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EXAMPLES OF NON-NOETHERIAN DOMAINS INSIDE POWER SERIES RINGS

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ABSTRACT. Given a power series ring R^* over a Noetherian integral domain R and an intermediate field L between R and the total quotient ring of R^*, the integral domain $A = L \cap R^*$ often (but not always) inherits nice properties from R^* such as the Noetherian property. For certain fields L it is possible to approximate A using a localization B of a particular nested union of polynomial rings over R associated to A; if B is Noetherian, then $B = A$. If B is not Noetherian, we can sometimes identify the prime ideals of B that are not finitely generated. We have obtained in this way, for each positive integer m, a three-dimensional local unique factorization domain B such that the maximal ideal of B is two-generated, B has precisely m prime ideals of height 2, each prime ideal of B of height 2 is not finitely generated and all the other prime ideals of B are finitely generated. We examine the structure of the map $\operatorname{Spec} A \to \operatorname{Spec} B$ for this example. We also present a generalization of this example to dimension four. This four-dimensional, non-Noetherian local unique factorization domain has exactly one prime ideal Q of height three, and Q is not finitely generated.

1. Introduction. In this paper, we analyze the prime ideal structure of particular non-Noetherian integral domains arising from a general construction developed in our earlier papers [**12–17**]. With this technique, two types of integral domains are constructed:

(1) the intersection of an ideal-adic completion R^* of a Noetherian integral domain R with an appropriate subfield of the total quotient ring of R^* yields an integral domain A as in the abstract, and

(2) an approximation of the domain A by a nested union B of localized polynomial rings has the second form described in the abstract.

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Recently there has been considerable interest in non-Noetherian analogues of Noetherian notions such as the concept of a "regular" ring. Rotthaus and Sega have shown that the rings A and B produced in the general construction are *coherent regular* local rings in the sense that every finitely generated submodule of a free module has a finite free resolution, see [**27**] and Remark 4.3.

We construct in this paper rings that are not Noetherian but are very close to being Noetherian, in that localizations at most prime ideals are Noetherian and most prime ideals are finitely generated; sometimes just one prime ideal is not finitely generated. If a ring has exactly one prime ideal that is not finitely generated, that prime ideal contains all non-finitely generated ideals of the ring.

This article expands upon previous work of the authors where we construct non-Noetherian local domains of dimension $d \geq 3$ with some of these properties [**16, 17**]. In the case of dimensions three and four, we give considerably more detail in this article about these non-Noetherian domains. In particular, we categorize the height 1 primes of the three-dimensional example in terms of the spectral map from A to B.

In Section 2 we describe examples of three-dimensional, non-Noetherian, non-catenary unique factorization domains. Another example of a three-dimensional, non-Noetherian unique factorization domain is given by David [**5**]. The examples given in Examples 2.1 are very close to being Noetherian. We give more details about a specific case where there is precisely one non-finitely generated prime ideal in Example 2.3. In Section 3 we give background results that apply in a more general setting. Our main results are in Sections 4 and 5. Section 4 contains the verification of the properties of the three-dimensional examples.

In Example 5.1 of Section 5, we construct a four-dimensional, non-Noetherian, non-catenary local unique factorization domain B that again is close to being Noetherian. The ring B has exactly one prime ideal Q of height three, and Q is not finitely generated. We leave open the question of whether there exist any prime ideals of B of height 2 that are not finitely generated. Following a suggestion of the referee, we use a "$D+M$" construction to obtain in Example 5.16 a four-dimensional, non-Noetherian, non-catenary local domain C; the maximal ideal of C is principal and is the only non-zero finitely generated prime ideal of C.

All rings we consider are assumed to be commutative with identity. A general reference for our notation and terminology is [**22**]. We abbreviate the *unique factorization domain* by UFD, the *regular local ring* by RLR and the *discrete rank one valuation domain* by DVR.

2. A family of examples in dimension 3. In this section, we construct examples as described in Examples 2.1. In the next section we give a diagram and more detail for a special case of the example with exactly one nonfinitely generated prime ideal.

Examples 2.1. For each positive integer m, we construct an example of a non-Noetherian local integral domain $(B, \mathbf{n})$ such that:

(1) $\dim B = 3$.

(2) The ring B is a UFD that is not catenary.

(3) The maximal ideal $\mathbf{n}$ of B is 2-generated.

(4) The $\mathbf{n}$-adic completion of B is a two-dimensional regular local domain.

(5) For every non-maximal prime ideal P of B, the ring B_P is Noetherian.

(6) The ring B has precisely m prime ideals of height 2.

(7) Each prime ideal of B of height 2 is not finitely generated; all other prime ideals of B are finitely generated.

To establish the existence of the examples in Examples 2.1, we use the following notation. Let k be a field, let x and y be indeterminates over k and set

$$R := k[x,y]_{(x,y)}, \qquad K := k(x,y) \quad \text{and} \quad R^* := k[y]_{(y)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let $\tau \in xk[[x]]$ be transcendental over $k(x)$. For each integer i with $1 \leq i \leq m$, let $p_i \in R \setminus xR$ be such that $p_1R^*, \dots, p_mR^*$ are m prime ideals. For example, if each $p_i \in R \setminus (x,y)^2R$, then each p_iR^* is prime in R^*. In particular, one could take $p_i = y - x^i$. Let $p := p_1 \cdots p_m$. We set $f := p\tau$ and consider the injective R-algebra homomorphism $S := R[f] \hookrightarrow R[\tau] =: T$. In this construction the polynomial rings S

and T have the same field of fractions $K(f) = K(\tau)$. The intersection domain

$$A := R^* \cap K(f) = R^* \cap K(\tau) \tag{2.1.0}$$

is a two-dimensional regular local domain with maximal ideal $(x, y)A$ and the $(x, y)A$-adic completion of A is $k[[x, y]]$, [**29**].

Let $\tau := c_1 x + c_2 x^2 + \cdots + c_i x^i + \cdots \in xk[[x]]$, where the $c_i \in k$ and, for each non-negative integer n, define the "nth endpiece" τ_n of τ by

$$\tau_n := \sum_{i=n+1}^{\infty} c_i x^{i-n} = \frac{\tau - \sum_{i=1}^{n} c_i x^i}{x^n}. \tag{2.1.a}$$

We have the following relation between τ_n and τ_{n+1} for each n:

$$\tau_n = c_{n+1} x + x\tau_{n+1}. \tag{2.1.b}$$

Define $f_n := p\tau_n$, set $U_n = R[f_n] = k[x, y]_{(x,y)}[f_n]$, a three-dimensional polynomial ring over R, and set $B_n = (U_n)_{(x,y,f_n)} = k[x, y, f_n]_{(x,y,f_n)}$, a three-dimensional localized polynomial ring. Similarly, set $U_{\tau n} = R[\tau_n] = k[x, y]_{(x,y)}[\tau_n]$, a three-dimensional polynomial ring containing U_n and $B_{\tau n} = k[x, y, \tau_n]_{(x,y,\tau_n)}$, a localized polynomial ring containing $U_{\tau n}$ and B_n. Let U, B, U_τ and B_τ be the nested union domains defined as follows:

$$U := \bigcup_{n=0}^{\infty} U_n \subseteq U_\tau := \bigcup_{n=0}^{\infty} U_{\tau n};$$

$$B := \bigcup_{n=0}^{\infty} B_n \subseteq B_\tau := \bigcup_{n=0}^{\infty} B_{\tau n} \subseteq A.$$

Remark 2.2. By equation (2.1.b), $k[x, y, f_{n+1}] \subseteq k[x, y, f_n][1/x]$ for each n. Thus, $k[x, y, f][1/x] = k[x, y, f_n][1/x]$ and

$$U[1/x] = R[f][1/x]; \qquad U_\tau[1/x] = R[\tau][1/x]. \tag{2.2.0}$$

Hence, for each n, the ring $B_n[1/x]$ is a localization of $S = U_0 = R[f]$. It follows that $B[1/x]$ is a localization of S and $B[1/x]$ is a localization of B_n. Similarly, $B_\tau[1/x]$ is a localization of $T = R[\tau]$.

We establish in Theorem 4.1 of Section 4 that the rings B of Examples 2.1 have properties (1)–(7) and also some additional properties.

Assuming properties (1)–(7) of Examples 2.1, we describe the ring B of Examples 2.1 in the case where $m = 1$ and $p = p_1 = y$.

Example 2.3. Let the notation be as in Examples 2.1. Thus,

$$R = k[x, y]_{(x,y)}, \quad f = y\tau, \ f_n = y\tau_n,$$
$$B_n = R[y\tau_n]_{(x,y,y\tau_n)}, \quad B = \bigcup_{n=0}^{\infty} B_n.$$

As we will show in Section 4, this ring B has exactly one prime ideal $Q := (y, \{y\tau_n\}_{n=0}^{\infty})B$ of height 2. Moreover, Q is not finitely generated and is the only prime ideal of B that is not finitely generated. We also have $Q = yA \cap B$ and $Q \cap B_n = (y, y\tau_n)B_n$ for each $n \geq 0$.

To identify the ring B up to isomorphism, we include the following details: By equation (2.1.b), we have $\tau_n = c_{n+1}x + x\tau_{n+1}$. Thus, we have

$$f_n = xf_{n+1} + yxc_{n+1}. \tag{2.3.1}$$

The family of equations (2.3.1) uniquely determines B as a nested union of the three-dimensional RLRs $B_n = k[x, y, f_n]_{(x,y,f_n)}$.

We recall the following terminology of [**30**, page 325].

Definition 2.4. If a ring C is a subring of a ring D, a prime ideal P of C is *lost* in D if $PD \cap C \neq P$.

Discussion 2.5. Assuming properties (1)–(7) of Examples 2.1, if q is a height 1 prime of B, then B/q is Noetherian if and only if q is not contained in Q. This is clear since q is principal and Q is the unique prime of B that is not finitely generated and a ring is Noetherian if each prime ideal of the ring is finitely generated; see [**23**, Theorem 3.4].

The height 1 primes q of B may be separated into several types as follows:

Type I. The primes $q \not\subseteq Q$ have the property that B/q is a one-dimensional Noetherian local domain. These primes are contracted from A, i.e., they are not lost in A. To see this, consider $q = gB$ where $g \notin Q$. Then gA is contained in a height 1 prime P of A. Hence, $g \in (P \cap B) \setminus Q$ so $P \cap B \neq Q$. Since $\mathbf{m}_B A = \mathbf{m}_A$, we have $P \cap B \neq \mathbf{m}_B$. Therefore, $P \cap B$ is a height 1 prime containing q, so $q = P \cap B$ and $B_q = A_P$.

There are infinitely many primes q of type I, because every element of $\mathbf{m}_B \setminus Q$ is contained in a prime q of type I. Thus $\mathbf{m}_B \subseteq Q \cup \bigcup \{q \text{ of Type I}\}$. Since $\mathbf{m}_B$ is not the union of finitely many strictly smaller prime ideals, there are infinitely many primes q of Type I.

Type I*. Among the primes of Type I, we label the prime ideal xB as Type I*. The prime ideal xB is special since it is the unique height 1 prime q of B for which R^*/qR^* is not complete. If q is a height 1 prime of B such that $x \notin qR^*$, then $x \notin q$ by Proposition 3.2.4. Thus, R^*/qR^* is complete with respect to the powers of the nonzero principal ideal generated by the image of x mod qR^*. Notice that $R^*/xR^* \cong k[y]_{yk[y]}$.

If q is a height 1 prime of B not of Type I, then $\overline{B} = B/q$ has precisely three prime ideals. These prime ideals form a chain: $(\overline{0}) \subset \overline{Q} \subset \overline{(x,y)B} = \overline{\mathbf{m}_B}$.

Type II. We define the primes of Type II to be the primes $q \subset Q$ such that q has height 1 and is contracted from a prime p of $A = k(x,y,f) \cap R^*$, i.e., q is not lost in A. For example, the prime $y(y+\tau)B$ is of Type II, by Lemma 4.5. For q of this type, B/q is dominated by the one-dimensional Noetherian local domain A/p. Thus, B/q is a non-Noetherian *generalized local ring* in the sense of Cohen; that is, B/q has a unique maximal ideal $\overline{\mathbf{n}}$ that is finitely generated and $\cap_{i=1}^{\infty} \overline{\mathbf{n}}^i = (0)$, [**4**].

For q of Type II, the maximal ideal of B/q is not principal. This follows because a generalized local domain having a principal maximal ideal is a DVR [**24**, (31.5)].

There are infinitely many height 1 primes of Type II, for example, $y(y + x^t\tau)B$ for each $t \in \mathbf{N}$; see Lemma 4.4. For q of Type II, the DVR B_q is birationally dominated by A_p. Hence, $B_q = A_p$ and the ideal $\sqrt{qA} = p \cap yA$.

That each element $y(y + x^t\tau)$ is irreducible, and thus generates a height 1 prime ideal, is shown in greater generality in Lemma 4.4.

Type III. The primes of Type III are the primes $q \subset Q$ such that q has height 1 and is not contracted from A, i.e., q is lost in A. For example, the prime yB and the prime $(y + x^t y\tau)B$ for $t \in \mathbf{N}$ are of Type III; see Lemma 4.5. Since the elements y and $y + x^t y\tau$ are in $\mathbf{m}_B$ and are not in $\mathbf{m}_B^2$ and, since B is a UFD, these elements are necessarily prime. There are infinitely many such prime ideals by Lemma 4.4. For q of Type III, we have $\sqrt{qA} = yA$.

