Proposition 2.1. *Let T be an R-module. Then T is a quasidualizing R-module if and only if T is a quasidualizing $\widehat{R}$-module.*

Proof. We need to check the equivalence of three conditions. For the first condition, T is an artinian R-module if and only if T is an artinian $\widehat{R}$-module by Fact 1.6. For the rest of the proof we assume without loss of generality that T is artinian.

For the second condition, we have the equality $\operatorname{Hom}_R(T,T) = \operatorname{Hom}_{\widehat{R}}(T,T)$ from the fact that T is $\mathfrak{m}$-torsion and [**6,** Lemma 1.5(a)]. This explains the equality in the following commutative diagram.

$$\begin{array}{ccc} \widehat{R} & \xrightarrow{\chi_T^{\widehat{R}}} & \operatorname{Hom}_R(T,T) \\ \cong\big\downarrow & & \big\downarrow = \\ \widehat{\widehat{R}} & \xrightarrow{\chi_T^{\widehat{\widehat{R}}}} & \operatorname{Hom}_{\widehat{R}}(T,T) \end{array}$$

Since $\widehat{R} \cong \widehat{\widehat{R}}$, we have $\chi_T^{\widehat{R}}$ is an isomorphism if and only if $\chi_T^{\widehat{\widehat{R}}}$ is an isomorphism.

For the last condition, Lemma 1.8 implies that there exists an injective resolution I of T such that for each $i \geqslant 0$ we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbf{N}$. For all $i \geqslant 0$, the modules T and I_i are artinian and hence $\mathfrak{m}$-torsion. By [**6,** Lemma 1.5(a)], we have the equality $\operatorname{Hom}_{\widehat{R}}(T, I_i) = \operatorname{Hom}_R(T, I_i)$ and I is an injective resolution of T over $\widehat{R}$. This explains the first and second steps in the next display:

$$\operatorname{Ext}^i_{\widehat{R}}(T,T) \cong \mathrm{H}_{-i}(\operatorname{Hom}_{\widehat{R}}(T,I_i)) \cong \mathrm{H}_{-i}(\operatorname{Hom}_R(T,I_i)) \cong \operatorname{Ext}^i_R(T,T).$$

The third step is by definition. Thus, we have $\operatorname{Ext}^i_{\widehat{R}}(T,T) = 0$ for all $i > 0$ if and only if $\operatorname{Ext}^i_R(T,T) = 0$ for all $i > 0$. ☐

Proposition 2.2. *The following conditions are equivalent*:

(i) *E is a semidualizing R-module*;
(ii) *R is a quasidualizing R-module*;
(iii) *E is a noetherian R-module*;
(iv) *R is an artinian ring*;
(v) $\mathfrak{Q}_0(R) = \mathfrak{S}_0(R)$; *and*
(vi) $\mathfrak{Q}_0(R) \cap \mathfrak{S}_0(R) \neq 0$.

Proof. (iii) $\Leftrightarrow$ (iv). By [**7,** Theorem 18.6 (ii)] we have $\operatorname{len}_R(R) = \operatorname{len}_R(R^\vee) = \operatorname{len}_R(E)$, where $\operatorname{len}_R(L)$ denotes the length of an R-module L. Since R is noetherian by assumption, we have R is artinian if and only if R has finite length if and only if $R^\vee = E$ has finite length (by the equalities above), if and only if E is noetherian over R (since E is artinian; see [**3,** Theorem 3.4.1] or [**2,** Theorem 10.2.5]). That is, R is artinian if and only if E is noetherian over R.

(i) $\Rightarrow$ (iii). If E is a semidualizing R-module, then E is noetherian over R by definition.

(iv) $\Rightarrow$ (i). Assume that R is artinian. Then E is finitely generated by the equivalence (iii) $\Leftrightarrow$ (iv). We have $R \cong \widehat{R}$ since R is artinian, and $\widehat{R} \cong \operatorname{Hom}_R(E,E)$ by [**7,** Theorem 18.6 (iv)] explaining the unspecified isomorphisms in the following commutative diagram.

$$\begin{array}{ccc} R & \xrightarrow{\chi_E^R} & \operatorname{Hom}_R(E,E) \\ {\scriptstyle\cong}\downarrow & \nearrow{\scriptstyle\cong} & \\ \widehat{R} & & \end{array}$$

Hence, we conclude that the homothety morphism χ_E^R is an isomorphism. Since E is injective, we have that $\operatorname{Ext}_R^i(E,E) = 0$ for all $i > 0$. Thus, E is a semidualizing R-module.

