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RESOLUTIONS OF DEFINING IDEALS OF ORBIT CLOSURES FOR QUIVERS OF TYPE A_3

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ABSTRACT. We construct explicitly a minimal free resolution of the defining ideal of an orbit closure arising from a representation of the non-equioriented A_3 quiver. The resolution is a generalization of Lascoux's resolution for determinantal ideals.

The case of non-equioriented A_3 quiver is made special by the fact that, in this case, every orbit closure admits a so-called 1-*step desingularization.* Using the resolution we give a description of the minimal set of generators of the defining ideal. The resolution also allows us to read off some geometric properties of the orbit closure, like normality and Cohen-Macaulay. In addition, we give a characterization for the orbit closure to be Gorenstein.

1. Introduction. Let K be a field of characteristic zero. Let $Q = (Q_0, Q_1)$ be a Dynkin quiver with a set of vertices Q_0 and a set of arrows Q_1. We use the notation $ta \xrightarrow{a} ha$ for arrows in Q.

The *representation space* $\operatorname{Rep}(Q, \underline{d})$ of a quiver Q is the collection of all representations of Q of fixed dimension vector $\underline{d}$ (see Section 2 for precise definitions). Note that we can think of $\operatorname{Rep}(Q, \underline{d})$ as the set $\prod_{a \in Q_1} \operatorname{Hom}(K^{d_{ta}}, K^{d_{ha}})$. Thus, $\operatorname{Rep}(Q, \underline{d})$ is a finite dimensional K-vector space with an affine structure.

The algebraic group $\prod_{x \in Q_0} \operatorname{GL}(\underline{d}(x))$ acts on $\operatorname{Rep}(Q, \underline{d})$ by a simultaneous change of basis at every vertex. For $V \in \operatorname{Rep}(Q, \underline{d})$, let $\overline{O}_V$ denote the (Zariski) closure of an orbit O_V. Then $\overline{O}_V$ is a subvariety of $\operatorname{Rep}(Q, \underline{d})$. It is an interesting problem to study the type of singularities that occur in these orbit closures.

When Q is the Dynkin quiver A_2 $(1 \xrightarrow{a} 2)$ the corresponding orbit closures are the well-studied determinantal varieties. Thus, the orbit

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closures $\overline{O}_V$ are natural generalizations of determinantal varieties. The geometry of these orbit closures was first studied by Abeasis, Del Fra and Kraft in [**1**]. They proved for the case of equioriented A_n (over fields of characteristic zero) that the orbit closures are normal, Cohen-Macaulay and have rational singularities. This result was generalized to fields of arbitrary characteristic by Lakshmibai and Magyar in [**10**]. They show using standard monomial theory that the defining ideals of orbit closures in the case of equioriented A_n are reduced, so the singularities of $\overline{O}_V$ are identical to those of Schubert varieties. This implies that the orbit closures are normal, Cohen-Macaulay, etc. This result was generalized to orbit closures for arbitrary quivers of type A_n and D_n by Bobinski and Zwara in [**5, 6**]. They make use of certain *hom-controlled* functors to reduce the general case to a special one and draw their conclusions by comparing the special case to Schubert varieties.

In this paper, we outline a method of constructing a $\mathbf{Z}$-graded complex $\mathbf{F}_\bullet$ supported in $\overline{O}_V$ whenever $\overline{O}_V$ admits a 1-*step desingularization.* This construction works for any Dynkin quiver Q. In the case when then $F_i = 0$ for $i < 0$, the *geometric technique* (also referred to as the Kempf-Lascoux-Weyman geometric technique in recent literature) asserts that $\mathbf{F}_\bullet$ is a minimal free resolution of the normalization of $\overline{O}_V$. For the quiver A_3 with *source-sink orientation*[1] we show that the above condition is satisfied for all orbit closures $\overline{O}_V$.

In effect, we have an algorithm for calculating a minimal resolution which depends only on the Littlewood-Richardson rule and Bott's theorem. The general idea is to construct a desingularization Z of $\overline{O}_V$ such that Z is the total space of a suitable vector bundle. Using the results of Kempf [**9**] on collapsing of vector bundles, Lascoux [**11**] gave the construction of a minimal resolution of determinantal ideals for generic matrices. He made effective use of the combinatorics of representations of the general linear group and Bott's vanishing theorem for the cohomology of homogeneous vector bundles. These results were later generalized to similar cases. We use this generalization for our case of representations of Dynkin quivers to prove our results. A good reference for these results is the book 'Cohomology of vector bundles and syzygies' by Jerzy Weyman [**13**].

In addition to giving us an explicit resolution of the coordinate ring, the geometric technique gives us a direct proof of the result of Bobinski and Zwara [**4**] that orbit closures are normal with rational singularities in the case of non-equioriented A_3. The key proposition is an estimate involving the Euler form of the quiver Q (Proposition 4.5). In principle it is possible to calculate every term of the complex, although it is difficult to find a closed formula for every syzygy. However, we find a closed formula for the first term of the resolution for our case of non-equioriented A_3 (Theorem 4.9). This formula allows us to calculate the minimal generators of defining ideals. We also give a characterization of Gorenstein orbits for our case (Theorem 4.17) and a sufficient condition for orbit closures to be Gorenstein for any Dynkin quiver Q (Theorem 4.14). The techniques described in this paper in the context of non-equioriented A_3 can be generalized to other classes of Dynkin quivers. We handle these cases in our forthcoming papers.

In order to find the resolution described above, we have used Reineke's desingularization [**12**] for the orbit closure Y. We restrict to orbit closures admitting a 1-step desingularization (subsection 2.2) in order to get semisimple vector bundles. This restriction does not induce an additional condition for non-equioriented A_3 since, in this case, every orbit closure admits a 1-step desingularization.

This paper is organized as follows:

• in Section 2, we list some basic definitions and results about representations of quivers, orbit closures and Reineke desingularization.

• In Section 3, we describe the geometric setup we are working in.

• Section 4 contains the main results for non-equioriented A_3; subsection 4.1 contains the calculation of the resolution $\mathbf{F}_\bullet$; in subsection 4.2, we describe the first term of $\mathbf{F}_\bullet$ which gives us the minimal generators of the defining ideal; in subsection 4.3 we investigate the last term of $\mathbf{F}_\bullet$ and obtain a classification of Gorenstein orbits for our case.

2. Preliminaries. First we recall some basic facts about representations of quivers.

A *representation* $((V_i)_{i\in Q_0},(V_\mathfrak{a})_{\mathfrak{a}\in Q_1})$ of Q is an assignment of a finite dimensional K-vector space V_i to every vertex $i \in Q_0$, and K-linear maps $V_{t\mathfrak{a}} \stackrel{V_\mathfrak{a}}{\to} V_{h\mathfrak{a}}$ to every arrow $\mathfrak{a} \in Q_1$. The *dimension vector* of a representation $((V_x)_{x\in Q_0},(V_\mathfrak{a})_{\mathfrak{a}\in Q_1})$ is defined as the function $\underline{d} : Q_0 \to \mathbf{Z}$ given by $\underline{d}(x) = \dim(V_x)$. Given two representations $V = ((V_i)_{i\in Q_0},(V_\mathfrak{a})_{\mathfrak{a}\in Q_1})$ and $W = ((W_i)_{i\in Q_0},(W_\mathfrak{a})_{\mathfrak{a}\in Q_1})$ of Q, a morphism $\Phi : V \to W$ is a collection of K-linear maps $\phi_i : V_i \to W_i$ such that, for every $\mathfrak{a} \in Q_1$, the square

$$\begin{tikzcd} V_{t\mathfrak{a}} \arrow[r, "V_{\mathfrak{a}}"] \arrow[d, "\phi_{t\mathfrak{a}}"] & V_{h\mathfrak{a}} \arrow[d, "\phi_{h\mathfrak{a}}"] \\ W_{t\mathfrak{a}} \arrow[r, "W_{\mathfrak{a}}"] & W_{h\mathfrak{a}} \end{tikzcd}$$

commutes.

With this definition of morphisms, the collection of all representations of a quiver Q (over K) forms a category which we denote by $\mathrm{Rep}_K(Q)$. Given a quiver Q, one can define its path algebra KQ as the K-algebra generated by the paths in Q. It is known that KQ is an associative algebra and is finite dimensional if and only if Q is finite and has no oriented cycles. An important result in the theory of representation theory of associative algebras asserts that, for Q being a finite, connected, acyclic quiver, there is an equivalence of categories $\mathrm{Mod}\, KQ$ and $\mathrm{Rep}_K(Q)$ (refer to [**2**] for details).

Gabriel [**7**] proved that the set $\mathrm{ind}\,(KQ)$ of isomorphism classes of indecomposable representations of Q is in bijective correspondence with the set of positive roots R^+ of the corresponding root systems. Under this correspondence, simple roots correspond to simple objects. In particular, $\mathrm{ind}\,(KQ)$ is a finite set. Every representation V of Q can be written uniquely (up to permutation of factors) as a direct sum of indecomposable representations

$$V = \bigoplus_{\alpha \in R^+} m_\alpha X_\alpha$$

(where m_α is the multiplicity of X_α in V). The indecomposable representations can be arranged as the vertices of a graph called the Auslander-Reiten quiver $\Gamma(Q)$ of Q. The arrows in this graph represent the irreducible maps between the indecomposable objects. Thus, the vertices and arrows of an Auslander-Reiten quiver $\Gamma(Q)$ constitute the building blocks of representations of the corresponding path algebra KQ and morphisms between them.

Given a quiver Q, one can define an Euler form $\langle \cdot, \cdot \rangle$ on the dimension vectors of Q as follows.

Definition 2.1. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two elements of $\mathbf{N}^{Q_0}$ ($|Q_0| = n$). Then the *Euler form* $\langle \cdot, \cdot \rangle$ is

$$\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{a \in Q_1} x_{ta} y_{ha}. \tag{2.1}$$

Remark 1. The Euler form can also be expressed in terms of the Cartan matrix C_Q of Q as

$$\langle x, y \rangle = x^t (C_Q^{-1})^t y.$$

Remark 2. We have the following useful dimension formula in terms of the Euler form (refer to [**2**]): if $V, W \in \mathrm{Rep}\,(Q, \underline{d})$, then

$$\langle \dim V, \dim W \rangle = \dim_K \mathrm{Hom}_{KQ}(V, W) - \dim_K \mathrm{Ext}^1_{KQ}(V, W).$$

2.1. Orbit closures. The group $\prod_{x \in Q_0} \mathrm{GL}\,(\underline{d}(x))$ acts on $\mathrm{Rep}\,(Q, \underline{d})$ by

$$((g_x)_{x \in Q_0}, (V_\mathfrak{a})_{\mathfrak{a} \in Q_1}) \longmapsto (g_{h\mathfrak{a}}\ V_\mathfrak{a}\ g_{t\mathfrak{a}}^{-1})_{\mathfrak{a} \in Q_1}.$$

The orbits of this action are the isomorphism classes of representations of Q.

Let $V, W \in \mathrm{Rep}\,(Q, \underline{d})$. We say that $V \leq_{\deg} W$ (i.e., V degenerates to W) if the orbit of W is contained in the closure of the orbit of V (i.e., $O_W \subset \overline{O}_V$). This introduces a partial order on the orbits. There is also Riedtmann's rank criterion: $V \leq W$ if $\dim \mathrm{Hom}_Q(X, V) \leq \dim \mathrm{Hom}_Q(X, W)$ for all indecomposables X in $\mathrm{Rep}\,(Q, \underline{d})$. The connection between these two partial orders is given by the following.