If $q = yB$ or $q = (y + x^t y\tau)B$, then the image $\overline{\mathbf{m}_B}$ of $\mathbf{m}_B$ in B/q is principal. It follows that the intersection of the powers of $\overline{\mathbf{m}_B}$ is Q/q, and so B/q is not a generalized local ring. To see that $\bigcap_{i=1}^{\infty}(\overline{\mathbf{m}_B})^i \neq (0)$, we argue as follows. If P is a principal prime ideal of a ring and P' is a prime ideal properly contained in P, then P' is contained in the intersection of the powers of P; see [**20**, page 7, Example 5].

The picture of Spec (B) is shown in Diagram 2.3.2.

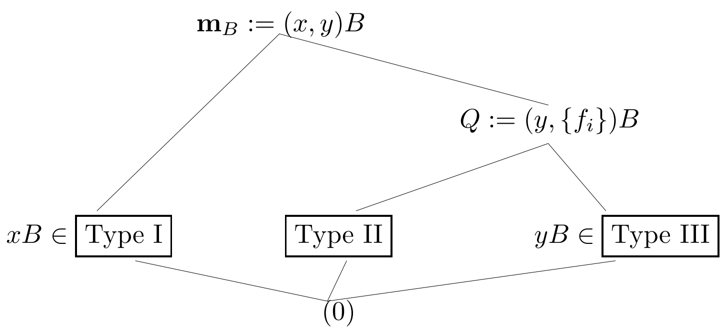

DIAGRAM 2.3.2.

In Remarks 2.6 we examine the height 1 primes of B from a different perspective.

Remarks 2.6. (1) Assume the notation of Example 2.3. If w is a nonzero prime element of B such that $w \notin Q$, then wA is a prime ideal in A and is the unique prime ideal of A lying over wB. To see this, observe that $w \notin yA$ since $w \notin Q = yA \cap B$. It follows that, if $p \in \operatorname{Spec} A$ is a minimal prime of wA, then $y \notin p$. Thus $p \cap B \neq Q$, and so, since we assume the properties of Examples 2.1 hold, $p \cap B$ has height 1. Therefore, $p \cap B = wB$. Hence, the DVR B_{wB} is birationally dominated by A_p, and thus $B_{wB} = A_p$. This implies that p is the unique prime of A lying over wB. We also have $wB_{wB} = pA_p$. Since A is a UFD and p is the unique minimal prime of wA, it follows that $wA = p$. In particular, q is not lost in A; see Definition 2.4.

If q is a height 1 prime of B that is contained in Q, then yA is a minimal prime of q, and q is of Type II or III depending on whether or not qA has other minimal prime divisors.

To see this, observe that if yA is the only prime divisor of qA, then qA has radical yA and $yA \cap B = Q$ implies that Q is the radical of $qA \cap B$. Thus, q is lost in A and q is of Type III.

On the other hand, if there is a minimal prime $p \in \operatorname{Spec} A$ of qA that is different from yA, then y is not in $p \cap B$, and hence $p \cap B \neq Q$. Since Q is the only prime of B of height 2, it follows that $p \cap B$ is a height 2 prime, and thus $p \cap B = q$. Thus, q is not lost in A and q is of Type II.

We observe that, for every Type II prime q, there are exactly two minimal primes of qA. One of these is yA and the other is a height 3 prime p of A such that $p \cap B = q$. For every height 1 prime ideal p of A such that $p \cap B = q$, we have B_q is a DVR that is birationally dominated by A_p, and hence $B_q = A_p$. The uniqueness of B_q implies that there is precisely one such prime ideal p of A.

An example of a height 1 prime ideal q of Type II is $q := (y^2 + y\tau)B$. Then $qA = (y^2 + y\tau)A$ has the two minimal primes yA and $(y + \tau)A$.

(2) The ring B/yB is a rank 2 valuation ring. This can be seen directly or else one may apply [**19**, Proposition 3.5 (iv)]. For other prime elements g of B with $g \in Q$, it need not be true that B/gB is a valuation ring. If g is a prime element contained in $\mathbf{m}_B^2$, then the maximal ideal of B/gB is 2-generated but not principal, and thus B/gB cannot be a valuation ring. For a specific example over the field $\mathbf{q}$, let $g = x^2 + y^2\tau$.

3. Background results. We use results from a general construction developed in our earlier papers. In particular, we use the following theorem in establishing Examples 2.1.

Theorem 3.1 [**14**, Theorem 1.1], [**15**, Theorem 3.2], [**19**]. *Let R be a Noetherian integral domain with field of fractions K. Let a be a* *nonzero nonunit of R, and let R^* be the (a)-adic completion of R. Let h be a positive integer, and let $\tau_1, \dots, \tau_h \in aR[[a]] = aR^*$, abbreviated by $\underline{\tau}$, be algebraically independent over K. Let $U_{\underline{\tau}}$ and $C_{\underline{\tau}}$ be defined as follows*:

$$U_{\underline{\tau}} := \bigcup_{r=0}^{\infty} U_{\underline{\tau}r} \quad \text{and} \quad C_{\underline{\tau}} := \bigcup_{r=0}^{\infty} C_{\underline{\tau}r},$$

where for each integer $r \geq 0$, $U_{\underline{\tau}r} := R[\tau_{1r}, \dots, \tau_{hr}]$, $C_{\underline{\tau}r} := (1 + aU_{\underline{\tau}r})^{-1}U_{\underline{\tau}r}$, and each τ_{ir} is the rth endpiece of τ_i defined as in equation (2.1.a)*. Then the following statements are equivalent*:

(1) *$A_{\underline{\tau}} := K(\underline{\tau}) \cap R^*$ is Noetherian and $A_{\underline{\tau}} = C_{\underline{\tau}}$.*

(2) *$A_{\underline{\tau}}$ is Noetherian and is a localization of a subring of $U_{\underline{\tau}0}[1/a]$.*

(3) *$A_{\underline{\tau}}$ is Noetherian and is a localization of a subring of $U_{\underline{\tau}}[1/a]$.*

(4) *$U_{\underline{\tau}}$ is Noetherian.*

(5) *$C_{\underline{\tau}}$ is Noetherian.*

(6) *$R[\underline{\tau}] \to R^*[1/a]$ is flat.*

(7) *$C_{\underline{\tau}} \to R^*[1/a]$ is flat.*

Propositions 3.2 and 3.4 are used for Examples 2.1.

Proposition 3.2. *With the notation of Theorem* 3.1*, let $U = U_{\underline{\tau}}$, $U_n = U_{\underline{\tau}n}$, $C = C_{\underline{\tau}}$ and $A = A_{\underline{\tau}}$. Then, for all $t \in \mathbf{N}$, we have*

$$\text{(3.2.0)} \quad a^tC = a^tR^* \cap C \quad \textit{and} \quad \frac{R}{a^tR} = \frac{U}{a^tU} = \frac{C}{a^tC} = \frac{A}{a^tA} = \frac{R^*}{a^tR^*}.$$

Moreover,

(1) *The (a)-adic completions of U, C and A are all equal to R^*, and a is in the Jacobson radical of C.*

(2) *The ring $U[1/a] = R[\underline{\tau}][1/a]$, and so $C[1/a]$ is a localization of $R[\underline{\tau}]$.*

(3) *If q is a prime ideal of R, then qU is a prime ideal of U, and either $qC = C$ or qC is a prime ideal of C.*

(4) *Let I be an ideal of C, and let $t \in \mathbf{N}$. If $a^t \in IR^*$, then $a^t \in I$.*

(5) *Let $P \in \operatorname{Spec} C$ with $a \notin P$. Then a is a non-zerodivisor on R^*/PR^*. Thus, $a \notin Q$ for each associated prime Q of the ideal PR^*.*

Since a is in the Jacobson radical of R^*, it follows that PR^* is contained in a nonmaximal prime ideal of R^*.

(6) *If R is local, then R^* and C are both local, we let $\mathbf{m}_R, \mathbf{m}_{R^*}$ and $\mathbf{m}_C$ denote the maximal ideals of R, R^* and C, respectively. In this case,*

• *$\mathbf{m}_C = \mathbf{m}_R C$ and each prime ideal P of C such that $\operatorname{ht}(\mathbf{m}_C/P) = 1$ is contracted from R^*.*

• *If an ideal I of C is such that IR^* is primary for $\mathbf{m}_{R^*}$, then I is primary for $\mathbf{m}_C$.*

Proof. The equalities in equation (3.2.0) follow from [**14**, Proposition 2.2.1], [**15**, Proposition 2.4.3], [**18**, Corollary 6.19], and these imply item (1) about (a)-adic completions. For the second statement, since $C_n = (1 + aU_N)^{-1}U_n$, it follows that $1 + ac$ is a unit of C_n for each $c \in C_n$. Therefore, a is in the Jacobson radical of C_n for each n, and thus a is in the Jacobson radical of C.

For item (2), the relation given in equation (2.1.b) for the case of one variable τ holds also in the case of several variables and implies that $U[1/a] = R[\underline{\tau}][1/a]$. Since C is a localization of U, we have $C[1/a]$ is a localization of $R[\underline{\tau}]$ by Remark 2.2.

For item (3), since each U_n is a polynomial ring over R, qU_n is a prime ideal of U_n, and thus $qU = \bigcup_{n=0}^{\infty} qU_n$ is a prime ideal of U. Since C is a localization of U, qC is either C or a prime ideal of C.

To see item (4), observe that there exist elements $b_1, \dots, b_s \in I$ such that $IR^* = (b_1, \dots, b_s)R^*$. If $a^t \in IR^*$, there exist $\alpha_i \in R^*$ such that

$$a^t = \alpha_1 b_1 + \cdots + \alpha_s b_s.$$

We have $\alpha_i = a_i + a^{t+1}\lambda_i$ for each i, where $a_i \in C$ and $\lambda_i \in R^*$. Thus,

$$\begin{aligned} a^t[1 - a(b_1\lambda_1 + \cdots + b_s\lambda_s)] &= a_1 b_1 + \cdots + a_s b_s \in C \cap a^t R^* \\ &= a^t C. \end{aligned}$$

Therefore, $\gamma := 1 - a(b_1\lambda_1 + \cdots + b_s\lambda_s) \in C$. Thus, $a(b_1\lambda_1 + \cdots + b_s\lambda_s) \in C \cap aR^* = aC$, and so $b_1\lambda_1 + \cdots + b_s\lambda_s \in C$. By item (1), the element a is in the Jacobson radical of C. Hence, γ is invertible in C. Since $\gamma a^t \in (b_1, \dots, b_s)C$, it follows that $a^t \in I$.

For item (5), assume that $P \in \text{Spec,}C$ and $a \notin P$. We have that

$$P \cap aC = aP \quad \text{and so} \quad \frac{P}{aP} = \frac{P}{P \cap aC} \cong \frac{P + aC}{aC}.$$

By equation (3.2.0), C/aC is Noetherian. Hence, the C-module C/aC is finitely generated. Let $g_1, \dots, g_t \in P$ be such that $P = (g_1, \dots, g_t)C + aP$. Then also $PR^* = (g_1, \dots, g_t)R^* + aR^* = (g_1, \dots, g_t)R^*$; the first equality is by equation (3.2.0), and the last equality is by Nakayama's lemma.

Let $\widehat{f} \in R^*$ be such that $a\widehat{f} \in PR^*$. We show that $\widehat{f} \in PR^*$.

Since $\widehat{f} \in R^*$, we have $\widehat{f} := \sum_{i=0}^{\infty} c_i a^i$, where each $c_i \in R$. For each $m > 1$, let $f_m := \sum_{i=0}^{m} c_i a^i$, the first $m+1$ terms of this expansion of $\widehat{f}$. Then $f_m \in R \subseteq C$, and there exists an element $\widehat{h}_1 \in R^*$ so that

$$\widehat{f} = f_m + a^{m+1}\widehat{h_1}.$$

Since $a\widehat{f} \in PR^*$, we have

$$a\widehat{f} = \widehat{a_1}g_1 + \cdots + \widehat{a_t}g_t,$$

for some $\widehat{a_i} \in R^*$. The $\widehat{a_i}$ have power series expansions in a over R, and thus there exist elements $a_{im} \in R$ such that $\widehat{a_i} - a_{im} \in a^{m+1}R^*$. Thus,

$$a\widehat{f} = a_{1m}g_1 + \cdots + a_{tm}g_t + a^{m+1}\widehat{h_2},$$

where $\widehat{h}_2 \in R^*$, and

$$af_m = a_{1m}g_1 + \cdots + a_{tm}g_t + a^{m+1}\widehat{h_3},$$

where $\widehat{h_3} = \widehat{h_2} - a\widehat{h_1} \in R^*$. Since the g_i are in C, we have $a^{m+1}\widehat{h_3} \in a^{m+1}R^* \cap C = a^{m+1}C$, the last equality by equation (3.2.0). Therefore $\widehat{h_3} \in C$. Rearranging the last set-off equation above, we obtain

$$a(f_m - a^m\widehat{h_3}) = a_{1m}g_1 + \cdots + a_{tm}g_t \in P.$$

Since $a \notin P$, we have $f_m - a^m\widehat{h_3} \in P$. It follows that $\widehat{f} \in P + a^mR^* \subseteq PR^* + a^mR^*$, for each $m > 1$. Hence, we have that $\widehat{f} \in PR^*$, as desired.