(iv) $\Rightarrow$ (v). Assume that R is artinian, and let L be an R-module. We show that L is a semidualizing module if and only if L is a quasidualizing module. We need to check the equivalence of three conditions. For the first condition, L is finitely generated if and only if L is artinian by Lemma 1.7. For the second condition, the fact that R is artinian implies that $\widehat{R} \cong R$. This explains the unlabeled isomorphism in the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\cong} & \widehat{R} \\ {\scriptstyle\chi_L^R}\downarrow & \swarrow{\scriptstyle\chi_L^{\widehat{R}}} & \\ \operatorname{Hom}_R(L,L). & & \end{array}$$

Thus, the map χ_L^R is an isomorphism if and only if the map $\chi_L^{\widehat{R}}$ is an isomorphism. The Ext vanishing conditions are equivalent by definition.

For the implication (v) $\Rightarrow$ (ii), assume that $\mathfrak{Q}_0(R) = \mathfrak{S}_0(R)$. The R-module R is always semidualizing. Then, by assumption, it is also a quasidualizing R-module.

The implication (ii) $\Rightarrow$ (iv) is evident since R is an artinian ring if and only if it is an artinian R-module. For the implication (ii) $\Rightarrow$ (vi), if R is a quasidualizing R-module, then the intersection $\mathfrak{Q}_0(R)\cap\mathfrak{S}_0(R)$ is nonempty since R is also a semidualizing R-module.

For the implication (vi) $\Rightarrow$ (ii), assume that the intersection $\mathfrak{Q}_0(R)\cap\mathfrak{S}_0(R)$ is nonempty. Let $L \in \mathfrak{Q}_0(R) \cap \mathfrak{S}_0(R)$. Then L is artinian and noetherian, so it has finite length. Since L is artinian, it is $\mathfrak{m}$-torsion and by [**6,** Fact 1.2(b)] we have $\operatorname{Supp}_R(L) \subseteq \{\mathfrak{m}\}$. Since L is a semidualizing R-module, the map $R \to \operatorname{Hom}_R(L, L)$ is an isomorphism so we have $\operatorname{Ann}_R(L) \subseteq \operatorname{Ann}_R(R) = 0$. This explains the second step in the following sequence

$$\operatorname{Supp}_R(L) = V(\operatorname{Ann}_R(L)) = V(0) = \operatorname{Spec}(R).$$

Thus, $\operatorname{Spec}(R) = \operatorname{Supp}_R(L) \subseteq \{\mathfrak{m}\} \subseteq \operatorname{Spec}(R)$, and we conclude that $\operatorname{Spec}(R) = \{\mathfrak{m}\}$. Thus, [**1,** Theorem 8.5] implies that R is artinian. $\square$

3. Classes of modules and Matlis duality. This section explores the connections between the class of quasidualizing R-modules and the class of semidualizing R-modules as well as connections between different subclasses of $\mathcal{A}_M(R)$, $\mathcal{B}_M(R)$ and $\mathcal{G}_M^{\text{full}}(R)$. The instrument used to detect these connections is Matlis duality.

Theorem 3.1. *Assume that R is complete. Then the maps*

$$\mathfrak{S}_0(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathfrak{Q}_0(R)$$

are inverse bijections.

Proof. Let $C \in \mathfrak{S}_0(R)$. We show that $C^\vee \in \mathfrak{Q}_0(R)$. Fact 1.3 implies that $C^\vee$ is artinian. In the following commutative diagram, the unspecified isomorphisms are from Hom-tensor adjointness and the commutativity of tensor product

$$\begin{array}{ccc}
R & \xrightarrow{\chi^R_{C^\vee}} & \operatorname{Hom}_R(C^\vee, \operatorname{Hom}_R(C,E)) \\
\Big\downarrow \chi^R_C & & \Big\downarrow \cong \\
\operatorname{Hom}_R(C,C) & & \operatorname{Hom}_R(C^\vee \otimes_R C, E) \\
\cong \Big\downarrow \operatorname{Hom}_R(C,\delta^E_C) & & \Big\downarrow \cong \\
\operatorname{Hom}_R(C, \operatorname{Hom}_R(C^\vee, E)) & \xrightarrow{\cong} & \operatorname{Hom}_R(C \otimes_R C^\vee, E).
\end{array}$$

Since $C \in \mathfrak{S}_0(R)$, it follows that χ^R_C is an isomorphism. Fact 1.3 implies that the map δ^E_C, and by extension the map $\operatorname{Hom}_R(C, \delta^E_C)$, is an isomorphism. Hence, we conclude from the diagram that $\chi^R_{C^\vee}$ is an isomorphism.

For the last condition, Lemma 1.4 explains the first step in the following sequence

$$\operatorname{Ext}^i_R(C^\vee, C^\vee) \cong \operatorname{Ext}^i_R(C,C) = 0.$$

The second step follows from the fact that C is a semidualizing module. Thus, $C^\vee$ is a quasidualizing module.

