JOURNAL OF COMMUTATIVE ALGEBRA
Volume 6, Number 1, Spring 2014

A NOTE ON RIGIDITY AND TRIANGULABILITY OF A DERIVATION

MANOJ K. KESHARI AND SWAPNIL A. LOKHANDE

ABSTRACT. Let A be a $\mathbf{Q}$-domain, $K = \mathrm{frac}(A)$, $B = A^{[n]}$ and $D \in \mathrm{LND}_A(B)$. Assume rank $D = \mathrm{rank}\, D_K = r$, where D_K is the extension of D to $K^{[n]}$. Then we show that

(i) If D_K is rigid, then D is rigid.

(ii) Assume $n = 3$, $r = 2$ and $B = A[X, Y, Z]$ with $DX = 0$. Then D is triangulable over A if and only if D is triangulable over $A[X]$. In case A is a field, this result is due to Daigle.

1. Introduction. Throughout this paper, *k is a field and all rings are $\mathbf{Q}$-domains.* We will begin by setting up some notations from [**4**]. Let $B = A^{[n]}$ be an A-algebra, i.e., B is A-isomorphic to the polynomial ring in n variables over A. A *coordinate system* of B over A is an ordered n-tuple $(X_1, X_2, \dots, X_n)$ of elements of B such that $A[X_1, X_2, \dots, X_n] = B$.

An A-derivation $D : B \to B$ is *locally nilpotent* if, for each $x \in B$, there exists an integer $s > 0$ such that $D^s(x) = 0$; D is *triangulable* over A if there exists a coordinate system $(X_1, \dots, X_n)$ of B over A such that $D(X_i) \in A[X_1, \dots, X_{i-1}]$ for $1 \le i \le n$; *rank* of D is the least integer $r \ge 0$ for which there exists a coordinate system $(X_1, \dots, X_n)$ of B over A satisfying $A[X_1, \dots, X_{n-r}] \subset \ker D$; $\mathrm{LND}_A(B)$ is the set of all locally nilpotent A-derivations of B.

Let $\Gamma(B)$ be the set of coordinate systems of B over A. Given $D \in \mathrm{LND}_A(B)$ of rank r, let $\Gamma_D(B)$ be the set of $(X_1, \dots, X_n) \in \Gamma(B)$ satisfying $A[X_1, \dots, X_{n-r}] \subset \ker D$; D is *rigid* if $A[X_1, \dots, X_{n-r}] = A[X'_1, \dots, X'_{n-r}]$ holds whenever $(X_1, \dots, X_n)$ and $(X'_1, \dots, X'_n)$ belong to $\Gamma_D(B)$.

For example, if $D \in \mathrm{LND}_A(B)$ has rank 1, then D is rigid. In this case $\ker D = A[X_1, \dots, X_{n-1}]$ for some coordinate system $(X_1, \dots, X_n)$ and $D = f\partial_{X_n}$ for some $f \in \ker D$. If rank $D = n$, then D is obviously

2010 AMS *Mathematics subject classification.* Primary 14L30, Secondary 13B25.
Keywords and phrases. Locally nilpotent derivation, rigidity, triangulability.
Received by the editors on May 14, 2012, and in revised form on August 6, 2012.

DOI:10.1216/JCA-2014-6-1-95

rigid, as no variable is in $\ker D$. If rank $D \neq 1, n$, then $\ker D$ is not generated by $n-1$ elements of a coordinate system and it is generally difficult to see whether D is rigid. For an example of a non-rigid triangular derivation of $k^{[4]}$, see Section 3. We remark that there is also a notion of a ring to be rigid. We say that a ring A is rigid if $\mathrm{LND}(A) = \{0\}$, i.e., there is no non-zero locally nilpotent derivation on A. Clearly, polynomial rings $k^{[n]}$ are non-rigid rings for $n \geq 1$.

We will state the following result of Daigle ([**4**, Theorem 2.5]) which is used later.

