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TRIPLETS OF PURE FREE SQUAREFREE COMPLEXES

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Dedicated to Jürgen Herzog on the occasion of his 70th birthday.

ABSTRACT. On the category of bounded complexes of finitely generated free squarefree modules over the polynomial ring S, there is the standard duality functor $\mathbf{D} = \mathrm{Hom}_S(-, \omega_S)$ and the Alexander duality functor $\mathbf{A}$. The composition $\mathbf{A} \circ \mathbf{D}$ is an endofunctor on this category, of order three up to translation. We consider complexes $F_\bullet$ of free squarefree modules such that both $F_\bullet, \mathbf{A} \circ \mathbf{D}(F_\bullet)$ and $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ are pure, when considered as singly graded complexes. We conjecture: i) the existence of such triplets of complexes for given triplets of degree sequences, and ii) the uniqueness of their Betti numbers, up to scalar multiple. We show that this uniqueness follows from the existence, and we construct such triplets if two of its degree sequences are linear.

Introduction. *Pure free resolutions* are free resolutions over the polynomial ring S of the form

$$S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_r)^{\beta_r}.$$

Their Betti diagrams have proven to be of fundamental importance in the study of Betti diagrams of graded modules over the polynomial ring. Their significance was put to light by the Boij-Söderberg conjectures, [**2**]. The existence of pure resolutions were first proved by Eisenbud, the author and Weyman in [**7**] in characteristic zero, and by Eisenbud and Schreyer in all characteristics, [**8**]. Later, the methods of [**8**] were made more explicit and put into a larger framework, called tensor complexes, by Berkesch et al. [**1**].

The Boij-Söderberg conjectures, settled in full generality in [**8**], concerns the stability theory of *Betti diagrams* of graded modules, i.e.,

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it describes such diagrams up to multiplication by a positive rational number, or alternatively the positive rational cone generated by such diagrams. The Betti diagrams of pure resolutions are exactly the extremal rays in this cone. Two introductory papers on this theory are [**9, 11**].

Homological invariants. The Betti diagram is, however, only part of the story when it comes to homological invariants of graded modules. A complex $F_\bullet$ of free modules over the polynomial ring S, for instance a free resolution, comes with three sets of numerical homological invariants:

- B: The graded Betti numbers $\{\beta_{ij}\}$,
- H: The Hilbert functions of the homology modules $H^i(F_\bullet)$,
- C: The Hilbert functions of the cohomology modules. These modules are the homology modules of the dual complex $\mathrm{Hom}_S(F_\bullet, \omega_S)$, where ω_S is the canonical module.

It is then natural to approach the stability theory of the triplet data set (B, H, C): Up to rational multiple, what triplets of such can occur? The recent article [**6**] has partial results in this direction. It describes the Betti diagrams of complexes $F_\bullet$ with specified nondecreasing codimensions of the homology modules. We do not investigate here the above question directly, but we believe the following will be of relevance.

Squarefree modules. The notion of pure resolution or pure complex, has a very natural extension into *triplets of pure complexes*, in the setting of squarefree modules over the polynomial ring. Squarefree modules are $\mathbf{N}^n$-graded modules over the polynomial ring $S = \mathbf{k}[x_1, \ldots, x_n]$ and form a module category including squarefree monomial ideals, and Stanley-Reisner rings. Both the category of singly graded S-modules as well as squarefree S-modules have the standard duality functor $\mathbf{D} = \mathrm{Hom}_S(-, \omega_S)$. However, for squarefree modules there is also another duality functor, Alexander duality $\mathbf{A}$. The composition $\mathbf{A} \circ \mathbf{D}$ becomes an endofunctor on the category of bounded complexes of finitely generated free squarefree S-modules. (This is in fact the Auslander-Reiten translate on the derived category of complexes of squarefree modules, see [**3**].) There are two amazing facts concerning this endofunctor.

• The third iterate $(\mathbf{A} \circ \mathbf{D})^3$ is isomorphic to the nth iterate of the translation functor on complexes, a result of Yanagawa [**21**].

• The composition functor cyclically rotates the homological invariants: If $F_\bullet$ has homological invariants (B, H, C), then

– $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ has homological invariants (H, C, B), and

– $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ has homological invariants (C, B, H). This is also implicit in [**21**].

Thus, the various homology modules of $F_\bullet$ are transferred to the various linear strands of $\mathbf{A} \circ \mathbf{D}(F_\bullet)$, and the cohomology modules of $F_\bullet$ are transferred to the linear strands of $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$.

The main idea of this paper is to consider complexes $F_\bullet$ of free squarefree modules such that (when considered as singly graded modules)

- $F_\bullet$ is pure,
- $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ is pure,
- $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ is pure.

We call this a *triplet of pure complexes*. That $F_\bullet$ is a pure resolution of a Cohen-Macaulay squarefree module, the classical case, corresponds to

- $F_\bullet$ is pure,
- $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ is linear,
- $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ is linear.

Construction of triplets. Squarefree complexes are $\mathbf{Z}^n$-graded or, equivalently, they are equivariant for the action of the diagonal matrices of $GL(n)$. That pure resolutions come with various group actions is the rule in the various constructions we have, [**1, 7**]. Sam and Weyman pursue this [**18**] in the context of other linear algebraic groups. However, being squarefree is something more than being $\mathbf{Z}^n$-graded. In particular, for a squarefree complex $F_\bullet$, it may happen that the only multidegree $\mathbf{b}$ such that $F_\bullet(-\mathbf{b})$ is squarefree, is the zero degree. It is therefore a priori not clear, even in the classical case, how to construct such complexes $F_\bullet$. As it turns out the tensor complexes of [**1**] make the perfect input for a construction, see in particular Remark 4.6. These tensor complexes are over a large polynomial ring $S(V \otimes W_0^* \otimes \cdots \otimes W_{r+1}^*)$. Letting V be the linear space $\langle x_1, \dots, x_n \rangle$ and taking a general map

$$V \otimes W_0^* \otimes \cdots \otimes W_{r+1}^* \longrightarrow V,$$

equivariant for the diagonal matrices in $GL(n)$, we may construct all cases of complexes $F_\bullet$ corresponding to the classical case, Theorem 4.8. The existence of triplets of pure complexes in full generality, we state as Conjecture 2.11. In a subsequent paper, [**12**], we give a conjecture on the existence of certain complexes of coherent sheaves on projective spaces, which implies Conjecture 2.11.

Uniqueness of Betti numbers. In the classical case the singly graded Betti numbers of $F_\bullet$ (and also of the linear complexes $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ and $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ are uniquely determined up to scalar multiple, by the degree sequence of $F_\bullet$. These Betti numbers are determined by the Herzog-Kühl equations [**14**], see also [**11**, subsection 1.3].

It now turns out that, for a triplet of pure complexes, given the degree sequences of each of the three complexes, the Betti numbers fulfill a number of homogeneous linear equations which is one less than the number of variables, i.e., the number of Betti numbers. We thus expect there to be a unique solution up to common rational multiple. Under the assumption that triplets of pure complexes exist (for all triplet of degree sequences fulfilling a simple necessary criterion), we show that the Betti numbers are uniquely determined up to common rational multiple, Theorem 3.10.

Pure resolutions in the squarefree setting have previously been considered by Bruns and Hibi for Stanley-Reisner rings. In [**4**] they describe all possible degree sequences $0 = d_0, d_1, \dots$ for pure resolutions of Stanley-Reisner rings with $d_1 = 2$ and classify the simplicial complexes where this occurs. When $d_1 = 3$, they give a thorough investigation of possible degree sequences and the possible simplicial complexes, as well as interesting examples when $d_1 \geq 4$. They also give a complete classification of simplicial complexes where $d_1 = m$ and $d_2 = 2m-1$ for $m \geq 2$. In [**5**] they classify Cohen-Macaulay posets where the Stanley-Reisner ring of the order complex has pure resolution. In [**10**] the author considers *Cohen-Macaulay designs* which, in the language of the present article, correspond to Cohen-Macaulay Stanley-Reisner rings with pure resolution and exactly three linear strands (so the Stanley-Reisner ideal has exactly two linear strands). Examples of such are cyclic polytopes and Alexander duals of Steiner systems.

However, from the perspective of the present article, approaches in those directions are severely hampered by the fact that, only for few degree sequences, by simple numerical considerations, can one hope that the first Betti number β_0 may be chosen to be 1. For degree sequences where this value may be achieved these articles also testify to the difficulty in constructing pure resolutions of Stanley-Reisner rings. Our construction avoids the restriction $\beta_0 = 1$, rather making β_0 large.

Organization of the article. In Section 1 we give the setting of squarefree modules and the functors $\mathbf{A}$ and $\mathbf{D}$. We show that they rotate the homological invariants of squarefree complexes. In Section 2 we develop the basic theory of triplets of pure complexes. We find a basic necessary condition, the balancing condition, on the triplet of degree sequences of such complexes. We conjecture the existence of triplets of pure complexes for all balanced triplets of degree sequences, and the uniqueness of their Betti numbers, up to common scalar multiple, Conjecture 2.11. In Section 3 we show this uniqueness of Betti numbers, under the assumption that triplets of pure complexes do exist. In Section 4 we use the tensor complexes of [**1**] to construct triplets of pure complexes $F_\bullet, \mathbf{A} \circ \mathbf{D}(F_\bullet)$ and $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ when the last two complexes are linear.