Theorem 2.2 [**6**]. *If A is a representation-directed, finite dimensional, associative K-algebra, then the partial orders $\leq_{\deg}$ and $\leq$ coincide.*

An algebra A is called *representation-directed* if every indecomposable A-module M is directing. This means M is not part of a sequence

$$M_0 \xrightarrow{f_1} M_1 \cdots \xrightarrow{f_t} M_t$$

of indecomposable A-modules $M_0, \dots, M_t$ and nonzero nonisomorphisms $f_1, \dots, f_t$ satisfying $M_0 = M_t$ (that is, the sequence is a cycle). If A is a representation-finite hereditary algebra, then it can be shown that every indecomposable A-module is directing [**2**, Chapter 9, Lemma 1.1]. Thus, every representation-finite hereditary algebra is representation-directed.

Since Q is a Dynkin quiver, the path algebra KQ is representation-finite and hereditary. Thus, Theorem 2.2 applies to modules over KQ or representations of Q. So the orbit of $V \in \mathrm{Rep}\,(Q, \underline{d})$ is given by

(2.2)
$$O_V = \{W \in \mathrm{Rep}\,(Q, \underline{d}) \mid \dim \mathrm{Hom}_Q(X, V) = \dim \mathrm{Hom}_Q(X, W)\},$$

and the corresponding orbit closure is

$$\overline{O}_V = \{W \in \mathrm{Rep}\,(Q, \underline{d}) \mid \dim \mathrm{Hom}_Q(X, V) \leq \dim \mathrm{Hom}_Q(X, W)\},$$

where X varies over all indecomposables in $\mathrm{Rep}\,(Q, \underline{d})$. Thus,

$$\overline{O}_V = \bigcup_{V \leq W} O_W.$$

2.2. Desingularization. In [**12**], Reineke describes an explicit method of constructing desingularizations of orbit closures of representations of Q. The desingularizations depend on certain directed partitions of the isomorphism classes of indecomposable objects.

Definition 2.3. A partition $\mathcal{I}_* = (\mathcal{I}_1, \dots, \mathcal{I}_s)$, where $R^+ = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_s$, is called directed if:

(1) $\mathrm{Ext}^1_Q(X_\alpha, X_\beta) = 0$ for all $\alpha, \beta \in \mathcal{I}_t$ for $t = 1, \dots, s$.

(2) $\mathrm{Hom}_Q(X_\beta, X_\alpha) = 0 = \mathrm{Ext}^1_Q(X_\alpha, X_\beta)$ for all $\alpha \in \mathcal{I}_t, \beta \in \mathcal{I}_u$, $t < u$.

These conditions can be expressed in terms of the Euler form as:

(1) $\langle \alpha, \beta \rangle = 0$ for $\alpha, \beta \in \mathcal{I}_t$ for $t = 1, \dots, s$.

(2) $\langle \alpha, \beta \rangle \geq 0 \geq \langle \beta, \alpha \rangle$ for $\alpha \in \mathcal{I}_t$, $\beta \in \mathcal{I}_u$, $t < u$.

A partition of indecomposables exists because the category of finite-dimensional representations is directed; in particular, we can choose a sectional tilting module and let $\mathcal{I}_t$ be its Coxeter translates. We fix a partition $\mathcal{I}_*$ of $\operatorname{ind}(KQ)$. For a representation $V = \oplus_{\alpha \in R^+} m_\alpha X_\alpha$, we define representations

$$V_{(t)} := \bigoplus_{\alpha \in \mathcal{I}_t} m_\alpha X_\alpha, \quad t = 1, \dots, s.$$

Then $V = V_{(1)} \oplus \cdots \oplus V_{(s)}$. Let $\underline{d}_t = \dim V_{(t)}$. Then $\dim V_x = d(x) = d_1(x) + \cdots + d_s(x)$ for every $x \in Q_0$. We consider the incidence variety

$$\begin{aligned} Z_{\mathcal{I}_*,V} \subset \operatorname{Rep}_K(Q, \underline{d}) \\ \times \prod_{x \in Q_0} \operatorname{Flag}(d_s(x), d_{s-1}(x) + d_s(x), \dots, \\ d_2(x) + \cdots + d_s(x), V_x) \end{aligned}$$

defined as

$$\begin{aligned} Z_{\mathcal{I}_*,V} = \{(V, (R_s(x) \subset R_{s-1}(x) \subset \cdots \subset R_2(x) \subset V_x)) \mid \\ \forall \mathfrak{a} \in Q_1, \forall t,\ V_\mathfrak{a}(R_t(t\mathfrak{a})) \subset R_t(h\mathfrak{a})\}. \end{aligned}$$

In this case, we say that Z is an $(s-1)$-step desingularization.

Theorem 2.4 [**12**]. *Let Q be a Dynkin quiver, $\mathcal{I}_*$ a directed partition of R^+. Then the projection*

$$q : Z_{\mathcal{I}_*,V} \longrightarrow Rep_K(Q, \underline{d})$$

makes $Z_{\mathcal{I}_*,V}$ a desingularization of the orbit closure $\overline{O}_V$. More precisely, the image of q equals $\overline{O}_V$ and q is a proper birational isomorphism of $Z_{\mathcal{I}_*,V}$ and $\overline{O}_V$.

In the next section, we will realize $Z_{\mathcal{I}_*,V}$ as the total space of a vector bundle η^* over $\prod_{x \in Q_0} \operatorname{Flag}(d_s(x), d_{s-1}(x) + d_s(x), \dots, d_2(x) + \cdots + d_s(x), V_x)$.

3. The geometric technique. The varieties of type $Z_{\mathcal{I}_*,V}$ described in subsection 2.2 are the total spaces of homogeneous vector bundles on the product of flag varieties. For $x \in Q_0$, let $\underline{d}_*(x)$ denote the sequence $(d_s(x), d_s(x)+d_{s-1}(x), \ldots, d_s(x)+d_{s-1}(x)+\cdots+d_1(x) = d(x))$. We will use shorthand notation

$$Z_{\underline{d}_*} \subset \operatorname{Rep}(Q, \underline{d}) \times \prod_{x \in Q_0} \operatorname{Flag}(\underline{d}_*(x), V_x)$$

to denote the incidence varieties described in subsection 2.2 for a fixed representation V and a fixed partition $\mathcal{I}_*$.

The space $\operatorname{Rep}(Q, \underline{d})$ has the structure of an affine variety. Let A denote the coordinate ring of $\operatorname{Rep}(Q, \underline{d})$.

Let $\mathcal{R}_t(x)$ denote the tautological subbundle of rank $(d_1(x) + \cdots + d_t(x))$ on $\operatorname{Flag}(\underline{d}_*(x), V_x)$ and $\mathcal{Q}_t(x)$ denote the corresponding tautological factor bundle. We define the following vector bundles:

$$\xi(\mathfrak{a}) = \sum_{t=1}^{s} \mathcal{R}_t(t\mathfrak{a}) \otimes \mathcal{Q}_t(h\mathfrak{a})^* \subset V_{d(t\mathfrak{a})} \otimes V^*_{d(h\mathfrak{a})} \tag{3.1}$$

$$\eta(\mathfrak{a}) = V_{d(t\mathfrak{a})} \otimes V^*_{d(h\mathfrak{a})} / \xi(\mathfrak{a}).$$

We set

$$\eta = \bigoplus_{\mathfrak{a} \in Q_1} \eta(\mathfrak{a}) \tag{3.3}$$

$$\xi = \bigoplus_{\mathfrak{a} \in Q_1} \xi(\mathfrak{a}). \tag{3.4}$$

Then $Z = Z_{\underline{d}_*}$ is the total space of η^*. We have the following setup:

$$\begin{array}{ccccc}
Z \subset & \operatorname{Rep}(Q, \underline{\mathrm{d}}) \times & \prod_{x \in Q_0} \operatorname{Flag}(\underline{\mathrm{d}}_*(x), K^{d(x)}) & & \\
\downarrow q' & \downarrow q & & \searrow p & \\
\overline{O}_V \subset & \operatorname{Rep}(Q, \underline{\mathrm{d}}) & & & \prod_{x \in Q_0} \operatorname{Flag}(\underline{\mathrm{d}}_*(x), K^{d(x)})
\end{array}$$

The structure sheaf $\mathcal{O}_Z$ can be resolved using the vector bundle ξ over $\prod_{x\in Q_0}\operatorname{Flag}(\underline{d}_*(x), V_x)$; this is a Koszul complex of sheaves on $\operatorname{Rep}(Q,\underline{d}) \times \prod_{x\in Q_0}\operatorname{Flag}(\underline{d}_*(x), V_x)$:

$$0 \longrightarrow \bigwedge^{t}(p^*\xi) \longrightarrow \cdots \longrightarrow \bigwedge^{2}(p^*\xi)$$
$$\longrightarrow p^*\xi \xrightarrow{\delta} \mathcal{O}_{\operatorname{Rep}(Q,\underline{d})\times\prod_{x\in Q_0}\operatorname{Flag}(\underline{d}_*(x),V_x)}.$$

Applying the direct image functor Rq_* to this complex gives a free resolution $\mathbf{F}_\bullet$ of $K[\overline{O}_V]$ in terms of cohomology bundles on V. The terms of this resolution are given by [**13**, Theorem 5.1.2]:

$$F_i = \bigoplus_{j\geq 0} H^j\left(\prod_{x\in Q_0}\operatorname{Flag}(\underline{d}_*(x), V_x), \bigwedge^{i+j}\xi\right) \otimes A(-i-j).$$

We identify A with the symmetric algebra

$$\bigotimes_{\mathfrak{a}\in Q_1}\operatorname{Sym}(V_{t\mathfrak{a}}\otimes V_{h\mathfrak{a}}^*).$$

Theorem 3.1 [**13**, Theorem 5.1.3]. *The normalization of $\overline{O}_V$ has rational singularities if and only if $F_i = 0$ for $i < 0$. The orbit closure $\overline{O}_V$ is normal with rational singularities if and only if $F_i = 0$ for $i < 0$ and $F_0 = A$.*

In the next section, we apply the above tool for calculations in the case of a non-equioriented quiver of type A_3. We will consider a family of incidence varieties which is more general in the sense that we take arbitrary dimension vectors $\underline{d}_1, \ldots, \underline{d}_s$ in place of the dimension vectors described by the partition above; on the other hand, we will restrict to 1-step desingularizations. In this case, ξ is semi-simple and has the form

$$\xi = \bigoplus_{\mathfrak{a}\in Q_1}\mathcal{R}_1(t\mathfrak{a})\otimes\mathcal{Q}_1(h\mathfrak{a})^*. \tag{3.5}$$

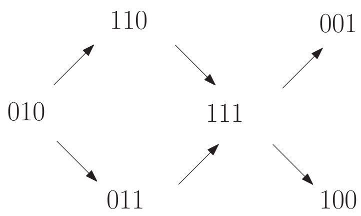


FIGURE 1. AR quiver for $1 \xrightarrow{a} 3 \xleftarrow{b} 2$.

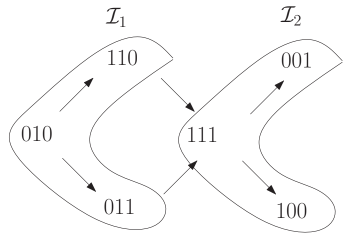


FIGURE 2. Partition.

4. Non-equioriented quiver of type A_3. We will work with non-equioriented quiver $Q = A_3$ in the form $1 \xrightarrow{a} 3 \xleftarrow{b} 2$. Note that we can choose this orientation without loss of generality (Remark 4.1).

Recall that any representation of Q can be expressed uniquely as a direct sum of indecomposable representations of Q. The AR quiver $\Gamma(Q)$ of $\mathrm{Rep}_K(Q)$ lists all the indecomposables along with the irreducible maps between them. Since Q is fixed we can denote the indecomposable representations by writing the dimension of V_i at the vertex i, for example, 110 denotes the representation $K \to K \leftarrow 0$. With this notation the AR quiver of $1 \xrightarrow{a} 3 \xleftarrow{b} 2$ is as shown in Figure 1.