A similar argument shows that, given a quasidualizing R-module T, the module $T^\vee$ is semidualizing. Fact 1.3 implies that $C \cong C^{\vee\vee}$ and $T \cong T^{\vee\vee}$, so that the given maps $\mathfrak{S}_0(R) \xrightarrow{(-)^\vee} \mathfrak{Q}_0(R)$ and $\mathfrak{Q}_0(R) \xrightarrow{(-)^\vee} \mathfrak{S}_0(R)$ are inverse equivalences. □

Example 3.2. Assume that R is Cohen-Macaulay and complete and admits a dualizing module D. The fact that D is dualizing means that D is semidualizing and has finite injective dimension. Therefore, by Theorem 3.1, we conclude that $D^\vee$ is quasidualizing.

Proposition 3.3. *Assume that R is complete, and let T be a quasidualizing R-module. Then the maps $\mathcal{B}^{mr}_{T^\vee}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}^{mr}_T(R)$ are inverse bijections.*

Proof. Let M be a Matlis reflexive R-module. We show that, if $M \in \mathcal{B}^{\mathrm{mr}}_{T^\vee}(R)$, then $M^\vee \in \mathcal{G}^{\mathrm{mr}}_T(R)$. Fact 1.2 implies that $M^\vee$ is Matlis reflexive. There are three remaining conditions to check.

First we show that $\mathrm{Ext}^i_R(M^\vee, T) = 0$ for all $i > 0$. Since T is artinian and R is complete, Fact 1.3 implies that T is Matlis reflexive, so we have

$$\mathrm{Ext}^i_R(M^\vee, T) \cong \mathrm{Ext}^i_R(T^\vee, M). \tag{3.3.1}$$

by Lemma 1.4. We have $\mathrm{Ext}^i_R(T^\vee, M) = 0$ for all $i > 0$ since $M \in \mathcal{B}^{\mathrm{mr}}_{T^\vee}(R)$. Thus, we conclude $\mathrm{Ext}^i_R(M^\vee, T) = 0$ for all $i > 0$.

Next we show that the map $\delta^T_{M^\vee}$ is an isomorphism. The fact that $M \in \mathcal{B}^{\mathrm{mr}}_{T^\vee}(R)$ implies the map $\xi^{T^\vee}_M$ is an isomorphism. Therefore, the map $\mathrm{Hom}_R(\xi^{T^\vee}_M, E)$ in the following commutative diagram is an isomorphism

$$\begin{array}{ccc} M^\vee & \xrightarrow[\cong]{\mathrm{Hom}_R(\xi^{T^\vee}_M, E)} & \mathrm{Hom}_R(\mathrm{Hom}_R(T^\vee, M) \otimes_R T^\vee, E) \\ \downarrow {\scriptstyle \delta^T_{M^\vee}} & & \downarrow {\scriptstyle \cong} \\ \mathrm{Hom}_R(\mathrm{Hom}_R(M^\vee, T), T) & \xrightarrow[\cong]{} & \mathrm{Hom}_R(\mathrm{Hom}_R(T^\vee, M), T). \end{array}$$

The unspecified isomorphisms are from Hom-tensor adjointness and the isomorphism (3.3.1). Hence, we conclude from the diagram that $\delta^T_{M^\vee}$ is an isomorphism.

For the last condition, let I be an injective resolution of T such that, for each $i \geqslant 0$, we have $I_i \cong E^{b_i}$ for some $b_i \in \mathbf{N}$. Lemma 1.8 implies that $I^\vee$ is a free resolution of $T^\vee$. This explains steps (2) and (6) in the following sequence:

$$\begin{aligned}
\operatorname{Ext}^i_R(\operatorname{Hom}_R(M^\vee,T),T) &\overset{(1)}{\cong} \operatorname{Ext}^i_R(\operatorname{Hom}_R(T^\vee,M),T)\\
&\overset{(2)}{\cong} \mathrm{H}_{-i}(\operatorname{Hom}_R(\operatorname{Hom}_R(T^\vee,M),I))\\
&\overset{(3)}{\cong} \mathrm{H}_{-i}(\operatorname{Hom}_R(\operatorname{Hom}_R(T^\vee,M),I^{\vee\vee}))\\
&\overset{(4)}{\cong} \mathrm{H}_{-i}(\operatorname{Hom}_R(\operatorname{Hom}_R(T^\vee,M)\otimes_R I^\vee,E))\\
&\overset{(5)}{\cong} \operatorname{Hom}_R(\mathrm{H}_i(I^\vee\otimes_R \operatorname{Hom}_R(T^\vee,M)),E)\\
&\overset{(6)}{\cong} \operatorname{Hom}_R(\operatorname{Tor}^R_i(T^\vee,\operatorname{Hom}_R(T^\vee,M)),E).
\end{aligned}$$

