JOURNAL OF COMMUTATIVE ALGEBRA
Volume 6, Number 1, Spring 2014

STABILIZATION OF BETTI TABLES

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ABSTRACT. Let $I \subseteq R = \mathbf{k}[x_1, \ldots, x_n]$ be a homogeneous ideal generated by forms of degree r. We show here that the shapes of the Betti tables of the ideals I^d stabilize, in the sense that there exists some D such that for all $d \geq D$, $\beta_{i,j+rd}(I^d) \neq 0 \Leftrightarrow \beta_{i,j+rD}(I^D) \neq 0$. We also produce upper bounds for the stabilization index $\mathrm{Stab}\,(I)$. This strengthens the result of Cutkosky, Herzog and Trung that the Castelnuovo-Mumford regularity $\mathrm{reg}\,(I^d)$ is eventually a linear function in d.

1. Background and results.

1.1. Asymptotics of regularity of I^d. Let $\mathbf{k}$ be a field. For an ideal $I \subseteq R = \mathbf{k}[x_1, \ldots, x_n]$, much work has been done on showing that the Castelnuovo-Mumford regularity of I^d is a linear function in terms of d for high powers. The following theorem is a result of Cutkosky, Herzog and Trung:

Theorem 1.1 ([**3**, Theorem 1.1]). *Let I be an arbitrary homogeneous ideal. Let $r(I)$ denote the maximum degree of the homogeneous generators of I. The following hold*:

(i) *There is a number e such that* $\mathrm{reg}\,(I^d) \leq d \cdot r(I) + e$ *for all* $d \geq 1$.

(ii) $\mathrm{reg}\,(I^d)$ *is a linear function for all d large enough.*

They provide a criterion for estimating this e in the case of an equigenerated ideal I, i.e., an ideal generated by homogeneous generators of the same degree. This result generalizes an earlier bound by Swanson

2010 AMS *Mathematics subject classification.* Primary 13A02,13D02, 13C99.
Keywords and phrases. Free resolutions, Betti tables, stabilization, powers of ideals, regularity.
Received by the editors on June 13, 2011, and in revised form on September 19, 2011.
DOI:10.1216/JCA-2014-6-1-113

giving the existence of k such that

$$\operatorname{reg}(I^d) \leq kd,$$

for homogeneous ideals in [**8**].

Let $I \subseteq R = \mathbf{k}[x_1, \ldots, x_n]$ be an ideal. The *graded Betti numbers* of a homogeneous ideal I are given by $\beta_{i,j}(I) = \dim_{\mathbf{k}} \operatorname{Tor}_i(\mathbf{k}, I)_j$. The graded Betti numbers also correspond to the ranks of the free modules in a minimal free resolution of I. We organize this data into the *Betti table of* I (in the style of Macaulay 2) displaying $\beta_{i,i+j}(R/I)$ in the ith column and jth row, as seen in Example 1.3.

Using techniques similar to those in [**1, 3, 6**], we produce here a sharper result on the asymptotics of Betti tables of powers I^d.

Theorem 1.2 (Theorem 4.1). *Let $I = (f_0, f_1, \ldots, f_k) \subseteq \mathbf{k}[x_1, \ldots, x_n] = R$ be an equigenerated ideal generated in degree r. Then there exists a D such that, for all $d > D$, we have*

$$\beta_{i,j+rd}(I^d) \neq 0 \Longleftrightarrow \beta_{i,j+rD}(I^D) \neq 0.$$

This gives us that the shape of the Betti tables of powers of an ideal I is eventually fixed, translated down by degree r of the ideal.

Example 1.3. Let $I = (x_3x_4x_5, x_1x_6x_7, x_3x_6x_8, x_1x_5x_9, x_2x_8x_9) \subseteq \mathbf{k}[x_1, \ldots, x_9]$. We consider the Betti diagrams of the resolutions of the first few powers I^d of I. The diagrams have been shifted to only show nonzero Betti numbers in the resolution of I^d.