Theorem 1.1. *All locally nilpotent derivations of $k^{[3]}$ are rigid.*

Our first result extends this as follows:

Theorem 1.2. *Let A be a ring, $B = A^{[n]}$, $K = \mathrm{frac}\,(A)$ and $D \in \mathrm{LND}_A(B)$. Assume that $\mathrm{rank}\, D = \mathrm{rank}\, D_K$, where D_K is the extension of D to $K^{[n]}$. If D_K is rigid, then D is rigid.*

In [**4**, Corollary 3.4], Daigle obtained the following triangulability criterion. Let D be an irreducible, locally nilpotent derivation of $R = k^{[3]}$ of rank at most 2. Let $(X, Y, Z) \in \Gamma(R)$ be such that $DX = 0$. Then D is triangulable over k if and only if D is triangulable over $k[X]$. Our second result extends this result as follows:

Theorem 1.3. *Let A be a ring, $B = A^{[3]}$, $K = \mathrm{frac}\,(A)$ and $D \in \mathrm{LND}_A(B)$. Let $(X, Y, Z) \in \Gamma(B)$ be such that $DX = 0$. Assume that $\mathrm{rank}\, D = \mathrm{rank}\, D_K = 2$. Then D is triangulable over A if and only if D is triangulable over $A[X]$.*

2. Preliminaries. Recall that a ring is called an *HCF*-ring if the intersection of two principal ideals is again a principal ideal. We state some results for later use.

Lemma 2.1 [**4**, 1.2]. *Let D be a k-derivation of $R = k^{[n]}$ of* rank 1, *and let $(X_1, X_2, \ldots, X_n) \in \Gamma(R)$ be such that $k[X_1, X_2, \ldots, X_{n-1}] \subset \ker D$. Then*:

(i) $\ker D = k[X_1, X_2, \dots, X_{n-1}]$;

(ii) *D is locally nilpotent if and only if $D(X_n) \in \ker D$.*

Proposition 2.2 [**1**, Proposition 4.8]. *Let R be a* HCF*-ring, A a ring of transcendence degree one over R and $R \subset A \subset R^{[n]}$ for some $n \geq 1$. If A is a factorially closed subring of $R^{[n]}$, then $A = R^{[1]}$.*

Lemma 2.3 [**1**, 1.7]. *Suppose $A^{[n]} = R = B^{[n]}$. If $b \in B$ is such that $bR \cap A \neq 0$, then $b \in A$.*

Theorem 2.4 [**6**, Theorem 4.11]. *Let R be an* HCF*-ring and $0 \neq D \in \mathrm{LND}_R(R[X, Y])$. Then there exists $P \in R[X, Y]$ such that $\ker D = R[P]$.*

Theorem 2.5 [**3**]. *Let A be a Noetherian ring containing the field $\mathbf{Q}$ and $B = A^{[2]}$. Then $b \in B$ is a variable of B over A if and only if, for every prime ideal $\mathfrak{p}$ of A, $\overline{b} \in \overline{B} := B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is a variable of $\overline{B}$ over $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.*

3. Rigidity.

Theorem 3.1. *Let A be a ring, $B = A^{[n]}$, $K = \mathrm{frac}\,(A)$ and $D \in \mathrm{LND}_A(B)$. Assume that $\mathrm{rank}\, D = \mathrm{rank}\, D_K$, where D_K is the extension of D to $K^{[n]}$. If D_K is rigid, then D is rigid.*

Proof. Assume rank $D = \mathrm{rank}\, D_K = r$ and D_K is rigid. We need to show that D is rigid, i.e., if $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are two coordinate systems of B satisfying $A[x_1, \dots, x_{n-r}] \subset \ker D$ and $A[y_1, \dots, y_{n-r}] \subset \ker D$, then we have to show that $A[x_1, \dots, x_{n-r}] = A[y_1, \dots, y_{n-r}]$. By symmetry, it is enough to show that $A[x_1, \dots, x_{n-r}] \subset A[y_1, \dots, y_{n-r}]$.