1. Duality functors and rotation of homological invariants. In this section we recall the notion of a squarefree module over the polynomial ring, $S = \mathbf{k}[x_1, \ldots, x_n]$, and the two duality functors we may define on the category of complexes of such modules, standard duality $\mathbf{D}$ and Alexander duality $\mathbf{A}$.

A striking result of Yanagawa [**21**] says that the composition $(\mathbf{A} \circ \mathbf{D})^3$ is naturally equivalent to the nth iterate of the translation functor on the derived category of squarefree modules. A complex of squarefree modules comes with three sets of homological invariants, the multigraded homology, cohomology modules and multigraded Betti spaces. We show that $\mathbf{A} \circ \mathbf{D}$ cyclically rotates these invariants (which is a rather well-known fact to experts).

1.1. Squarefree modules and dualities. Let S be the polynomial ring $\mathbf{k}[x_1, \dots, x_n]$ where $\mathbf{k}$ is a field. Let ε_i be the ith coordinate vector in $\mathbf{N}^n$. An $\mathbf{N}^n$-graded S-module is called *squarefree*, introduced by Yanagawa in [**20**], if M is finitely generated and the multiplication map $M_{\mathbf{b}} \stackrel{\cdot x_i}{\to} M_{\mathbf{b}+\varepsilon_i}$ is an isomorphism of vector spaces whenever the ith coordinate $b_i \geq 1$. We denote the category of finitely generated squarefree S-modules by $\mathrm{sq}-S$.

There is a one-one correspondence between subsets $R \subseteq [n] = \{1, 2, \dots, n\}$ and multidegrees R in $\{0, 1\}^n$, by letting R be the set of coordinates of R equal to 1. By abuse of notation, we shall often write R when strictly speaking we mean R. For instance, the degree R part of M, which is, M_R may be written M_R. Also, if R is a set, we shall, if no confusion arises, denote its cardinality by the smaller case letter r. We also denote $(1, 1, \dots, 1)$ as $\mathbf{1}$. Note that a squarefree module is completely determined, up to isomorphism, by the graded pieces M_R and the multiplication maps between them

$$M_R \xrightarrow{x_v} M_{R\cup\{v\}}$$

where $v \notin R$.

If M is a squarefree module and $0 \leq d \leq n$, its *squarefree part of degree* d is

$$\bigoplus_{|R|=d} M_R.$$

Note that taking squarefree parts is an exact functor from squarefree modules to vector spaces. In particular, note that the squarefree part of $S(-\mathbf{b})$ in degree d has dimension $\binom{n-|\mathbf{b}|}{d-|\mathbf{b}|} = \binom{n-|\mathbf{b}|}{n-d}$.

For a squarefree module M there is a notion of Alexander dual module $A(M)$, defined by Römer [**17**] and Miller [**16**]. For R a subset of $[n]$, let R^c be its complement. Then $A(M)_R$ is the dual $\mathrm{Hom}_{\mathbf{k}}(M_{R^c}, k)$. If v is not in R, the multiplication

$$A(M)_R \xrightarrow{\cdot x_v} A(M)_{R\cup\{v\}}$$

is the dual of the multiplication

$$M_{(R\cup\{v\})^c} \xrightarrow{\cdot x_v} M_{R^c}.$$

By obvious extension, this defines $A(M)_{\mathbf{b}}$ for all $\mathbf{b}$ in $\mathbf{N}^n$ and all multiplications.

Example 1.1. If $S = \mathbf{k}[x_1, x_2, x_3, x_4]$, then the Alexander dual of $S(-(1,0,1,1))$ is $S/(x_1, x_3, x_4)$. The Alexander dual of $S(-\mathbf{1})$ is the simple quotient module $\mathbf{k}$.

For a multidegree $\mathbf{b}$ in $\mathbf{Z}^n$, the free S-module $S(-\mathbf{b})$ is a squarefree module if and only if $\mathbf{b} \in \{0,1\}^n$, i.e., all coordinates of $\mathbf{b}$ are 0 or 1. Direct sums of such modules are the *free squarefree S-modules*. Denote by $\mathrm{fsq}{-}S$ the category of such finitely generated modules.

Let $C^b(\mathrm{sq}{-}S)$ and $C^b(\mathrm{fsq}{-}S)$ be the categories of bounded complexes of finitely generated squarefree, respectively, free squarefree modules. There is a natural duality

$$\mathbf{D} : C^b(\mathrm{fsq}{-}S) \longrightarrow C^b(\mathrm{fsq}{-}S)$$

defined by

$$\mathbf{D}(F_\bullet) = \mathrm{Hom}_S(F_\bullet, S(-\mathbf{1})),$$

so in particular $\mathbf{D}(S(-\mathbf{b})) = S(\mathbf{b} - \mathbf{1})$. We would also like to define Alexander duality on the category $C^b(\mathrm{fsq}{-}S)$. However, there is a slight problem in that Alexander duality as defined above does not take free modules to free modules.

To remedy this, any bounded complex of squarefree modules $X_\bullet$ has a minimal resolution $F_\bullet \to X_\bullet$ by free squarefree modules. This defines a functor $\mathrm{res} : C^b(\mathrm{sq}{-}S) \to C^b(\mathrm{fsq}{-}S)$. (There is of course also a natural inclusion $\iota : C^b(\mathrm{fsq}{-}S) \to C^b(\mathrm{sq}{-}S)$.) We now define Alexander duality

$$\mathbf{A} : C^b(\mathrm{fsq}{-}S) \longrightarrow C^b(\mathrm{fsq}{-}S)$$

by letting $\mathbf{A}$ be the composition $\mathrm{res} \circ A$ where A is the Alexander duality defined above.

Example 1.2. Continuing the example above, a free resolution of $S/(x_1, x_2, x_4)$ is

$$\begin{array}{ccccccc}
S & \longleftarrow & S^3 & \longleftarrow & S^3 & \longleftarrow & S \\
(0,0,0,0) & & (1,0,0,0) & & (1,0,1,0) & & (1,0,1,1) \\
 & & (0,0,1,0) & & (1,0,0,1) & & \\
 & & (0,0,0,1) & & (0,0,1,1) & &
\end{array}$$

where we have written below the multidegrees of the generators. Then the Alexander dual $\mathbf{A}(S(-(1,0,1,1)))$ is the above resolution.

By composing with the resolution we may also consider $\mathbf{A}$ and $\mathbf{D}$ as functors on $C^b(\text{sq}-S)$

$$C^b(\text{sq}-S) \xrightarrow{\text{res}} C^b(\text{fsq}-S) \xrightarrow{\mathbf{A},\mathbf{D}} C^b(\text{fsq}-S).$$

For a complex $X_\bullet$, let $X_\bullet[p]$ be its pth translate, i.e., $(X_\bullet[p])_q = X_{q-p}$. Yanagawa [**21**], shows that $(\mathbf{A} \circ \mathbf{D})^3$ is isomorphic to the nth iterate $[n]$ of the translation functor.

1.2. Homological invariants. The complex $X_\bullet$ comes with three sets of squarefree homological invariants. First there is the homology

$$H_i(X_\bullet)_R$$

where $i \in \mathbf{Z}$ and $R \subseteq [n]$. For a vector space V, denote by V^* its dual $\mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$. We define the cohomology as

$$C_i(X_\bullet)_R := (H_{-i}(\mathbf{D}(X_\bullet))_{R^c})^*.$$

Note that, by local duality, if $X_\bullet$ is a module M, then this relates to local cohomology by

$$C_{n-i}(M)_R = H^i_{\mathfrak{m}}(M)_{\mathbf{r}-\mathbf{1}}$$

where $\mathbf{r}$ is the $0,1$-vector with support R. Thirdly a minimal free squarefree resolution $F_\bullet$ of $X_\bullet$ has terms which may be written $F_i = \oplus_{R\subseteq[n]} S \otimes_{\mathbf{k}} B_{i,R}$, and we define the Betti spaces to be

$$B_i(X_\bullet)_R := (\mathrm{Tor}^S_i(X_\bullet, k)_R) = (B_{i,R}).$$

Now a basic and very interesting fact is that the functors $\mathbf{A}$ and $\mathbf{D}$ interchange the homology, cohomology and Betti spaces. First we consider $\mathbf{D}$.

Lemma 1.3. *The functor* $\mathbf{D}$ *interchanges the homological invariants of* $X_\bullet$ *as follows.*

- $B_i(\mathbf{D}(X_\bullet))_R = (B_{-i}(X_\bullet)_{R^c})^*$.
- $H_i(\mathbf{D}(X_\bullet))_R = (C_{-i}(X_\bullet)_{R^c})^*$.
- $C_i(\mathbf{D}(X_\bullet))_R = (H_{-i}(X_\bullet)_{R^c})^*$.

Proof. This is clear. □

Before describing how the functor $\mathbf{A}$ interchanges the homological invariants, we recall a basic fact from [**21**]. For a square-free module M, one may define a complex $\mathcal{L}(M)$ (see [**21**, page 9] where it is denoted by $\mathcal{F}(M)$) by

$$\mathcal{L}_{n-i}(M) = \bigoplus_{|R|=i} (M_R)^\circ \otimes_{\mathbf{k}} S$$

where $(M_R)^\circ$ is M_R but considered to have multidegree R^c. The differential is

$$m^\circ \otimes s \longmapsto \sum_{j \notin R} (-1)^{\alpha(j,R)} (x_j m)^\circ \otimes x_j s,$$

where $\alpha(j, R)$ is the number of i in R such that $i < j$.

