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MULTIPLICITIES ASSOCIATED TO GENERALIZED SYMBOLIC POWERS

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Dedicated to Juergen Herzog on the occasion of his 70th birthday

1. Introduction. Suppose that R is a Noetherian local ring of dimension d and I, J are ideals in R. Let

$$I_n(J) = I^n : J^\infty = \bigcup_{i=1}^{\infty} I^n : J^i,$$

be the "nth symbolic power of I with respect to J."

In the introduction to paper [**7**] by Herzog, Puthenpurakal and Verma, the following interesting question is raised.

Let s be the limit dimension of family $I_n(J)/I^n$. When does

$$\lim_{n\to\infty} \frac{e_{m_R}(I_n(J)/I^n)}{n^{d-s}}$$

exist?

In this paper we review some results in [**7**] and give a very general answer to this question, using some recent results from [**3, 4**].

2. Notation. m_R will denote the maximal ideal of a local ring R. $Q(R)$ will denote the quotient field of a domain R. $\ell_R(N)$ will denote the length of an R-module N. $\mathbf{Z}_+$ denotes the positive integers and $\mathbf{N}$ the nonnegative integers. Suppose that $x \in \mathbf{R}$. $\lceil x \rceil$ is the smallest integer n such that $x \le n$. $\lfloor x \rfloor$ is the largest integer n such that $n \le x$.

We recall some notation on multiplicity from [**10**, Chapter VIII, Section 10], [**8**, Section V-2] and [**2**, Section 4.6]. Suppose that (R, m_R)

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is a (Noetherian) local ring, N is a finitely generated R-module with $r = \dim N$ and a is an ideal of the definition of R. Then

$$e_a(N) = \lim_{k\to\infty} \frac{\ell_R(N/a^kN)}{k^r/r!}.$$

We write $e(a) = e_a(R)$.

If $s \geq r = \dim N$, then we define

$$e_s(a, N) = \begin{cases} e_a(N) & \text{if } \dim N = s \\ 0 & \text{if } \dim N < s. \end{cases}$$

3. Asymptotic multiplicity. Suppose that R is a Noetherian local ring of dimension d and I, J are ideals in R. Let

$$I_n(J) = I^n : J^\infty = \bigcup_{i=1}^{\infty} I^n : J^i,$$

be the "nth symbolic power of I with respect to J."

The saturation of an ideal I is $I^{\text{sat}} = I :_R m_R^\infty$. Thus,

$$I_n(m_R) = (I^n)^{\text{sat}}.$$

If R_p has depth ≥ 2 for all $p \in V(J)$, then $I_n(J)/I^n \cong H^1_J(I^n)$ for all n.

Lemma 3.1. *The dimension of $I_n(J)/I^n$ is a constant for $n \gg 0$.*

Proof. There exists a positive integer n_0 such that the set of associated primes of R/I^n stabilizes for $n \geq n_0$ by [**1**]. Let $\{p_1, \dots, p_t\}$ be this set of associated primes. We thus have irredundant primary decompositions for $n \geq n_0$,

$$I^n = q_1(n) \bigcap \cdots \bigcap q_t(n), \tag{1}$$

where $q_i(n)$ are p_i-primary.

We further have that

$$I^n : J^\infty = \bigcap_{J \not\subset p_i} q_i(n). \tag{2}$$

Thus $\dim I_n(J)/I^n$ is constant for $n \geq n_0$. ◻

We will call, as in [**7**], the *dimension*

$$s = \dim\,(I_n(J)/I^n)$$

for $n \gg 0$ the limit dimension of the family of R-modules $I_n(J)/I^n$.

The following result is proven in [**7**].

Theorem 3.2 [**7**, Theorem 3.2]. *Suppose that I has positive grade in R, and*

$$S_J(I) = \bigoplus_{n \geq 0} I_n(J)$$

is a finitely generated R-algebra. Then polynomials $P_0, \ldots, P_{g-1}$ exist such that

$$P_i(m) = e_{m_R}(I_{mg+i}(J)/I^{mg+i}),$$

for all $m \gg 0$, so that, for $n \gg 0$, $e_{m_R}(I_n(J)/I^n)$ is a quasi-polynomial of period g. Further, all the P_i have the same degree and leading coefficient.

As a consequence, they prove the following result.

Theorem 3.3 [**7**, Theorem 2.5, Corollary 2.6]. *Suppose that R is a regular local ring and I, J are monomial ideals in R. Then for $n \gg 0$, $e_{m_R}(I_n(J)/I^n)$ is a quasi-polynomial, with constant coefficient leading term.*

Further applications of Theorem 3.2 are given in [**6, 7**].

Suppose that R is universally catenary. With the assumptions of Theorems 3.2 and 3.3, we have that the limit

$$\lim_{n \to \infty} \frac{e_{m_R}(I_n(J)/I^n)}{n^{d-s}}$$

exists and is a rational number, where s is the limit dimension of the family $I_n(J)/I^n$.

In [**5**], an example is given, showing that this limit can be an irrational number. In the example, $J = m_R$, so that $I_k(J)/I^k$ is the saturation $(I^k)^{\text{sat}}$ of I^k. The limit dimension is $s = 0$, so that

$$\lim_{n\to\infty} \frac{\ell_R((I^n)^{\text{sat}})}{n^d}$$

is an irrational number.