For item (6), if R is local, then C is local since $C/aC = R/aR$ and a is in the Jacobson radical of C. Hence, also $\mathbf{m}_C = \mathbf{m}_R C$. If $a \notin P$, then item (4) implies that no power of a is in PR^*. Hence, PR^* is contained in a prime ideal Q of R^* that does not meet the multiplicatively closed set $\{a^n\}_{n=1}^{\infty}$. Hence, $P \subseteq Q \cap C \subsetneq \mathbf{m}_C$. Since $\operatorname{ht}(\mathbf{m}_C/P) = 1$, we have $P = Q \cap C$, so P is contracted from R^*. If $a \in P$, then (3.2.0) implies that PR^* is a prime ideal of R^* and $P = PR^* \cap C$.

For the second part of item (6), if IR^* is $\mathbf{m}$-primary then $a^t \in IR^*$. Thus, $a^t \in I$ by item (4). By equation (3.2.0), $C/a^tC = R^*/a^tR^*$ and so I/a^tC is primary for the maximal ideal of C/a^tC. Therefore I is primary for the maximal ideal of C. □

The definition of B as a directed union as given in Examples 2.1 and later in this article is not the same as the definition of C as a directed union given in Theorem 3.1 and Propositions 3.2 and 3.4. However, the ring B *is* the same as the ring C for R as in Examples 2.1. We show this more generally in Remark 3.3.1 for R a Noetherian local domain.

Remarks 3.3. (1) Assume the setting of Theorem 3.1 with the additional assumption that R is a Noetherian local domain with maximal ideal $\mathbf{m}$. We observe in this case that $C_{\underline{\tau}}$ as defined in Theorem 3.1 is the directed union of the localized polynomial rings $B_r := (U_{\underline{\tau}r})_{P_r}$, where $P_r := (\mathbf{m}, \tau_{1r}, \dots, \tau_{hr})U_{\underline{\tau}r}$.

Proof. It is clear that $B_r \subseteq B_{r+1}$, and $P_r \cap (1 + aU_{\underline{\tau}r}) = \varnothing$ implies that $C_{\underline{\tau}r} \subseteq B_r$. We show that $B_r \subseteq C_{\underline{\tau}r+1}$. Let $u/d \in B_r$, where $u \in U_{\underline{\tau}r}$ and $d \in U_{\underline{\tau}r} \setminus P_r$. Then $d = d_0 + \sum_{i=1}^{h} \tau_{ir} b_i$, where $d_0 \in R$ and each $b_i \in U_{\underline{\tau}r}$. Notice that $d_0 \notin \mathbf{m}$ since $d \notin P_r$, and so $d_0^{-1} \in R$. Thus, $dd_0^{-1} = 1 + \sum_{i=1}^{h} \tau_{ir} b_i d_0^{-1} \in (1 + aU_{\underline{\tau}r+1})$ since each $\tau_{ir} \in aU_{\underline{\tau}r+1}$ by (2.1.b). Hence $u/d = (ud_0)/(dd_0) \in C_{\underline{\tau}r+1}$, and so $C_{\underline{\tau}} = \bigcup_{r=1}^{\infty} C_{\underline{\tau}r} = \bigcup_{r=1}^{\infty} B_r$. □

(2) With the notation of Examples 2.1, where R is the localized polynomial ring $k[x,y]_{(x,y)}$ over a field k, $R^* = k[y]_{(y)}[[x]]$ is the (x)-adic completion of R and $\tau \in xR^*$ is transcendental over K, the proof in item (1) shows that $C_\tau = \bigcup B_r$, where $B_r = (U_r)_{P_r}$, $U_r = k[x,y,\tau_r]$ and $P_r = (x,y,\tau_r)U_r$. A similar remark applies to C_f with appropriate modifications to B_r, U_r and P_r.

(3) Thus, the results of Theorem 3.1 and Propositions 3.2 and 3.4 hold for the ring B of Examples 2.1 and also the examples later in this article. □

Proposition 3.4. *Assume the notation of Theorem* 3.1 *and set* $C := C_{\underline{\tau}}$.

(1) *If* R *is a* UFD *and* a *is a prime element of* R, *then* aC *is a prime ideal,* $C[1/a]$ *is a Noetherian* UFD *and* C *is a* UFD.

(2) *If in addition* R *is regular, then* $C[1/a]$ *is a regular Noetherian UFD.*

Proof. By Proposition 3.2.3, aC is a prime ideal. Since R is a Noetherian UFD and $S = R[\tau_1, \dots, \tau_h]$ is a polynomial ring extension of R, it follows that S is a Noetherian UFD. By Remark 2.2, the ring $C[1/a]$ is a localization of S, and thus a Noetherian UFD; moreover, $C[1/a]$ is regular if R is. The (a)-adic completion of C is R^* by Proposition 3.2.1. Since R^* is Noetherian and a is in the Jacobson radical of R^*, see [**22**, Theorem 8.2 (i)], it follows that $\bigcap_{n=1}^{\infty} a^n R^* = (0)$. Thus, $\bigcap_{n=1}^{\infty} a^n C = (0)$ by equation 3.2.0. It follows that C_{aC} is Noetherian [**24**, (31.5)], and hence C_{aC} is a DVR. We use the following fact:

Fact 3.5. *If* D *is an integral domain and* c *is a nonzero element of* D *such that* cD *is a prime ideal, then* $D = D[1/c] \cap D_{cD}$.

Proof. Let $\beta \in D[1/c] \cap D_{cD}$. Then $\beta = b/c^n = b_1/s$ for some $b, b_1 \in D$, $s \in D \setminus cD$ and integer $n \geq 0$. If $n > 0$, we have $sb = c^n b_1 \implies b \in cD$. Thus, we may reduce to the case where $n = 0$; it follows that $D = D[1/c] \cap D_{cD}$. This proves the fact. □

We return to the proof of Proposition 3.4. By Fact 3.5, $C = C[1/a] \cap C_{aC}$, and therefore C is a Krull domain. Since $C[1/a]$ is a UFD and C is a Krull domain, it follows that C is a UFD [**28**, page 21]. □

In order to examine more closely the prime ideal structure of the ring B of Examples 2.1, we establish in Proposition 3.6 some properties of its overring A and of the map $\operatorname{Spec} A \to \operatorname{Spec} B$.

Proposition 3.6. *With the notation of Examples* 2.1, *we have*:

(1) $A = B_\tau$ *and* $A[1/x]$ *is a localization of* $R[\tau]$.

(2) *For $P \in \operatorname{Spec} A$ with $x \notin P$, the following are equivalent*: (a) $A_P = B_{P\cap B}$, (b) $\tau \in B_{P\cap B}$, (c) $p \notin P$.

Proof. For item (1), to see that $A = B_\tau$, we first show that the map

$$\varphi : R[\tau] \longrightarrow R^*[1/x] = k[y]_{(y)}[[x]][1/x]$$

is flat. By [**22**, page 46], the field of fractions L of $k[x]_{(x)}[\tau]$ is flat over $k[x]_{(x)}[\tau]$ since it is a localization. The field $k[[x]][1/x]$ contains L and is flat over L since it has a vector space basis over L. Thus, the map $\psi : k[x]_{(x)}[\tau] \to k[[x]][1/x]$ is flat. We use the following:

Fact 3.7. *Let C be a commutative ring, let D, E and F be C-algebras, and let $\psi : D \to E$ be a flat C-algebra homomorphism; equivalently, E is a flat D-module via the C-algebra homomorphism ψ. Then $\psi \otimes_C 1_F : D \otimes_C F \to E \otimes_C F$ is a flat C-algebra homomorphism. Equivalently, $E \otimes_C F$ is a flat $D \otimes_C F$-module via the C-algebra homomorphism $\psi \otimes_C 1_F$.*

Proof. Since E is a flat D-module, $E \otimes_D (D \otimes_C F)$ is a flat $(D \otimes_C F)$-module by [**22**, page 46, Change of coefficient ring]. The fact follows because $E \otimes_D (D \otimes_C F) = E \otimes_C F$. □

We return to the proof of Proposition 3.6. We have $R = k[x,y]_{(x,y)}$. Consider the following composition:

$$R[\tau] = k[x]_{(x)}[\tau] \otimes_{k[x]_{(x)}} R \xrightarrow{\alpha} k[[x]][1/x] \otimes_{k[x]_{(x)}} R \overset{\gamma}{\hookrightarrow} k[[x]][y]_{(x,y)}[1/x].$$

By Fact 3.7, the map α is flat. The map γ is a localization. Hence, the composition $\gamma \circ \alpha$ is flat. The extension $k[[x]][y]_{(x,y)} \to k[y]_{(y)}[[x]]$ is flat since it is the map taking a Noetherian ring to an ideal-adic completion [**23**, Corollary 1, page 170]. Therefore, the localization map $\beta : k[[x]][y]_{(x,y)}[1/x] \to k[y]_{(y)}[[x]][1/x]$ is flat. Thus, the map $\varphi = \beta \circ \gamma \circ \alpha : R[\tau] \to k[y]_{(y)}[[x]][1/x]$ is flat. Theorem 3.1 implies that $A = B_\tau$. By Remark 2.2, the ring $A[1/x]$ is a localization of $R[\tau]$.

For item (2), since $\tau \in A$, (a) $\Longrightarrow$ (b) is clear.

For (b) $\Longrightarrow$ (c), we show that $p \in P \implies \tau \notin B_{P\cap B}$. By Remark 2.2, $B[1/x]$ is a localization of $R[f]$. Since $x \notin P$, the ring $B_{P\cap B}$ is a localization of $R[f]$, and thus $B_{P\cap B} = R[f]_{P\cap R[f]}$. The assumption that $p \in P$ implies that some $p_i \in P$, and so $R[f]_{P\cap R[f]}$ is contained in the DVR $V := R[f]_{p_i R[f]}$. Since $R[f]$ is a polynomial ring over R, f is a unit in V. Hence, $\tau = f/p \notin V$, and thus $\tau \notin R[f]_{P\cap R[f]}$. This shows that (b) $\Longrightarrow$ (c).

For (c) $\Longrightarrow$ (a), notice that $f = p\tau$ implies that $R[f][1/xp] = R[\tau][1/xp]$. By item (1), $A[1/x]$ is a localization of $R[\tau][1/x]$ and so $A[1/xp]$ is a localization of $R[\tau][1/xp] = R[f][1/xp]$. Thus, $A[1/xp]$ is a localization of $R[f]$. By Remark 2.2, $B[1/x]$ is a localization of $R[f]$. Since $xp \notin P$ and $x \notin P \cap B$, we have that A_P and $B_{P\cap B}$ are both localizations of $R[f]$. Thus, we have

$$A_P = R[f]_{PA_P\cap R[f]} = R[f]_{(P\cap B)B_{P\cap B}\cap R[f]} = B_{P\cap B}.$$

This completes the proof of Proposition 3.6. □

We observe in Proposition 3.8 that, over a perfect field k of characteristic $p > 0$ (so that $k = k^{1/p}$), a one-dimensional form of the construction yields a DVR that is not a Nagata ring, and thus not excellent; see [**22**, page 264], [**23**, Theorem 78, Definition 34.8].

Proposition 3.8. *Let k be a perfect field of characteristic $p > 0$, and let $\tau \in xk[[x]]$ be such that x and τ are algebraically independent over k. Let $V := k(x, \tau) \cap k[[x]]$. Then V is a* DVR *for which the integral closure $\overline{V}$ of V in the purely inseparable field extension $k(x^{1/p}, \tau^{1/p})$ is not a finitely generated V-module. Hence, V is not a Nagata ring.*

Proof. It is clear that V is a DVR with maximal ideal xV. Since x and τ are algebraically independent over k, $[k(x^{1/p}, \tau^{1/p}) : k(x, \tau)] = p^2$. Let W denote the integral closure of V in the field extension $k(x^{1/p}, \tau)$ of degree p over $k(x, \tau)$. Notice that

$$W = k(x^{1/p}, \tau) \cap k[[x^{1/p}]] \quad \text{and} \quad \overline{V} = k(x^{1/p}, \tau^{1/p}) \cap k[[x^{1/p}]]$$

are both DVRs having residue field k and maximal ideal generated by $x^{1/p}$. Thus, $\overline{V} = W + x^{1/p}\overline{V}$. If $\overline{V}$ were a finitely generated W-module, then by Nakayama's lemma, it would follow that $W = \overline{V}$. This is impossible because $\overline{V}$ is not birational over W. It follows that $\overline{V}$ is not a finitely generated V-module, and hence V is not a Nagata ring. □

4. Verification of the three-dimensional examples. In Theorem 4.2, we record and establish the properties asserted in Examples 2.1 and other properties of the ring B.