For a minimal complex $F_\bullet$ of free squarefree S-modules, define its ith *linear strand* $F_\bullet^{\langle i \rangle}$ to have terms

$$F_j^{\langle i \rangle} = \bigoplus_{|R|=i+j} S \otimes_{\mathbf{k}} B_{j,R}.$$

Since $F_\bullet$ is minimal, the ith linear strand is naturally a complex. The following is [**21**, Theorem 3.8].

Proposition 1.4. *The* $(-i)$*th linear strand of* $\mathbf{A} \circ \mathbf{D}(X_\bullet)$ *is*

$$\mathcal{L}(H_i(X_\bullet))[i].$$

This gives the following.

Lemma 1.5. *The functor* $\mathbf{A}$ *interchanges the homological invariants of* $X_\bullet$ *as follows* (*denoting the cardinality of* R *by* r).

a. $B_i(\mathbf{A}(X_\bullet))_R = (C_{r-i}(X_\bullet)_R)^*$.
b. $H_i(\mathbf{A}(X_\bullet))_R = (H_{-i}(X_\bullet)_{R^c})^*$.
c. $C_i(\mathbf{A}(X_\bullet))_R = (B_{r-i}(X_\bullet)_R)^*$.

Proof. Part b. is clear. By the proposition above,

$$\mathcal{L}(H_i(\mathbf{D}(X_\bullet)))[i] \cong \mathbf{A}(X_\bullet)^{\langle -i\rangle}. \tag{1}$$

The first complex has terms which are direct sums over R of

$$(H_i(\mathbf{D}(X_\bullet))_R)^\circ \otimes_{\mathbf{k}} S,$$

where the generating space has internal degree R^c and is in homological position $n-r-i = |R^c| - i$. By Lemma 1.3, the generating space here is

$$((C_{-i}(X_\bullet)_{R^c})^*)^\circ.$$

Hence, by (1), this equals

$$B_{|R^c|+i}(\mathbf{A}(X_\bullet))_{R^c},$$

which is equivalent to part a.

Part c. follows from a. by replacing $X_\bullet$ by $\mathbf{A}(X_\bullet)$. $\square$

Putting these two lemmata together, we get the following.

Corollary 1.6. *The composition* $\mathbf{A} \circ \mathbf{D}$ *cyclically rotates the homological invariants as follows.*

- $B_i(\mathbf{A} \circ \mathbf{D}(X_\bullet))_R = H_{i-r}(X_\bullet)_{R^c}$.
- $H_i(\mathbf{A} \circ \mathbf{D}(X_\bullet))_R = C_i(X_\bullet)_R$.
- $C_i(\mathbf{A} \circ \mathbf{D}(X_\bullet))_R = B_{i-r}(X_\bullet)_{R^c}$.

We may depict the rotation of homological invariants by the diagram:

$$\begin{tikzcd}
 & B \arrow[dd, "\mathbf{A}\circ\mathbf{D}"] \\
H \arrow[ur, "\mathbf{A}\circ\mathbf{D}"] & \\
 & C. \arrow[ul, "\mathbf{A}\circ\mathbf{D}"]
\end{tikzcd}$$

Remark 1.7. Composing $\mathbf{A}$ and $\mathbf{D}$ alternately, and applying it to $F_\bullet$, we get six distinct complexes up to translation, corresponding to all permutations of the triplet data set (B, H, C). In the squarefree setting, we thus get a situation of perfect symmetry between the homological invariants. In contrast, in the singly graded case we get a "symmetry breakdown" where only H and C may be transferred into each other by the functor $\mathbf{D}$, while the Betti spaces have a distinct position.

Remark 1.8. For a positive multidegree $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, Alexander duality may also be defined for the more general class of $\mathbf{a}$-determined modules, see [**16**]. (Squarefree modules are $\mathbf{1}$-determined.) The composition $\mathbf{A} \circ \mathbf{D}$ then has order the least common multiple $\operatorname{lcm}\{a_i + 2 \mid i = 1, \ldots, n\}$, see [**3**]. In that paper all the multigraded homology and Betti spaces of the iterates $(\mathbf{A} \circ \mathbf{D})^i(S/I)$ are computed for an $\mathbf{a}$-determined ideal $I \subseteq S$.

The following will be of particular interest and motivation in the following Section 2.

Lemma 1.9. *The complexes $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ and $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ are both linear if and only if $F_\bullet$ is a resolution of a Cohen-Macaulay module.*

Proof. The various homology modules of $F_\bullet$ are translated to the various linear strands of $(\mathbf{A} \circ \mathbf{D})(F_\bullet)$. So $F_\bullet$ has only one nonzero homology module if and only if $(\mathbf{A} \circ \mathbf{D})(F_\bullet)$ is linear. Similarly the cohomology of $F_\bullet$ is translated to the Betti spaces of $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ so $F_\bullet$ has only one nonzero cohomology module if and only if $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ is linear. But the fact that $F_\bullet$ has only one nonzero homology module M and $\mathbf{D}(F_\bullet)$ has only one nonzero homology module is equivalent to M being a Cohen-Macaulay module. $\square$

1.3. The functor $\mathbf{A} \circ \mathbf{D}$ on a basic class of modules. For $A \subseteq [n]$, the module $S(-A)$ is a projective module. Denote by $S/A = S/(x_i)_{i \in A}$. (This is an injective module in $\mathrm{sq}{-}S$.)

More generally, for a partition $A \cup B \cup C$ of $[n]$, the module $(S/A)(-B)$ will be a squarefree module. Let us denote it as $S/A(-B; C)$. These form a basic simple class of squarefree modules closed with respect to the functors $\mathbf{A}$ and $\mathbf{D}$ when we identify modules with their minimal resolutions.

Lemma 1.10. *Let $A \cup B \cup C$ be a partition of $[n]$.*

1. *There is a quasi-isomorphism*
$$\mathbf{D}(S/A(-B;C)) \xrightarrow{\simeq} S/A(-C;B)[-a].$$

2. *There is a quasi-isomorphism*
$$\mathbf{A}(S/A(-B;C)) \xrightarrow{\simeq} S/B(-A;C).$$

Proof. For $A \subseteq [n]$, denote by $\mathbf{k}A$ the vector space generated by x_i, $i \in A$. The projective resolution of $S/A(-B)$ is

$$P_\bullet : S(-B) \leftarrow S(-B) \otimes (\mathbf{k}A) \leftarrow S(-B) \otimes \wedge^2(\mathbf{k}A)\varphi \leftarrow \cdots \leftarrow S(-B) \otimes \wedge^a(\mathbf{k}A).$$

The dual complex $\mathrm{Hom}_S(P_\bullet, S(-\mathbf{1}))$ is $\mathbf{D}(S/A(-B))$. Since the last term $S(-B) \otimes \wedge^a(\mathbf{k}A)$ in $P_\bullet$ is generated in degree $A \cup B$, the dual complex is

$$S(-C) \leftarrow S(-C) \otimes (\mathbf{k}A) \leftarrow S(-C) \otimes \wedge^2(\mathbf{k}A) \leftarrow \cdots \leftarrow S(-C) \otimes (\mathbf{k}A)^a,$$

a resolution of $S/A(-C;B)$.

To see the second part of the lemma, it is not difficult to verify that the Alexander dual module of $S/A(-B;C)$ is $S/B(-A;C)$. □

We then get the following diagram.

$$\begin{tikzcd}
& S/C(-B;A)[-a-c] \arrow[dr, "\mathbf{A}"] & \\
S/C(-A;B)[a] \arrow[ur, "\mathbf{D}"] & & S/B(-C;A)[a+c] \arrow[d, "\mathbf{D}"] \\
S/A(-C;B)[-a] \arrow[u, "\mathbf{A}"] & & S/B(-A;C)[-a-b-c] \arrow[dl, "\mathbf{A}[-n]"] \\
& S/A(-B;C) \arrow[ul, "\mathbf{D}"] &
\end{tikzcd}$$

A particular case is the following diagram.

$$\begin{tikzcd}
& \mathbf{k}[-n] \arrow[dr, "\mathbf{A}"] & \\
\mathbf{k} \arrow[ur, "\mathbf{D}"] & & S(-\mathbf{1})[n] \arrow[d, "\mathbf{D}"] \\
S(-\mathbf{1}) \arrow[u, "\mathbf{A}"] & & S[-n] \arrow[dl, "{\mathbf{A}[-n]}"] \\
& S \arrow[ul, "\mathbf{D}"] &
\end{tikzcd}$$

2. Triplets of pure complexes. As stated in the introduction, the importance of pure free resolutions of Cohen-Macaulay S-modules is established with the Boij-Söderberg conjectures and their subsequent demonstration in [**8**].

A complex of free S-modules $F_\bullet$ is *pure* if it has the form

$$F_\bullet : S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_r)^{\beta_r}$$

for some integers $d_0 < d_1 < \cdots < d_r$. These integers are the *degree sequence* of the pure complex.

We shall investigate the condition that *all three complexes* $F_\bullet$, $(\mathbf{A} \circ \mathbf{D})(F_\bullet)$ and $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ are *pure* when considered as singly graded complexes. By Lemma 1.9, the special case that $F_\bullet$ is a pure resolution of a Cohen-Macaulay module corresponds to the case that $F_\bullet$ is pure while $(\mathbf{A} \circ \mathbf{D})(F_\bullet)$ and $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ are both linear complexes.

2.1. Basic properties and examples. We now give an example of a triplet of pure complexes, but let us first give a lemma telling how $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ may be computed.