Since D_K is rigid and $\mathrm{rank}\, D_K = r$, we get $K[x_1, \dots, x_{n-r}] = K[y_1, \dots, y_{n-r}]$. If $f \in A[x_1, \dots, x_{n-r}]$, then $f \in K[y_1, \dots, y_{n-r}]$. We can choose $a \in A$ such that $af \in A[y_1, \dots, y_{n-r}]$, and hence $fB \cap A[y_1, \dots, y_{n-r}] \neq 0$. Applying (2.3) to $A[x_1, \dots, x_{n-r}]^{[r]} = B = A[y_1, \dots, y_{n-r}]^{[r]}$, we get $f \in A[y_1, y_2, \dots, y_{n-r}]$. Therefore, $A[x_1, \dots, x_{n-r}] \subset A[y_1, \dots, y_{n-r}]$. This completes the proof. □

The following result is immediate from (3.1) and (1.1).

Corollary 3.2. *Let A be a ring, $B = A^{[3]}$, $D \in \mathrm{LND}_A(B)$. If* $\operatorname{rank} D = \operatorname{rank} D_K$*, then D is rigid.*

Remark 3.3. (1) If $D \in \mathrm{LND}_A(B)$, then rank D and rank D_K need not be the same. For example, consider $A = \mathbf{Q}[X]$ and $B = A[T, Y, Z]$. Define $D \in \mathrm{LND}_A(B)$ as $DT = 0$, $D(Y) = X$ and $D(Z) = Y$. Then rank $D = 2$ and rank $D_K = 1$. Further, $(T' = T - Y^2 + 2XZ, Y, Z) \in \Gamma_D(B)$ and $A[T] \neq A[T']$. Therefore, D is not rigid, whereas D_K is rigid, by (1.1).

The above example gives a $D \in \mathrm{LND}(k^{[4]})$ which is not rigid. Hence, Daigle's result (1.1) is the best possible. Note that D is a triangular derivation and, by [**2**], $\ker D$ is a finitely generated k-algebra.

(2) The condition in (3.1) is sufficient but not necessary, i.e., $D \in \mathrm{LND}_A(B)$ may be rigid even if $\operatorname{rank} D \neq \operatorname{rank} D_K$. For example, consider $A = \mathbf{Q}[X]$ and $B = A[Y, Z]$. Define $D \in \mathrm{LND}_A(B)$ as $D(Y) = X$ and $D(Z) = Y$. Then rank $D = 2$, and hence, D is rigid. Further, rank $D_K = 1$ and D_K is also rigid, by (1.1).

(3) It will be interesting to know if $D \in \mathrm{LND}(k^{[n]})$ being rigid implies that $\ker D$ is a finitely generated k-algebra. The following example could provide an answer.

Let $D = X^3\partial_S + S\partial_T + T\partial_U + X^2\partial_V \in \mathrm{LND}(B)$, where $B = k^{[5]} = k[X, S, T, U, V]$. Daigle and Freudenberg [**5**] have shown that $\ker D$ is not a finitely generated k-algebra. We do not know if D is rigid. We will show that rank $D = 3$.

Clearly $X, S - XV \in \ker D$ is a part of a coordinate system. Hence, rank $D \leq 3$. If rank $D = 1$, then there exists a coordinate system $(X_1, \ldots, X_4, Y)$ of B over k such that $X_1, \ldots, X_4 \in \ker D$. Hence, $D = f\partial_Y$ for some $f \in k[X_1, \ldots, X_4]$ and $\ker D = k[X_1, \ldots, X_4]$ is a finitely generated k-algebra, a contradiction. If rank $D = 2$, then there exists a coordinate system (X_1, X_2, X_3, Y, Z) of B over k such that $X_1, X_2, X_3 \in \ker D$. If we write $A = k[X_1, X_2, X_3]$, then $D \in \mathrm{LND}_A(A[Y, Z])$. Since A is UFD, by [**6**, Theorem 4.11], $\ker D = A^{[1]}$; hence, $\ker D$ is a finitely generated k-algebra, a contradiction. Therefore, rank of D is 3.