I

–	1	2	3	4	5
total:	5	10	9	3	·
2:	5	·	·	·	·
3:	·	6	·	·	·
4:	·	4	9	3	·

I^2

–	1	2	3	4	5
total	15	41	39	12	·
5:	15	·	·	·	·
6:	·	33	12	·	·
7:	·	8	27	12	·

I^3

–	1	2	3	4	5
total:	35	117	121	39	1
8:	35	·	·	·	·
9:	·	105	67	9	·
10:	·	12	54	30	1

I^4

–	1	2	3	4	5
total:	70	271	302	105	5
11:	70	·	·	·	·
12:	·	255	212	45	·
13:	·	16	90	60	5

I^5

–	1	2	3	4	5
total:	126	545	645	240	15
14:	126	·	·	·	·
15:	·	525	510	135	·
16:	·	20	135	105	15

I^6

–	1	2	3	4	5
total:	210	990	1229	483	35
17:	210	·	·	·	·
18:	·	996	1040	315	·
19:	·	24	189	168	35

We can see the stabilized shape of the powers of I^d will be:

I^d

-	1	2	3	4	5
total:	*	*	*	*	*
3d-1:	*	·	·	·	·
3d:	·	*	*	*	·
3d+1:	·	*	*	*	*

Unfortunately, Theorem 4.1 does not guarantee that powers of our ideals I^d will have linear resolutions if the resolution of I^l is linear for some l with $d > l$. As a counterexample, we have the following example (due to Sturmfels):

Example 1.4 [**7**]. Set $I = (def, cef, cdf, cde, bef, bcd, acf, ade) \subseteq \mathbf{k}[a, b, c, d, e, f]$. The ideal I has linear resolution and linear quotients with respect to the ordering given above, but I^2 fails to be linear. We include the Betti tables of I and I^2 here.

I

–	0	1	2	3
total:	1	8	11	4
0:	1	·	·	·
1:	·	·	·	·
2:	·	8	11	4

I^2

–	0	1	2	3	4	5	6
total:	1	36	85	79	38	10	1
0:	1	·	·	·	·	·	·
1:	·	·	·	·	·	·	·
2:	·	·	·	·	·	·	·
3:	·	·	·	·	·	·	·
4:	·	·	·	·	·	·	·
5:	·	36	84	75	32	6	·
6:	·	·	1	4	6	4	1

More generally, Conca provided a class of ideals I_k which have linear quotients (and hence, linear resolutions) until the kth power, then have nonlinear resolutions for all powers higher than k [**2**]. This implies that, for an ideal I, the shapes of Betti tables of $I, I^2, \dots, I^d$ and I^{d+1} need not satisfy any chain of inclusions, though they eventually stabilize for some I^D.

We also provide an upper bound for the Betti numbers of powers of an equigenerated ideal I in terms of the Betti numbers of the Rees ideal of I as follows.

Theorem 1.5 (Theorem 3.1). *Let* $I = (f_0, f_1, \dots, f_k) \subseteq R = \mathbf{k}[x_1, \dots, x_N]$ *with* f_i *homogeneous of degree* r. *Let* $\mathcal{R}(I)$ *be the Rees algebra of* I *over the ring* $S = \mathbf{k}[x_1, \dots, x_N, w_0, \dots, w_k]$ *with bigrading* $\deg(x_i) = (1, 0)$ *and* $\deg(w_i) = (0, 1)$. *Then*

$$\beta_{i,j+rd}(I^d) \leq \sum_{m=0}^{d} \binom{d+k-m}{d-m} \beta_{i,(j,m)}(\mathcal{R}(I))$$

holds for all i, j, d.

The proof follows from a careful examination of the restriction of a minimal resolution of $\mathcal{R}(I)$ to bidegrees $(*, d)$. We give a name to the smallest D for which this stabilization occurs:

Definition 1.6. Let I be a homogeneous equigenerated ideal in a polynomial ring R. Let the *stabilization index* $\operatorname{Stab}(I)$ *of* I be the smallest such D such that, for all $d \geq D$,

$$\beta_{i,j+rd}(I^d) \neq 0 \Longleftrightarrow \beta_{i,j+rD}(I^D) \neq 0.$$

Finding $\mathrm{Stab}\,(I)$ directly in terms algebraic properties of I remains open. Areas of future research include producing explicit $\mathrm{Stab}\,(I)$ for other classes of ideals or providing sharper bounds for $\mathrm{Stab}\,(I)$ than those offered here.

After the initial submission of this work, the author has been informed that one of Herzog's students, Pooja Singla, obtained in her thesis a very similar result.

2. Rees algebras of equigenerated ideals.

2.1. Rees algebras and degree restrictions. One common technique used in investigating powers I^n of an ideal I involves passing to the Rees algebra of I. The Rees algebra $\mathcal{R}(I)$ of an ideal I is an object which captures the ideal I and all of its powers.