Lemma 2.1. *Let*

$$F_\bullet : \cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots .$$

Then $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ *is homotopy equivalent to the total complex of*

$$\cdots \longrightarrow \mathbf{A} \circ \mathbf{D}(F_i) \longrightarrow \mathbf{A} \circ \mathbf{D}(F_{i-1}) \longrightarrow \cdots .$$

Proof. Recall the Alexander duality A on the category of squarefree modules. The complex $A \circ \mathbf{D}(F_\bullet)$ is simply the complex of modules

$$\cdots \longrightarrow A \circ \mathbf{D}(F_i) \longrightarrow A \circ \mathbf{D}(F_{i-1}) \longrightarrow \cdots .$$

Now, if

$$\cdots \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \cdots \tag{2}$$

is a sequence of modules and $F_{i,\bullet} \to M_i$ is a free resolution, we may lift the differentials $M_i \to M_{i-1}$, to differentials $F_{i,\bullet} \to F_{i-1,\bullet}$. Then the total complex of

$$\cdots \longrightarrow F_{i,\bullet} \longrightarrow F_{i-1,\bullet} \longrightarrow \cdots$$

will be a resolution of (2), and hence it is homotopy equivalent to a minimal free resolution of this complex. Whence the result follows since $\mathbf{A} \circ \mathbf{D}(F_i)$ is a free resolution of $A \circ \mathbf{D}(F_i)$. □

Example 2.2. Let $S = \mathbf{k}[x_1, x_2, x_3]$. Consider the complex

$$F_\bullet : S \xleftarrow{[x_1x_2, x_1x_3, x_2x_3]} S(-2)^3.$$

First we find $(\mathbf{A} \circ \mathbf{D})(F_\bullet)$. By the figures of subsection 1.3, $\mathbf{A} \circ \mathbf{D}(S)$ is isomorphic to $\mathbf{k}$, and the resolution is the Koszul complex

$$S \leftarrow S(-1)^3 \leftarrow S(-2)^3 \leftarrow S(-3).$$

(It is really multigraded, but for simplicity we only depict it as singly graded.) Also, $\mathbf{A} \circ \mathbf{D}(S(-([3]\backslash\{i\})))$ is isomorphic to $S/(x_i)$ and so has resolution $S \xleftarrow{x_i} S(-1)$.

Therefore, $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ is a minimal version of the total complex of

$$\begin{tikzcd}
S^3 \arrow[d] & S(-1)^3 \arrow[l] \arrow[d] & & \\
S & S(-1)^3 \arrow[l] & S(-2)^3 \arrow[l] & S(-3). \arrow[l]
\end{tikzcd}$$

It is easily seen that such a minimal version is

$$S^2 \xleftarrow{\begin{bmatrix} x_2x_3 & -x_1x_3 & 0 \\ 0 & x_1x_3 & -x_1x_2 \end{bmatrix}} S(-2)^3 \xleftarrow{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} S(-3).$$

Now consider $(\mathbf{A}\circ\mathbf{D})^2(F_\bullet)$. By Lemma 1.10, $(\mathbf{A}\circ\mathbf{D})^2(S)$ is isomorphic to $S(-3)[3]$ and $(\mathbf{A}\circ\mathbf{D})^2(S(-\{1,2\}))$ is isomorphic to $S/(x_1,x_2)(-\{3\})[1]$. Therefore, $(\mathbf{A}\circ\mathbf{D})^2(F_\bullet)$ is a minimal version of the total complex of

$$\begin{tikzcd} S(-1)^3 & S(-2)^6 \arrow[l] & S(-3)^3 \arrow[l] \arrow[d] \\ & & S(-3). \end{tikzcd}$$

Such a minimal version is then

$$S(-1)^3 \xleftarrow{\begin{bmatrix} x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & x_1 \end{bmatrix}} S(-2)^6 \xleftarrow{\begin{bmatrix} x_2 & -x_1 & -x_3 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & -x_2 & -x_1 & x_3 \end{bmatrix}^t} S(-3)^2.$$

In summary,

$$\begin{aligned} F_\bullet &: S \leftarrow S(-2)^3 \\ \mathbf{A}\circ\mathbf{D}(F_\bullet) &: S^2 \leftarrow S(-2)^3 \leftarrow S(-3) \\ (\mathbf{A}\circ\mathbf{D})^2(F_\bullet) &: S(-1)^3 \leftarrow S(-2)^6 \leftarrow S(-3)^2, \end{aligned}$$

so all complexes are pure, and two of them are not linear.

Lemma 2.3. *Let*

$$F_\bullet : S(-a_0)^\alpha \leftarrow S(-a_1)^{\alpha'} \leftarrow \cdots$$

be a pure complex of squarefree modules with final term $S(-a_0)^\alpha$ in homological position t. If $\mathbf{A}\circ\mathbf{D}(F_\bullet)$ is also a pure complex, then

$$\mathbf{A}\circ\mathbf{D}(F_\bullet) :\ \cdots \leftarrow S(-n+a_0)^\alpha$$

where the initial term $S(-n+a_0)^\alpha$ is in homological position $n-a_0+t$.

As a consequence the initial terms of $\mathbf{D}(F_\bullet)$ and its Alexander dual $\mathbf{A}\circ\mathbf{D}(F_\bullet)$ are both equal to $S(-n+a_0)^\alpha$.

Proof. Considered as a complex of graded modules, $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ is the total complex of (we display only the last two rows)

$$\begin{tikzcd} S^{\alpha'} \arrow[d] & S(-1)^{\alpha'(n-a_1)} \arrow[l] \arrow[d] & \cdots \arrow[l] & \\ S^{\alpha} & S(-1)^{\alpha(n-a_0)} \arrow[l] & \cdots \arrow[l] & S(-n+a_0)^{\alpha}. \arrow[l] \end{tikzcd}$$

When making a minimal complex of the total complex, $S(-n+a_0)^\alpha$ cannot cancel out, so it must be the last term. Since S^α is in homological position t, the last term must be in homological position $n - a_0 + t$. $\square$

In a pure complex,

$$F_\bullet : S(-a_0)^{\alpha_0} \leftarrow S(-a_1)^{\alpha_1} \leftarrow \cdots \leftarrow S(-a_r)^{\alpha_r} \tag{3}$$

an integer d is called a *degree* of this complex if $d = a_i$ for some i. Otherwise, it is called a nondegree. If the nondegree is in $[a_0, a_r]$, it is an *internal* nondegree.

Now suppose we have a situation where $(\mathbf{A} \circ \mathbf{D})^i(F_\bullet)$ are pure complexes for $i = 0, 1$ and 2. Write the complexes as:

$$\begin{aligned} F_\bullet &: S(-a_0)^{\alpha_0} \leftarrow S(-a_1)^{\alpha_1} \leftarrow \cdots \leftarrow S(-a_{r_0})^{\alpha_{r_0}} \\ \mathbf{A} \circ \mathbf{D}(F_\bullet) &: S(-b_0)^{\beta_0} \leftarrow S(-b_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-b_{r_1})^{\alpha_{r_1}} \\ (\mathbf{A} \circ \mathbf{D})^2(F_\bullet) &: S(-c_0)^{\gamma_0} \leftarrow S(-c_1)^{\gamma_1} \leftarrow \cdots \leftarrow S(-c_{r_2})^{\gamma_{r_2}}. \end{aligned}$$

We denote by A the set of degrees of $F_\bullet$, and similarly B and C for the degrees of $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ and $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$. The triplet (A, B, C) is the *degree triplet* of the triplet of pure complexes. Let e_A be the number of internal nondegrees of $F_\bullet$, and correspondingly we define e_B and e_C. Let e be the total number of internal nondegrees for the triplet, $e_A + e_B + e_C$. As they turn out to be central invariants, we let $c = a_0$, $a = b_0$ and $b = c_0$.

Proposition 2.4. a. *The degrees in the last terms of the complexes above are $a_{r_0} = n - b$, $b_{r_1} = n - c$ and $c_{r_2} = n - a$.*

b. *The number of variables $n = a + b + c + e$.*

Proof. Part a. is by Lemma 2.3 above. Also, by the lemma above, if $S(-a_0)^{\alpha_0}$ in $F_\bullet$ is in homological position t, then $S(-b_{r_1})^{\beta_{r_1}}$ in $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ is in position $t + b_{r_1}$, and so the first term $S(-b_0)^{\beta_0}$ is in position $t + b_{r_1} - r_1$. But, $r_1 + e_B = b_{r_1} - b_0$, and so this position is $t + a + e_B$. Applying the lemma again, we get that $S(-c_0)^{\gamma_0}$ in $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ is in position $t + a + b + e_B + e_C$. And then, again, we get that $S(-a_0)^{\alpha_0}$ in $(\mathbf{A} \circ \mathbf{D})^3(F_\bullet)$ is in position $t + a + b + c + e$.

But, since $(\mathbf{A} \circ \mathbf{D})^3$ is isomorphic to the nth iterate of the translation functor, we get that $n = a + b + c + e$. $\square$

We can represent the degrees of the complex $F_\bullet$ as a string of circles indexed by the integers from $a_0 = c$ to $a_{r_0} = n - b$ by letting a circle be filled $\bullet$ if it is at a position a_i and be a blank circle $\circ$ otherwise.