We can construct a partition $(\mathcal{I}_1, \mathcal{I}_2)$ of this quiver as described in subsection 2.2 which has the form shown in Figure 2.

The fact that we can partition the $\Gamma(Q)$ into two parts means that every orbit will admit a 1-step desingularization. This important point distinguishes the case of the non-equioriented A_3 quiver. Note that this is the only 2-part partition possible for the AR quiver of $1 \xrightarrow{a} 3 \xleftarrow{b} 2$.

Remark 4.1. Reversing all the arrows of Q gives the opposite quiver Q^{op}. The vector space duality $D = \mathrm{Hom}_k(-, k)$ induces a duality functor

$$D : \mathrm{Rep}_K(Q) \longrightarrow \mathrm{Rep}_K(Q^{op}).$$

If $V = ((V_i)_{i\in Q_0}, (V_{\mathfrak{a}})_{\mathfrak{a}\in Q_1}) \in \mathrm{Rep}_K(Q)$, then the dual representation

$$V^* = ((DV_i)_{i\in Q_0^{op}}, (DV_{\mathfrak{a}})_{\mathfrak{a}\in Q_1^{op}})$$

is an element of $\mathrm{Rep}_K(Q^{op})$ (for details refer to [**3**, Chapter 3]). In particular, $\dim(V^*) = \dim(V)$. A resolution of $\overline{O}_V$ gives a resolution of $\overline{O}_{V^*}$ and vice-versa (see Example 4.8).

The duality maps projectives (respectively, injectives) in $\Gamma(Q)$ to injectives (respectively, projectives) in $\Gamma(Q^{op})$. Thus, the AR quiver $\Gamma(Q^{op})$ is the mirror image of $\Gamma(Q)$ in which all the arrows are reversed. A partition of $\Gamma(Q)$ then gives a corresponding partition of $\Gamma(Q^{op})$. Using this partition, it is not difficult to see the correspondence between resolutions of $\overline{O}_V$ and $\overline{O}_{V^*}$. By uniqueness of minimal free resolutions, the resolution obtained is independent of the choice of partition.

4.1. Calculation of $\mathbf{F}_\bullet$. Let $V = V_1 \overset{V_{\mathfrak{a}}}{\to} V_3 \overset{V_{\mathfrak{b}}}{\leftarrow} V_2$ be a representation of Q. By the unique decomposition theorem, $V = a(010) \oplus b(110) \oplus c(011) \oplus d(111) \oplus e(001) \oplus f(100)$ where the non-negative integers a, b, c, d, e, f denote the multiplicities with which the corresponding indecomposable representations appear as a summand of V. Then, the dimension vector of V written as $(\dim V_1, \dim V_2, \dim V_3)$ is $\alpha = (b+d+f,\ c+d+e,\ a+b+c+d)$. Reineke's construction of the desingularization Z dictates that $\beta = (d+f,\ d+e,\ d)$. Using the above partition we get the desingularization $Z \subset \mathrm{Rep}(Q,\alpha) \times \mathrm{Gr}(d+f, V_1) \times \mathrm{Gr}(d+e, V_2) \times \mathrm{Gr}(d, V_3)$ of $\overline{O}_V$, given by

(4.1)
$$Z = \{(R_1, R_2, R_3) \in \mathrm{Gr}(d+f, V_1) \times \mathrm{Gr}(d+e, V_2) \times \mathrm{Gr}(d, V_3) \mid ((R_x)_{x\in Q_0}, V_{\mathfrak{a}}, V_{\mathfrak{b}}) \in \mathrm{Rep}(Q, \beta)\}$$

or equivalently by

(4.2)
$$Z = \{(V_{\mathfrak{a}}, V_{\mathfrak{b}}) \in \mathrm{Hom}(V_1, V_3) \times \mathrm{Hom}(V_2, V_3) \mid V_{\mathfrak{a}}(R_1) \subset R_3 \quad \text{and} \quad V_{\mathfrak{b}}(R_2) \subset R_3\}.$$

We may visualize Z as being of the form:

$$\begin{array}{ccccc} V_1 & \longrightarrow & V_3 & \longleftarrow & V_2 \\ \bigcup & & \bigcup & & \bigcup \\ R_1 & \longrightarrow & R_3 & \longleftarrow & R_2 \end{array}$$

with dimension vectors of the rows being $\alpha = (b+d+f,\ c+d+e,\ a+b+c+d)$ and $\beta = (d+f,\ d+e,\ d)$. Let $Q_x := V_x/R_x$ and $\gamma_x = \alpha_x - \beta_x$ be such that $\dim Q_x = \gamma_x$ (in the notation of subsection 2.2, $\beta = \underline{d}_2$, $\gamma = \underline{d}_1$ and $\alpha = \underline{d}_1 + \underline{d}_2$).

Let $\mathcal{R}_x$ and $\mathcal{Q}_x$ denote, respectively, the tautological subbundle and factorbundle of the trivial vector bundle $V_x \times \mathrm{Gr}\,(\beta_x, V_x) \xrightarrow{p} \mathrm{Gr}\,(\beta_x, V_x)$ for $1 \leq x \leq 3$. By definition, the fibers of a point $R_x \in \mathrm{Gr}\,(\beta_x, V_x)$ with respect to vector bundles $\mathcal{R}_x$ and $\mathcal{Q}_x$ are R_x and Q_x, respectively. Identify the vector space $\mathrm{Hom}\,(V, W)$ with $V^* \otimes W$. Under this identification, the desingularization Z can be viewed as being the total space of a vector bundle η which is a subbundle of the trivial vector bundle

$$\mathcal{E} = (V_1^* \otimes V_3 \oplus V_2^* \otimes V_3) \times \prod_{x \in Q_0} \mathrm{Gr}\,(\beta_x, V_x) \longrightarrow \prod_{x \in Q_0} \mathrm{Gr}\,(\beta_x, V_x).$$

In order to calculate the complex $\mathbf{F}_\bullet$ we consider the vector bundle which is dual to the factorbundle $\mathcal{E}/\eta$ given by

$$\xi = \mathcal{R}_1 \otimes \mathcal{Q}_3^* \oplus \mathcal{R}_2 \otimes \mathcal{Q}_3^*. \tag{4.3}$$

Let $\mathcal{V}$ denote $\prod_{x \in Q_0} \mathrm{Gr}\,(\beta_x, V_x)$. From [**13**, Theorem 5.1.2], we know that the terms of the free resolution $\mathbf{F}_\bullet$ resolving the structure sheaf of Z are

$$F_i = \bigoplus_{j \geq 0} H^j\left(\mathcal{V}, \bigwedge^{i+j} \xi\right) \otimes A(-i-j). \tag{4.4}$$

By Cauchy's formula [**13**, Corollary 2.3.3], we have

$$\bigwedge^t \xi = \bigoplus_{|\lambda|+|\mu|=t} S_\lambda \mathcal{R}_1 \otimes S_\mu \mathcal{R}_2 \otimes S_{\lambda'} \mathcal{Q}_3^* \otimes S_{\mu'} \mathcal{Q}_3^*, \tag{4.5}$$

where S_λ is the Schur functor corresponding to partition λ and λ' denotes the transpose (or conjugate) of λ.

To calculate $H^j(\mathcal{V}, \bigwedge^{i+j} \xi)$, we apply Bott's algorithm to the weights corresponding to each summand $S_\lambda \mathcal{R}_1 \otimes S_\mu \mathcal{R}_2 \otimes S_\nu \mathcal{Q}_3^*$ for all S_ν occurring in $S_{\lambda'} \otimes S_{\mu'}$. This consists of applying an exchange rule (which we will call the *Bott exchange*) to the weights.

Theorem 4.1 [**13**, Remark 4.1.5]. *Let $\mathcal{V}$ be a nonsingular projective variety and $\mathcal{E}$ a vector bundle of rank n over $\mathcal{V}$. Let $\mathrm{Flag}_{\mathcal{V}}(\mathcal{E})$ denote the flag variety associated to $\mathcal{E}$ and $h : \mathrm{Flag}_{\mathcal{V}}(\mathcal{E}) \to \mathcal{V}$ be the corresponding vector bundle. Let $\mathcal{L}(\alpha)$ be a line bundle over $\mathrm{Flag}_{\mathcal{V}}(\mathcal{E})$ of weight $\alpha = (\alpha_1, \dots, \alpha_n)$. The permutation $\sigma_i = (i, i+1) \in \Sigma_n$ acts on the set of weights as follows:*

$$\sigma_i \cdot \alpha = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1} - 1, \alpha_i + 1, \alpha_{i+2}, \dots, \alpha_n).$$

Then one of two mutually exclusive possibilities can occur:

(1) *if, for some i, $\alpha_{i+1} = \alpha_i + 1$ then $R^i h_* \mathcal{L}(\alpha) = 0$ for all $i \geq 0$;*

(2) *if, after j exchanges, α is transformed into a non-increasing sequence β, then $R^i h_* \mathcal{L}(\alpha) = 0$ for $i \neq j$ and*

$$R^j h_* \mathcal{L}(\alpha) = S_\beta \mathcal{E}.$$

We are interested in those weights which transform into non-increasing sequences after some number of Bott exchanges. For details we refer the reader to [**13**, Theorem 4.1.4].

We use the following notation: the rectangular partition with a columns and b rows will be denoted by (a^b). The weight of $S_\lambda \mathcal{R}_x$ is written as $(0^{\dim Q_x}, \lambda)$ and the weight of $S_\nu \mathcal{Q}^*$ is written as $(-\nu^{op}, 0^{\dim R_x})$. Here 0^n is the n-tuple consisting of zeroes and $-\nu^{op}$ is the partition dual to ν; it is obtained by writing ν in reverse order with minus signs.

We will denote by N_λ the number of exchanges applied to a weight $(0^p, \lambda)$. Note that, if the first k terms of λ are involved in the exchanges, then $N_\lambda = pk$.

Thus, for applying Bott's algorithm to a summand of $S_\lambda \mathcal{R}_1 \otimes S_\mu \mathcal{R}_2 \otimes S_\nu \mathcal{Q}_3^*$, we apply the Bott exchanges to the weights

$$(0^{\gamma_1}, \lambda), (0^{\gamma_2}, \mu), (-\nu^{op}, 0^{\beta_3}).$$

Now suppose the first u terms of λ, the first v terms of μ and the first w terms of ν are involved in the exchanges. Then $N_\lambda = u\gamma_1$, $N_\mu = v\gamma_2$ and $N_\nu = w\beta_3$. Let $[0^{\gamma_1}, \lambda]$ denote the sequence obtained *after* all exchanges are applied. The application of Bott's algorithm works as follows:

$$
\begin{aligned}
(0^{\gamma_1}, \lambda) \xrightarrow[\text{Bott exchanges}]{\text{after } u\gamma_1} & [0^{\gamma_1}, \lambda] \\
& = (\lambda_1 - \gamma_1, \dots, \lambda_u - \gamma_1,\ u^{\gamma_1},\ \lambda_{u+1}, \dots).
\end{aligned}
$$

$$
(0^{\gamma_2}, \mu) \xrightarrow[\text{Bott exchanges}]{\text{after } v\gamma_2} [0^{\gamma_2}, \mu] = (\mu_1 - \gamma_2, \dots, \mu_v - \gamma_2,\ v^{\gamma_2},\ \mu_{v+1}, \dots).
$$

We write the third weight in its dual form:

$$
\begin{aligned}
(-\nu^{op}, 0^{\beta_3}) \xrightarrow[\text{Bott exchanges}]{\text{after } w\beta_3} & [-\nu^{op}, 0^{\beta_3}] \\
& = (\dots, -\nu_{w+1},\ w^{\beta_3},\ -\nu_w - \beta_3, \dots, \nu_1 - \beta_3).
\end{aligned}
$$

Then the total number of exchanges N equals $u\gamma_1 + v\gamma_2 + w\beta_3$. We summarize this in:

Proposition 4.2. *Let $V = ((V_1, V_2, V_3), (V_{\mathfrak{a}}, V_{\mathfrak{b}}))$ be a representation of Q. The terms of the complex $\mathbf{F}_\bullet$ are given by*

$$
F_i = \bigoplus_{t=1}^{\operatorname{rank}\xi} \bigoplus_{|\lambda|+|\mu|=t} c^{\nu}_{\lambda',\mu'} (S_{[0^{\gamma_1},\lambda]} V_1 \otimes S_{[0^{\gamma_2},\mu]} V_2 \otimes S_{[-\nu^{op},0^{\beta_3}]} V_3^*),
$$

where $S_\nu \subset S_{\lambda'} \otimes S_{\mu'}$, $c^{\nu}_{\lambda',\mu'}$ is the corresponding Littlewood-Richardson coefficient and $|\lambda| + |\mu| - N = i$.