In the introduction to [**7**], the following question is raised.

Question 3.4. Let s be the limit dimension of the family $I_n(J)/I^n$. When does

$$\lim_{n\to\infty} \frac{e_{m_R}(I_n(J)/I^n)}{n^{d-s}} \tag{3}$$

exist?

At the time that [**7**] was written, it was known from [**5**] that the question has a positive answer if I is a homogeneous ideal in a polynomial ring over a field and J is the graded maximal ideal. After this, the question was given a positive answer in [**6**] for ideals I defining an isolated singularity in a regular local ring, with J being the maximal ideal of R. More recently, the following two general results have been proved for saturated powers.

Theorem 3.5 [**3**, Corollary 1.5]. *Suppose that (R, m_R) is a local domain of dimension $d \geq 1$ which is essentially of finite type over a field k of characteristic zero (or over a perfect field k such that R/m_R is algebraic over k). Suppose that I is an ideal in R. Then the limit*

$$\lim_{n\to\infty} \frac{\ell_R((I^n)^{\text{sat}}/I^n)}{n^d} \in \mathbf{R}$$

exists.

Theorem 3.6 [**4**, Corollary 6.3]. *Suppose that R is a local ring of dimension $d > 0$ such that one of the following holds:*

1) *R is regular or*

2) *R is analytically irreducible and excellent with algebraically closed residue field or*

3) *R is normal, excellent and equicharacteristic with perfect residue field.*

Suppose that I is an ideal in R. Then the limit

$$\lim_{i\to\infty} \frac{\ell_R((I^i)^{\rm sat}/I^i)}{i^d}$$

exists.

Since $(I^n)^{\rm sat}/I^n \cong H^0_{m_R}(R/I^n)$, these results show that the epsilon multiplicity of Ulrich and Validashti [**9**]

$$\varepsilon(I) = \limsup \frac{\ell_R(H^0_{m_R}(R/I^n))}{n^d/d!}$$

exists as a limit, under the assumptions of Theorems 3.5 and 3.6. In particular, Theorem 3.6 is valid whenever R is a complex analytic local domain.

Theorem 3.5 is proven for more general families of modules when R is a local domain which is essentially of finite type over a perfect field k such that R/m_R is algebraic over k in [**3**], proving that the epsilon multiplicity exists in these cases.

As a consequence of Theorems 3.5 and 3.6, we show that the limit of Question 3.4 exists under very general conditions.

Theorem 3.7. *Suppose that R is a local domain of dimension d such that one of the following holds*:

1) *R is regular or*

2) *R is normal and excellent of equicharacteristic* 0 *or*

3) *R is essentially of finite type over a field of characteristic zero.*

Suppose that I and J are ideals in R. Let

$$I_n(J) = I^n : J^\infty = \cup_{i=1}^\infty I^n : J^i,$$

be the "nth symbolic power of I with respect to J." Let s be the constant limit dimension of $I_n(J)/I^n$ for $n \gg 0$. Then

$$\lim_{n\to\infty} \frac{e_{m_R}(I_n(J)/I^n)}{n^{d-s}}$$

exists.

Proof. We use the notation of the proof of Lemma 3.1 Let s be the limit dimension of the family $I_n(J)/I^n$. The set

$$A = \left\{ p \in \bigcup_{n\geq n_0} \mathrm{Ass}\,(I_n(J)/I^n) \mid n \geq n_0 \text{ and } \dim R/p = s \right\}$$

is a finite set. Moreover, every such prime is in $\mathrm{Ass}\,(I_n(J)/I^n$ for all $n \geq n_0$. For $n \geq n_0$, we have by the associativity formula ([**8**, V-2] or [**2**, page 189, Corollary 4.6.8]), that

$$e_{m_R}(I_n(J)/I^n) = \sum_p \ell_{R_p}((I_n(J)/I^n)_p)e(m_{R/p})$$

where the sum is over the finite set of primes $p \in \mathrm{Spec}\,(R)$ such that $\dim R/p = s$. This sum is thus over the finite set A.

Suppose that $p \in A$ and $n \geq n_0$. Then

$$I_p^n = \bigcap q_i(n)_p$$

where the intersection is over the $q_i(n)$ such that $p_i \subset p$, and

$$I_n(J) = \bigcap q_i(n)_p$$

where the intersection is over the $q_i(n)$ such that $J \not\subset p_i$ and $p_i \subset p$. Thus, an index i_0 exists such that $p_{i_0} = p$ and

$$I_p^n = q_{i_0}(n)_p \bigcap I_n(J)_p.$$

By (1),

$$(I_p^n)^{\mathrm{sat}} = I_n(J)_p$$

for $n \geq n_0$. Since R_p satisfies one of conditions 1) or 3) of Theorem 3.6, or the conditions of Theorem 3.5, and $\dim R_p = d - s$ (as R is universally catenary), the limit

$$\lim_{n\to\infty} \frac{\ell_R((I_n(J)/I_n)_p)}{n^{d-s}}$$

exists. □

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