Example 2.5. A complex

$$S(-1)^6 \leftarrow S(-3)^{27} \leftarrow S(-4)^{24} \leftarrow S(-7)^3$$

with $n = 9$ gives rise to the diagram

$$\overset{1}{\bullet} \leftarrow \overset{2}{\circ} \leftarrow \overset{3}{\bullet} \leftarrow \overset{4}{\bullet} \leftarrow \overset{5}{\circ} \leftarrow \overset{6}{\circ} \leftarrow \overset{7}{\bullet}.$$

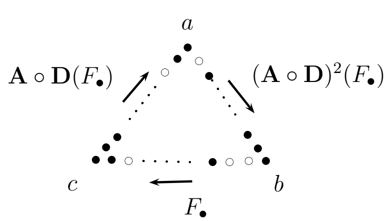
The dual complex $\mathbf{D}(F_\bullet) = \mathrm{Hom}_S(F_\bullet, S(-\mathbf{1}))$, which is

$$S(-8)^6 \longrightarrow S(-6)^{27} \longrightarrow S(-5)^{24} \longrightarrow S(-2)^3,$$

gives a diagram by switching the orientation above and letting the numbering be

$$\overset{8}{\bullet} \longrightarrow \overset{7}{\circ} \longrightarrow \overset{6}{\bullet} \longrightarrow \overset{5}{\bullet} \longrightarrow \overset{4}{\circ} \longrightarrow \overset{3}{\circ} \longrightarrow \overset{2}{\bullet}.$$

All three complexes may be represented in a triangle, called the *degree triangle* of the three complexes.



Note that the degrees of $F_\bullet$ start with c, then proceed in ascending order and end with $n-b$.

Lemma 2.6. *In the degree triangle above, the following hold*:

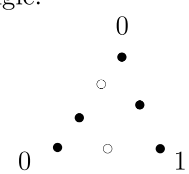
a. *The length, i.e., the number of circles* (*both filled and blank*), *of the side corresponding to the set* A, *the degree sequence of* $F_\bullet$, *is* $a+e+1$. *Similar relations hold for the other sides.*

b. *The number of circles in the degree triangle is* $a+b+c+3e$. *In particular at most a third of the circles are blank circles.*

c. *The Koszul complexes given in subsection* 1.3 *give all cases of degree triangles where there are no internal nondegrees, i.e., no blank circles.*

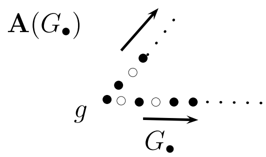
Proof. a. This is because the number of circles is the cardinality of the interval $[c, n-b]$ which is this number by Proposition 2.4 b. Part b. above follows immediately from part a. Concerning part c., there are three numerical parameters for these Koszul complexes, the cardinalities of $|A|, |B|$ and $|C|$, and these correspond to $a+1, b+1$ and $c+1$. $\square$

Example 2.7. The minimal complexes in Example 2.2 give rise to the following degree triangle.



2.2. A balancing condition. Suppose $F_\bullet, \mathbf{A}\circ\mathbf{D}(F_\bullet)$ and $(\mathbf{A}\circ\mathbf{D})^2(F_\bullet)$ is a triplet of pure free squarefree complexes. The interior nondegrees of these complexes cannot be arbitrarily distributed. There is a certain balancing condition which we now give.

Let $G_\bullet$ be one of the three complexes, so $G_\bullet$ and its Alexander dual $\mathbf{A}(G_\bullet)$ are pure complexes. In particular they have the same initial term $S(-n+g)^\gamma$. We can display their degrees as



The balancing condition is the following.

Proposition 2.8. *Suppose $G^\bullet$ belongs to a triplet of pure free squarefree complexes, and let $S(-n+g)^\gamma$ be the initial term of $G_\bullet$ and its Alexander dual $\mathbf{A}(G_\bullet)$. Then, for each $0 \leq v \leq n-g$, the number of degrees of $G_\bullet$ in the interval $[v, n-g]$ is greater than the number of nondegrees of $\mathbf{A}(G_\bullet)$ in the interval $[v, n-g]$.*

Proof. We may let $F_\bullet = \mathbf{D}(G_\bullet)$, so $F_\bullet, \mathbf{A} \circ \mathbf{D}(F_\bullet)$, and $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$ is a triplet of pure free squarefree complexes. With this notation,

$$G_\bullet = \mathbf{D}(F_\bullet) : S(-n+c)^\alpha \longrightarrow \cdots \longrightarrow S(-b)^{\alpha'}$$

so

$$\mathbf{A}(G_\bullet) = \mathbf{A} \circ \mathbf{D}(F_\bullet) : S(-n+c)^\alpha \longrightarrow \cdots S(-a)^{\alpha''}.$$

Let $\phi(v)$ be the sum of the number of degrees of $G_\bullet$ in $[v, n-c]$ and the number of degrees of $\mathbf{A}(G_\bullet)$ in this interval. The statement of the proposition is equivalent to: $\phi(v)$ is greater than the cardinality of $[v, n-c]$.

Case 1. In the range $0 \leq v \leq \max\{a, b\}$, the difference $\phi(v) - |[v, n-c]|$ is weakly decreasing as v decreases. So, in order to prove the statement in this range, it is enough to prove that

$$\phi(0) > |[0, n-c]| = n - c + 1 = a + b + e + 1. \tag{4}$$

But

$$\begin{aligned}\phi(0) &= |A| + |B| \\ &= a + e + 1 - e_A + b + e + 1 - e_B \\ &= a + b + 2 + e + e_C,\end{aligned}$$

so clearly (4) holds.

Case 2. Now suppose $v > \max\{a, b\}$. We may as well assume that $a \geq b$, so $v > a$. Let $c = a_0, a_1, \ldots$ be the degrees of $F_\bullet = \mathbf{D}(G_\bullet)$, with $S(-a_i)^{\alpha_i}$ in homological degree i. The homology module $H_i(F_\bullet)$ is transferred to the $(-i)$th linear strand of $\mathbf{A}(G_\bullet)$ by Proposition 1.4. Note that, if $i > 0$, the least nonzero degree of this homology module, if this module is nonzero, is $\geq a_i + 1$. Hence, the largest degree occurring in the $-i$th linear strand of $\mathbf{A}(G_\bullet)$, if this strand is nonzero, is $\leq n - a_i - 1$.

Note that $S(-(n-c))$ belongs to the 0th linear strand of $\mathbf{A}(G_\bullet)$ (since this term comes from $H^0(F_\bullet)$). If v is a degree of $\mathbf{A}(G_\bullet)$, let $-l$ be the linear strand to which it belongs. We must have $l \geq 0$. The number of degrees of $\mathbf{A}(G_\bullet)$ in $[v, n-c]$ is then $n - c - v + 1 - l$.

Now, if $v - 1$ is an interior nondegree of $\mathbf{A}(G_\bullet)$, then

$$\phi(v-1) - |[v-1, n-c]| \leq \phi(v) - |[v, n-c]|.$$

Therefore, we might as well prove the statement for $v - 1$. Since a is a degree of $\mathbf{A}(G_\bullet)$, we may continue this way and in the end come to a situation where $v - 1$ is a degree of $\mathbf{A}(G_\bullet)$. Let $-l$ be its linear strand in $\mathbf{A}(G_\bullet)$. When $l > 0$, by what was said above, $v - 1 \leq n - a_l - 1$ or, equivalently, $a_l \leq n - v$. But this also holds when $l = 0$. Hence, the degrees $a_0, a_1, \ldots, a_l$ of $F_\bullet$ all belong to $[c, n-v]$, and so

$$\phi(v) \geq (n - c - v + 1 - l) + (l + 1) > n - c - v + 1. \qquad \square$$

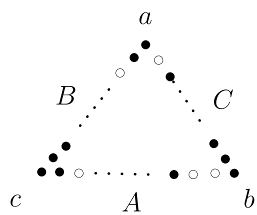
Let there be a natural number n. For an integer d, let $\overline{d} = n - d$ and, for a subset of integers D, let $\overline{D} = \{\overline{d} \mid d \in D\}$.

Definition 2.9. A triplet of nonempty subsets (A, B, C) of $\mathbf{N}_0$ is a *balanced degree triplet of type* n if there are integers $0 \leq a, b, c \leq n$ such that

1.
$$A \subseteq [c, \overline{b}], \qquad B \subseteq [a, \overline{c}], \qquad C \subseteq [b, \overline{a}],$$
and the endpoints of each interval are in the respective subsets A, B or C.

2. Let e_A be the cardinality of $[c, \overline{b}] \backslash A$ and correspondingly define e_B and e_C. Then $n = a + b + c + e_A + e_B + e_C$.

3. A and $\overline{B}$ are balanced with respect to the common endpoint c, i.e., for each $c \leq v \leq n$, the number of elements of $[c, v]$ in A is greater than the number of elements of $[c, v]$ not in $\overline{B}$. Similarly for B and $\overline{C}$ with respect to a and C and $\overline{A}$ with respect to b.



Remark 2.10. Note that parts a and b of Lemma 2.6 may be deduced solely from the properties 1 and 2 above.

Conjecture 2.11. a. *For each balanced degree triplet (A, B, C) of type n, there exists a triplet of pure free squarefree complexes over the polynomial ring in n variables whose degree sequences are given by A, B, and C.*

b. *The Betti numbers of this triplet of complexes are uniquely determined by the degree triplet, up to common scalar multiple.*

3. Constraints on the Betti numbers. In this section we give linear equations fulfilled by the Betti numbers in a triplet of pure complexes. The number of equations is one less than the number of Betti numbers, so we expect a unique set of Betti numbers up to multiplication by a common scalar. We prove that this is the case, provided part a of Conjecture 2.11 holds. In other words, we prove that part a of the conjecture implies part b.