Theorem 4.1. *With the notation of Example* 2.1, *let* $Q_i := p_iR^* \cap B$, *for each* i *with* $1 \le i \le m$. *We have*:

(1) *The ring* B *is a three-dimensional, non-Noetherian local* UFD *with maximal ideal* $\mathbf{n} = (x, y)B$, *and the* $\mathbf{n}$-*adic completion of* B *is the two-dimensional regular local ring* $k[[x, y]]$.

(2) *The rings* $B[1/x]$ *and* B_P, *for each nonmaximal prime ideal* P *of* B, *are regular Noetherian* UFD*s, and the ring* B/xB *is a* DVR.

(3) *The ring* A *is a two-dimensional regular local domain with maximal ideal* $\mathbf{m}_A := (x, y)A$, *and* $A = B_\tau$. *The ring* A *is excellent if the field* k *has characteristic zero. If* k *is a perfect field of characteristic* p, *then* A *is not excellent.*

(4) *The ideal* $\mathbf{m}_A$ *is the only prime ideal of* A *lying over* $\mathbf{n}$.

(5) *The ideals* Q_i *are the only height* 2 *prime ideals of* B.

(6) *The ideals* Q_i *are not finitely generated and they are the only nonfinitely generated prime ideals of* B.

(7) *The ring* B *has saturated chains of prime ideals from* (0) *to* $\mathbf{n}$ *of length two and of length three, and hence is not catenary.*

Proof. For item (1), since B is a directed union of three-dimensional regular local domains, $\dim B \le 3$. By Proposition 3.2, B is local with maximal ideal $(x, y)B$, xB and p_iB are prime ideals and the (x)-adic completion of B is equal to R^*, the (x)-adic completion of R. Thus, the $\mathbf{n}$-adic completion of B is $k[[x, y]]$. Since each $Q_i = \bigcup_{i=1}^{\infty} Q_{in}$, where $Q_{in} = p_iR^* \cap B_n$, we see that each Q_i is a prime ideal of B with $p_i, f \in Q_i$ and $x \notin Q_i$. Since $p_iB = \bigcup p_iB_n$, we have $f \notin p_iB$. Thus,

$$(0) \subsetneq p_iB \subsetneq Q_i \subsetneq (x, y)B.$$

This chain of prime ideals of length at least three yields that $\dim B = 3$ and that the height of each Q_i is 2.

The map $S = R[f] \to R^*[1/x]$ is not flat since flat extensions satisfy the going-down property [**22**, Theorem 9.5, page 68], and $p_iR^*[1/x]$ is a height 1 prime whereas $p_iR^*[1/x] \cap S = (p_i, f)S$ is a height 2 prime. Therefore, Theorem 3.1 implies that the ring B is not Noetherian. By Proposition 3.4, B is a UFD, and so item (1) holds.

For item (2), by equation (3.2.0), B/xB is a DVR. By Proposition 3.4, $B[1/x]$ is a regular Noetherian UFD. If $x \in P$ and P is nonmaximal, then, again by equation (3.2.0), $P = xB$. If $x \notin P$, the ring B_P is a localization of $B[1/x]$ and so is a regular Noetherian UFD. Thus, item (2) holds.

The statement in item (3) that A is a two-dimensional regular local domain with maximal ideal $\mathbf{m}_A = (x, y)A$ follows by a result of Valabrega [**29**], see equation (2.1.0). By Proposition 3.6.1, we have $A = B_\tau$. The ring $V := k[[x]] \cap k(x, \tau)$ is a DVR by [**24**, (33.7)]. If the field k has characteristic zero, then V is excellent by [**10**, Chapter IV], [**26**, Folgerung 3]. Since A is a localization of $V[y]$, it follows that A is also excellent if k has characteristic zero.

Assume the field k is perfect with characteristic $p > 0$. By Proposition 3.8, the ring V is not excellent. Since $A = V[y]_{(x,y)}$, the ring V is a homomorphic image of A. Since excellence is preserved under homomorphic image, the ring A is not excellent. This completes the proof of item (3).

By equation (3.2.0), $B/xB = A/xA = R^*/xR^*$. Hence, $\mathbf{m}_A = (x, y)A$ is the unique prime of A lying over $\mathbf{n} = (x, y)B$. Thus, item (4) holds and for item (5) we see that x is not in any height 2 prime ideal of B.

To complete the proof of item (5), it remains to consider $P \in \operatorname{Spec} B$ with $x \notin P$ and $\operatorname{ht} P > 1$. By Proposition 3.2.4, we have $x^n \notin PR^*$ for each $n \in \mathbf{N}$. Thus, $\operatorname{ht}(PR^*) \leq 1$. Since $A \hookrightarrow R^*$ is faithfully flat, $\operatorname{ht}(PA) \leq 1$. Let P' be a height 1 prime ideal of A containing PA. Since $\dim B = 3$, $\operatorname{ht} P > 1$ and $x \notin P' \cap B$, it follows that $P = P' \cap B$. If $p \notin P$, then Proposition 3.6 implies that $A_{P'} = B_P$. Since P' is a height 1 prime ideal of A, it follows that P is a height 1 prime ideal of B.

Now suppose that $p_i \in P$ for some i. Then p_iR^* is a height 1 prime ideal contained in PR^* and so $p_iR^* = PR^*$. Hence, P is squeezed between p_iB and $Q_i = p_iR^* \cap B \neq (x, y)B$. Since $\dim B = 3$, either P has height 1 or $P = Q_i$ for some i. This completes the proof of item (5).

For item (6), we show that each Q_i is not finitely generated by showing for each $n \geq 0$, that $f_{n+1} \notin (p_i, f_n)B$. By equation (2.1.b), we have $\tau_n = c_{n+1}x + x\tau_{n+1}$, and hence $f_n = xf_{n+1} + pxc_{n+1}$. Assume that $f_{n+1} \in (p_i, f_n)B$. Then

$$(p_i, f_n)B = (p_i, xf_{n+1} + pxc_{n+1})B \implies f_{n+1} = ap_i + b(xf_{n+1} + pxc_{n+1}),$$

for some $a, b \in B$. Thus, $f_{n+1}(1 - xb) \in p_iB$. Since $1 - xb$ is a unit of B, it follows that $f_{n+1} \in p_iB$, and thus $f_{n+1} \in p_iB_{n+r}$, for some $r \geq 1$. The relations $f_t = xf_{t+1} + pxc_{t+1}$, for each $t \in \mathbf{N}$, imply that

$$\begin{aligned} f_{n+1} &= xf_{n+2} + pxc_{n+2} = x^2 f_{n+3} + px^2 c_{n+3} + pxc_{n+2} = \cdots \\ &= x^{r-1} f_{n+r} + p\alpha, \end{aligned}$$

where $\alpha \in R$. Thus, $x^{r-1}f_{n+r} \in (p, f_{n+1})B_{n+r}$. Since $f_{n+1} \in p_iB_{n+r}$, we have $x^{r-1}f_{n+r} \in p_iB_{n+r}$. This implies $f_{n+r} \in p_iB_{n+r}$, a contradiction because the ideal $(p_i, f_{n+r})B_{n+r}$ has height 2. We conclude that Q_i is not finitely generated.

Since B is a UFD, the height 1 primes of B are principal, and since the maximal ideal of B is two-generated, every nonfinitely generated prime ideal of B has height 2 and thus is in the set $\{Q_1, \dots, Q_m\}$. This completes the proof of item (6).

For item (7), the chain $(0) \subset xB \subset (x, y)B = \mathbf{m}_B$ is saturated and has length two, while the chain $(0) \subset p_1B \subset Q_1 \subset \mathbf{m}_B$ is saturated and has length three. □

Remark 4.2. With the notation of Examples 2.1 and Theorem 4.1, we obtain the following additional details about the prime ideals of B.

(1) If $P \in \operatorname{Spec} B$ is nonzero and nonmaximal, then $\operatorname{ht}(PR^*) = 1$ and $\operatorname{ht}(PA) = 1$. Thus, every nonmaximal prime of B is contained in a nonmaximal prime of A.

(2) If $P \in \operatorname{Spec} B$ is such that $P \cap R = (0)$, then $\operatorname{ht}(P) \leq 1$ and P is principal.

(3) If $P \in \operatorname{Spec} B$, $\operatorname{ht} P = 1$ and $P \cap R \neq 0$, then $P = (P \cap R)B$.

(4) Let p_i be one of the prime factors of p. Then p_iB is prime in B. Moreover, the ideals p_iB and $Q_i := p_iA \cap B = (p_i, f_1, f_2, \dots)B$ are the

only nonmaximal prime ideals of B that contain p_i. Thus, they are the only prime ideals of B that lie over p_iR in R.

(5) The constructed ring B has Noetherian spectrum.

Proof. For the proof of item (1), if $P = Q_i$ for some i, then $PR^* \subseteq p_iR^*$ and $\operatorname{ht} PR^* = 1$. If P is not one of the Q_i, then by Theorem 4.1, P is a principal height 1 prime and $\operatorname{ht} PR^* = 1$. Since A is Noetherian and local, R^* is faithfully flat over A, and hence $\operatorname{ht} PA = 1$.

The proof of item (1) is contained in the proof of item (5) of Theorem 4.1.

For item (2), $\operatorname{ht} P \leq 1$, because the field of fractions $K(f)$ of B has transcendence degree one over the field of fractions K of R. Since B is a UFD, P is principal.

For item (3), if $x \in P$, then $P = xB$ and the statement is clear. Assume $x \notin P$. By Remark 2.2, $B[1/x]$ is a localization of B_n, and so $\operatorname{ht}(P \cap B_n) = 1$ for all integers $n \geq 0$. Thus, $(P \cap R)B_n = P \cap B_n$, for each n, and so $P = (P \cap R)B$.

For item (4), each p_iB is prime by Proposition 3.2.3. By Theorem 4.1, $\dim B = 3$ and the Q_i are the only height 2 primes of B. Since for $i \neq j$, the ideal $p_iR + p_jR$ is $\mathbf{m}_R$-primary, it follows that $p_iB + p_jB$ is $\mathbf{n}$-primary, and hence p_iB and Q_i are the only nonmaximal prime ideals of B that contain p_i.

Item (5) follows from Theorem 4.1, since the prime spectrum is Noetherian if it satisfies the ascending chain condition and if, for each finite set in the spectrum, there are only finitely many points minimal with respect to containing all of them. Thus, the proof is complete. □

Remark 4.3. Rotthaus and Sega proved that the rings B of Theorems 3.1, 4.1 and 5.8 are coherent and regular in the sense that every finitely generated submodule of a free module has a finite free resolution [**27**]. For the ring $B = \bigcup_{n=1}^{\infty} B_n$ of these constructions, it is stated in [**27**] that $B_n[1/x] = B_{n+k}[1/x] = B[1/x]$ and that B_{n+k} is generated over B_n by a single element for all positive integers n and k. This is not correct for the local rings B_n. However, if, instead of using the localized polynomial rings B_n and their union B of the construction for these theorems, one uses the underlying polynomial rings U_n and their union U defined in Theorem 3.1, then one does have that $U_n[1/x] = U_{n+k}[1/x] = U[1/x]$ and that U_{n+k} is generated over U_n by a single element for all positive integers n and k.

We use the following lemma.

Lemma 4.4. *Let the notation be as in Examples* 2.1 *and Theorem* 4.1.

(1) *For every element $c \in \mathbf{m}_R \setminus xR$ and every $t \in \mathbf{N}$, the element $c + x^t f$ is a prime element of the* UFD B.

(2) *For every fixed element $c \in \mathbf{m}_R \setminus xR$, the set $\{c + x^t f\}_{t \in \mathbf{N}}$ consists of infinitely many nonassociate prime elements of B, and so there exist infinitely many distinct height* 1 *primes of B of the form $(c + x^t f)B$.*

Proof. For the first item, since $f = p\tau$, equation (2.1.b) implies that

$$f_r = pc_{r+1}x + xf_{r+1}.$$

In $B_0 = k[x, y, f]_{(x,y,f)}$, the polynomial $c + x^t f$ is linear in the variable $f = f_0$, and the coefficient x^t of f is relatively prime to the constant term c. Thus, $c + x^t f$ is irreducible in B_0. Since $f = f_0 = pc_1x + xf_1$ in $B_1 = k[x, y, f_1]_{(x,y,f_1)}$, the polynomial $c + x^t f = c + x^t pc_1 x + x^{t+1} f_1$ is linear in the variable f_1 and the coefficient x^{t+1} of f_1 is relatively prime to the constant term c. Thus, $c + x^t f$ is irreducible in B_1. To see that this pattern continues, observe that, in B_2, we have

$$\begin{aligned} f = \ & pc_1x + xf_1 = pc_1x + pc_2x^2 + x^2 f_2 \\ & \qquad \Longrightarrow\ c + x^t f = c + pc_1x^{t+1} + pc_2x^{t+2} + x^{t+2} f_2, \end{aligned}$$

a linear polynomial in the variable f_2. Thus, $c + x^t f$ is irreducible in B_2 and a similar argument shows that $c + x^t f$ is irreducible in B_r for each positive integer r. Therefore, for each $t \in \mathbf{N}$, the element $c + x^t f$ is prime in B.