Since the term $|\lambda| + |\mu| - N$ occurs often, we give it a name.

Definition 4.3. Let $\lambda(a)$ be a partition associated to an arrow $a \in Q_1$, and let $\underline{\lambda} = (\lambda(a))_{a \in Q_1}$. Define

$$
D(\underline{\lambda}) := \sum_{a \in Q_1} |\lambda(a)| - N. \tag{4.6}
$$

In our case the tuple $\underline{\lambda}$ will be (λ, μ), that is, we associate partition λ to arrow a and μ to arrow b of $Q : 1 \xrightarrow{a} 3 \xleftarrow{b} 2$. We denote by ν a partition occurring in the Littlewood-Richardson product of λ and μ. From the earlier discussion, it is clear that the triple (u, v, w) depends on the partitions (λ, μ). We denote the triple (u, v, w) by $\underline{u}(\underline{\lambda})$. From Proposition 4.2, it is clear that, in order to calculate the terms F_i of the resolution, we need to calculate $D(\underline{\lambda})$. Due to the number of variables involved and the peculiar form of exchanges required, the calculation of a closed formula for $D(\underline{\lambda})$ is not easy in general. Our key result is Proposition 4.5 which gives us a lower bound for $D(\underline{\lambda})$ in terms of the Euler form of quiver Q. First we prove a lemma which is an easy exercise in counting boxes.

Lemma 4.4. *Let λ be a partition. Then, for all a and b,*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_a \leq ab + (\lambda'_{b+1} + \cdots + \lambda'_{\text{last}}).$$

Proof. We consider three cases.

Case (1). $\lambda'_{b+1} = a$. Then

$$\lambda_1 + \lambda_2 + \cdots \lambda_a = ab + \lambda'_{b+1} + \cdots + \lambda'_{last}.$$

Case (2). $\lambda'_{b+1} > a$. In this case $\lambda'_{b+1}, \lambda'_{b+2}, \ldots \lambda'_{last}$ contribute more boxes so that

$$\lambda_1 + \lambda_2 + \cdots + \lambda'_a \leq ab + \lambda'_{b+1} + \cdots + \lambda'_{\text{last}}.$$

Case (3). $\lambda'_{b+1} < a$. Here the rectangle ab contributes more boxes, so that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_a \leq ab + \lambda'_{b+1} + \cdots + \lambda'_{last}.$$

By symmetry, we also have for all a and b:

$$\lambda'_1 + \lambda'_2 + \cdots + \lambda'_a \leq ab + (\lambda_{b+1} + \cdots + \lambda_{last}). \qquad \square$$

Proposition 4.5. *Let Q be the non-equioriented quiver A_3. Let $\underline{\lambda}$ be a tuple of partitions associated to arrows of Q, and let $\underline{u}(\underline{\lambda}) \in \mathbf{N}^{|Q_0|}$ be a vector that depends on $\underline{\lambda}$. If $\langle \cdot, \cdot \rangle$ denotes the Euler form on Q, then*

$$D(\underline{\lambda}) \geq \langle \underline{u}(\underline{\lambda}), \underline{u}(\underline{\lambda}) \rangle .$$

Proof. Since $[0^{\gamma_1}, \lambda] = (\lambda_1 - \gamma_1, \lambda_2 - \gamma_1, \dots, \lambda_u - \gamma_1, u^{\gamma_1}, \lambda_{u+1}, \dots)$ is a non-increasing sequence, we have that each of $\lambda_1 - \gamma_1, \dots, \lambda_u - \gamma_1$ is greater than (or equal to) u, which means each of $\lambda_1, \dots, \lambda_u$ is greater than (or equal to) $u + \gamma_1$. Thus $\lambda_1 + \cdots + \lambda_u \geq u^2 + u\gamma_1$. Similarly, $\mu_1 + \cdots + \mu_v \geq v^2 + v\gamma_2$ and $\nu_1 + \cdots + \nu_w \geq w^2 + w\beta_3$. By Lemma 4.4, we get

$$\begin{aligned} w.u &\geq (\lambda_1' + \cdots + \lambda_w') - (\lambda_{u+1} + \cdots + \lambda_{\text{last}}) \\ w.v &\geq (\mu_1' + \cdots + \mu_w') - (\mu_{v+1} + \cdots + \mu_{\text{last}}) \\ \text{Adding } w(u+v) &\geq (\lambda_1' + \cdots + \lambda_w' + \mu_1' + \cdots + \mu_w') \\ &\quad - (\lambda_{u+1} + \cdots \lambda_{\text{last}} + \mu_{v+1} + \cdots + \mu_{\text{last}}) \\ &\geq \nu_1 + \cdots + \nu_w \\ &\quad - (\lambda_{u+1} + \cdots + \lambda_{\text{last}} + \mu_{v+1} + \cdots + \mu_{\text{last}}) \end{aligned}$$

so that

$$\nu_1 + \cdots + \nu_w \leq w(u+v) + (\lambda_{u+1} + \cdots + \lambda_{\text{last}} + \mu_{v+1} + \cdots + \mu_{\text{last}}).$$

Therefore,

$$\begin{aligned} &(u^2 + u\gamma_1) + (v^2 + v\gamma_2) + (w^2 + w\beta_3) \\ &\qquad \leq \lambda_1 + \cdots + \lambda_u + \mu_1 + \cdots + \mu_v + \nu_1 + \cdots + \nu_w \\ &\qquad \leq \lambda_1 + \cdots + \lambda_u + \mu_1 + \cdots + \mu_v + w(u+v) + \lambda_{u+1} + \cdots \\ &\qquad\quad + \lambda_{\text{last}} + \mu_{v+1} + \cdots + \mu_{\text{last}} \\ &\qquad = w(u+v) + |\lambda| + |\mu|, \\ \text{So } |\lambda| + |\mu| &\geq (u^2 + u\gamma_1) + (v^2 + v\gamma_2) + w(w + \beta_3 - u - v) \\ &= u\gamma_1 + v\gamma_2 + w\beta_3 + (u^2 + v^2 + w^2 - uw - vw) \\ &= u\gamma_1 + v\gamma_2 + w\beta_3 + \langle (u,v,w), (u,v,w) \rangle . \qquad \square \end{aligned}$$

In their paper [**4**], Bobinski and Zwara proved the normality of orbit closures for Dynkin quivers of type A_n with arbitrary orientation. Using the above proposition we can derive the normality of orbit closures in our case.

Corollary 4.6. *In the case of quiver $Q : 1 \to 2 \leftarrow 3$ the orbit closures are normal, Cohen-Macaulay with rational singularities.*

Proof. We have that $\langle (u,v,w),(u,v,w)\rangle \geq 0$ since it is the Euler form of a Dynkin quiver Q. Then, from Propositions 4.2 and 4.5, $F_i = 0$ for $i < 0$. Also, $\langle (u,v,w),(u,v,w)\rangle = 0$ if and only if $u = v = w = 0$ in which case $\lambda = \mu = \nu = 0$. Thus, $F_0 = A$. This, together with Theorem 3.1, implies that the orbit closure is normal with rational singularities. $\square$

Remark 4.2. For purposes of calculation, it is useful to record some simple observations regarding the sizes of partitions λ, μ and ν. From equation 4.5, it is clear that, when calculating $\bigwedge^t \xi$, we only need to consider those partitions λ, μ, ν such that λ is contained in a $\dim R_1 \times \dim Q_3$ rectangle, μ is contained in a $\dim R_2 \times \dim Q_3$ rectangle and ν is contained in a $\dim Q_3 \times \dim (R_1 + R_2)$ rectangle. Thus, the largest possible contributing triples are $(\lambda,\mu,\nu) = (\gamma_3^{\beta_1}, \gamma_3^{\beta_2}, (\beta_1+\beta_2)^{\gamma_3})$ (the notation α^β stands for a $\beta \times \alpha$ rectangle, i.e., the rectangular partition $(\alpha, \alpha, \dots, \alpha)$ of length β).

Example 4.7. Let $V = 010 \oplus 011 \oplus 110 \oplus 111 \oplus 100 \oplus 001$ and I be the defining ideals of $\overline{O}_V$. Then, $\alpha = (3,3,4)$ and $\beta = (2,2,1)$. The representation space $\mathrm{Rep}\,(Q, \underline{d})$ is $\mathrm{Hom}\,(K^3, K^4) \times \mathrm{Hom}\,(K^3, K^4)$ and the coordinate ring is

$$A = \mathrm{Sym}\,(V_1 \otimes V_3^*) \otimes \mathrm{Sym}\,(V_2 \otimes V_3^*).$$

The vector bundle $\xi = \mathcal{R}_1 \otimes \mathcal{Q}_3{}^* \oplus \mathcal{R}_2 \otimes \mathcal{Q}_3{}^*$ and $\mathrm{rank}\,\xi = 12$. Hence, we need to calculate $\bigwedge^0 \xi, \bigwedge^1 \xi, \dots, \bigwedge^{12} \xi$.

By Cauchy's formula,

$$\begin{aligned} \bigwedge^t \xi &= \bigoplus_{|\lambda|+|\mu|=t} S_\lambda \mathcal{R}_1 \otimes S_{\lambda'} \mathcal{Q}_3^* \otimes S_\mu \mathcal{R}_2 \otimes S_{\mu'} \mathcal{Q}_3^* \\ &= \bigoplus_{|\lambda|+|\mu|=t} c^{\nu}_{\lambda',\mu'} (S_\lambda \mathcal{R}_1 \otimes S_\mu \mathcal{R}_2 \otimes S_\nu \mathcal{Q}_3^*). \end{aligned} \tag{4.7}$$

A weight associated to $S_\lambda \mathcal{R}_1 \otimes S_\mu \mathcal{R}_2 \otimes S_\nu \mathcal{Q}_3^*$ is of the form

$$(0, \lambda_1, \lambda_2),\ (0, \mu_1, \mu_2),\ (-\nu_3, -\nu_2, -\nu_1, 0).$$

Let $\xi_1 = \mathcal{R}_1 \otimes \mathcal{Q}_3{}^*$ and $\xi_2 = \mathcal{R}_2 \otimes \mathcal{Q}_3{}^*$,

$$\begin{aligned}\bigwedge^1 \xi &= (\bigwedge^1 \xi_1 \otimes \bigwedge^0 \xi_2) \oplus \bigwedge^0 \xi_1 \otimes \bigwedge^1 \xi_2)\\ &= [(S_1\mathcal{R}_1 \otimes S_1\mathcal{Q}_3{}^*) \otimes (S_0\mathcal{R}_2 \otimes S_0\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_0\mathcal{R}_1 \otimes S_0\mathcal{Q}_3{}^*) \otimes (S_1\mathcal{R}_2 \otimes S_1\mathcal{Q}_3{}^*)]\\ &= [S_1\mathcal{R}_1 \otimes S_0\mathcal{R}_2 \otimes S_1\mathcal{Q}_3{}^*]\\ &\quad \oplus [S_0\mathcal{R}_1 \otimes S_1\mathcal{R}_2 \otimes S_1\mathcal{Q}_3{}^*].\end{aligned}$$