3.1. Some elementary relations for binomial coefficients. For nonnegative integers p, we have the binomial coefficient $\binom{x}{p}$. When p is a negative integer, we set this coefficient to be zero. The following identities hold in $\mathbf{Q}[x,y]$ and are repeatedly used in the proof of the below lemma.

1. $\binom{x+y}{p} = \sum_{i=0}^{p} \binom{x}{p-i}\binom{y}{i}$, [**15**, Example 4.3.3].
2. $\binom{x}{p} = (-1)^p \binom{p-1-x}{p}$.

Let $A = (a_{ij})$ be the $(n+1) \times (n+1)$-matrix with $a_{ij} = (-1)^j \binom{n-j}{i}$ for $i, j = 0, \ldots, n$. For instance, when $n = 2$ this is the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Lemma 3.1. $A^3 = (-1)^n \cdot I$.

Proof. First we show that $A^2 = (b_{ij})$ where $b_{ij} = (-1)^j \binom{j}{n-i}$. For instance, when $n = 2$, this is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

The ith row in A is

$$\binom{n}{i}, \; -\binom{n-1}{i}, \; \binom{n-2}{i}, \ldots .$$

Now

$$\binom{n-j}{i} = \binom{n-j}{n-j-i} = (-1)^{n-i-j}\binom{-i-1}{n-j-i}.$$

The ith row of A is then $(-1)^{n-i}$ multiplied with:

$$\binom{-i-1}{n-i}, \; \binom{-i-1}{n-1-i}, \; \binom{-i-1}{n-2-i}, \ldots .$$

The jth column in A is $(-1)^j$ multiplied with the following:

$$\binom{n-j}{0},\ \binom{n-j}{1},\ \binom{n-j}{2},\ldots .$$

From this, b_{ij} is $(-1)^{n-i-j}$ multiplied with

$$\binom{n-j}{0}\binom{-i-1}{n-i}+\binom{n-j}{1}\binom{-i-1}{n-1-i}+\cdots ,$$

which, by property 1 in the beginning of this section, equals

$$\binom{n-j-i-1}{n-i}=(-1)^{n-i}\binom{j}{n-i}. \tag{6}$$

Hence, $b_{ij}=(-1)^j\binom{j}{n-i}$.

To find A^3, note that the ith row in A^2 is

$$\binom{0}{n-i},\ -\binom{1}{n-i},\ \binom{2}{n-i},\ldots .$$

Note that

$$\binom{j}{n-i}=\binom{j}{j+i-n}=(-1)^{j+i-n}\binom{i-n-1}{j+i-n}.$$

Hence, row i is $(-1)^{n-i}$ multiplied with

$$\binom{i-n-1}{i-n},\ \binom{i-n-1}{i-n+1},\ \binom{i-n-1}{i-n+2},\ldots .$$

The jth column in A is $(-1)^j$ multiplied with

$$\binom{n-j}{0},\ \binom{n-j}{1},\ \binom{n-2}{2},\ldots .$$

Since

$$\binom{n-j}{i}=\binom{n-j}{n-j-i},$$

this column becomes

$$\binom{n-j}{n-j}, \binom{n-j}{n-j-1}, \binom{n-j}{n-j-2}, \dots .$$

The first nonzero position in the ith row is $n-i$. The last nonzero position in the jth column is $n-j$. Hence, if $n-j < n-i$, equivalently $i < j$, the product of the ith row and jth column is zero. On the other hand, if $i \geq j$, the product is $(-1)^{n-i-j}$, multiplied with

$$\binom{i-1-j}{i-j} = (-1)^{i-j}\binom{0}{i-j} = \begin{cases} 1 & i=j \\ 0 & i>j. \end{cases}$$

Hence, we obtain $A^3 = (-1)^n \cdot I$. $\square$

3.2. Linear equations for the Betti numbers. Let $F_\bullet$ be the pure free squarefree complex

$$F_\bullet : S(-a_0)^{\alpha_0} \leftarrow S(-a_1)^{\alpha_1} \leftarrow \cdots \leftarrow S(-a_r)^{\alpha_r}. \tag{7}$$

Let $\widehat{\alpha}_{a_i} = (-1)^{l(a_i)} \cdot \alpha_i$, where $l(a_i)$ is the linear strand containing the term $S(-a_i)^{\alpha_i}$, be the *signadjusted* Betti numbers. We set $\widehat{\alpha}_d = 0$ if $d \in [0,n]$ is not a degree of $F_\bullet$. Note that these signadjusted Betti numbers are parametrized by the internal degrees. Note also that

$$l(a_i) = l(a_0) - a_0 + a_i - i. \tag{8}$$

Assume, in addition, that $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ is a pure complex

$$S(-b_0)^{\beta_0} \leftarrow \cdots \leftarrow S(-b_{r'})^{\beta_{r'}}.$$

Recall that the ith homology module of $F_\bullet$ is transferred to the ith linear strand of $\mathbf{A} \circ \mathbf{D}(F_\bullet)$.

Suppose the ith homology module of $F_\bullet$ is nonzero, and let d be a degree for which the dth graded part of this module is nonzero. This module is squarefree, and the dimension of its squarefree part in degree d (recall this notion in subsection 1.1) is

$$(-1)^{i+l(a_0)-a_0}\left[\alpha_0\binom{n-a_0}{n-d} - \alpha_1\binom{n-a_1}{n-d} + \alpha_2\binom{n-a_2}{n-d} + \cdots\right]. \tag{9}$$

By Proposition 1.4, this will be equal to $(-1)^i\widehat{\beta}_{n-d}$. Using (8), we see that, when d varies, the equations (9) are the same as

$$\widehat{\beta} = A \cdot \widehat{\alpha}. \tag{10}$$

If, furthermore, $(\mathbf{A}\circ\mathbf{D})^2(F_\bullet)$ is pure, we get in the same way

$$\widehat{\gamma} = A \cdot \widehat{\beta} \tag{11}$$

$$\widehat{\widehat{\alpha}} = A \cdot \hat{\gamma}, \tag{12}$$

where $\widehat{\widehat{\alpha}} = (-1)^n\widehat{\alpha}$ due to the shift n of linear strands of the functor $(\mathbf{A}\circ\mathbf{D})^3$.

If $F_\bullet$, $\mathbf{A}\circ\mathbf{D}(F_\bullet)$ and $(\mathbf{A}\circ\mathbf{D})^2(F_\bullet)$ are all pure, then clearly the following equations hold:

(13) $\quad \widehat{\alpha}_i = 0$ for all nondegrees i of $F_\bullet$ in $[0,n]$,

(14) $\quad \widehat{\beta}_i = 0$ for all nondegrees i of $(\mathbf{A}\circ\mathbf{D})(F_\bullet)$ in $[0,n]$,

(15) $\quad \widehat{\gamma}_i = 0$ for all nondegrees i of $(\mathbf{A}\circ\mathbf{D})^2(F_\bullet)$ in $[0,n]$.

In addition, we must have the equations (10), (11) and (12) above (where any two of these determine the third by Lemma 3.1).

Lemma 3.2. *The equations $\widehat{\alpha}_i = 0$, for $i = 0,\dots,c-1$, are equivalent to the equations $\widehat{\beta}_{n-i} = 0$, for $i = 0,\dots,c-1$.*

Similarly the equations $\widehat{\beta}_i = 0$ and $\widehat{\gamma}_{n-i} = 0$, for $i = 0,\dots,a-1$ are equivalent, and $\widehat{\gamma}_i = 0$ and $\widehat{\alpha}_{n-i} = 0$, for $i = 0,\dots,b-1$, are equivalent.

Proof. This is due to the transition matrix A having the triangular form

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & * & 0 & \cdots \\ * & * & 0 & \cdots & \\ * & 0 & \cdots & & \end{bmatrix}. \qquad \square$$

Corollary 3.3. *Given a balanced degree triangle, the $3n+3$ signadjusted Betti numbers $\widehat{\alpha}_i, \widehat{\beta}_i, \widehat{\gamma}_i$, $i = 0,\dots,n$, fulfill equations* (10)–(15)*, which may be reduced to $3n+2$ natural equations.*

Remark 3.4. We expect these equations to be linearly independent. Hence, there would be a unique solution up to scalar multiple.

Proof. There are $c+b+e_A$ equations of the form $\widehat{\alpha}_i = 0$. Similarly, there are $a+c+e_B$ equations of the form $\widehat{\beta}_i = 0$, and $a+b+e_C$ equations of the form $\widehat{\gamma}_i = 0$. This gives a total of $2a+2b+2c+e$ equations. However, by the above Lemma 3.2, there are $a+b+c$ dependencies among them, giving $a+b+c+e=n$ equations. In addition, transition equations (11) and (12) give $2n+2$ further equations, a total of $3n+2$. ◻

The complex $F_\bullet$ is

$$S(-n+b)^{\alpha_{r_0}} \longrightarrow \cdots \longrightarrow S(-c)^{\alpha_0}.$$

Its Alexander dual $\mathbf{A}(F_\bullet)$ equals (up to translation) $\mathbf{D} \circ (\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$, which is

$$S(-n+b)^{\gamma_0} \longrightarrow \cdots \longrightarrow S(-a)^{\gamma_{r_2}}.$$

(Note that $\gamma_0 = \alpha_{r_0}$.) Let $v_1 < \cdots < v_{e_C}$ be the internal nondegrees of $\mathbf{A}(F_\bullet)$.