4. Triangulability. We begin with the following result which is of independent interest.

Lemma 4.1. *Let A be a* UFD*, $K = \text{frac}\,(A)$, $B = A^{[n]}$ and $D \in \text{LND}_A(B)$. Let D_K be the extension of D on $K^{[n]}$. If D is irreducible, then D_K is irreducible.*

Proof. We prove that, if D_K is reducible, then so is D. Let $D_K(K^{[n]}) \subset fK^{[n]}$ for some $f \in B$. If $B = A[x_1, \dots, x_n]$, then we can write $Dx_i = fg_i/c_i$ for some $g_i \in B$ and $c_i \in A$ with $\gcd_B(g_i, c_i) = 1$. Since $Dx_i \in B$, we get c_i divides f in B. If c is the lcm of c_i's, then c divides f. If we take $f' = f/c \in B$, then $Dx_i \in f'B$ and hence D is reducible. $\square$

Proposition 4.2. *Let A be a Noetherian ring, $B = A^{[3]}$, and $D \in \text{LND}_A(B)$ be of rank one. Let $(X, Y, Z) \in \Gamma(B)$ be such that $DX = 0$. Then D is triangulable over $A[X]$.*

Proof. As rank $D = 1$, there exists $(X', Y', Z') \in \Gamma(B)$ such that $DX' = DY' = 0$. By (2.1), $\ker D = A[X', Y']$ and $DZ' \in \ker D$.

The following shows that X is a variable of $A[X', Y']$ over A. By (2.5), it is enough to prove that, for every prime ideal $\mathfrak{p}$ of A, if $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, then $\overline{X}$ is a variable of $\kappa(\mathfrak{p})[X', Y']$ over $\kappa(\mathfrak{p})$. Extend D on $A_\mathfrak{p}[X, Y, Z]$ and let $\overline{D}$ be D modulo $\mathfrak{p}A_\mathfrak{p}$. Then $\ker \overline{D} = \kappa(\mathfrak{p})[X', Y']$. By (2.2), $\ker \overline{D} = \kappa(\mathfrak{p})[X]^{[1]}$. Therefore, X is a variable of $A[X', Y']$, i.e., $A[X', Y'] = A[X, P]$ for some $P \in B$. Hence, $B = A[X, P, Z']$. Thus, D is triangulable over $A[X]$. $\square$

Proposition 4.3. *Let A be a ring, $K = \text{frac}\,(A)$, $B = A^{[3]}$ and $D \in \text{LND}_A(B)$. Let $(X, Y, Z) \in \Gamma(B)$ be such that $DX = 0$. Assume* $\text{rank}\, D = \text{rank}\, D_K = 2$*. Then D is triangulable over A if and only if D is triangulable over $A[X]$.*

Proof. We need to show only ($\Rightarrow$). Suppose that D is triangulable over A. Then there exists $(X', Y', Z') \in \Gamma(B)$ such that $DX' \in A$, $DY' \in A[X']$ and $DZ' \in A[X', Y']$. If $a = DX' \neq 0$, then $D_K(X'/a) = 1$, which implies that rank $D_K = 1$, a contradiction. Hence, $DX' = 0$.

Since D_K is rigid, by (3.1), D is rigid of rank 2. Therefore, $A[X] = A[X']$ and D is triangulable over $A[X]$. $\square$

Acknowledgments. We sincerely thank the referee for his/her remarks which improved the exposition.

REFERENCES

1. S.S. Abhyankar, P. Eakin and W. Heinzer, *On the uniqueness of the coefficient ring in a polynomial ring*, J. Algebra **23** (1972), 310–342.

2. S.M. Bhatwadekar and D. Daigle, *On finite generation of kernels of locally nilpotent R-derivations of* $R[X, Y, Z]$, J. Algebra **322** (2009), 2915–2916.

3. S. M. Bhatwadekar and Amartya K. Dutta, *On residual variables and stably polynomial algebras*, Comm. Algebra **21** (1993), 635–645.

4. D. Daigle, *A necessary and sufficient condition for triangulability of derivations of* $k[X, Y, Z]$, J. Pure Appl. Alg. **113** (1996), 297–305.

5. D. Daigle and G. Freudenburg, *A counterexample to Hilbert's fourteenth problem in dimension* 5, J. Algebra **221** (1999), 528–535.

6. G. Freudenburg, *Algebraic theory of locally nilpotent derivations*, Encyclopedia of Mathematical Sciences, vol. 136, Springer-Verlag, Berlin, 2006.

Department of Mathematics, IIT Bombay, Mumbai, 400076, India
Email address: keshari@math.iitb.ac.in

Department of Mathematics, IIT Bombay, Mumbai, 400076, India
Email address: swapnil@math.iitb.ac.in