The weight associated to the first summand is $(0,1,0;0,0,0;0,0,-1,0)$ and the weight associated to the second summand is $(0,0,0;0,1,0;0,0,-1,0)$. Applying Bott's algorithm, we see that none of these terms contribute to any of the F_i. For an example of a contributing weight we calculate $\bigwedge^3 \xi$. From Remark 4.2, we know that λ is contained in the rectangle (3^2), μ is contained in (3^2) and ν is contained in (4^3).

$$\begin{aligned}\bigwedge^3 \xi &= (\bigwedge^3 \xi_1 \otimes \bigwedge^0 \xi_2) \oplus (\bigwedge^2 \xi_1 \otimes \bigwedge^1 \xi_2) \oplus (\bigwedge^1 \xi_1 \otimes \bigwedge^2 \xi_2) \oplus (\bigwedge^0 \xi_1 \otimes \bigwedge^3 \xi_2)\\ &= [(S_{(2,1)}\mathcal{R}_1 \otimes S_{(0)}\mathcal{R}_2 \otimes S_{(2,1)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(3)}\mathcal{R}_1 \otimes S_{(0)}\mathcal{R}_2 \otimes S_{(1,1,1)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(2)}\mathcal{R}_1 \otimes S_{(1)}\mathcal{R}_2 \otimes S_{(2,1)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(2)}\mathcal{R}_1 \otimes S_{(1)}\mathcal{R}_2 \otimes S_{(1,1,1)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(1,1)}\mathcal{R}_1 \otimes S_{(1)}\mathcal{R}_2 \otimes S_{(2,1)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(1,1)}\mathcal{R}_1 \otimes S_{(1)}\mathcal{R}_2 \otimes S_{(3)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(1)}\mathcal{R}_1 \otimes S_{(2)}\mathcal{R}_2 \otimes S_{(1,1,1)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(1)}\mathcal{R}_1 \otimes S_{(2)}\mathcal{R}_2 \otimes S_{(2,1)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(1)}\mathcal{R}_1 \otimes S_{(1,1)}\mathcal{R}_2 \otimes S_{(2,1)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(1)}\mathcal{R}_1 \otimes S_{(1,1)}\mathcal{R}_2 \otimes S_{(3)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(0)}\mathcal{R}_1 \otimes S_{(3)}\mathcal{R}_2 \otimes S_{(1,1,1)}\mathcal{Q}_3{}^*)]\\ &\quad \oplus [(S_{(0)}\mathcal{R}_1 \otimes S_{(2,1)}\mathcal{R}_2 \otimes S_{(2,1)}\mathcal{Q}_3{}^*)].\end{aligned}$$

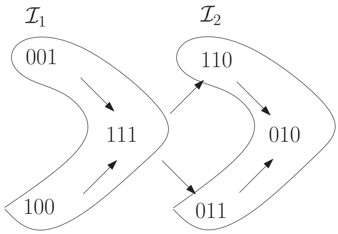
The weights associated to the summands in that order are:

$$\begin{array}{lll} (0\,2\,1;0\,0\,0;0-1-2\ \ 0), & (0\,3\,0;0\,0\,0;-1-1-1\,0), & (0\,2\,0;0\,1\,0;0-1-2\,0)\\ (0\,2\,0;0\,1\,0;-1-1-1\,0), & (0\,1\,1;0\,1\,0;0-1-2\,0), & (0\,1\,1;0\,1\,0;0\,0-3\,0)\\ (0\,1\,0;0\,2\,0;-1-1-1\,0), & (0\,1\,0;0\,2\,0;0-1-2\,0), & (0\,1\,0;0\,1\,1;0-1-2\,0)\\ (0\,1\,0;0\,1\,1;0\,0-3\,0), & (0\,0\,0;0\,3\,0;-1-1-1\,0)\ , & (0\,0\,0;0\,2\,1;0-1-2\,0) \end{array}$$

Applying Bott exchanges to each weight we see that only the first and last summands contribute the non-zero terms $(\bigwedge^3 V_1 \otimes \bigwedge^3 V_3^* \otimes A(-3))$ and $(\bigwedge^3 V_2 \otimes \bigwedge^3 V_3^* \otimes A(-3))$ to F_1. Continuing in this manner, we get the resolution:

$$
\begin{gathered}
A \\
\uparrow \\
(\bigwedge^3 V_1 \otimes \bigwedge^3 V_3^* \otimes A(-3)) \oplus (\bigwedge^3 V_2 \otimes \bigwedge^3 V_3^* \otimes A(-3)) \oplus (\bigwedge^2 V_1 \otimes \bigwedge^2 V_2 \otimes \bigwedge^4 V_3^* \otimes A(-4)) \\
\uparrow \\
(S_{211} V_1 \otimes \bigwedge^4 V_3^* \otimes A(-4)) \oplus (S_{211} V_2 \otimes \bigwedge^4 V_3^* \otimes A(-4)) \oplus \\
(\bigwedge^3 V_1 \otimes \bigwedge^2 V_2 \otimes S_{2111} V_3^* \otimes A(-5)) \oplus (\bigwedge^2 V_1 \otimes \bigwedge^3 V_2 \otimes S_{2111} V_3^* \otimes A(-5)) \oplus \\
\bigwedge^3 V_1 \otimes \bigwedge^3 V_2 \otimes S_{222} V_3^* \otimes A(-6) \\
\uparrow \\
(S_{211} V_1 \otimes \bigwedge^3 V_2 \otimes S_{2221} V_3^* \otimes A(-7)) \oplus (\bigwedge^3 V_1 \otimes S_{211} V_2 \otimes S_{2221} V_3^* \otimes A(-7)) \oplus \\
(\bigwedge^2 V_1 \otimes S_{222} V_2 \otimes S_{2222} V_3^* \otimes A(-8)) \oplus (S_{222} V_1 \otimes \bigwedge^2 V_2 \otimes S_{2222} V_3^* \otimes A(-8)) \oplus \\
\bigwedge^3 V_1 \otimes \bigwedge^3 V_2 \otimes S_{3111} V_3^* \otimes A(-6) \\
\uparrow \\
(S_{211} V_1 \otimes S_{211} V_2 \otimes S_{2222} V_3^* \otimes A(-8)) \oplus (S_{222} V_1 \otimes \bigwedge^3 V_2 \otimes S_{3222} V_3^* \otimes A(-9)) \oplus \\
(\bigwedge^3 V_1 \otimes S_{222} V_2 \otimes S_{3222} V_3^* \otimes A(-9)) \\
\uparrow \\
(S_{222} V_1 \otimes S_{222} V_2 \otimes S_{3333} V_3^* \otimes A(-12)).
\end{gathered}
$$

Example 4.8. Let Q be the quiver $1 \leftarrow 3 \rightarrow 2$. The AR quiver of Q along with its partition is



Let $V = 001 \oplus 100 \oplus 111 \oplus 110 \oplus 011 \oplus 010$ be a representation of Q (note that this representation is the dual of the representation in Example 4.7). Then $\alpha = (3,3,4)$, $\beta = (1,1,3)$ and $\gamma = (2,2,1)$. The coordinate ring is

$$A = \operatorname{Sym}(V_3 \otimes V_1^*) \otimes \operatorname{Sym}(V_3 \otimes V_2^*)$$

and the vector bundle is $\xi = (\mathcal{R}_3 \otimes \mathcal{Q}_1{}^*) \oplus (\mathcal{R}_3 \otimes \mathcal{Q}_2{}^*)$. By Cauchy's formula,

$$\begin{aligned} \bigwedge^t \xi &= \bigoplus_{|\lambda|+|\mu|=t} S_\lambda \mathcal{R}_3 \otimes S_{\lambda'} \mathcal{Q}_1^* \otimes S_\mu \mathcal{R}_3 \otimes S_{\mu'} \mathcal{Q}_2^* \\ &= \bigoplus_{|\lambda|+|\mu|=t} c^{\kappa}_{\lambda,\mu} (S_{\lambda'} \mathcal{Q}_1^* \otimes S_{\mu'} \mathcal{Q}_2^* \otimes S_\kappa \mathcal{R}_3) \end{aligned} \tag{4.8}$$

By Remark 4.2, the partitions λ, μ and κ are such that $\lambda' \subset (3^2)$, $\mu' \subset (3^2)$ and $\kappa \subset (4^3)$. Note that, for each t, the set of triples (λ', μ', κ) occurring in equation (4.8) is equal to the set of triples (λ, μ, ν) which occur in equation (4.7).

A weight associated to $S_{\lambda'} \mathcal{Q}_1^* \otimes S_{\mu'} \mathcal{Q}_2^* \otimes S_\kappa \mathcal{R}_3$ is of the form

$$(-\lambda'_2, -\lambda'_1, 0),\ (-\mu'_2, -\mu'_1, 0),\ (0, \kappa_1, \kappa_2, \kappa_3).$$

The resolution of $\overline{O}_V$ is -

$$
\begin{gathered}
A \\
\uparrow \\
(\bigwedge^3 V_1^* \otimes \bigwedge^3 V_3 \otimes A(-3)) \oplus (\bigwedge^3 V_2^* \otimes \bigwedge^3 V_3 \otimes A(-3)) \oplus (\bigwedge^2 V_1^* \otimes \bigwedge^2 V_2^* \otimes \bigwedge^4 V_3 \otimes A(-4)) \\
\uparrow \\
(S_{211} V_1^* \otimes \bigwedge^4 V_3 \otimes A(-4)) \oplus (S_{211} V_2^* \otimes \bigwedge^4 V_3 \otimes A(-4)) \oplus \\
(\bigwedge^3 V_1^* \otimes \bigwedge^2 V_2^* \otimes S_{2111} V_3 \otimes A(-5)) \oplus (\bigwedge^2 V_1^* \otimes \bigwedge^3 V_2^* \otimes S_{2111} V_3 \otimes A(-5)) \oplus \\
\bigwedge^3 V_1^* \otimes \bigwedge^3 V_2^* \otimes S_{222} V_3 \otimes A(-6) \\
\uparrow \\
(S_{211} V_1^* \otimes \bigwedge^3 V_2^* \otimes S_{2221} V_3 \otimes A(-7)) \oplus (\bigwedge^3 V_1^* \otimes S_{211} V_2^* \otimes S_{2221} V_3 \otimes A(-7)) \oplus \\
(\bigwedge^2 V_1^* \otimes S_{222} V_2^* \otimes S_{2222} V_3 \otimes A(-8)) \oplus (S_{222} V_1^* \otimes \bigwedge^2 V_2^* \otimes S_{2222} V_3 \otimes A(-8)) \oplus \\
\bigwedge^3 V_1^* \otimes \bigwedge^3 V_2^* \otimes S_{3111} V_3 \otimes A(-6) \\
\uparrow \\
(S_{211} V_1^* \otimes S_{211} V_2^* \otimes S_{2222} V_3 \otimes A(-8)) \oplus (S_{222} V_1^* \otimes \bigwedge^3 V_2^* \otimes S_{3222} V_3 \otimes A(-9)) \oplus \\
(\bigwedge^3 V_1^* \otimes S_{222} V_2^* \otimes S_{3222} V_3 \otimes A(-9)) \\
\uparrow \\
(S_{222} V_1^* \otimes S_{222} V_2^* \otimes S_{3333} V_3 \otimes A(-12)).
\end{gathered}
$$

4.2. Minimal generators of the defining ideal. Let $V \in \operatorname{Rep}(Q, \underline{d})$, $V = a(010) \oplus b(110) \oplus c(011) \oplus d(111) \oplus e(001) \oplus f(100)$. Then

$$\operatorname{rank} V_{\mathfrak{a}} = b + d,\ \operatorname{rank} V_{\mathfrak{b}} = c + d,\ \operatorname{rank}(V_{\mathfrak{a}} | V_{\mathfrak{b}}) = b + c + d.$$

In the notation of subsection 4.1, $\alpha = (b+d+f,\ c+d+e,\ a+b+c+d)$, $\beta = (d+f,\ d+e,\ d)$ and $\gamma = (b,\ c,\ a+b+c)$. Hence, $N = ub+vc+wd$.