Definition 2.1. Let $I = (f_0, f_1, \dots, f_k) \subseteq R = \mathbf{k}[x_1, \dots, x_N]$. The *Rees algebra* $\mathcal{R}(I)$ *of* I is

$$\mathcal{R}(I) = R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots \oplus I^nt^n \oplus \cdots .$$

This is occasionally denoted $R[It]$.

In general, we will use a presentation of $\mathcal{R}(I)$ as a quotient module of the ring $S = R[w_0, w_1, \dots, w_k] = \mathbf{k}[x_1, \dots, x_N, w_0, w_1, \dots, w_k]$.

Proposition 2.2 [**9**]. *Let $I = (f_1, \dots, f_k) \subseteq R = \mathbf{k}[x_1, \dots, x_N]$, and let $\mathcal{R}(I)$ be its Rees algebra. Then $\mathcal{R}(I) = R[w_1, \dots, w_k]/L = \mathbf{k}[x_1, \dots, x_N, w_0, w_1, \dots, w_k]/L$, with presentation ideal*

$$L = (f_i - w_it : 1 \leq i \leq k)S[t] \cap S.$$

If $S = \mathbf{k}[x_1, \dots, x_N, w_1, \dots, w_k]$, and $\mathcal{R}(I) = S/L$, then L is the Rees ideal *of I.*

2.2. Resolutions and bigradings of Rees algebras. Taking a resolution (with an appropriately chosen bigrading) of L gives resolutions of all powers of L and can be used to bound or explicitly compute Betti numbers $\beta_{i,j}(I^n)$ for all n.

We will assume throughout this paper that $I = (f_0, f_1, \dots, f_k)$ is an equigenerated ideal of degree r in $R = \mathbf{k}[x_1, \dots, x_N]$. Notationally, we set $\mathcal{R}(I) = S/L$ with L the Rees ideal of I and $S = \mathbf{k}[x_1, \dots, x_N, w_0, w_1, \dots, w_k]$.

We bigrade $\mathcal{R}(I)$ by $\deg(x_i) = (1,0)$ and $\deg(w_i) = (0,1)$ and take the minimal graded free resolution of $\mathcal{R}(I)$ with respect to this grading.

$$\mathcal{F}:\ \mathcal{R}(I) \leftarrow S \leftarrow \bigoplus_{(j,m)} S(-j,-m)^{\beta_{1,(j,m)}} \leftarrow \cdots$$
$$\leftarrow \bigoplus_{(j,m)} S(-j,-m)^{\beta_{p,(j,m)}} \leftarrow 0.$$

Restricting to the strand $(*,d)$, we obtain a (possibly nonminimal) resolution of I^d:

$$\mathcal{F}_\lceil:\ I^d \leftarrow S_{(*,d)} \leftarrow \bigoplus_{(j,m)} S(-j,-m)^{\beta_{1,(j,m)}}_{(*,d)} \leftarrow \cdots$$
$$\leftarrow \bigoplus_{(j,m)} S(-j,m)^{\beta_{p,(j,m)}}_{(*,d)} \leftarrow 0.$$

Tensoring this resolution with $\mathbf{k}$ and taking the homology of the maps gives us $\dim \operatorname{Tor}_i^R(\mathbf{k}, I^d)_{j+rd} = \beta_{i,j+rd}(I^d)$. This shift in the indices of $\beta_{i,j+rd}(I^d)$ accounts for the shift in grading to agree with that of R while viewing I^d as an R module.

Alternately, we could have first tensored with S/M for $M = (x_1, \dots, x_N)$, taken homology of our maps, then restricted in degrees. This will give us modules $\operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_j$, and as these two actions commute, we have that

$$\begin{aligned}\operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)} &= \operatorname{Tor}_i^R(S/M, I^d)_{j+rd}\\ &= \operatorname{Tor}_i^R(\mathbf{k}, I^d)_{j+rd},\end{aligned}$$

where the second equality follows from $S/M \cong \mathbf{k}$ as an R-module.

Hence, we have that all Betti numbers of higher powers can be written in terms of the dimensions of the bigraded modules $\mathrm{Tor}_i^S(S/M, \mathcal{R}(I))$, as

$$\beta_{i,j+rd}(I^d) = \dim\, \mathrm{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)}.$$

3. Bounds on Betti numbers of powers of ideals. We resolve the Rees algebra $\mathcal{R}(I)$ and restrict to fixed w-degree strands to produce explicit bounds on the Betti numbers of I^d.