For item (2), observe that $(c + x^t f)B \neq (c + x^m f)B$, for positive integers $t > m$. If $(c + x^t f)B = (c + x^m f)B := q$, a height 1 prime ideal of B, then

$$(x^t - x^m)f = x^m(x^{t-m} - 1)f \in q.$$

Since $c \notin xB$, we have $q \neq xB$. Thus, $x^m \notin q$. Also $x^{t-m} - 1$ is a unit of B. It follows that $f \in q$, and thus $(c, f)B \subseteq q$.

By Remark 2.2, $B[1/x]$ is a localization of $R[f] = S$, and $x \notin q$ implies that $B_q = S_{q \cap S}$. This is a contradiction since the ideal $(c, f)S$ has height 2. Thus, there exist infinitely many distinct height 1 primes of the form $(c + x^t f)B$. $\square$

Lemma 4.5 is useful for giving a more precise description of $\operatorname{Spec} B$ for B as in Examples 2.1. For each nonempty finite subset H of $\{Q_1, \dots, Q_m\}$, we show there exist infinitely many height 1 prime ideals contained in each $Q_i \in H$, but not contained in Q_j if $Q_j \notin H$.

Lemma 4.5. *Let the notation be as in Theorem* 4.1*. Let G be a nonempty subset of $\{1, \dots, m\}$, and let $H = \{Q_i \mid i \in G\}$. Let $p_G = \prod\{p_i \mid i \in G\}$. Then, for each $t \in \mathbf{N}$, we have*

(1) *$(p_G + x^t f)B$ is a prime ideal of B that is lost in A.*

(2) *$(p_G^2 + x^t f)B$ is a prime ideal of B that is not lost in A.*

The sets $\{(p_G + x^t f)B\}_{t \in \mathbf{N}}$ and $\{(p_G^2 + x^t f)B\}_{t \in \mathbf{N}}$ are both infinite. Moreover, the prime ideals in both items (1) *and* (2) *are contained in each Q_i such that $Q_i \in H$, but are not contained in Q_j if $Q_j \notin H$.*

Proof. For item (1), we have

$$(p_G + x^t f)A \cap B = p_G\Big(1 + x^t \tau \prod_{j \notin G} p_j\Big)A \cap B = p_G A \cap B = \bigcap_{i \in G} Q_i. \tag{4.5.1}$$

Thus, each prime ideal of B of the form $(p_G + x^t f)B$ is lost in A and R^*. By the second item of Lemma 4.4, there exist infinitely many height 1 primes $(p_G + x^t f)B$ of B that are lost in A and R^*.

For item (2), we have

$$(p_G^2 + x^t f)A \cap B = \Big(p_G^2 + x^t p_G\Big(\prod_{j \notin G} p_j\Big)\tau\Big)A \cap B \tag{4.5.2}$$

$$
\begin{aligned}
&= p_G\Big(p_G + x^t\Big(\prod_{j\notin G} p_j\Big)\tau\Big)A \cap B \\
&\subsetneq p_G A \cap B = \bigcap_{i\in G} Q_i.
\end{aligned}
$$

The strict inclusion is because $p_G + x^t(\prod_{j\notin G} p_j)\tau \in \mathbf{m}_A$. This implies that prime ideals of B of form $(p_G^2 + x^t f)B$ are not lost. By Lemma 4.4, there are infinitely many distinct prime ideals of that form.

The "moreover" statement for the prime ideals in item (1) follows from equation (4.5.1). Equation (4.5.2) implies that the prime ideals in item (2) are contained in each $Q_i \in H$. For $j \notin G$, if $p_G^2 + x^t f \in Q_j$, then $p_j + x^t f \in Q_j$ implies that $p_G^2 - p_j \in Q_j$ by subtraction. Since $p_j \in Q_j$, this would imply that $p_G^2 \in Q_j$, a contradiction. This completes the proof of Lemma 4.5. $\square$

Remark 4.6. With the notation of Examples 2.1 consider the birational inclusion $B \hookrightarrow A$ and the faithfully flat map $A \hookrightarrow R^*$. The following statements hold concerning the inclusion maps $R \hookrightarrow B \hookrightarrow A \hookrightarrow R^*$, and the associated maps in the opposite direction of their spectra.

(1) The map $\operatorname{Spec} R^* \to \operatorname{Spec} A$ is surjective, while the maps $\operatorname{Spec} R^* \to \operatorname{Spec} B$ and $\operatorname{Spec} A \to \operatorname{Spec} B$ are not surjective. All the induced maps to $\operatorname{Spec} R$ are surjective since the map $\operatorname{Spec} R^* \to \operatorname{Spec} R$ is surjective.

(2) By Lemma 4.5, each of the prime ideals Q_i of B contains infinitely many height 1 primes of B that are the contraction of prime ideals of A and infinitely many that are not.

Since an ideal contained in a finite union of prime ideals is contained in one of the prime ideals by [**2**, page 8, Proposition 1.11], there are infinitely many non-associate prime elements of the UFD B that are not contained in the union $\bigcup_{i=1}^m Q_i$. We observe that, for each prime element q of B with $q \notin \bigcup_{i=1}^m Q_i$, the ideal qA is contained in a height 1 prime $\mathbf{q}$ of A, and $\mathbf{q} \cap B$ is properly contained in $\mathbf{m}_B$ since $\mathbf{m}_A$ is the unique prime ideal of A lying over $\mathbf{m}_B$. Hence $\mathbf{q} \cap B = qB$. Thus, each qB is contracted from A and R^*.

In the four-dimensional example B of Theorem 5.8, each height 1 prime of B is contracted from R^*, but there are infinitely many height 2

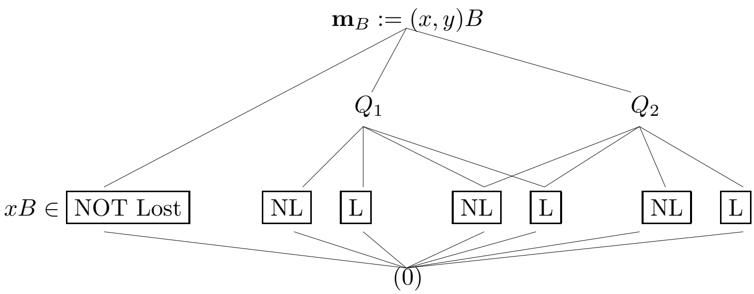

DIAGRAM 4.6.0.

primes of B that are lost in R^*, i.e., are not contracted from R^*; see Section 5.

(3) Among the prime ideals of the domain B of Example 2.1 that are not contracted from A are the p_iB. Since $p_iA \cap B = Q_i$ properly contains p_iB, the prime ideal p_iB is lost in A.

(4) Since x and y generate the maximal ideals of B and A, and since B is integrally closed, a version of Zariski's main theorem [**6, 25**], implies that A is not essentially finitely generated as a B-algebra.

Using the information above, we display below a picture of Spec (B) in the case $m = 2$.

Comments on Diagram 4.6.0. Here we have $Q_1 = p_1R^* \cap B$ and $Q_2 = p_2R^* \cap B$, and each box represents an infinite set of height 1 prime ideals. We label a box "NL" for "not lost" and "L" for "lost." An argument similar to that given for the Type I primes in Example 2.3 shows that the height 1 primes q such that $q \notin Q_1 \cup Q_2$ are not lost. That the other boxes are infinite follows from Lemma 4.5.

5. A four-dimensional prime spectrum. In Example 5.1, we present a four-dimensional example analogous to Example 2.3.

Example 5.1. Let k be a field, let x, y and z indeterminates over k. Set

$$R := k[x,y,z]_{(x,y,z)} \quad \text{and} \quad R^* := k[y,z]_{(y,z)}[[x]],$$

and let $\mathbf{m}_R$ and $\mathbf{m}_{R^*}$ denote the maximal ideals of R and R^*, respectively. The power series ring R^* is the xR-adic completion of R.

Consider τ and σ in $xk[[x]]$

$$\tau := \sum_{n=1}^{\infty} c_n x^n \quad \text{and} \quad \sigma := \sum_{n=1}^{\infty} d_n x^n,$$

where the c_n and d_n are in k and τ and σ are algebraically independent over $k(x)$. Define

$$f := y\tau + z\sigma \quad \text{and} \quad A := A_f = R^* \cap k(x, y, z, f),$$

that is, A is the intersection domain associated with f. For each integer $n \geq 0$, let τ_n and σ_n be the nth endpieces of τ and σ as in equation (2.1.a). Then the nth endpiece of f is $f_n = y\tau_n + z\sigma_n$. As in equation (2.1.b), we have

$$\tau_n = x\tau_{n+1} + xc_{n+1} \quad \text{and} \quad \sigma_n = x\sigma_{n+1} + xd_{n+1},$$

where c_{n+1} and d_{n+1} are in the field k. Therefore,

$$\begin{aligned} f_n &= y\tau_n + z\sigma_n = yx\tau_{n+1} + yxc_{n+1} + zx\sigma_{n+1} + zxd_{n+1} \\ &= xf_{n+1} + yxc_{n+1} + zxd_{n+1}. \end{aligned} \tag{5.1.1}$$

The approximation domains U_n, B_n, U and B for A are as follows: for $n \geq 0$,

$$\begin{aligned} U_n &:= k[x, y, z, f_n] & B_n &:= k[x, y, z, f_n]_{(x,y,z,f_n)} \\ U &:= \bigcup_{n=0}^{\infty} U_n \qquad \text{and} & B &:= B_f = \bigcup_{n=0}^{\infty} B_n. \end{aligned}$$

Thus, B is the directed union of four-dimensional localized polynomial rings. It follows that $\dim B \leq 4$.

The rings A and B are constructed inside the intersection domain $A_{\tau,\sigma} := R^* \cap k(x, y, z, \tau, \sigma)$. By [**15**, Proposition 4.1] or [**18**, Theorem 9.2], the domain $A_{\tau,\sigma}$ is Noetherian and equals its approximation domain $B_{\tau,\sigma}$. Here $B_{\tau,\sigma}$ is the nested union of the regular local domains $B_{\tau,\sigma,n} = k[x, y, z, \tau_n, \sigma_n]_{(x,y,z,\tau_n,\sigma_n)}$. By Theorem 3.1, the extension $T := R[\tau, \sigma] \hookrightarrow R^*[1/x]$ is flat. It follows that $A_{\tau,\sigma}$ is a three-dimensional RLR that is a directed union of five-dimensional RLRs.

Before we list and establish the other properties of Example 5.1 in Theorem 5.8, we discuss the Jacobian ideal of a map and its relation to flatness.

Discussion 5.2. Let R be a Noetherian ring, let m and n be positive integers, let $z_1, \dots, z_n$ be indeterminates over R and let $f_1, \dots, f_m$ be polynomials in $R[z_1, \dots, z_n]$ that are algebraically independent over R. Let

$$S := R[f_1, \dots, f_m] \overset{\varphi}{\hookrightarrow} R[z_1, \dots, z_n] =: T \tag{5.2.0}$$

be the inclusion map.

We define the *Jacobian ideal* J of the extension $S \hookrightarrow T$ to be the ideal of T generated by the $m \times m$ minors of the $m \times n$ matrix $\mathcal{J}$ defined as follows:

$$\mathcal{J} := \left(\frac{\partial f_i}{\partial z_j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}.$$

For the extension $\varphi : S \hookrightarrow T$, the *non-flat locus* of φ is the set $\mathcal{F}$, where

$$\mathcal{F} := \{Q \in \operatorname{Spec}(T) \mid \text{the map} \quad \varphi_Q : S \to T_Q \text{ is not flat}\}.$$

The non-flat locus of φ is a closed subset of $\operatorname{Spec} T$, [**20**, Theorem 24.3]. We say that an ideal F of T *defines the non-flat locus* of φ if F is such that, for every $Q \in \operatorname{Spec}(T)$, we have $F \subseteq Q$ if and only if the associated map $\varphi_Q : S_{Q \cap S} \to T_Q$ is not flat.

Proposition 5.3 [**12**, Propositions 2.4.2, 2.7.2]. *With notation as in Discussion* 5.2*, let $Q \in \operatorname{Spec} T$ and consider $\varphi_Q : S \to T_Q$. Then*

(1) *φ_Q is flat if and only if, for each prime ideal $P \subseteq Q$ of T, we have* $\operatorname{ht}(P) \ge \operatorname{ht}(P \cap S)$*.*

(2) *If Q does not contain J, then φ_Q is flat. Thus, $J \subseteq F$.*

We use the following proposition concerning flatness to justify Example 5.1.

Proposition 5.4. *With the notation of Example* 5.1*, we have*

(1) *for the extension $\varphi : S = R[f] \hookrightarrow T = R[\tau, \sigma]$, the Jacobian ideal J is the ideal $(y, z)T$. Thus, the non-flat locus F of φ contains J.*

(2) *For every* $P \in \operatorname{Spec}(R^*[1/x])$, *the ideal* $(y,z)R^*[1/x] \not\subseteq P$ *if and only if the map* $B_{P\cap B} \hookrightarrow (R^*[1/x])_P$ *is flat. Thus, the ideal* $(y,z)R^*[1/x]$ *defines the non-flat locus of the map* $B \hookrightarrow R^*[1/x]$.

(3) *For every height* 1 *prime ideal* $\mathbf{p}$ *of* R^*, *we have* $\operatorname{ht}(\mathbf{p}\cap B) \le 1$.