We consider orbits admitting a Reineke desingularization given by the partition in Figure 2. The following result is the main theorem of this section. It describes the first term F_1 of the resolution $\mathbf{F}_\bullet$. In particular, it says that the summands of F_1 are obtained by contributions from $\bigwedge^{\operatorname{rank}(V_{\mathfrak{a}})+1}\xi$, $\bigwedge^{\operatorname{rank}(V_{\mathfrak{b}})+1}\xi$ and $\bigwedge^{\operatorname{rank}(V_{\mathfrak{a}}|V_{\mathfrak{b}})+1}\xi$. As a result, we will have that the generators of the defining ideal are minors of $V_{\mathfrak{a}}$, $V_{\mathfrak{b}}$ and $(V_{\mathfrak{a}} \mid V_{\mathfrak{b}})$ of sizes given by their ranks. Let p, q, r denote $\operatorname{rank} V_{\mathfrak{a}}$, $\operatorname{rank} V_{\mathfrak{b}}$, $\operatorname{rank}(V_{\mathfrak{a}} \mid V_{\mathfrak{b}})$, respectively.

Theorem 4.9. *$F_1 = H^p(\mathcal{V}, \bigwedge^{p+1}\xi) \oplus H^q(\mathcal{V}, \bigwedge^{q+1}\xi) \oplus H^r(\mathcal{V}, \bigwedge^{r+1}\xi)$.*

Proof. From Proposition 4.2, we have that

$$F_1 = \bigoplus_{t=1}^{\operatorname{rank}\xi} \bigoplus_{|\lambda|+|\mu|=t} c^{\nu}_{\lambda',\mu'}(S_{[0^b,\lambda]}V_1 \otimes S_{[0^c,\mu]}V_2 \otimes S_{[-\nu^{op},0^d]}V_3^*),$$

where $S_\nu \subset S_{\lambda'} \otimes S_{\mu'}$ and $D(\underline{\lambda}) = 1$. Also, by Proposition 4.5,

$$D(\underline{\lambda}) \geq \langle (u,v,w),(u,v,w) \rangle$$

i.e.,

$$1 \geq \langle (u,v,w),(u,v,w) \rangle .$$

But Q is Dynkin, so the Euler form $\langle (u,v,w),(u,v,w) \rangle > 0$, which implies

$$\langle (u,v,w),(u,v,w) \rangle = 1. \tag{4.9}$$

By Gabriel's theorem [**8**], there is a one-to-one correspondence between the roots of the quadratic from in equation (4.9) and dimension vectors of indecomposables in $\operatorname{mod} KQ$ when KQ is representation-finite. Thus, (u, v, w) is one of $(1, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$. We analyze these triples to prove our proposition. Recall that the weights of $\bigwedge^i \xi$ are of the form

$$(0^b, \lambda), (0^c, \mu), (-\nu^{op}, 0^d),$$

where $|\lambda| + |\mu| = i$. Also $N = ub + vc + wd$ and $D(\underline{\lambda}) = 1$ implies $|\lambda| + |\mu| = |\nu| = N + 1$.

(1) $(u, v, w) = (1, 0, 0)$. In this case $N = b$, so $|\lambda| + |\mu| = b + 1$. $u = 1$ implies that $\lambda = (b+1, 0 \ldots, 0)$, so $|\lambda| = b+1$ and $|\mu| = 0$. This implies $\nu = \lambda'$, but $w = 0$, so we will get a contributing triple only when $d = 0$. In that case $p = \gamma_1$ and

$$H^p\left(\mathcal{V}, \bigwedge^{p+1} \xi\right) = \bigwedge^{p+1} V_1 \otimes \bigwedge^{p+1} V_3^*$$

is the only contribution to F_1.

(2) $(u, v, w) = (0, 0, 1)$. Here $N = d$. So $|\lambda| + |\mu| = |\nu| = d + 1$. Also, $w = 1$ implies ν must be $(d+1, 0, \ldots, 0)$. So a contributing triple occurs only when $b = c = 0$. Then $r = d$, and we get contributing triples $(1^k;\ 1^l;\ d + 1)$ where $k + l = d + 1$. The contribution to F_1 is

$$H^r\left(\mathcal{V}, \bigwedge^{r+1} \xi\right) = \bigoplus_{k+l=d+1} \bigwedge^{k} V_1 \otimes \bigwedge^{l} V_2 \otimes \bigwedge^{r+1} V_3^*.$$

(3) $(u, v, w) = (0, 1, 0)$. This case is analogous to the first one. A contributing triple occurs only when $d = 0$, in which case the contribution to F_1 is

$$H^q(\mathcal{V}, \bigwedge^{q+1} \xi) = \bigwedge^{q+1} V_2 \otimes \bigwedge^{q+1} V_3^*.$$

(4) $(u, v, w) = (1, 0, 1)$. This implies $N = b + d = p$. So $|\lambda| + |\mu| = |\nu| = b + d + 1$. $u = 1$ implies λ is of the form $(b+1, 1^k, 0, \ldots)$, similarly $w = 1$ implies ν is of the form $(d + 1, 1^l, 0, \ldots)$ (thus both λ and ν are hooks). Then $|\nu| = b + d + 1$ implies $l = b$.

Since $v = 0$, we know that there are zero exchanges for the weight $(0^c, \mu)$. This can happen if either $\mu = 0$ or $c = 0$. If $\mu = 0$, then $\nu = \lambda'$ and

$$\begin{aligned} H^p(\mathcal{V}, \bigwedge^{p+1} \xi) &= S_{[0^b,\lambda]} V_1 \otimes S_{[-\nu^{op},0^d]} V_3^* \\ &= \bigwedge^{p+1} V_1 \otimes \bigwedge^{p+1} V_3^*. \end{aligned}$$

If $c = 0$, then $\mu = \nu \setminus \lambda = (1^{d-k})$. In this case,

$$\begin{aligned}H^p(\mathcal{V}, \bigwedge^{p+1} \xi) &= S_{[0^b,\lambda]}V_1 \otimes S_\mu V_2 \otimes S_{[-\nu^{op},0^d]}V_3^* \\ &= \bigoplus_{k=0}^{d+f-1} \bigwedge^{b+k+1} V_1 \otimes \bigwedge^{d-k} V_2 \otimes \bigwedge^{p+1} V_3^*.\end{aligned}$$

(5) $(u, v, w) = (0, 1, 1)$. This case is analogous to the previous one. $u = 0$ implies either $\lambda = 0$ or $b = 0$. If $\lambda = 0$, then $\nu = \mu'$ and

$$\begin{aligned}H^q(\mathcal{V}, \bigwedge^{q+1} \xi) &= S_{[0^c,\mu]}V_2 \otimes S_{[-\nu^{op},0^d]}V_3^* \\ &= \bigwedge^{q+1} V_2 \otimes \bigwedge^{q+1} V_3^*.\end{aligned}$$

If $b = 0$, then $\lambda = \nu \setminus \mu = (1^{d-k})$. In this case,

$$\begin{aligned}H^q(\mathcal{V}, \bigwedge^{q+1} \xi) &= S_\lambda V_1 \otimes S_{[0^c,\mu]}V_2 \otimes S_{[-\nu^{op},0^d]}V_3^* \\ &= \bigoplus_{k=0}^{d+e-1} \bigwedge^{d-k} V_1 \otimes \bigwedge^{c+k+1} V_2 \otimes \bigwedge^{q+1} V_3^*.\end{aligned}$$

(6) $(u, v, w) = (1, 1, 1)$. In this case, $N = b + c + d = r$. λ and μ are hooks of the form:

$$\lambda = (b + 1, 1^k, 0, \dots), \qquad \mu = (c + 1, 1^l, 0, \dots).$$

Since ν is such that $S_\nu \subset S_{\lambda'} \otimes S_{\mu'}$, ν is also a hook of the form $(d + 1, 1^m, 0, \dots)$. Since $|\lambda| + |\mu| = |\nu| = b + c + d + 1$, we must have $k + l = d - 1$ and $m = b + c$. Thus,

$$\begin{aligned}H^r(\mathcal{V}, \bigwedge^{r+1} \xi) &= S_{[0^b,\lambda]}V_1 \otimes S_{[0^c,\mu]}V_2 \otimes S_{[-\nu^{op},0^d]}V_3^* \\ &= \bigoplus_{k+l=d-1} \bigwedge^{b+k+1} V_1 \otimes \bigwedge^{c+l+1} V_2 \otimes \bigwedge^{b+c+d+1} V_3^*.\end{aligned}$$

By Cauchy's formula, this term is a direct summand of $\bigwedge^{r+1}([V_1 \oplus V_2] \otimes V_3^*)$. ◻

Corollary 4.10. *Let* $\operatorname{rank}(V_\mathfrak{a}) = p$, $\operatorname{rank}(V_\mathfrak{b}) = q$, $\operatorname{rank}(V_\mathfrak{a} \mid V_\mathfrak{b}) = r$. *The minimal generators of the defining ideal are determinantal:* $(p+1) \times (p+1)$ *minors of* $V_\mathfrak{a}$*, the* $(q+1) \times (q+1)$ *minors of* $V_\mathfrak{b}$ *and the* $(r+1) \times (r+1)$ *minors of* $(V_\mathfrak{a} \mid V_\mathfrak{b})$*, taken by choosing* $b+k+1$ *columns of* $V_\mathfrak{a}$ *and* $c+l+1$ *columns of* $V_\mathfrak{b}$*, where* $k+l = d-1$.

Proof. The defining ideal of the orbit closure $\overline{O}_V$ is generated by the image of the map $F_1 \xrightarrow{\delta} A$. By Theorem 4.9, the image of the differential map δ is generated by $(p+1)\times(p+1)$-minors of the matrix corresponding to the linear map $V_\mathfrak{a}$, $(q+1) \times (q+1)$-minors of the matrix corresponding to the linear map $V_\mathfrak{b}$ and $(r+1)\times(r+1)$-minors of the matrix corresponding to the linear map $(V_\mathfrak{a} \mid V_\mathfrak{b})$. ◻

In Example 4.7, we found

$$\begin{aligned} F_1 = {} & (\bigwedge^3 V_1 \otimes \bigwedge^3 V_3^* \otimes A(-3)) \oplus (\bigwedge^3 V_2 \otimes \bigwedge^3 V_3^* \otimes A(-3)) \\ & \oplus (\bigwedge^2 V_1 \otimes \bigwedge^2 V_2 \otimes \bigwedge^4 V_3^* \otimes A(-4)). \end{aligned}$$

Fixing a basis for vector spaces V_1, V_2 and V_3, the minimal generators of the defining ideal are 3×3 minors of the 4×3 matrices $V_\mathfrak{a}$ and $V_\mathfrak{b}$ and 4×4 minors of the map $(V_\mathfrak{a} \mid V_\mathfrak{b}) : V_1 \oplus V_2 \to V_3$, obtained by choosing 2 columns of $V_\mathfrak{a}$ and 2 columns of $V_\mathfrak{b}$.