Theorem 3.1. *Let $I = (f_0, f_1, \ldots, f_k) \subseteq R = \mathbf{k}[x_1, \ldots, x_N]$ with all f_i homogeneous of degree r. Let $\mathcal{R}(I)$ be the Rees algebra of I in ring $S = \mathbf{k}[x_1, \ldots, x_N, w_0, \ldots, w_k]$ with bigrading* $\deg(x_i) = (1, 0)$ *and* $\deg(w_i) = (0, 1)$. *Then*

$$\beta_{i,j+dr}(I^d) \le \sum_{m=0}^{d} \binom{d+k-m}{d-m} \beta_{i,(j,m)}(\mathcal{R}(I))$$

holds for all i, j, d.

Proof. We take a minimal free resolution of $\mathcal{R}(I)$ and consider the degree restricted strand used in Section 2:

$$\mathcal{F}_\lceil :\ I^d \leftarrow S_{(*,d)} \leftarrow \bigoplus_{(j,m)} S(-j,-m)_{(*,d)}^{\beta_{1,(j,m)}} \leftarrow \cdots$$
$$\leftarrow \bigoplus_{(j,m)} S(-j,-m)_{(*,d)}^{\beta_{p,(j,m)}} \leftarrow 0.$$

Let $T = \mathbf{k}[w_0, w_1, \ldots, w_k]$ be the polynomial ring in the w_i-variables. Then we can rewrite our bigraded pieces $S(-j,-m) = R(-j) \otimes T(-m)$. Then, in a fixed strand $(*, d)$, we have:

$$\mathcal{F}_\lceil :\ I^d \leftarrow R \otimes T_d \leftarrow \bigoplus_{(j,m)} R(-j) \otimes T(-m)_d^{\beta_{1,(j,d)}} \leftarrow \cdots$$
$$\leftarrow \bigoplus_{(j,m)} R(-j) \otimes T(-m)_d^{\beta_{p,(j,d)}} \leftarrow 0.$$

It remains to count the dimension over R of the ith module

$$F_i = \bigoplus_{(j,m)} R(-j) \otimes T(-m)_d^{\beta_{i,(j,m)}(\mathcal{R}(I))}$$

in a fixed degree $j + rd$ of the resolution. Finally, the dimension of $T(-m)_d$ is the number of degree $d-m$ monomials in a polynomial ring in $k+1$ variables, or

$$\binom{d+k-m}{k}.$$

So we have that

$$\beta_{i,j+rd}(I^d) \leq \sum_{m=0}^{d} \binom{d+k-m}{k} \beta_{i,(j,m)}(\mathcal{R}(I)),$$

proving the theorem. □

We see from this that the Betti diagram of I^d sits inside an (appropriately degree shifted) table coming from the Betti diagram of the resolution of $\mathcal{R}(I)$. This implies that the number of nonzero graded Betti numbers of I^d is bounded independent of the power d. We refine this rough bound in the following section.

4. Betti diagrams of powers of equigenerated ideals. We are now ready to prove the main theorem:

Theorem 4.1 (Betti tables of powers of equigenerated ideals). *Let $I = (f_0, f_1, \ldots, f_k) \subseteq \mathbf{k}[x_1, \ldots, x_N] = R$ be an equigenerated ideal of degree r. Then there exists a D such that, for all $d > D$, we have*

$$\beta_{i,j+rd}(I^d) \neq 0 \Longleftrightarrow \beta_{i,j+rD}(I^D) \neq 0.$$

Proof. From the calculation in Section 2, we have that

$$\beta_{i,j+rd}(I^d) = \dim \operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)}.$$

The $\operatorname{Tor}_i(S/M, \mathcal{R}(I))$ are finitely generated bigraded S-modules. We decompose them into bigraded components in the following way.

Let $M_i := \operatorname{Tor}_i(S/M, \mathcal{R}(I))$ and $M_{ij} := (M_i)_{(j,*)}$.

Lemma 4.2. *The M_{ij} are finitely generated graded T-modules, where*

$$T = \mathbf{k}[w_0, w_1, \dots, w_k]$$

is the polynomial ring in the w_i-variables.