(4) *For every prime element* w *of* B, $wR^* \cap B = wB$.

Proof. For item (1), the Jacobian ideal is the ideal of T generated by the 1×1 minors of the matrix $(y \quad z)$ by Discussion 5.2, and so $J = (y,z)T$. By Proposition 5.3.2, $(y,z)T \subseteq F$.

The two statements of item (2) are equivalent by the definition of non-flat locus in Discussion 5.2. To compute the non-flat locus of $B \hookrightarrow R^*[1/x]$, we use that $T := R[\tau,\sigma] \hookrightarrow R^*[1/x]$ is flat as noted in Example 5.1. Let $P \in \operatorname{Spec}(R^*[1/x])$, and let $Q := P\cap T$. The map $B \hookrightarrow R^*[1/x]_P$ is flat if and only if the composition

$$k[x,y,z,f] \hookrightarrow k[x,y,z,\tau,\sigma] \hookrightarrow R^*[1/x]_P \text{ is flat } \iff$$
$$S := k[x,y,z,f] \overset{\varphi}{\hookrightarrow} T_Q = k[x,y,z,\tau,\sigma]_Q \text{ is flat.}$$

From item (1), the Jacobian ideal of the extension $S \hookrightarrow T$ is the ideal $J = (y,z)T$. Since $(y,z)T\cap S = (y,z,f)S$ has height 3, φ_Q is not flat for every $Q \in \operatorname{Spec}(T)$ such that $(y,z)T \subseteq Q$. Thus the non-flat locus of $B \hookrightarrow R^*[1/x]$ is $(y,z)R^*[1/x]$ as stated in item (2).

For item (3), let $\mathbf{p}$ be a height 1 prime of R^*. Since $\mathbf{p}$ does not contain $(y,z)R^*$, the map $B_{\mathbf{p}\cap B} \hookrightarrow (R^*)_{\mathbf{p}}$ is faithfully flat. Thus $\operatorname{ht}(\mathbf{p}\cap B) \le 1$. This establishes item (3).

Item (4) is clear if $wB = xB$. Assume that $wB \ne xB$, and let $\mathbf{p}$ be a height 1 prime ideal of R^* that contains wR^*. Then $\mathbf{p}R^*[1/x]\cap R^* = \mathbf{p}$, and by item (3), $\mathbf{p}\cap B$ has height at most one. We have $\mathbf{p}\cap B \supseteq wR^*\cap B \supseteq wB$. Thus, item (4) follows. □

Next we prove a proposition about homomorphic images of the constructed ring B. This result enables us in Corollary 5.6 to relate ring B of Example 5.1 to ring B of Example 2.3.

Proposition 5.5. *Assume the notation of Example* 5.1, *and let* w *be a prime element of* $R = k[x,y,z]_{(x,y,z)}$ *with* $wR \ne xR$. *Let* π :

$R^* \to R^*/wR^*$ be the natural homomorphism, and let $^{-}$ denote image in R^*/wR^*. Let B' be the approximation domain formed by considering $\overline{R}$ and the endpieces $\overline{f}_n$ of $\overline{f}$, defined analogously to equation (2.1.a). *That is, B' is defined by setting*

$$U'_n = \overline{R}[\overline{f}_n], \qquad B'_n = (U'_n)_{\mathbf{n}'_n},$$
$$U' = \bigcup_{n=1}^{\infty} U'_n, \quad \textit{and} \quad B' = \bigcup_{n=1}^{\infty} B's_n,$$

where $\mathbf{n}'_n$ is the maximal ideal of U'_n that contains $\overline{f}_n$ and the image of $\mathbf{m}_R$. Then $B' = \overline{B}$.

Proof. By Proposition 3.2.3, wB is a prime ideal of B. By Proposition 5.4.3, $wR^* \cap B = wB$. Hence, $\overline{B} = B/(wR^* \cap B) = B/wB$. We have
$$\overline{R}/x\overline{R} = \overline{B}/x\overline{B} = \overline{R^*}/x\overline{R^*},$$
and the ring $\overline{R^*}$ is the $(\overline{x})$-adic completion of $\overline{R}$. Since the ideal $(y, z)R$ has height 2 and the kernel of π has height 1, at least one of $\overline{y}$ and $\overline{z}$ is non-zero. Since τ and σ are algebraically independent over $k(x, y, z)$, the element $\overline{f} = \overline{y} \cdot \overline{\tau} + \overline{z} \cdot \overline{\sigma}$ of the integral domain $\overline{B}$ is transcendental over $\overline{R}$. Similarly the endpieces $\overline{f}_n$ are transcendental over $\overline{R}$. The fact that $\overline{R}^*$ may fail to be an integral domain does not affect the algebraic independence of these elements that are inside the integral domain $\overline{B}$.

By Proposition 3.2.2 and Remarks 3.3.2, we have $U_n[1/x] = U[1/x]$, and thus $wU \cap U_n = wU_n$ for each $n \in \mathbf{N}$. Since B_n is a localization of U_n, we also have $wB \cap B_n = wB_n$. Since $wR^* \cap B = wB$, it follows that $wR^* \cap B_n = wB_n$. Thus, we have

$$\overline{R} \subseteq \overline{B_n} = B_n/wB_n \subseteq \overline{B} = B/wB \subseteq \overline{R^*} = R^*/wR^*.$$

We conclude that $\overline{B} = \bigcup_{n=0}^{\infty} \overline{B_n}$. Since $B'_n = \overline{B}_n$, we have $B' = \overline{B}$. □

Corollary 5.6. *The homomorphic image B/zB of the ring B of Example* 5.1 *is isomorphic to the three-dimensional ring B of Example* 2.3.

Proof. Assume the notation of Example 5.1 and Proposition 5.5, and let $w = z$. We show that the ring $B/zB \cong C$, where C is the ring called

B in Example 2.3. By Proposition 5.5, we have $B' = B/zB$, where B' is the approximation domain over $\overline{R} = R/zR$ using the element $\overline{f}$, transcendental over $\overline{R}$. Let R_C denote the base ring $k[x,y]_{(x,y)}$ for C in Example 2.3, and let $\psi_0 : \overline{R} \to R_C$ denote the k-isomorphism defined by $\overline{x} \mapsto x$ and $\overline{y} \mapsto y$. Then ψ_0 extends to an isomorphism $\psi : (\overline{R})^* \to (R_C)^*$ that agrees with ψ_0 on $\overline{R}$ and such that $\psi(\overline{\tau}) = \tau$. Furthermore, $\psi(\overline{f}) = \psi(\overline{y} \cdot \overline{\tau} + \overline{z} \cdot \overline{\sigma}) = y\tau$, which is the transcendental element f used in the construction of C. Thus, ψ is an isomorphism from $\overline{B} = B/zB$ to C, the ring constructed in Example 2.3. □

In the proof of Theorem 5.8, we use the following proposition regarding a birational extension of a Krull domain.

Proposition 5.7. *Let $S \hookrightarrow T$ be a birational extension of commutative rings, where S is a Krull domain and each height* 1 *prime ideal of S is contracted from T. Then $S = T$.*

Proof. Recall that S is Krull implies that $S = \cap\{S_{\mathbf{p}} \mid \mathbf{p}$ is a height 1 prime ideal of $S\}$. We show that $T \subseteq S_{\mathbf{p}}$, for each height 1 prime ideal of S. Since $\mathbf{p}$ is contracted from T, there exists a prime ideal $\mathbf{q}$ of T such that $\mathbf{q} \cap S = \mathbf{p}$. Then $S_{\mathbf{p}} \subseteq T_{\mathbf{q}}$ and $T_{\mathbf{q}}$ birationally dominates $S_{\mathbf{p}}$. Since $S_{\mathbf{p}}$ is a DVR, we have $S_{\mathbf{p}} = T_{\mathbf{p}}$. Therefore, $T \subseteq S_{\mathbf{p}}$, for each $\mathbf{p}$. It follows that $T = S$. □

We record in Theorem 5.8 properties of the ring B and its prime spectrum.

Theorem 5.8. *As in Example* 5.1*, let $R := k[x,y,z]_{(x,y,z)}$ with k a field, let x, y and z be indeterminates, and let $R^* := k[y,z]_{(y,z)}[[x]]$ be the xR-adic completion of R. Let τ and $\sigma \in xk[[x]]$ be algebraically independent over $k(x)$. Set $f := y\tau + z\sigma$, $A := R^* \cap k(x,y,z,f)$ and $B := \bigcup_{n=0}^{\infty} B_n = \bigcup_{n=0}^{\infty} k[x,y,z,f_n]_{(x,y,z,f_n)}$ as in* (5.1.2)*. Let $Q := (y,z)R^* \cap B$. Then*:

(1) *The rings A and B are equal.*

(2) *The ring B is a four-dimensional, non-Noetherian local* UFD *with maximal ideal $\mathbf{m}_B = (x,y,z)B$, and the $\mathbf{m}_B$-adic completion of B is the three-dimensional* RLR *$k[[x,y,z]]$.*

(3) *The ring* $B[1/x]$ *is a Noetherian regular* UFD, *the ring* B/xB *is a two-dimensional* RLR *and, for every non-maximal prime ideal* P *of* B, *the ring* B_P *is an* RLR.

(4) *The ideal* Q *is the unique prime ideal of* B *of height* 3.

(5) *The ideal* Q *equals* $\bigcup_{n=0}^{\infty} Q_n$ *where* $Q_n := (y, z, f_n)B_n$, Q *is a non-finitely generated prime ideal and* $QB_Q = (y, z, f)B_Q$.

(6) *There exist infinitely many height* 2 *prime ideals of* B *not contained in* Q, *and each of these prime ideals is contracted from* R^*.

(7) *For certain height* 1 *primes* $\mathfrak{p}$ *contained in* Q, *there exist infinitely many height* 2 *primes between* $\mathfrak{p}$ *and* Q *that are contracted from* R^*, *and infinitely many that are not contracted from* R^*. *Hence, the map* $\operatorname{Spec} R^* \to \operatorname{Spec} B$ *is not surjective.*

(8) *Every saturated chain of prime ideals of* B *has length either* 3 *or* 4, *and there exist saturated chains of prime ideals of lengths both* 3 *and* 4. *Thus,* B *is not catenary.*

(9) *Each height* 1 *prime ideal of* B *is the contraction of a height* 1 *prime ideal of* R^*.

(10) B *has Noetherian spectrum.*

We prove Theorem 5.8 below. Assuming Theorem 5.8, we display a picture of Spec (B) in Diagram 5.8.0 and make some remarks.

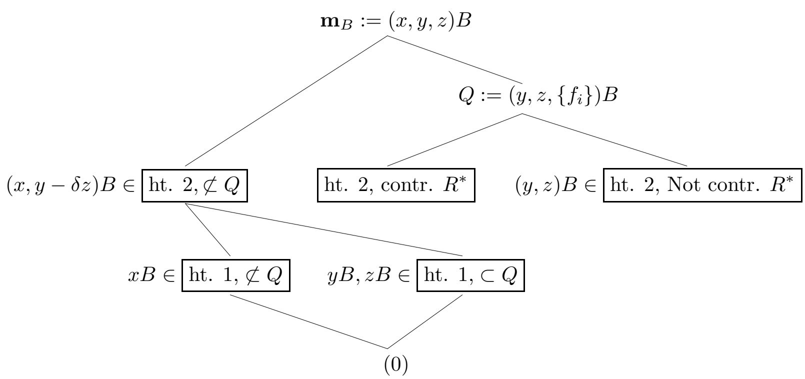

DIAGRAM 5.8.0.

Comments on Diagram 5.8.0. A line going from a box at one level to a box at a higher level indicates that every prime ideal in the lower level box is contained in at least one prime ideal in the higher level box. Thus, as indicated in the diagram, every height 1 prime gB of B is contained in a height 2 prime of B that contains x and so is not contained in Q. This is obvious if $gB = xB$ and can be seen by considering minimal primes of $(g, x)B$ otherwise. Thus, B has no maximal saturated chain of length 2. We have not drawn any lines from the lower level righthand box to higher boxes that are contained in Q because we are uncertain about what inclusion relations exist for these primes. We discuss this situation in Remarks 5.15.

Proof of Theorem 5.8. For convenience, we prove item (2) first.

Since B is a directed union of four-dimensional RLRs, we have $\dim B \leq 4$. By Corollary 5.6 and Theorem 4.1, $\dim (B/zB) = 3$, and so $\dim B \geq 4$. Thus, $\dim B = 4$. By Proposition 3.2, B is local with maximal ideal $\mathbf{m}_B = (x, y, z)B$, and the (x)-adic completion of B is R^*. Thus, the $\mathbf{m}_B$-adic completion of B is $k[[x, y, z]]$. By Krull's altitude theorem, the ring B is not Noetherian [**22**, Theorem 13.5]. The ring B is a UFD by Proposition 3.4.

For item (1), the ring B is a UFD by item (2), and hence a Krull domain, and the extension $B \hookrightarrow A$ is birational. Thus, it suffices to show that each height 1 prime P of B is the contraction of a prime ideal of A by Proposition 5.7.