4.3. F_{top} **and classification of Gorenstein orbits.** Let Q be a Dynkin quiver. We denote the last term of the resolution $\mathbf{F}_\bullet$ by F_{top}. Let $t = \operatorname{rank}\xi$, where ξ is the vector bundle defined in equation (3.4). The top exterior power of $\xi(a)$ contributes the term

$$\begin{aligned} (4.10) \quad & S_{[0^{d_1(ta)}, d_1(ha)^{d_2(ta)}, \dots, (d_1(ha)+\dots+d_{s-1}(ha))^{d_s(ta)}]}(ta) \\ & \quad \otimes S_{[(-d_2(ta)-\dots-d_s(ta))^{d_1(ha)}, \dots, -d_s(ta)^{d_{s-1}(ha)}, 0^{d_s(ha)}]}(ha)^*. \end{aligned}$$

Thus, the contribution of the top exterior power of ξ is given by

$$(4.11) \qquad \bigotimes_{x \in Q_0} S_{(k_1(x)^{d_1(x)}, \dots, k_s(x)^{d_s(x)})}(x)$$

where

$$k_p(x) = \sum_{a\in Q_1;ta=x}\sum_{u<p} d_u(ha) - \sum_{a\in Q_1;ha=x}\sum_{u>p} d_u(ta). \tag{4.12}$$

First we give a sufficient condition for the orbit closure $\overline{O}_V$ to be Gorenstein in case of any Dynkin quiver Q. The condition that, for every $x \in Q_0$, the number

$$k_p(x) - \sum_{u<p} d_u(x) + \sum_{u>p} d_u(x) \tag{4.13}$$

is independent of p ($p = 1, 2, \dots, s$), is equivalent to the condition that $\bigwedge^t \xi$, the top exterior power of ξ, contributes a trivial representation to F_{top}. We show that the latter condition, together with normality, implies that the corresponding orbit closure is Gorenstein. First we show that the condition (4.13) is equivalent to the property that the Coxeter orbits in the Auslander-Reiten quiver are constant.

Lemma 4.11. *Suppose $\underline{d}(x) = (d_u(x))$ (for $u = 1, 2, \dots, s$) are dimensions of the flag at vertex x in the desingularization Z. Then*:

$$\langle \underline{e}_x, \underline{d}_p(x)\rangle = -\langle \underline{d}_{p+1}(x), \underline{e}_x\rangle,$$

for all $x \in Q_0$ and $p = 1, 2, \dots, s-1$, where $\underline{e}_x$ is the dimension vector of the simple representation supported at x.

Proof. Condition (4.13) translates to the equation

$$k_{p+1}(x) - k_p(x) = d_p(x) + d_{p+1}(x) \tag{4.14}$$

for $x \in Q_0$ and $p = 1, 2, \dots, s-1$. This is equivalent to

$$\sum_{a\in Q_1;ta=x} d_t(ha) + \sum_{a\in Q_1;ha=x} d_{p+1}(ta) = d_{p+1}(x) + d_p(x) \tag{4.15}$$

for all $x \in Q_0$ and $p = 1, 2, \dots, s-1$. These conditions can be expressed in terms of Euler form as follows:

$$\begin{aligned}\langle \underline{e}_x, \underline{d}_p\rangle &= d_p(x) - \sum_{\substack{a\in Q_1\\ ta=x}} d_p(ha)\\ &= \sum_{\substack{a\in Q_1\\ ha=x}} d_{p+1}(ta) - d_{p+1}(x)\\ &= -\langle \underline{d}_{p+1}, \underline{e}_x\rangle.\end{aligned}$$

This proves the claim. $\square$

Lemma 4.12. *Let* $m = \dim \mathcal{V}$ *and* $t = \operatorname{rank} \xi$. *Then*

$$\operatorname{codim} \overline{O}_V = t - m.$$

Proof.

$$\begin{aligned}\operatorname{codim} \overline{O}_V &= \dim X - \dim \overline{O}_V \\ &= \dim X - \dim Z \\ &= \dim X - (\dim X + m - t) \\ &= t - m. \qquad \square\end{aligned}$$

Lemma 4.13. *Suppose* $\bigwedge^t \xi$ *contributes a trivial representation to* F_{t-m}. *Then the resolution* $\mathbf{F}_\bullet$ *is self-dual. In particular,* $F_{t-m} \cong F_0^*$.

Proof. If $H^m(\mathcal{V}, \bigwedge^t \xi)$ is a trivial representation, then $\bigwedge^t \xi \cong \omega_{\mathcal{V}}$, where $\omega_{\mathcal{V}}$ denotes the canonical sheaf on $\mathcal{V}$. This implies that $\omega_{\mathcal{V}} \otimes \bigwedge^t \xi^* \cong \bigwedge^0 \xi \cong K$. Then, for $0 \le i \le m$,

$$\begin{aligned}F_{t-m-i} &= \bigoplus_{j\ge 0} H^{m-j}(\mathcal{V}, \bigwedge^{t-i-j} \xi) \\ &\cong \bigoplus_{j\ge 0} H^{j}(\mathcal{V}, \omega_{\mathcal{V}} \otimes \bigwedge^{t-i-j} \xi^*)^* \quad \text{(by Serre duality)} \\ &\cong \bigoplus_{j\ge 0} H^{j}(\mathcal{V}, \omega_{\mathcal{V}} \otimes \bigwedge^{t} \xi^* \otimes \bigwedge^{i+j} \xi)^* \\ &\cong \bigoplus_{j\ge 0} H^{j}(\mathcal{V}, \bigwedge^{i+j} \xi)^* \\ &= F_i^*. \qquad \square\end{aligned}$$

Theorem 4.14. *Let* τ *denote the Auslander-Reiten translate. Assume that, for each* $p = 1, 2, \ldots, s-1$, *we have* $\underline{d}_{p+1} = \tau^+ \underline{d}_p$. *Then the* *complex* $\mathbf{F}_\bullet$ *is self-dual. If the incidence variety comes from Reineke desingularization and the corresponding orbit closure is normal with rational singularities, then it is also Gorenstein.*

Proof. If the Coxeter orbits of an AR quiver are constant, then by Lemma 4.11 $\bigwedge^t \xi$ contributes a trivial representation to F_{t-m}. Then, applying Lemma 4.13, we get that $F_{t-m} \cong F_0^* \cong A^*$; therefore, $\dim F_{t-m} = 1$. □

In particular, for the case of non-equioriented A_3, Theorem 4.14 asserts that the orbits with multiplicities satisfying $a = d$, $b = e$ and $c = f$ are Gorenstein.

Next, we investigate necessary conditions for the orbit closure $\overline{O}_V$ to be Gorenstein in the case of non-equioriented A_3. Recall that for our case of non-equioriented A_3, we have the desingularization

$$\begin{tikzcd} Z \subset \arrow[d, "q'"'] & \mathrm{Rep}\,(Q,\underline{\mathrm{d}}) \times \arrow[d, "q"'] & \prod_{x\in Q_0} \mathrm{Flag}\,(\underline{\mathrm{d}}_*(x), K^{d(x)}) \arrow[d, "p"] \\ \overline{O}_V \subset & \mathrm{Rep}\,(Q,\underline{\mathrm{d}}) & \prod_{x\in Q_0} \mathrm{Flag}\,(\underline{\mathrm{d}}_*(x), K^{d(x)}) \end{tikzcd}$$

As before, let $V = a(010) \oplus b(110) \oplus c(011) \oplus d(111) \oplus e(001) \oplus f(100)$ be a representation of A_3. Then,

$$\underline{d}_1 = (b,\ c,\ a+b+c); \qquad \underline{d}_2 = (d+f,\ d+e,\ d).$$

From (4.11), the weights for $\bigwedge^t \xi$ are:

$$(0^b, (a+b+c)^{d+f}), \qquad (0^c, (a+b+c)^{d+e}), \qquad ((-2d-e-f)^{a+b+c}, 0^d).$$

For the case of non-equioriented A_3, we investigate the following question: in what cases does $\bigwedge^t \xi$ contribute a non-zero representation? To which term F_i does $\bigwedge^t \xi$ contribute? First we show that a contribution from $\bigwedge^t \xi$ always goes to F_{t-m}.

Lemma 4.15. *If the weight of the* $\bigwedge^t \xi$ *gives a non-zero partition after Bott exchanges, then the corresponding representation is a summand of* F_{t-m}.

Proof. It is enough to show that $D(\underline{\lambda}) = \operatorname{codim} \overline{O}_V$ for $\lambda = ((a+b+c)^{d+f})$ and $\mu = ((a+b+c)^{d+e})$. We apply Bott's algorithm to each weight to get:

$$\begin{aligned}
[0^b, (a+b+c)^{d+f}] &= ((a+c)^{d+f}, (d+f)^b) \\
&\qquad \text{after } b(d+f) \text{ Bott exchanges,} \\
[0^c, (a+b+c)^{d+e}] &= ((a+b)^{d+e}, (d+e)^c) \\
&\qquad \text{after } c(d+e) \text{ Bott exchanges,} \\
[(-2d-e-f)^{a+b+c}, 0^d] &= ((-a-b-c)^d, (-d-e-f)^{a+b+c}) \\
&\qquad \text{after } d(a+b+c) \text{ Bott exchanges.}
\end{aligned}$$

$$\begin{aligned}
D(\underline{\lambda}) &= [(d+f)(a+b+c)] + [(d+e)(a+b+c)] \\
&\quad - [b(d+f) + c(d+e) + d(a+b+c)] \\
&= ad + ae + af + be + cf \\
&= \operatorname{codim} \overline{O}_V \\
&= t - m. \qquad \square
\end{aligned}$$

Next we list the cases in which $\bigwedge^t \xi$ contributes a non-zero term. Observe that a contribution will occur whenever the Bott exchanges give a non-increasing sequence for every term of

$$(0^b, (a+b+c)^{d+f}), \qquad (0^c, (a+b+c)^{d+e}), \qquad ((-2d-e-f)^{a+b+c}, 0^d).$$

Also note that if any of b, c or d are zero, then there are no exchanges for the corresponding term in the weight. We base our cases on this observation.

For the cases listed in Table 1, we calculate the representation that $\bigwedge^t \xi$ contributes to F_{t-m} and list these in Table 2.

Proposition 4.16. *$\bigwedge^t \xi$ contributes to F_{t-m} in the following cases when the corresponding conditions are satisfied:*

TABLE 1. Cases when $\bigwedge^t \xi$ contributes to F_{t-m}.

Cases	Conditions		
$b \neq 0,\ c \neq 0,\ d \neq 0$	$a+c \geq d+f$,	$a+b \geq d+e$,	$d+e+f \geq a+b+c$
$b \neq 0,\ c \neq 0,\ d = 0$	$a+c \geq d+f$,	$a+b \geq d+e$	
$b \neq 0,\ c = 0,\ d \neq 0$	$a+c \geq d+f$,		$d+e+f \geq a+b+c$
$b = 0,\ c \neq 0,\ d \neq 0$		$a+b \geq d+e$,	$d+e+f \geq a+b+c$
$b = 0,\ c = 0,\ d \neq 0$	$d+e+f \geq a+b+c$		
$b = 0,\ c \neq 0,\ d = 0$		$a+b \geq d+e$	
$b \neq 0,\ c = 0,\ d = 0$	$a+c \geq d+f$		
$b = 0,\ c = 0,\ d = 0$	no condition		

TABLE 2. Term contributed by $\bigwedge^t \xi$.