Step (1) follows from the isomorphism (3.3.1). Step (3) follows from the fact that any finite direct sum of artinian modules is artinian; thus, I_j is artinian for all j and we can apply Fact 1.3. Step (4) follows from Hom-tensor adjointness, and step (5) follows from the fact that E is injective and homology commutes with exact functors. Since $M\in\mathcal{B}^{\mathrm{mr}}_{T^\vee}(R)$, we have $\operatorname{Tor}^R_i(T^\vee,\operatorname{Hom}_R(T^\vee,M))=0$ for all $i>0$. Hence, we conclude that

$$\operatorname{Ext}^i_R(\operatorname{Hom}_R(M^\vee,T),T)\cong\operatorname{Hom}_R(\operatorname{Tor}^R_i(T^\vee,\operatorname{Hom}_R(T^\vee,M)),E)=0$$

for all $i>0$.

Given an R-module $M'\in\mathcal{G}^{\mathrm{mr}}_T(R)$, the argument to show that $M'^\vee\in\mathcal{B}^{\mathrm{mr}}_{T^\vee}(R)$ is similar. Since M and M' are Matlis reflexive, that is, $M\cong M^{\vee\vee}$ and $M'\cong M'^{\vee\vee}$, we conclude that the maps $\mathcal{B}^{\mathrm{mr}}_{T^\vee}(R)\xrightarrow{(-)^\vee}\mathcal{G}^{\mathrm{mr}}_T(R)$ and $\mathcal{G}^{\mathrm{mr}}_T(R)\xrightarrow{(-)^\vee}\mathcal{B}^{\mathrm{mr}}_{T^\vee}(R)$ are inverse equivalences. □

Corollary 3.4. *Assume that R is complete, and let T be a quasidualizing R-module. Then the following maps are inverse bijections*:

$$\mathcal{B}^{\mathrm{noeth}}_{T^\vee}(R)\underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}}\mathcal{G}^{\mathrm{artin}}_T(R)\qquad\textit{and}\qquad\mathcal{B}^{\mathrm{artin}}_{T^\vee}(R)\underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}}\mathcal{G}^{\mathrm{noeth}}_T(R).$$

Proof. Fact 1.3 implies that, if N is a noetherian R-module, then $N^\vee$ is an artinian R-module and $N\cong N^{\vee\vee}$. Furthermore, if A is an artinan R-module, then $A^\vee$ is a noetherian R-module and

$A \cong A^{\vee\vee}$. Together with Proposition 3.3, this implies that the maps $\mathcal{B}^{\text{noeth}}_{T^\vee}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}^{\text{artin}}_{T}(R)$ are inverse bijections. The proof for $\mathcal{B}^{\text{artin}}_{T^\vee}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}^{\text{noeth}}_{T}(R)$ is similar. □

Proposition 3.5. *Assume that R is complete, and let T be a quasidualizing R-module. Then the maps* $\mathcal{B}^{mr}_{T}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}^{mr}_{T^\vee}(R)$ *are inverse bijections.*

Proof. Let M be a Matlis reflexive R-module. We show that if $M \in \mathcal{G}^{\text{mr}}_{T^\vee}(R)$, then $M^\vee \in \mathcal{B}^{\text{mr}}_{T}(R)$. First we show that the map $\xi^T_{M^\vee}$ is an isomorphism. The fact that M is Matlis reflexive implies that the map δ^E_M in the following commutative diagram is an isomorphism:

$$\begin{array}{ccc} M & \xrightarrow[\cong]{\delta^E_M} & M^{\vee\vee} \\ \downarrow{\scriptstyle \delta^{T^\vee}_M} & & \downarrow{\scriptstyle (\xi^T_{M^\vee})^\vee} \\ \mathrm{Hom}_R(\mathrm{Hom}_R(M,T^\vee),T^\vee) & & \mathrm{Hom}_R(\mathrm{Hom}_R(T,M^\vee)\otimes_R T, E) \\ \downarrow{\scriptstyle \cong} & \nearrow{\scriptstyle \cong} & \\ \mathrm{Hom}_R(\mathrm{Hom}_R(T,M^\vee),T^\vee). & & \end{array}$$

The unspecified isomorphisms are from Hom-tensor adjointness and Lemma 1.4. Since $M \in \mathcal{G}^{\text{mr}}_{T^\vee}(R)$, we have that the map $\delta^{T^\vee}_M$ is an isomorphism. Hence, $(\xi^T_{M^\vee})^\vee$ is an isomorphism. Since E is faithfully injective, this implies that $\xi^T_{M^\vee}$ is an isomorphism.

Next we show that $\mathrm{Ext}^i_R(T,M^\vee) = 0$ for all $i > 0$. Since M is Matlis reflexive, Lemma 1.4 explains the first step in the following sequence $\mathrm{Ext}^i_R(T,M^\vee) \cong \mathrm{Ext}^i_R(M,T^\vee) = 0$. The second step follows from the fact that $M \in \mathcal{G}^{\text{mr}}_{T^\vee}(R)$.