Proof. As S is a Noetherian ring, we have that S/M and $\mathcal{R}(I)$ are Noetherian S-modules. From this, we have that $\operatorname{Tor}_i^S(S/M, \mathcal{R}(I))$ is a finitely generated S/M-module. We also note that, in a fixed strand degree strand $(j, *)$, we have

$$(S/M)_{(j,*)} \cong (R/M)_j \otimes T_* \cong \mathbf{k} \otimes T_* \cong T,$$

so in a fixed x-degree strand $(j, *)$, we have that $\operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,*)}$ is finitely generated as a T-module. □

As a T-module, each M_{ij} has a Hilbert polynomial

$$P_{ij}(d) := P_{M_{ij}}(d) = \dim(M_i)_{(j,d)}$$

for all $d \geq d_{ij}$, with d_{ij} the regularity of M_{ij}. Hence, for all $d \geq d_{ij}$, $P_{M_{ij}}$ is either identically zero or not.

Note that $D = \max_{i,j}\{d_{ij}\}$ will be an upper bound for $\operatorname{Stab}(I)$, provided such a maximum exists.

Lemma 4.3. *There are only finitely many nonzero M_{ij}.*

Proof. That only finitely many M_i are nonzero follows from

$$\beta_{i,j+rd}(I^d) = \dim \operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)}.$$

As the projective dimension of all powers I^d is bounded by N, the number of variables in our original ring, $\operatorname{Tor}_i^S(S/M, \mathcal{R}(I)) = 0$ for all $i > N$.

We now consider a fixed M_i. Theorem 3.1 gave a bound on the Betti numbers of I^d depending on the Betti numbers of $\mathcal{R}(I)$,

$$\beta_{i,j+rd}(I^d) \leq \sum_{m=0}^{d} \binom{d+k-m}{d-m} \beta_{i,(j,m)}(\mathcal{R}(I)).$$

As for a fixed i, the number of nonzero Betti numbers of $\mathcal{R}(I)$ must be finite, there can be only finitely many j such that $\beta_{i,(j,m)}(\mathcal{R}(I)) \neq 0$. This implies that, for j outside of this set, $\beta_{i,j+rd}(I^d) \leq \sum_{m=0}^{d} 0$ for all d, which implies $\beta_{i,j+rd}(I^d) = 0$. So $M_{ij} = 0$ except for a finite number of cases.

This completes the proof of the lemma. $\square$

By Lemma 4.3, we have that the maximum

$$D = \max_{i,j} \left\{ d_{ij} \right\}$$

exists. Hence, we have that

$$\dim \operatorname{Tor}_i(S/M, \mathcal{R}(I))_{(*,d)} = P_{M_i}(d)$$

is a polynomial function for all $d > D$. We note that, for all $d > D$,

$$\beta_{i,j+dr}(I^d) = \dim (M_i)_{(j,d)} = P_{M_{i,j}}(d) > 0$$

if and only if

$$\beta_{i,j+Dr}(I^D) = \dim (M_i)_{(j,D)} = P_{M_{i,j}}(D) > 0,$$

completing the proof. $\square$

The techniques used throughout the proof of Theorem 4.1 were similar to those seen in [**1, 3, 8**], but extend their results to a classification of all possible nonzero graded Betti numbers of powers of an equigenerated ideal I.

5. Stabilization index of I. The bound D produced in Theorem 4.1 is not sharp, and finding the smallest such D, which we will call the *stabilization index of* I, $\operatorname{Stab}(I)$, in terms of combinatorial data of I is a subject of future research.

Definition 5.1. Let I be a homogeneous ideal equigenerated in degree r in a polynomial ring R. Let $\operatorname{Stab}(I)$ be the smallest such D such that for all $d \geq D$,

$$\beta_{i,j+rd}(I^d) \neq 0 \Longleftrightarrow \beta_{i,j+rD}(I^D) \neq 0.$$

Finding a sharp formula for Stab(I) for ideals I remains open, even in the case of square-free monomial ideals generated in degree 2 (the so-called *edge ideals*) with a linear resolution.

One reason that this class initially appears a good candidate for calculation of Stab(I) is due to a result of Herzog, Hibi and Zheng [**5**] which states the following:

Theorem 5.2 [**5**]. *Let I be a quadratic monomial ideal of the polynomial ring. The following are equivalent*:

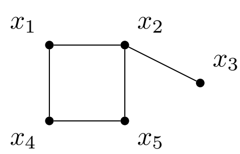
(1) *I has a linear resolution,*

(2) *I has linear quotients,*

(3) *I^k has a linear resolution for all $k \geq 1$.*

So for ideals generated in degree 2 with a linear resolution, it is known that the regularity of such ideals stabilizes at the first power, i.e., $\operatorname{reg}(I_G^d) = 2d$ for all I_G with $\operatorname{reg}(I_G) = 2$. This stabilization of regularity does *not* guarantee a stabilization of the Betti tables, even in this instance.