Let $\mathbf{p}$ be a height 1 prime ideal of B. Then $\mathbf{p}R^* \cap B = \mathbf{p}$ by Proposition 5.4.4. Also, $B \setminus \mathbf{p}$ is a multiplicatively closed subset of R^*, and so, if P is an ideal of R^* maximal with respect to $P \cap (B \setminus \mathbf{p}) = \varnothing$, then P is a prime ideal of R^* and $P \cap B = \mathbf{p}$. Then also $P \cap A$ is a prime ideal of A with $(P \cap A) \cap B = \mathbf{p}$, and so $\mathbf{p}$ is contracted from A. Thus $A = B$ as desired for item (1).

For item (3), the ring $B[1/x]$ is a Noetherian regular UFD by Proposition 3.4.1. By equation (3.2.0), we have $R/xR = B/xB$. Thus, B/xB is a two-dimensional RLR.

For the last part of item (3) if $x \notin P$, then B_P is a localization of $B[1/x]$, which is Noetherian and regular, and so B_P is a regular local ring. In particular, this proves that B_Q is a regular local ring. If $x \in P$ and $\operatorname{ht} P = 1$, then $P = (x)$ and B_{xB} is a DVR. If $x \in P$

and $\operatorname{ht}(P) = 2$, the ideal P is finitely generated since B/xB is an RLR. Since B is a UFD from item (2), it follows that B_P is a local UFD of dimension 2 with finitely generated maximal ideal. Thus, B_P is Noetherian by Cohen's theorem [**22**, Theorem 3.4]. This, combined with B/xB a regular local ring, implies that B_P is a regular local ring. Since $\operatorname{ht} P \leq 2$ for every non-maximal prime ideal P of R with $x \in P$, this completes the proof of item (3).

For item (4), since $(y,z)R^*$ is a prime ideal of R^*, the ideal $Q = (y,z)R^* \cap B$ is prime. By Proposition 3.2, the ideals yB and $(y,z)B$ are prime. Consider the chain of prime ideals

$$(0) \subset yB \subset (y,z)B \subset Q \subset \mathbf{m}_B.$$

The list y, z, f, x shows that each of the inclusions is strict; for example, we have $f \in Q \backslash (y,z)B$, since $f \notin (y,z)B_n$ for every $n \in \mathbf{N}$. By item (2) we have $\operatorname{ht} \mathbf{m}_B = 4$. Thus, $\operatorname{ht} Q = 3$. This also implies that $(y,z)B$ is a height 2 prime ideal of B.

For the uniqueness in item (4) let P be a non-maximal prime ideal of B. We first consider the case that $x \notin P$. Then, by Proposition 3.2.4, $x^n \notin PR^*$ for each positive integer n. Hence, $PR^*[1/x] \neq R^*[1/x]$. Let P_1 be a prime ideal of $R^*[1/x]$ such that $P \subseteq P_1$. If both y and z are in P_1, then $(y,z)R^*[1/x] \subseteq P_1$. Since $(y,z)R^*[1/x]$ is maximal, we have $(y,z)R^*[1/x] = P_1$. Therefore, $P \subseteq (y,z)R^*[1/x] \cap B = Q$, and so either $\operatorname{ht}(P) \leq 2$ or $P = Q$.

Next suppose that $x \notin P$ and y or z is not in P_1. By Propositions 5.4.1 and 5.3.2, the map $\psi : B \to R^*[1/x]_{P_1}$ is flat. Since $\dim R^*[1/x] = 2$, we have $\operatorname{ht}(P_1) \leq 2$. Flatness of ψ implies $\operatorname{ht}(P_1 \cap B) \leq 2$; see [**22**, Theorem 9.5]. Hence, $\operatorname{ht} P \leq 2$.

To complete the proof of item (4) we consider the case that $x \in P$. We have $\operatorname{ht} P \leq 3$, since $\dim B = 4$ and P is not maximal. If $\operatorname{ht} P \geq 3$, there exists a chain of primes of the form

$$(0) \subsetneq P_1 \subsetneq P_2 \subsetneq P \subsetneq (x,y,z)B. \tag{5.8.1}$$

By equation (3.2.0), $B/xB \cong R/xR$, and so $\dim(B/xB) = 2$. If $x \in P_2$, then $\operatorname{ht} P_2 \geq 2$ implies that $(0) \subsetneq xB \subsetneq P_2 \subsetneq P \subsetneq (x,y,z)B$, a contradiction to $\dim(B/xB) = 2$. Thus, $x \notin P_2$. Since $x \in P$ and P is non-maximal, we have that y or z is not in P. Hence, y or z is not in P_2.

By equation (3.2.0), P corresponds to a non-maximal prime ideal P' of R^* containing PR^*. Let P_2' be a prime ideal of R^* inside P' that is minimal over P_2R^*. If both y and z are in P_2', then, $(x,y,z)R^* \subseteq P'$, a contradiction to P' non-maximal. By Proposition 3.2.5, P_2' does not contain x. Thus, $P_2' \subsetneq P' \subsetneq (x,y,z)R^*$. Also $P_2' = P_2'' \cap R^*$, where P_2'' is a prime ideal of $R^*[1/x]$, and one of y and z is not an element of P_2''.

By Proposition 5.4.2, the map $\psi : B \to R^*[1/x]_{P_2''}$ is flat. This implies $\operatorname{ht}(P_2'') \geq \operatorname{ht}(P_2'' \cap B) \geq \operatorname{ht} P_2 \geq 2$, that is, $\operatorname{ht}(P_2'') \geq 2$. Also, P_2'' intersects R^* in P_2', and so $\operatorname{ht} P_2' \geq 2$. Thus, in R^*, we have a chain of primes $P_2' \subsetneq P' \subsetneq (x,y,z)R^*$, where $\operatorname{ht} P_2' \geq 2$, a contradiction, since R^*, a localization of $k[y,z][[x]]$, has dimension 3. This proves item (4).

For item (5), let $Q' = \bigcup_{n=0}^{\infty} Q_n$, where each $Q_n = (y,z,f_n)B_n$. Each Q_n is a prime ideal of height 3 in the four-dimensional RLR B_n. Therefore, Q' is a prime ideal of B of height ≤ 3 that is contained in Q. The ideal $(y,z)B$ is a prime ideal of height 2 by the proof of item (3). Hence, $\operatorname{ht}(Q') = 3$, and we have $Q' = Q$.

To show the ideal Q is not finitely generated, we show for each positive integer n that $f_{n+1} \notin (y,z,f_n)B$. By equation (5.1.1), $f_n = xf_{n+1} + yxc_{n+1} + zxd_{n+1}$. If $f_{n+1} \in (y,z,f_n)B$, then $f_{n+1} = ay + bz + c(xf_{n+1} + yxc_{n+1} + zxd_{n+1})$, where $a,b,c \in B$. This implies $f_{n+1}(1-cx)$ is in the ideal $(y,z)B$. By Proposition 3.2.1, $x \in \mathcal{J}(B)$, and so $1-cx$ is a unit of B. This implies $f_{n+1} \in (y,z)B \cap B_{n+1}$.

For each positive integer j we show that $(y,z)B \cap B_j = (y,z)B_j$. It is clear that $(y,z)B_j \subseteq (y,z)B \cap B_j$. To show the reverse inclusion, it suffices to show for each integer $j \geq 0$ that $(y,z)B_{j+1} \cap B_j \subseteq (y,z)B_j$. We have $B_j[f_{j+1}] \subseteq (B_j)_{(y,z)B_j}$ since $f_{j+1} = (f_j/x) + yc_{j+1} + zd_{j+1}$ by (5.1.1). The center of the two-dimensional RLR $(B_j)_{(y,z)B_j}$ on $B_j[f_{j+1}]$ is the prime ideal $(y,z)B_j[f_{j+1}]$. This prime ideal is contained in the maximal ideal $(x,y,z,f_{j+1})B_j[f_{j+1}]$; it follows that $B_{j+1} \subseteq (B_j)_{(y,z)B_j}$ and so $(y,z)B_{j+1} \cap B_j \subseteq (y,z)B_j$.

Thus, $(y,z)B \cap B_{n+1} = (y,z)B_{n+1}$, and $f_{n+1} \in (y,z)B_{n+1}$. Since x,y,z and f_{n+1} are algebraically independent variables over k, and $B_{n+1} = k[x,y,z,f_{n+1}]_{(x,y,z,f_{n+1})}$, this is a contradiction. We conclude that Q is not finitely generated.

By item (3), the ring B_Q is a three-dimensional regular local ring. Since x is a unit of B_Q and since $Q = (y,z,f,f_1,f_2,\dots)B$, it follows from Proposition 3.2.2 ($a = x$ and $C = B$) that $QB_Q = (y, z, f)B_Q$. This establishes item (5).

For item (6), since $x \notin Q$ and $B/xB \cong R/xR$, there are infinitely many height 2 primes of B containing xB. This proves there are infinitely many height 2 primes of B not contained in Q. If P is a height 2 prime of B not contained in Q, then $\operatorname{ht}(\mathbf{m}_B/P) = 1$, by item (4) above, and so, by Proposition 3.2.5, P is contracted from R^*. This completes item (6).

For item (7), we show that $p = zB$ has the stated properties. By Corollary 5.6, the ring B/zB is isomorphic to the ring called B in Example 2.3. For convenience, we relabel the ring of Example 2.3 as B'. By Theorem 4.1, B' has exactly one non-finitely generated prime ideal, which we label Q' and $\operatorname{ht} Q' = 2$. It follows that $Q/zB = Q'$. By Discussion 2.5, there are infinitely many height 1 primes contained in Q' of Type II (that is, primes that are contracted from R^*/zR^*) and infinitely many height 1 primes contained in Q' of Type III (that is, primes that are not contracted from R^*/zR^*). The preimages in R^* of these primes are height 2 primes of B that are contained in Q and contain zB. It follows that there are infinitely many contracted from R^* and there are infinitely many not contracted from R^*, as desired for item (7).

For item (8), we have a saturated chain of prime ideals

$$(0) \subset xB \subset (x, y)B \subset (x, y, z)B = \mathbf{m}_B$$

of length 3 by equation (3.2.0). We have a saturated chain of prime ideals

$$(0) \subset yB \subset (y, z)B \subset Q \subset \mathbf{m}_B$$

of length 4 from the proof of item (4). Hence, B is not catenary. By item (2), $\dim B = 4$, so there is no saturated chain of prime ideals of B of length greater than 4. By Comments 5.8.0, no saturated chain of prime ideals of B has length less than 3.

For item (9), since R^* is a Krull domain and $B = A = \mathcal{Q}(B) \cap R^*$, it follows that B is a Krull domain, and each height 1 prime of B is the contraction of a height 1 prime of R^*. Since B/xB and B[1/x] are Noetherian, item (10) follows from [**11**, Corollary 1.3]. □

Remarks 5.9. Let the notation be as in Theorem 5.8.

(1) It follows from Theorem 5.8 that the localization $B[1/x]$ has a unique maximal ideal $QB[1/x] = (y, z, f)B[1/x]$ of height 3 and has infinitely many maximal ideals of height 2. We observe that $B[1/x]$ has no maximal ideal of height 1. To show this last statement it suffices to show for each irreducible element p of B with $pB \neq xB$ that there exists $P \in \operatorname{Spec} B$ with $pB \subsetneq P$ and $x \notin P$. Assume there does not exist such a prime ideal P. Consider the ideal $(p, x)B$. This ideal has height 2 and has only finitely many minimal primes since B/xB is Noetherian. Let g be an element of $\mathbf{m}_B$ not contained in any of the minimal primes of $(p, x)B$. Every prime ideal of B that contains $(g, p)B$ also contains x and hence has height > 2. Since $x \notin Q$, it follows that $(g, p)B$ is $\mathbf{m}_B$-primary, and hence that $(g, p)R^*$ is $\mathbf{m}_{R^*}$-primary. Since R^* is Noetherian and $\operatorname{ht} \mathbf{m}_{R^*} = 3$, this contradicts the altitude theorem of Krull [**24**, Theorem 9.3].

(2) Every ideal I of B such that IR^* is $\mathbf{m}_{R^*}$-primary is $\mathbf{m}_B$-primary by Proposition 3.2.5.

(3) Define

$$C_n := \frac{B_n}{(y, z)B_n} \quad \text{and} \quad C := \frac{B}{(y, z)B}.$$

We have $C = \bigcup_{n=0}^{\infty} C_n$ by item (1). We show that C is a rank 2 valuation domain with principal maximal ideal generated by the image of x. For each positive integer n, let $g_n \in C_n$ denote the image of the element f_n, and let x denote the image of x. Then $C_n = k[x, g_n]_{(x, g_n)}$ is a two-dimensional RLR. By (5.1.1), $f_n = xf_{n+1} + x(c_n y + d_n z)$. It follows that $g_n = xg_{n+1}$ for each $n \in \mathbf{N}$. Thus, C is an infinite directed union of quadratric transformations of two-dimensional regular local rings. Thus, C is a valuation domain of dimension at most 2 by [**1**]. By items (2) and (4) of Theorem 5.8, $\dim C \geq 2$, and therefore C is a valuation domain of rank 2. The maximal ideal of C is xC.

By Corollary 5.6, $B/zB \cong D$, where D is the ring B of Example 2.3. By an argument similar to that of Proposition 5.5 and Corollary 5.6, we see that the above ring C is isomorphic to D/yD.