Lastly, we show that $\mathrm{Tor}^R_i(T, \mathrm{Hom}_R(T,M^\vee)) = 0$ for all $i > 0$. The commutativity of tensor product explains the first step in the following sequence:

$$\begin{aligned}\operatorname{Tor}_i^R(T, \operatorname{Hom}_R(T, M^\vee))^\vee &\cong \operatorname{Tor}_i^R(\operatorname{Hom}_R(T, M^\vee), T)^\vee\\ &\cong \operatorname{Ext}_R^i(\operatorname{Hom}_R(T, M^\vee), T^\vee)\\ &\cong \operatorname{Ext}_R^i(\operatorname{Hom}_R(M, T^\vee), T^\vee)\\ &= 0.\end{aligned}$$

The second step follows from [**6,** Remark 1.9], and the third step follows from Lemma 1.4. The last step follows from the fact that $M \in \mathcal{G}_{T^\vee}^{\mathrm{mr}}(R)$.

Given an R-module $M' \in \mathcal{B}_T^{\mathrm{mr}}(R)$, the argument to show that $M'^\vee \in \mathcal{G}_{T^\vee}^{\mathrm{mr}}(R)$ is similar but easier. Since M and M' are Matlis reflexive, we conclude that the maps $\mathcal{B}_T^{\mathrm{mr}}(R) \xrightarrow{(-)^\vee} \mathcal{G}_{T^\vee}^{\mathrm{mr}}(R)$ and $\mathcal{G}_{T^\vee}^{\mathrm{mr}}(R) \xrightarrow{(-)^\vee} \mathcal{B}_T^{\mathrm{mr}}(R)$ are inverse equivalences. □

Corollary 3.6. *Assume that R is complete, and let T be a quasidualizing R-module. Then the following maps are inverse bijections*:

$$\mathcal{B}_T^{\mathrm{noeth}}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}_{T^\vee}^{\mathrm{artin}}(R) \qquad \textit{and} \qquad \mathcal{B}_T^{\mathrm{artin}}(R) \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \mathcal{G}_{T^\vee}^{\mathrm{noeth}}(R).$$

The next proposition establishes the relationship between a subclass of the Auslander class and a subclass of the derived reflexive modules.

Proposition 3.7. *If R is complete and T is a quasidualizing R-module, then*

$$\mathcal{G}_{T^\vee}^{\mathrm{mr}}(R) = \mathcal{A}_T^{\mathrm{mr}}(R).$$

Proof. Let M be a Matlis reflexive R-module. We show that M satisfies the defining conditions of $\mathcal{G}_{T^\vee}^{\mathrm{mr}}(R)$ if and only if M satisfies the defining conditions of $\mathcal{A}_T^{\mathrm{mr}}(R)$. For the isomorphisms, consider the following commutative diagram:

$$\begin{array}{ccc}
M & \xrightarrow{\delta_M^{T^\vee}} & \mathrm{Hom}_R(\mathrm{Hom}_R(M, T^\vee), T^\vee) \\
\Big\downarrow {\scriptstyle \gamma_M^T} & & \Big\downarrow {\scriptstyle \cong} \\
\mathrm{Hom}_R(T, T \otimes_R M) & & \mathrm{Hom}_R(\mathrm{Hom}_R(M, T^\vee) \otimes_R T, E) \\
{\scriptstyle \cong} \Big\downarrow {\scriptstyle \mathrm{Hom}_R(T, \delta^E_{T \otimes_R M})} & & \Big\downarrow {\scriptstyle \cong} \\
\mathrm{Hom}_R(T, \mathrm{Hom}_R((T \otimes_R M)^\vee, E)) & \xrightarrow[\cong]{} & \mathrm{Hom}_R(T, \mathrm{Hom}_R(\mathrm{Hom}_R(M, T^\vee), E)).
\end{array}$$

The unspecified isomorphisms are Hom-tensor adjointness. The module $T \otimes_R M$ is artinian by [**6,** Lemma 1.19 and Theorem 3.1]. Fact 1.3 implies that the map $\delta^E_{T\otimes_R M}$, and hence the map $\mathrm{Hom}_R(T, \delta^E_{T\otimes_R M})$, is an isomorphism. Therefore, the map γ_M^T is an isomorphism if and only if the map $\delta_M^{T^\vee}$ is an isomorphism.