The complex $\mathbf{D}(F_\bullet)$ is

$$S(-n+c)^{\alpha_0} \longrightarrow \cdots \longrightarrow S(-b)^{\alpha_{r_0}},$$

and then its Alexander dual $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ is

$$S(-n+c)^{\beta_{r_1}} \longrightarrow \cdots \longrightarrow S(-a)^{\beta_0}.$$

(Note that $\beta_{r_1} = \alpha_0$.) Let $u_1 < \cdots < u_{e_B}$ be the internal nondegrees of $\mathbf{A} \circ \mathbf{D}(F_\bullet)$.

Proposition 3.5. *Given a triplet of pure free squarefree complexes, let $a_0 < \cdots < a_r$ be the degrees of the first complex $F_\bullet$ and $\overline{a}_r < \cdots < \overline{a}_0$ the degrees of the dual $\mathbf{D}(F_\bullet)$. By transition equations* (10)–(12), *equations* (13)–(15) *are equivalent to the following equations for the*

$(r+1)$ *nonzero Betti numbers* α_i.

(16)
$$\alpha_0\binom{a_0}{v_i} - \alpha_1\binom{a_1}{v_i} + \cdots + (-1)^r\alpha_r\binom{a_r}{v_i} = 0, \quad i = 1,\ldots,e_C$$
(17)
$$\alpha_0\binom{\overline{a}_0}{u_i} - \alpha_1\binom{\overline{a}_1}{u_i} + \cdots + (-1)^r\alpha_r\binom{\overline{a}_r}{u_i} = 0, \quad i = 1,\ldots,e_B$$
(18)
$$\alpha_0\binom{\overline{a_0}}{j} - \alpha_1\binom{\overline{a_1}}{j} + \cdots + (-1)^r\alpha_r\binom{\overline{a_r}}{j} = 0, \quad j = 0,\ldots,a-1.$$

The total number of these equations $e_C + e_B + a$ *equals* r.

Remark 3.6. These equations then generalize the Herzog-Kühl equations of [**14**], see also [**11**, subsection 1.3], which determines the Betti numbers of pure resolutions of Cohen-Macaulay modules.

Proof. The last part is because

$$r + e_A = n - b - c, \quad \text{and} \quad a + b + c + e_A + e_B + e_C = n.$$

By transition equation (10), the set of equations (17) is equivalent to $\widehat{\beta}_{u_i} = 0$ for each nondegree u_i of $\mathbf{A}\circ\mathbf{D}(F_\bullet)$ in the interval $[a, n-c]$. The vanishing of $\widehat{\beta}_j$ for $j \in [n-c+1, n]$ is, by Lemma 3.2, equivalent to $\widehat{\alpha}_j = 0$ for $j \leq c-1$. The vanishing of $\widehat{\beta}_j$ for $j \in [0, a-1]$ is, again by transition equation (10), equivalent to equation (18).

In the same way, the vanishing of $\widehat{\gamma}_j$ for each nondegree j of $(\mathbf{A}\circ\mathbf{D})^2(F_\bullet)$ in interval $[b, n-a]$ is equivalent to the equation (16). The vanishing of $\widehat{\gamma}_j$ for j in $[n-a+1, n]$ is, by Lemma 3.2, equivalent to the vanishing of $\widehat{\beta}_j$ for j in $[0, a-1]$ which is again equivalent to equation (18). □

Remark 3.7. It is easy to see that equation (18) with each $\overline{a_i}$ replaced by a_i becomes an equivalent set of equations.

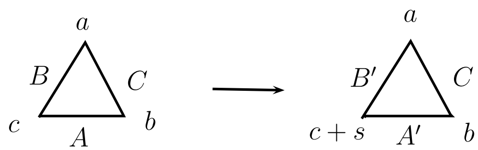
We also get corresponding equations for β_i and γ_i.

Corollary 3.8. *Given a balanced degree triplet, if equations* (10)–(12) *for Betti numbers* α_i *of the pure complex* $F_\bullet$ *has a* k*-dimensional solution set, then the corresponding equations for Betti numbers* β_i *of the pure complex* $\mathbf{A} \circ \mathbf{D}(F_\bullet)$ *has a* k*-dimensional solution set, and similarly for Betti numbers* γ_i *of* $(\mathbf{A} \circ \mathbf{D})^2(F_\bullet)$.

Proof. By transition equations (10)–(12), all these equation systems are equivalent to equations (13)–(15). □

3.3. Uniqueness of Betti numbers. Given a balanced degree triplet $\Delta = (A, B, C)$, set A is a subset of $[c, n-b]$, containing the end points of this interval. Let us suppose that there is an internal nondegree of A, i.e., A is a proper subset of $[c, n-b]$. Let A contain $[c, c+t-1]$ but not $c+t$. Set B is a subset of $[a, n-c]$ containing the endpoints. Let $s \geq 1$ be maximal such that $\overline{B} \subseteq [c, n-a]$ is disjoint from the interval $[c+1, c+s-1]$. Since the degree triangle is balanced we have $s \leq t$. Let $\Delta' = (A', B', C)$ where

$$A' = A \cup \{c+t\} \backslash [c, c+s-1], \quad \overline{B'} = \overline{B} \backslash \{c\}.$$



Lemma 3.9. *If* Δ *is a balanced degree triplet, then* Δ' *is a balanced degree triplet.*

Proof. If Δ has e internal nondegrees, then clearly Δ' has $e-s$ internal nondegrees. (We remove one nondegree from A and $s-1$ from $\overline{B}$.) Since Δ' has parameters $c+s$, b and a, the equation $a+b+c+e = n$ continues to hold when passing from Δ to Δ'. Viewing Δ' from the corner $c+s$, we see it is balanced here since Δ was. Viewing Δ' from corner a, we see that it is balanced in the interval $[v, n-a]$ for $v \geq \max\{c+s, b\}$ since Δ was, and when $v \leq \max\{c+s, b\}$, we can use the same argument as in Case 1 in the proof of Proposition 2.8. The last case of corner b goes in the same way. □

Proposition 3.10. *Let Δ be a balanced triplet, with an internal nondegree on one of its sides, say side A. Then if*: i) *there exists a triplet of pure free squarefree complexes for the degree triplet Δ', and* ii) *the equation system for Δ' has a one-dimensional solution, then the equation system for Δ also has a one-dimensional solution.*

Proof. Let X be the coefficient matrix for the system of equations given in Proposition 3.5 for Betti numbers α_i, of a pure complex associated to A, in the degree triplet Δ. Let X' be the corresponding coefficient matrix for the triplet Δ'. By hypothesis, the solution set of X' is one-dimensional. The coordinates of a solution vector $(\alpha'_0, \ldots, \alpha'_{r'})$ may be taken as the minors of the matrix X'. By hypothesis, there exists a pure free squarefree complex $F'_\bullet$, part of a triplet, whose Betti numbers are a multiple of this solution vector, and hence all the α'_i will be nonzero.

Now note that the columns in X' have columns parametrized by the degrees of A' in $[c+s, n-b]$. These are exactly the degrees of A which are in $[c+s, n-b]$, together with degree $c+t$. Write the coefficient matrix X such that equation (17) are the first rows and equation (16) the second group of rows, and then (18). If we remove the first column in X, indexed by c, then X will have a form

$$\begin{bmatrix} T & 0 \\ Z & Y \end{bmatrix},$$

where T is a triangular matrix of size $(s-1)\times(s-1)$. This is due to the hypotheses we have on the forms of A and $\overline{B}$ in the interval $[c, c+s-1]$. If, on the other hand, we remove the column of X' indexed by $c+t$, we will simply get matrix Y. Hence the determinant of Y, which is one of the α'_i, is nonzero. So matrix X will have full rank, and hence a one-dimensional solution set. $\square$

We then get the following.

Theorem 3.11. *Part* a *of Conjecture* 2.11 *implies part* b *of the conjecture*: *If there exist triplets of pure free squarefree complexes for every balanced degree triplet, then the Betti numbers of each of these triplets of complexes are uniquely determined up to a common scalar multiple.*

Proof. This follows from the previous proposition once we know it is true for the induction start. And the induction start is a degree triplet with no internal nondegrees. But, in any degree triplet where all internal nondegrees are on only one edge, the uniqueness of Betti numbers follows by the Herzog-Kühl equations, see [**11**, subsection 1.3] since, if this edge corresponds to a complex $F_\bullet$, then this complex is a resolution of a Cohen-Macaulay module, Lemma 1.9. The uniqueness of all Betti numbers up to common scalar multiple follows by transition equations (10)–(12). □

Remark 3.12. We have not proved that if, for a given degree triplet, there exists a triplet of pure free squarefree complexes, then their Betti numbers are uniquely determined up to common scalar multiple. Our proof is inductive so it relies on the existence of triplets of complexes associated to all "smaller" triplets.

4. Construction of triplets when the internal nondegrees are on only one side of the degree triangle. In this section we construct triplets of pure squarefree complexes in the case that two of the complexes are linear. These correspond to degree triangles where two of the sides consists solely of degrees (so filled circles, no internal nondegrees).

4.1. Auxiliary results on subspaces of vector spaces. Let E be a vector space and $E_1, \ldots, E_r$ subspaces of E. For I a subset of $[r] = \{1, \ldots, r\}$, we let E_I be the intersection $\cap_{i \in I} E_i$.