Case	Weight of $\bigwedge^t \xi$	Corresponding term in F_{t-m}
$b \neq 0,\ c \neq 0,\ d \neq 0$	$(0^b, (a+b+c)^{d+f};$ $0^c, (a+b+c)^{d+e};$ $(-2d-e-f)^{a+b+c}, 0^d)$	$S_{((a+c)^{d+f},(d+f)^b)}V_1$ $\otimes S_{((a+b)^{d+e},(d+e)^c)}V_2$ $\otimes S_{((-a-b-c)^d,(-d-e-f)^{a+b+c})}V_3^*$
$b \neq 0,\ c \neq 0,\ d = 0$	$(0^b, (a+b+c)^f;$ $0^c, (a+b+c)^e;$ $(-e-f)^{a+b+c})$	$S_{((a+c)^f,f^b)}V_1 \otimes S_{((a+b)^e,e^c)}V_2$ $\otimes S_{((-e-f)^{a+b+c})}V_3^*$
$b \neq 0,\ c = 0,\ d \neq 0$	$(0^b, (a+b)^{d+f};\ (a+b)^{d+e};$ $(-2d-e-f)^{a+b}, 0^d)$	$S_{(a^{d+f},(d+f)^b)}V_1 \otimes S_{((a+b)^{d+e})}V_2$ $\otimes S_{((-a-b)^d,(-d-e-f)^{a+b})}V_3^*$
$b = 0,\ c \neq 0,\ d \neq 0$	$((a+c)^{d+f};\ 0^c, (a+c)^{d+e};$ $(-2d-e-f)^{a+c}, 0^d)$	$S_{((a+c)^{d+f})}V_1 \otimes S_{(a^{d+e},(d+e)^c)}V_2$ $\otimes S_{((-a-c)^d,(-d-e-f)^{a+c})}V_3^*$
$b = 0,\ c = 0,\ d \neq 0$	$(a^{d+f};\ a^{d+e};$ $(-2d-e-f)^a, 0^d)$	$S_{(a^{d+f})}V_1 \otimes S_{(a^{d+e})}V_2$ $\otimes S_{(-a^d,(-d-e-f)^a)}V_3^*$
$b = 0,\ c \neq 0,\ d = 0$	$((a+c)^f;\ 0^c, (a+c)^e;$ $(-e-f)^{a+c})$	$S_{((a+c)^f)}V_1 \otimes S_{(a^e,e^c)}V_2$ $\otimes S_{((-e-f)^{a+c})}V_3^*$
$b \neq 0,\ c = 0,\ d = 0$	$(0^b, (a+b)^f;\ (a+b)^e;$ $(-e-f)^{a+b})$	$S_{(a^f,f^b)}V_1 \otimes S_{((a+b)^e)}V_2$ $\otimes S_{((-e-f)^{a+b})}V_3^*$
$b = 0,\ c = 0,\ d = 0$	$(a^f;\ a^e;\ (-e-f)^a)$	$S_{(a^f)}V_1 \otimes S_{(a^e)}V_2$ $\otimes S_{((-e-f)^a)}V_3^*$

Since $\overline{O}_V$ is Cohen-Macaulay by Corollary 4.6, it is Gorenstein if and only if F_{t-m} is one-dimensional. We consider two classes of orbit closures: those generated by minors of 2 or more maps and those generated by minors of exactly 1 map. Theorem 4.17 is about orbit closures of the former type (we refer to them as *non-determinantal*). In the latter case, the orbit closures are determinantal varieties. It is well known that determinantal varieties are Gorenstein if and only if they are generated by minors of a square matrix. We list these cases after Theorem 4.17.

Theorem 4.17. *A non-determinantal orbit closure* $\overline{O}_V$ *is Gorenstein if and only if* V *is in an orbit with multiplicities satisfying one of the following conditions*:

(1) $a = d$, $b = e$, $c = f$;

(2) $a = d + e$, $b = 0$, $c = f$;

(3) $a = d + e$, $b = f = 0$;

(4) $a = d + f$, $c = 0$, $b = e$;

(5) $a = d + f$, $c = e = 0$.

Proof. Part (1) follows from Theorem 4.14 and Table 2. For instance, in the case $b \neq 0$, $c \neq 0$, $d \neq 0$ the term $H^m(\mathcal{V}, \bigwedge^t \xi)$ is one-dimensional if and only if $a + c = d + f$, $a + b = d + e$ and $a + b + c = d + e + f$, that is, if and only if $a = d$, $b = e$ and $c = f$. For the remaining parts, note that (2) is similar to (4) and (3) is similar to (5), so it suffices to prove (2) and (3).

For part (2), note that the weight of $\bigwedge^t \xi$ is

$$((d+e+c)^{d+c};\ 0^c, (d+e+c)^{d+e};\ (-2d-e-c)^{d+e+c}, 0^d).$$

Calculating $D(\underline{\lambda})$ shows that $H^m(\mathcal{V}, \bigwedge^t \xi)$ is non-zero and $\dim H^m(\mathcal{V}, \bigwedge^t \xi) = 1$. So, by Lemma 4.13, the complex $\mathbf{F}_\bullet$ is self-dual in this case. $F_0 = A$ implies F_{t-m} is one-dimensional, hence Gorenstein.

Finally, to prove part (3), we show combinatorially that there exists a unique triple $\underline{\lambda} = (\lambda, \mu, \nu)$ for which $D(\underline{\lambda}) = t - m$. Notice that, for this case, we have

$$t - m = (d+e+c)(2d+e) - d(d+e+c) - c(d+e) = (d+e)^2.$$

Claim 1. $D((d+e)^d;(d+e+c)^{d+e};(2d+e)^{d+e},(d+e)^c) = t-m$. By definition,

$$\begin{aligned}
&D((d+e)^d;(d+e+c)^{d+e};(2d+e)^{d+e},(d+e)^c)\\
&\qquad\qquad\qquad = (d+e)(2d+e+c) - c(d+e) - d(d+e)\\
&\qquad\qquad\qquad = (d+e)^2\\
&\qquad\qquad\qquad = t-m.
\end{aligned}$$

Also note that $((2d+e)^{d+e},(d+e)^c)$ is the unique term in the Littlewood-Richardson product of $((d+e)^d)$ and $((d+e+c)^{d+e})$, which satisfies conditions of Remark 4.2.

Claim 2. If $\underline{\widehat{\lambda}} = (\widehat{\lambda},\widehat{\mu},\widehat{\nu})$ is any other contributing triple, then $D(\underline{\widehat{\lambda}}) < t-m$.

Observe that ν has two corner boxes, either of which can be removed to obtain a smaller $\widehat{\nu}$. Suppose we remove the first corner box. This corresponds to removing one corner box from μ. The next triple contributing a one-dimensional representation is $(\widehat{\lambda},\widehat{\mu},\widehat{\nu}) = ((d+e-1)^d;\ (d+e+c)^{d+e-1}, d+e-1;\ (2d+e-1)^{d+e-1},(d+e-1)^{c+1})$ with the number of exchanges decreased by $c+d$. Then

$$\begin{aligned}
D(\underline{\widehat{\lambda}}) &= (d+e-1)(2d+e+c-1) - c(d+e-1) + d(d+e-1)\\
&= (d+e-1)^2 < t-m.
\end{aligned}$$

On the other hand, if we remove the second corner box, this corresponds to removing a box from μ and the next contributing triple is again $((d+e-1)^d;\ (d+e+c)^{d+e-1}, d+e-1;\ (2d+e)^{d+e-1},(d+e-1)^{c+1})$. Thus, removing boxes from either corner results in a triple with $D(\underline{\widehat{\lambda}}) < t-m$.

Thus, the $((d+e)^d;\ (d+e+c)^{d+e};\ (2d+e)^{d+e},(d+e)^c)$ is the unique triple that contributes to F_{t-m}; applying Bott exchanges to the corresponding weight we get that the contribution is a trivial representation. By Lemma 4.13 and the fact that $\overline{O}_V$ is Cohen-Macaulay, we are done. □

Finally, we give a list of orbits whose closures are Gorenstein determinantal varieties (i.e. orbit closures arising from 1 map). Since it is enough to specify the multiplicities a,b,c,d,e,f to specify an orbit, we present the orbits in the shape of the AR quiver (Figure 1) with multiplicities in place of indecomposables.

$\begin{array}{cccc} & b & & e \\ a & & 0 & \\ & 0 & & f=a \end{array}$	$\begin{array}{cccc} & b & & e \\ 0 & & 0 & \\ & 0 & & f \end{array}$	$\begin{array}{cccc} & b & & e \\ a & & 0 & \\ & 0 & & 0 \end{array}$
$\begin{array}{cccc} & 0 & & e=a \\ a & & 0 & \\ & c & & f \end{array}$	$\begin{array}{cccc} & 0 & & 0 \\ a & & 0 & \\ & c & & f \end{array}$	$\begin{array}{cccc} & 0 & & e \\ 0 & & 0 & \\ & c & & f \end{array}$
$\begin{array}{cccc} & 0 & & e \\ a=d+e+f & & d & \\ & 0 & & f \end{array}$	$\begin{array}{cccc} & 0 & & e \\ 0 & & d & \\ & c & & f=c \end{array}$	$\begin{array}{cccc} & b & & e=b \\ 0 & & d & \\ & 0 & & f \end{array}$
$\begin{array}{cccc} & b & & e=b \\ a & & 0 & \\ & c & & f=c \end{array}$	$\begin{array}{cccc} & b & & e=a+b \\ a & & 0 & \\ & c & & 0 \end{array}$	$\begin{array}{cccc} & b & & 0 \\ a & & 0 & \\ & c & & f=a+c \end{array}$
$\begin{array}{cccc} & b & & e=b \\ 0 & & d & \\ & c & & f=c \end{array}$	$\begin{array}{cccc} & b & & 0 \\ a & & d=a & \\ & c & & 0 \end{array}$	$\begin{array}{cccc} & b & & 0 \\ 0 & & d & \\ & c & & f=c \end{array}$
	$\begin{array}{cccc} & b & & e=b \\ 0 & & d & \\ & c & & 0 \end{array}$	

We present the analysis of a few cases here, and the rest are similar. The orbit

$$\begin{array}{cccc} & b & & e \\ a & & 0 & \\ & 0 & & f=a \end{array}$$

FIGURE 3. Example of determinantal orbit closure.

corresponds to the representation $V = a(010)\oplus b(110)\oplus e(001)\oplus a(100)$. The dimension vector of V is $\underline{d} = (a+b, e, a+b)$ so that V is a representation of the form $K^{a+b} \underset{\text{rank}=b}{\overset{\phi}{\rightarrow}} K^{a+b} \underset{\text{rank}=0}{\overset{\psi}{\leftarrow}} K^{e}$. Thus, $\overline{O}_V$ is the determinantal variety generated by $(b+1)\times(b+1)$ minors of $V_{\mathfrak{a}}$.

For another example, consider the orbit in Figure 4. A representation in this orbit is given by $W = (d+e+f)(010)\oplus d(111)\oplus e(001)\oplus f(100)$ and has the form $K^{d+f} \underset{\text{rank}=d}{\overset{\phi}{\rightarrow}} K^{2d+e+f} \underset{\text{rank}=d}{\overset{\psi}{\leftarrow}} K^{d+e}$. $\overline{O}_W$ is the determinantal variety generated by $(d+1)\times(d+1)$ minors of the $(2d+e+f)\times(2d+e+f)$ minors of the matrix $(W_{\mathfrak{a}} \mid W_{\mathfrak{b}}) : W_1 \oplus W_2 \to W_3$.

$0 \qquad e$

$a = d + e + f \quad d$

$0 \qquad f$

FIGURE 4. Example of determinantal orbit closure.

$b \qquad 0$

$a \qquad 0$

$c \qquad f = a + c$

FIGURE 5. Example of determinantal orbit closure.

As a final example, consider the orbit in Figure 5. A representation in this orbit is of the form $V = K^{a+b+c} \underset{\text{rank}=b}{\overset{\phi}{\rightarrow}} K^{a+b+c} \underset{\text{rank}=c}{\overset{\psi}{\leftarrow}} K^c$. The corresponding orbit closure is a determinantal variety generated by $(b+1) \times (b+1)$ minors of $V_{\mathfrak{a}}$.

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ENDNOTES

1. Source-sink orientation refers to an orientation of the arrows such that every vertex is a source or a sink. In case of the A_3 quiver, source-sink orientation means the same as 'non-equioriented A_3.'

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