Next we show that, for all $i > 0$, we have $\mathrm{Ext}^i_R(M, T^\vee) = 0$ if and only if $\mathrm{Tor}^R_i(M, T) = 0$. By [**6,** Remark 1.9], we have $\mathrm{Ext}^i_R(M, T^\vee) \cong \mathrm{Tor}^R_i(M, T)^\vee$. Because the Matlis dual of a module is zero if and only if the module is zero, we conclude that $\mathrm{Ext}^i_R(M, T^\vee) = 0$ if and only if $\mathrm{Tor}^R_i(M, T) = 0$ for all $i > 0$.

Next we show that, for all $i > 0$, we have $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M, T^\vee), T^\vee) = 0$ if and only if $\mathrm{Ext}^i_R(T, M \otimes_R T) = 0$. Hom-tensor adjointness explains the first step in the following sequence:

$$\begin{aligned}
\mathrm{Ext}^i_R(\mathrm{Hom}_R(M, T^\vee), T^\vee) &\cong \mathrm{Ext}^i_R((M \otimes_R T)^\vee, T^\vee) \\
&\cong \mathrm{Ext}^i_R(T^{\vee\vee}, (M \otimes_R T)^{\vee\vee}) \\
&\cong \mathrm{Ext}^i_R(T, M \otimes_R T).
\end{aligned}$$

The second step follows from Lemma 1.4 and the fact that T is artinian and thus Matlis reflexive. The third step follows from the fact that T and $M \otimes_R T$ are artinian and hence Matlis reflexive, see [**6,** Corollary 3.9]. $\square$

Corollary 3.8. *Assume that R is complete, and let T be a quasi-dualizing R-module. Then $\mathcal{G}^{\mathrm{noeth}}_{T^\vee}(R) = \mathcal{A}^{\mathrm{noeth}}_T(R)$ and $\mathcal{G}^{\mathrm{artin}}_{T^\vee}(R) = \mathcal{A}^{\mathrm{artin}}_T(R)$.*

Proposition 3.9. *If R is complete and T is a quasidualizing R-module, then*

$$\mathcal{G}_T^{\mathrm{mr}}(R) = \mathcal{A}_{T^\vee}^{\mathrm{mr}}(R).$$

Proof. Let M be a Matlis reflexive R-module. We show that M satisfies the defining conditions of $\mathcal{G}_T^{\mathrm{mr}}(R)$ if and only if M satisfies the defining conditions of $\mathcal{A}_{T^\vee}^{\mathrm{mr}}(R)$. For the isomorphisms, consider the following commutative diagram:

$$\begin{array}{ccc}
M & \xrightarrow{\delta_M^T} & \mathrm{Hom}_R(\mathrm{Hom}_R(M,T),T) \\
\downarrow{\scriptstyle \gamma_M^{T^\vee}} & & \\
\mathrm{Hom}_R(T^\vee, T^\vee \otimes_R M) & & \\
\downarrow{\scriptstyle \mathrm{Hom}_R(T^\vee, \delta^E_{T^\vee \otimes M})} & & \\
\mathrm{Hom}_R(T^\vee, (T^\vee \otimes_R M)^{\vee\vee}) & & {\scriptstyle \mathrm{Hom}_R(\mathrm{Hom}_R(M,T),\delta^E_T)}\ \downarrow\ {\scriptstyle \cong} \\
\downarrow{\scriptstyle \cong} & & \\
\mathrm{Hom}_R(\mathrm{Hom}_R(T^\vee \otimes_R M, E), T^{\vee\vee}) & & \\
\downarrow{\scriptstyle \cong} & & \\
\mathrm{Hom}_R(\mathrm{Hom}_R(M, T^{\vee\vee}), T^{\vee\vee}) & \xrightarrow[\mathrm{Hom}_R(\mathrm{Hom}_R(M,\delta^E_T),T^{\vee\vee})]{\cong} & \mathrm{Hom}_R(\mathrm{Hom}_R(M,T),T^{\vee\vee})
\end{array}$$

where the unlabeled isomorphisms are Hom-tensor adjointness and Hom-swap. Since T is artinian, and hence Matlis reflexive, both the right hand map and the bottom map are isomorphisms. The module $T^\vee \otimes_R M$ is Matlis reflexive by [**6,** Corollary 3.6]. Thus, the map $\delta^E_{T^\vee \otimes M}$, and hence the map $\mathrm{Hom}_R(T^\vee, \delta^E_{T^\vee \otimes M})$ is an isomorphism. Therefore, the map $\gamma_M^{T^\vee}$ is an isomorphism if and only if the map δ_M^T is an isomorphism.

Next we show that, for all $i > 0$, we have $\mathrm{Ext}^i_R(M,T) = 0$ if and only if $\mathrm{Tor}^R_i(T^\vee, M) = 0$. The fact that T is artinian, and hence Matlis reflexive, explains the first step in the following sequence

$$\mathrm{Ext}^i_R(M,T) \cong \mathrm{Ext}^i_R(M,T^{\vee\vee}) \cong \mathrm{Tor}^R_i(M,T^\vee)^\vee \cong \mathrm{Tor}^R_i(T^\vee, M)^\vee.$$

The second step follows from [**6,** Remark 1.9], and the last step follows from the commutativity of the tensor product. Because the Matlis dual of a module is zero if and only if the module is zero, we conclude that $\operatorname{Ext}^i_R(M,T)=0$ if and only if $\operatorname{Tor}^R_i(T^\vee,M)=0$ for all $i>0$.