Example 5.3. Consider the graph G of a 4-cycle with a single attached leaf,



given by ideal

$$I = (x_1x_2, x_1x_4, x_2x_3, x_2x_5, x_4x_5) \subseteq \mathbf{k}[x_1, \dots, x_5].$$

This has a linear resolution (and hence, all powers have linear resolutions), but Stab$(I_G) > 1$:

I

–	0	1	2	3
total:	1	5	6	2
0:	1	·	·	·
1:	·	5	6	2

I^2

–	0	1	2	3	4
total:	1	14	24	13	2
0:	1	·	·	·	·
1:	·	·	·	·	·
2:	·	·	·	·	·
2:	·	14	24	13	2

Through examination of subsequent powers, we note that $\mathrm{Stab}\,(I) = 2$.

This indicates that known formulas bounding (or explicitly providing) regularity of powers of ideals are unlikely to provide bounds on $\mathrm{Stab}\,(I)$.

5.1. Areas of future research. We would like to answer the following questions in subsequent work on these stabilization indices:

(1) Do formulas for $\mathrm{Stab}\,(I)$ exist for squarefree monomial ideals? Do they relate to the dimensions of the facet complex or the Stanley-Reisner complex?

(2) For such squarefree monomial ideals, does $\mathrm{Stab}\,(I_\Delta)$ have a topological interpretation in terms of $\Delta_{\mathrm{pol}(I^n)}$, the Stanley-Reisner complex of the polarization of I^n?

(3) Does there exist a class of ideals for which the D produced in Theorem 4.1 is the sharp bound, i.e., $D = \mathrm{Stab}\,(I)$?

Aside from the stabilization index, the shapes of chain of Betti tables leading up to the stabilized Betti table appear fairly interesting. Generally, the shapes of Betti tables of powers of homogeneous equigenerated ideals seem to be unimodal, in the following sense:

Conjecture 5.4. *Let $I \subseteq R$ be a homogeneous ideal generated in degree r. Then for each pair of indices (i, j) there exist $1 \le D_1 \le D_2 \le \infty$ such that, for all d with $D_1 \le d \le D_2$,*

$$\beta_{i,j+dr}(I^d) \neq 0$$

and for all $d < D_1$ or $D_2 < d$,

$$\beta_{i,j+dr}(I^d) = 0.$$

Note that this conjecture does *not* state that there exists a pair of powers D_1, D_2 such that for *all* pairs (i,j), $\beta_{i,j+dr}$ are nonzero for $D_1 \le d \le D_2$ and zero outside of this range.

The behavior which motivated the conjecture was observing, for *every* example the author has encountered to date, the following: For an ideal I, let

$$b_d = \beta_{i,j+dr}(I^d).$$

The Betti number sequence of powers I^d for each pair of indices (i,j),

$$\{b_1, b_2, b_3, \dots, b_{d-1}, b_d, b_{d+1}, \dots\}$$

has been of the form

$$\{0, 0, \dots, 0, b_d, b_{d+1}, \cdots\}$$

where all $b_i \neq 0$ for $d \le i$, or of the form

$$\{b_0, b_1, \dots, b_{d-1}, 0, 0, \cdots\}$$

where $b_i \neq 0$ for all $0 \le i \le d-1$.

This amounts to the T-modules $M_{ij} = \operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,*)}$ defined above never vanishing in some degree d with another generator of M_{ij} occurring in a higher degree $d+s$.

To date, several hundred thousand randomly generated monomial, binomial and other ideals have been run through Macaulay 2 without a counterexample arising. Code to produce random ideals of various forms and examine these Betti sequences is available on the author's website.

Making progress on this conjecture (or providing a counterexample with pathological Betti diagram behavior) would require a much tighter understanding of the $M_{ij} = \operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,*)}$. These M_{ij} seem to carry interesting structure, and investigating the connections between M_{ij} and the geometry of the ideal I and its Rees algebra $\mathcal{R}(I)$ is another area of future interest.

Acknowledgments. The author would like to thank her advisor, Mike Stillman, for his many helpful conversations and comments throughout the course of this work. This paper would not have been possible without his many suggestions. The author would also like to thank Greg Muller for discussions on some of the most delicate points of the homological algebra involved.

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