Question 5.10. For the ring B constructed as in Example 5.1, we ask: is Q the only prime ideal of B that is not finitely generated?

Theorem 5.8 implies that the only possible non-finitely generated prime ideals of B other than Q have height 2. We do not know whether every height 2 prime ideal of B is finitely generated. We show in Corollary 5.13 and Theorem 5.14 that certain of the height 2 primes of B are finitely generated.

Lemma 5.11 is the key to the proof of Theorem 3.1 and is also useful below. We are grateful to Roger Wiegand for observing it.

Lemma 5.11 [**15**, Lemma 3.1], [**18**, Lemma 8.2]. *Let S be a subring of a ring T, and let $b \in S$ be a regular element of both S and T. Assume that $bS = bT \cap S$ and $S/bS = T/bT$. Then*:

(1) *$T[1/b]$ is flat over S if and only if T is flat over S.*

(2) *If T and $S[1/b]$ are both Noetherian and T is flat over S, then S is Noetherian.*

The following theorem shows that the non-flat locus of the map $\varphi : B \to R^*[1/a]$ yields flatness for certain homomorphic images of B, if R, a, R^* and B are as in the general construction outlined in Theorem 3.1.

Theorem 5.12. *Let R be a Noetherian integral domain with field of fractions K, let $a \in R$ be a non-zero, non-unit, and let R^* denote the (a)-adic completion of R. Let s be a positive integer, and let $\underline{\tau} = \{\tau_1, \dots, \tau_s\}$ be a set of elements of R^* that are algebraically independent over K, so that $R[\underline{\tau}]$ is a polynomial ring in s variables over R. Define $A := K(\underline{\tau}) \cap R^*$. Let U_n, B_n, B and U be defined as follows*:

$$U := \bigcup_{r=0}^{\infty} U_n \quad \textit{and} \quad B := \bigcup_{n=0}^{\infty} B_n,$$

where, for each integer $n \geq 0$, $U_n := R[\tau_{1n}, \dots, \tau_{sn}]$, $B_n := (1 + aU_n)^{-1}U_n$, and each τ_{in} is the nth endpiece of τ_i defined as in equation (2.1.a). *Assume that F is an ideal of $R^*[1/a]$ that defines the non-flat locus of the map $\varphi : B \to R^*[1/a]$. Let I be an ideal in B such that $IR^* \cap B = I$ and a is regular on R^*/IR^*.*

(1) *If $IR^*[1/a] + F = R^*[1/a]$, then $\varphi \otimes_B (B/I)$ is flat.*

(2) *If $R^*[1/a]/IR^*[1/a]$ is flat over B/I, then R^*/IR^* is flat over B/I.*

(3) *If $\varphi \otimes_B (B/I)$ is flat, then B/I is Noetherian.*

Proof. The hypothesis of item (1) implies that φ_P is flat for each $P \in \operatorname{Spec} R^*[1/a]$ with $I \subseteq P$. Hence, for each such P, we have $\varphi_P \otimes_B (B/I)$ is flat. Since flatness is a local property, it follows that $\varphi \otimes_B (B/I)$ is flat.

For items (2) and (3), apply Lemma 5.11 with $S = B/I$ and $T = R^*/IR^*$; the element b of Lemma 5.11 is the image in B/IB of the element a from the setting of Theorem 3.1. Since $IR^* \cap B = I$, the ring B/I embeds into R^*/IR^*, and since $B/aB = R^*/aR^*$, the ideal $a(R^*/IR^*) \cap (B/I) = a(B/I)$. Thus, items (2) and (3) of Theorem 5.12 follow from items (1) and (2), respectively, of Lemma 5.11. □

Corollary 5.13. *Assume the notation of Example* 5.1*. Let w be a prime element of B. Then B/wB is Noetherian if and only if $w \notin Q$. Thus, every non-finitely generated ideal of B is contained in Q.*

Proof. If $w \in Q$, then B/wB is not Noetherian since Q is not finitely generated. Assume $w \notin Q$. Since B/xB is known to be Noetherian, we may assume that $wB \neq xB$. By Proposition 5.4.1, $QR^*[1/x] = (y,z)R^*[1/x]$ defines the non-flat locus of $\varphi : B \to R^*[1/x]$. Since $wR^*[1/x] + (y,z)R^*[1/x] = R^*[1/x]$, Theorem 5.12 with $I = wB$ and $a = x$ implies that B/wB is Noetherian.

For the second statement, we use that every non-finitely generated ideal is contained in an ideal maximal with respect to not being finitely generated, and the latter ideal is prime. Thus, it suffices to show every prime ideal P not contained in Q is finitely generated. If $P \not\subseteq Q$, then, since B is a UFD, there exists a prime element $w \in P \setminus Q$. By the first statement, B/wB is Noetherian, and so P is finitely generated. □

Theorem 5.14. *Assume the notation of Example* 5.1*. Let w be a prime element of R with $w \in (y,z)k[x,y,z]$. If w is linear in either y or z, then Q/wB is the unique non-finitely generated prime ideal of B/wB. Thus, Q is the unique non-finitely generated prime ideal of B that contains w.*

Proof. Let $\overline{}$ denote the image under the canonical map of R^* onto R^*/wR^*. We may assume that w is linear in z, that the coefficient of z is 1, and therefore that $w = z - yg(x,y)$, where $g(x,y) \in k[x,y]$. Thus, $\overline{R} \cong k[x,y]_{(x,y)}$. By Proposition 5.5, $\overline{B}$ is the approximation domain over $\overline{R}$ with respect to the transcendental element

$$\overline{f} = \overline{y} \cdot \overline{\tau} + \overline{z} \cdot \overline{\sigma} = \overline{y} \cdot \overline{\tau} + \overline{y} \cdot \overline{g(x,y)} \cdot \overline{\sigma}.$$

The setting of Proposition 3.4 applies with $C = \overline{B}$, the underlying ring R replaced by $\overline{R}$, and $a = \overline{x}$. Thus, the ring $\overline{B}$ is a UFD, and so every height 1 prime ideal of $\overline{B}$ is principal. Since $w \in Q$ and Q is not finitely generated, it follows that $\operatorname{ht}(\overline{Q}) = 2$ and that $\overline{Q}$ is the unique non-finitely generated prime ideal of $\overline{B}$. Hence, the theorem holds. $\square$

Remarks 5.15. It follows from Proposition 3.2.5 that every height 2 prime of B that is not contained in Q is contracted from a prime ideal of R^*. As we state in item (7) of Theorem 5.8, there are infinitely many height 2 prime ideals of B that are contained in Q and contracted from R^*, and there are infinitely many height 2 prime ideals of B that are contained in Q and are *not* contracted from R^*. In particular, infinitely many of each type exist between zB and Q, and similarly also infinitely many of each type exist between yB and Q.

Since B_Q is a three-dimensional regular local ring, for each height 1 prime p of B with $p \subset Q$, the set

$$\mathcal{S}_p = \{P \in \operatorname{Spec} B \mid p \subset P \subset Q \text{ and } \operatorname{ht} P = 2\}$$

is infinite. The infinite set $\mathcal{S}_p$ is the disjoint union of the sets $\mathcal{S}_{pc}$ and $\mathcal{S}_{pn}$, where the elements of $\mathcal{S}_{pc}$ are contracted from R^* and the elements of $\mathcal{S}_{pn}$ are not contracted from R^*.

We do not know whether there exists a height 1 prime p contained in Q having the property that one of the sets $\mathcal{S}_{pc}$ or $\mathcal{S}_{pn}$ is empty. Furthermore, if one of these sets is empty, which one is empty? If there are some such height 1 primes p with one of the sets $\mathcal{S}_{pc}$ or $\mathcal{S}_{pn}$ empty, which height 1 primes are they? It would be interesting to know the answers to these questions.

The referee of this article asked how Example 5.1 compares to a specific ring constructed using the popular "$D + M$" technique of multiplicative ideal theory; see, for example, [**3, 9**] or [**8**, page 95]. The "$D+M$" construction involves an integral domain D and a prime ideal M of an extension domain E of D such that $D \cap M = (0)$. Then $D+M = \{a+b \mid a \in D,\ b \in M\}$. Moreover, for $a, a' \in D$ and $b, b' \in M$, if $a + b = a' + b'$, then $a = a'$ and $b = b'$. Since D embeds in E/M, the ring $D + M$ may be regarded as a pullback as in [**7**] or [**21**, page 42]. The ring suggested by the referee is an interesting example that contrasts nicely with Example 5.1. We describe it in Example 5.16.

Example 5.16. Let B be the ring of Example 2.3. Then $B = k+\mathbf{m}_B$ in the notation of Example 2.3. Assume the field k is the field of fractions of a DVR V, and let t be a generator of the maximal ideal of V. Define

$$C := V + \mathbf{m}_B = \{a + b \mid a \in V,\ b \in \mathbf{m}_B\}.$$

The integral domain C has the following properties:

(1) The maximal ideal $\mathbf{m}_B$ of B is also a prime ideal of C, and $C/\mathbf{m}_B \cong V$.

(2) C has a unique maximal ideal $\mathbf{m}_C$; moreover, $\mathbf{m}_C = tC$.

(3) $\mathbf{m}_B = \bigcap_{n=1}^{\infty} t^n C$ and $B = C_{\mathbf{m}_B} = C[1/t]$.

(4) Each $P \in \operatorname{Spec} C$ with $P \neq \mathbf{m}_C$ is contained in $\mathbf{m}_B$; thus, $P \in \operatorname{Spec} B$.

(5) $\dim C = 4$ and C has a unique prime ideal of height h, for $h = 2, 3$ or 4.

(6) $\mathbf{m}_C$ is the only non-zero prime ideal of C that is finitely generated. Indeed, every non-zero proper ideal of B is an ideal of C that is not finitely generated.

Thus, C is a non-Noetherian, non-catenary, four-dimensional local domain.

Proof. Since C is a subring of B, $\mathbf{m}_B \cap V = (0)$ and $V\mathbf{m}_B = \mathbf{m}_B$, item (1) holds. We have $C/(tV + \mathbf{m}_B) = V/tV$. Thus, $tV + \mathbf{m}_B$ is a maximal ideal of C. Let $b \in \mathbf{m}_B$. Since $1 + b$ is a unit of the local ring B, we have

$$\frac{1}{1+b} = 1 - \frac{b}{1+b} \quad \text{and} \quad \frac{b}{1+b} \in \mathbf{m}_B.$$

Hence, $1+b$ is a unit of C. Let $a+b \in C \setminus (tV+\mathbf{m}_B)$, where $a \in V \setminus tV$ and $b \in \mathbf{m}_B$. Then a is a unit of V and thus a unit of C. Moreover, $a^{-1}(a+b) = 1+a^{-1}b$ and $a^{-1}b \in \mathbf{m}_B$. Therefore, $a+b$ is a unit of C. We conclude that $\mathbf{m}_C := tV + \mathbf{m}_B$ is the unique maximal ideal of C. Since t is a unit of B, we have $\mathbf{m}_B = t\mathbf{m}_B$. Hence, $\mathbf{m}_C = tV + \mathbf{m}_B = tC$. This proves item (2).

For item (3), since t is a unit of B, we have $\mathbf{m}_B = t^n\mathbf{m}_B \subseteq t^nC$ for all $n \in \mathbf{N}$. Thus, $\mathbf{m}_B \subseteq \bigcap_{n=1}^{\infty} t^nC$. If $a+b \in \bigcap_{n=1}^{\infty} t^nC$ with $a \in V$ and $b \in \mathbf{m}_B$, then

$$b \in \bigcap_{n=1}^{\infty} t^nC \implies a \in \left(\bigcap_{n=1}^{\infty} t^nC\right) \cap V = \bigcap_{n=1}^{\infty} t^nV = (0).$$

Hence, $\mathbf{m}_B = \bigcap_{n=1}^{\infty} t^nC$. Again, using $t\mathbf{m}_B = \mathbf{m}_B$, we obtain

$$C[1/t] = V[1/t] + \mathbf{m}_B = k + \mathbf{m}_B = B.$$

Since $t \notin \mathbf{m}_B$, we have $B = C[1/t] \subseteq C_{\mathbf{m}_B} \subseteq B_{\mathbf{m}_B} = B$. This proves item (3).

For P as in item (4), we have $P \subsetneq tC$. Since P is a prime ideal of C, it follows that $P = t^nP$ for each $n \in \mathbf{N}$. By item (3), $P \subseteq \mathbf{m}_B$, and it follows that $P \in \operatorname{Spec} B$. Item (5) now follows from item (4) and the structure of $\operatorname{Spec} B$.

For item (6), let J be a non-zero proper ideal of B. Since t is a unit of B, we have $J = tJ$. This implies by Nakayama's lemma that J as an ideal of C is not finitely generated; see [**3**, Lemma 1]. Thus, item (6) follows from item (4).

By item (6), C is non-Noetherian. Since $(0) \subsetneq xB \subsetneq \mathbf{m}_B \subsetneq tC$ is a saturated chain of prime ideals of C of length 3, and $(0) \subsetneq yB \subsetneq Q \subsetneq \mathbf{m}_B \subsetneq tC$ is a saturated chain of prime ideals of C of length 4, the ring C is not catenary. $\square$

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