Next we show that, for all $i>0$, we have $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M,T),T)=0$ if and only if $\operatorname{Ext}^i_R(T^\vee,T^\vee\otimes_R M)=0$. The fact that T is artinian, and hence Matlis reflexive, explains the first and third steps in the following sequence:

$$\begin{aligned}\operatorname{Ext}^i_R(\operatorname{Hom}_R(M,T),T) &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(M,T^{\vee\vee}),T)\\ &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(M\otimes_R T^\vee,E),T)\\ &\cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(M\otimes_R T^\vee,E),T^{\vee\vee})\\ &\cong \operatorname{Ext}^i_R(T^\vee,M\otimes_R T^\vee).\end{aligned}$$

The second step follows from Hom-tensor adjointness, and the last step follows from Lemma 1.4. □

Corollary 3.10. *Assume that R is complete and let T be a quasidualizing R-module. Then $\mathcal{G}^{\text{noeth}}_T(R)=\mathcal{A}^{\text{noeth}}_{T^\vee}(R)$ and $\mathcal{G}^{\text{artin}}_T(R)=\mathcal{A}^{\text{artin}}_{T^\vee}(R)$.*

The above results show that the classes $\mathcal{G}^{\text{mr}}_T(R)$, $\mathcal{G}^{\text{artin}}_T(R)$, and $\mathcal{G}^{\text{noeth}}_T(R)$ do not exhibit some of the same properties as the class $\mathcal{G}^{\text{noeth}}_C(R)$, where C is semidualizing. For instance, we consider the following property. We say a class of R-modules $\mathcal{C}$ satisfies the two-of-three condition if, given an exact sequence of R-module homomorphisms $0\to L_1\to L_2\to L_3\to 0$, when any two of the modules are in $\mathcal{C}$, so is the third. The two-of-three condition holds for some classes of modules and not for others. For example, the class of noetherian modules and the class of artinian modules both satisfy the two-of-three condition. On the other hand, the class $\mathcal{G}^{\text{noeth}}_C(R)$ does not satisfy the two-of-three condition when C is semidualizing. In contrast, the next result shows that the class $\mathcal{G}^{\text{full}}_T(R)$ satisfies the two-of-three condition when the ring is complete. This is somewhat surprising since the definitions of $\mathcal{G}^{\text{noeth}}_C(R)$ and $\mathcal{G}^{\text{full}}_T(R)$ are so similar. First we need a lemma. In the language of **[4]** it says that quasidualizing implies faithfully quasidualizing.

Lemma 3.11. *Let L and T be R-modules such that T is quasidualizing. If one has $\operatorname{Hom}_R(L,T)=0$, then $L=0$.*

Proof. Assume that $\operatorname{Hom}_R(L,T)=0$.

Case 1. $T=E$. Because $\operatorname{Hom}_R(L,E)=0$, we have $L^{\vee\vee}=0$. Since the map $\delta_L^E: L\to L^{\vee\vee}$ is injective by Fact 1.2, we conclude that $L=0$.

Case 2. R is complete. Then T is Matlis reflexive and we have $0=\operatorname{Hom}_R(L,T)\cong\operatorname{Hom}_R(T^\vee,L^\vee)$ from Lemma 1.4. Since $T^\vee$ is semidualizing by Proposition 3.1, we have $L^\vee=0$ by [**4,** Proposition 3.6]. By Case 1, we conclude that $L=0$.

Case 3. the general case. The first step in the following sequence is by assumption:

$$0=\operatorname{Hom}_R(L,T)\cong\operatorname{Hom}_R(L,\operatorname{Hom}_{\widehat{R}}(\widehat{R},T))\cong\operatorname{Hom}_{\widehat{R}}(\widehat{R}\otimes_R L,T).$$

The second step follows from the fact that T is artinian and hence has an $\widehat{R}$ structure, and the third step is from Hom-tensor adjointness. Since T is a quasidualizing $\widehat{R}$-module, we can apply Case 2 to conclude that $\widehat{R}\otimes_R L=0$. Then $L=0$ because $\widehat{R}$ is faithfully flat over R. $\square$

Question 3.12. Does a version of Lemma 3.11 hold for $T\otimes_R -$ as in [**4**]?

Theorem 3.13. *Assume that R is complete, and let T be a quasidualizing R-module. Then $\mathcal{G}_T^{\text{full}}(R)$ satisfies the two-of-three condition.*

Proof. Let

$$0\longrightarrow L_1\xrightarrow{f}L_2\xrightarrow{g}L_3\longrightarrow 0 \tag{3.13.1}$$

be an exact sequence of R-module homomorphisms, and let $(-)^T=\operatorname{Hom}_R(-,T)$. There are two conditions to check and three cases. We will deal with the case when $L_1,L_2\in\mathcal{G}_T^{\text{full}}(R)$. The case where $L_2,L_3\in\mathcal{G}_T^{\text{full}}(R)$ is similar. The case where $L_1,L_3\in\mathcal{G}_T^{\text{full}}(R)$ is also similar but easier.

Assume that $L_1,L_2\in\mathcal{G}_T^{\text{full}}(R)$. Then we have $\operatorname{Ext}_R^i(L_1,T)=0=\operatorname{Ext}_R^i(L_2,T)$ for all $i>0$. The following portion of the long exact sequence in $\operatorname{Ext}_R^i(-,T)$ associated to the short exact sequence (3.13.1)

$$\cdots\to\operatorname{Ext}_R^{i-1}(L_1,T)\to\operatorname{Ext}_R^i(L_3,T)\to\operatorname{Ext}_R^i(L_2,T)\to\operatorname{Ext}_R^i(L_1,T)\to\cdots \tag{3.13.2}$$

shows that $\operatorname{Ext}^i_R(L_3,T)=0$ for all $i>1$. For the case where $i=1$, we apply $(-)^T$ to the following portion of the long exact sequence

$$0 \longrightarrow (L_3)^T \longrightarrow (L_2)^T \longrightarrow (L_1)^T \longrightarrow \operatorname{Ext}^1_R(L_3,T) \longrightarrow 0$$

to obtain exactness in the top row of the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (\operatorname{Ext}^1_R(L_3,T))^T & \longrightarrow & (L_1)^{TT} & \xrightarrow{f^{TT}} & (L_2)^{TT} \\
 & & & & {\cong}\Big\uparrow{\delta^T_{L_1}} & & {\cong}\Big\uparrow{\delta^T_{L_2}} \\
 & & 0 & \longrightarrow & L_1 & \xrightarrow{f} & L_2.
\end{array}$$

Since f is an injective map, the diagram shows that f^{TT} is an injective map. Hence, we have $(\operatorname{Ext}^1_R(L_3,T))^T=0$. From Lemma 3.11, we conclude that $\operatorname{Ext}^1_R(L_3,T)=0$.

Next we show that $\operatorname{Ext}^i_R(\operatorname{Hom}_R(L_3,T),T)=0$ for all $i>0$. From the argument above, we have the exact sequence

$$0 \longrightarrow (L_3)^T \longrightarrow (L_2)^T \longrightarrow (L_1)^T \longrightarrow 0. \tag{3.13.3}$$

In a similar, but easier, manner than above, the long exact sequence in $\operatorname{Ext}^i_R(-,T)$ shows that, if $L_1, L_2 \in \mathcal{G}^{\text{full}}_T(R)$, then $\operatorname{Ext}^i_R(\operatorname{Hom}_R(L_3,T),T)$ $=0$ for all $i>0$.

Lastly, we show that the map $\delta^T_{L_3}$ is an isomorphism. From the short exact sequence (3.13.1) and as a consequence of the above argument together with the short exact sequence (3.13.3), we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_1 & \xrightarrow{f} & L_2 & \xrightarrow{g} & L_3 & \longrightarrow & 0 \\
 & & {\cong}\Big\downarrow{\delta^T_{L_1}} & & {\cong}\Big\downarrow{\delta^T_{L_2}} & & \Big\downarrow{\delta^T_{L_3}} & & \\
0 & \longrightarrow & (L_1)^{TT} & \xrightarrow{f^{TT}} & (L_2)^{TT} & \xrightarrow{g^{TT}} & (L_3)^{TT} & \longrightarrow & 0.
\end{array}$$

Since L_1, L_2 are in $\mathcal{G}^{\text{full}}_T(R)$, the maps $\delta^T_{L_1}$ and $\delta^T_{L_2}$ are isomorphisms. By the Snake lemma, we conclude that $\delta^T_{L_3}$ is an isomorphism. □

Corollary 3.14. *Assume that R is complete, and let T be a quasidualizing R-module. Then $\mathcal{G}^{\text{artin}}_T(R)=\mathcal{A}^{\text{artin}}_{T^\vee}(R)$, $\mathcal{G}^{\text{noeth}}_T(R)=\mathcal{A}^{\text{noeth}}_{T^\vee}(R)$, and $\mathcal{G}^{\text{mr}}_T(R)$ satisfy the two-of-three condition.*

Proof. Apply Theorem 3.13 and Corollary 3.10. □

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