Lemma 4.1. *Suppose $E_1, \ldots, E_r$ are general subspaces of E of codimension one, where $r \leq \dim_{\mathbf{k}} E$. Then the $E_{[r]\setminus\{i\}}$ as i varies through $i = 1, \ldots, r$, generate E.*

Proof. By dividing out by $E_{[r]}$, we may as well assume that $r = \dim_{\mathbf{k}} E$. Then each $E_{[r]\setminus\{i\}}$ corresponds to a one-dimensional vector space. To construct the E_i, we may chose general vectors $v_1, \ldots, v_r$ and let E_i be spanned by the $(r-1)$-subsets of this r-set we get by successively omitting the v_i. □

Lemma 4.2. *Let E_i be a subspace of E of codimension e_i for $i = 1, \dots, r$. Suppose for each proper subset J of I that the codimension of E_J is $\sum_{i \in J} e_i$. If* $\operatorname{codim} E_I < \sum_{i \in I} e_i$, *then the $E_{I \setminus \{i\}}$ do not generate E as i varies through I.*

Proof. Let the codimension of E_I be $(\sum_{i \in I} e_i) - r$ where $r > 0$. By dividing out by E_I, we may assume $E_I = 0$, and so this number is the dimension of E. Then the dimension of $E_{I \setminus \{i\}}$ is $e_i - r$, and so if $|I| \geq 2$, these cannot generate the whole space E. $\square$

Notation. We shall in the following denote by $S^r(E)$ the rth symmetric power of E and by $D^r(E)$ the rth divided power of E. Also, let $\widetilde{D}^r(E) = \wedge^{\dim_{\mathbf{k}} E} E \otimes_{\mathbf{k}} D^r(E)$.

4.2. Construction of tensor complexes. We start with a degree triplet (A, B, C) where $B = [a, \overline{c}]$ and $C = [b, \overline{a}]$ are intervals, i.e., contain no nondegrees. We partition the complement of A in $[0, n]$ into successive intervals

$$[u_0 + 1, u_0 + w_0 - 1], [u_1 + 1, u_1 + w_1 - 1], \dots,$$
$$[u_r + 1, u_r + w_r - 1], [u_{r+1} + 1, u_{r+1} + w_{r+1} - 1],$$

where for the first and last interval we have $u_0 = -1, w_0 = c + 1$ and $u_{r+1} = n - b$ and $w_{r+1} = b + 1$, and for the middle intervals $c \leq u_1$, $u_i + w_i \leq u_{i+1}$, and $u_r + w_r \leq n - b$. Let W_i be a vector space of dimension w_i, and $W = \otimes_{i=0}^{r+1} W_i$. Denote by $\vec{W}$ the tuple $(W_0, \dots, W_{r+1})$. Let V be a vector space of dimension n and $S(V \otimes W^*)$ the symmetric algebra. In the language of [**1**, Section 5], $(0; u_0, \dots, u_{r+1})$ is a *pinching weight* for $V, \vec{W}$.

Berkesch et al. [**1**] construct a resolution $F_\bullet(V; \vec{W})$ of pure free $S(V \otimes W^*)$-modules with degree sequence A such that the term with free generators of degree $d \in A$ has the form:

(19)
$$\bigwedge^d V \bigotimes (\otimes_{d \leq u_i} S^{u_i - d}(W_i)) \bigotimes (\otimes_{d \geq u_i + w_i} \widetilde{D}^{d - u_i - w_i}(W_i)) \bigotimes S(V \otimes W^*).$$

This complex is a resolution of a Cohen-Macaulay module and is equivariant for the group

$$GL(V) \times GL(W_0) \times \cdots \times GL(W_{r+1}).$$

The construction of this complex follows the method of Lascoux, presented in [**19**, subsection 5.1]. Let $\mathbf{P}(\vec{W})$ be the product $\mathbf{P}(W_0) \times \cdots \times \mathbf{P}(W_{r+1})$. There is a tautological sequence:

$$0 \longrightarrow \mathcal{S} \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}(\vec{W})} \longrightarrow \mathcal{O}_{\mathbf{P}(\vec{W})}(1, \ldots, 1) \longrightarrow 0.$$

Dualizing this sequence and tensoring with V, we get a sequence (let $\mathcal{Q} = \mathcal{S}^*$)

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}(\vec{W})}(-1, \ldots, -1) \longrightarrow V \otimes W^* \otimes \mathcal{O}_{\mathbf{P}(\vec{W})} \longrightarrow V \otimes \mathcal{Q} \longrightarrow 0.$$

Constructing the affine bundles over $\mathbf{P}(\vec{W})$ of the last two terms in this complex, we get a diagram

$$\begin{tikzcd} Z = \mathbf{V}(V \otimes \mathcal{Q}) \arrow[r, hook] \arrow[d] & \mathbf{V}((V \otimes W^*) \otimes \mathcal{O}_{\mathbf{P}(\vec{W})}) \arrow[r, equal] \arrow[d, "\pi"] & \mathbf{V}(V \otimes W^*) \times \mathbf{P}(\vec{W}) \\ Y \arrow[r, hook] & \mathbf{V}(V \otimes W^*) & \end{tikzcd}$$

where Y is the image of Z by projection π. The projection of the structure sheaf $\pi_*(\mathcal{O}_Z)$ is the sheaf on the affine space Y associated to the $S(V \otimes W^*)$-module $H^0(\mathbf{P}(\vec{W}),\ \mathrm{Sym}\,(V \otimes \mathcal{Q}))$.

Let p be the projection of $\mathbf{V}(V \otimes W^*) \times \mathbf{P}(\vec{W}))$ to the second factor. Let $\mathcal{L}$ be the line bundle $\mathcal{O}_{\mathbf{P}(\vec{W})}(u_0, \ldots, u_{r+1})$ on $\mathbf{P}(\vec{W})$. Then $M = H^0(\mathbf{P}(\vec{W}), \mathrm{Sym}\,(V \otimes \mathcal{Q}) \otimes p^*\mathcal{L})$ is an $S(V \otimes W^*)$-module and the complex $F(V; \vec{W})$ is a resolution of this module, by [**19**, Prop. 5.1.2.b]. The sheafification of this module on the affine space is in fact $\pi_*(\mathcal{O}_Z \otimes p^*\mathcal{L})$.

Fact. $\dim Y = \dim Z$. This is argued for in [**1**], see for instance the proof of Proposition 3.3. First note that

$$\dim Z = \dim \mathbf{P}(\vec{W}) + n \cdot \mathrm{rk}\, \mathcal{Q}.$$

Since $F(V; \vec{W})$ is a resolution of a module supported on Y, the length of this resolution is at least the codimension of Y. Hence,

$$\begin{aligned} \dim Y &\geq n \dim_{\mathbf{k}} W - |A| + 1 \\ &= n \dim_{\mathbf{k}} W - n + \sum_i (w_i - 1). \end{aligned}$$

Since $\operatorname{rk} \mathcal{Q} = \dim_{\mathbf{k}} W - 1$ and $\dim \mathbf{P}(\vec{W}) = \sum_i (w_i - 1)$, we get $\dim Y \geq \dim Z$ and we obviously also have the opposite inequality.

4.3. Degeneracy loci of bundles. Let $\mathcal{E}$ be a vector bundle, i.e., a locally free sheaf of finite rank e, on a scheme S. Let T be a subspace of the sections $\Gamma(S, \mathcal{E})$. The map $T \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{E}$ defines a map and an exact sequence

$$T \otimes_{\mathbf{k}} \operatorname{Sym}(\mathcal{E}) \longrightarrow \operatorname{Sym}(\mathcal{E}) \longrightarrow \mathcal{R} \longrightarrow 0, \tag{20}$$

where cokernel $\mathcal{R}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. The space T gives global sections of the affine bundle $\mathbf{V} = \mathbf{V}_S(\mathcal{E})$, and they generate a sheaf of ideals of $\mathcal{O}_{\mathbf{V}}$ defining a subscheme $\mathcal{X} = \operatorname{Spec}_{\mathcal{O}_S} \mathcal{R}$.

Now we may stratify S according to the rank of the map $T \otimes_{\mathbf{k}} \mathcal{O}_S \to \mathcal{E}$. Let U_i be the open subset where the rank is $\geq \dim_{\mathbf{k}} T - i$. Then, if $x \in U_i \backslash U_{i-1}$, we get an exact sequence

$$T \otimes_{\mathbf{k}} \operatorname{Sym}(\mathcal{E}_{\mathbf{k}(x)}) \longrightarrow \operatorname{Sym}(\mathcal{E}_{\mathbf{k}(x)}) \longrightarrow \mathcal{R}_{\mathbf{k}(x)} \longrightarrow 0,$$

where $\mathcal{R}_{\mathbf{k}(x)}$ is the quotient symmetric algebra generated by a vector space of dimension $e-t+i$. Hence, the fiber $\mathcal{X}_{\mathbf{k}(x)}$ has dimension $e-t+i$. We observe that the dimension of $\mathcal{X}$ is less than or equal to the maximum of

$$\max\{\dim(S \backslash U_{i-1}) + e - t + i\}. \tag{21}$$

We adapt this to the situation of subsection 4.2 so $S = \mathbf{P}(\vec{W})$. Let V be a vector space with a basis $x_1, \ldots, x_n$ and $\mathcal{E} = V \otimes_{\mathbf{k}} \mathcal{Q}$. For each x_i, choose a general subspace $E_i \subseteq W^*$ of codimension one. Let T be the subspace

$$\bigoplus_i x_i \otimes E_i \subseteq \bigoplus_i x_i \otimes W^* = V \otimes_{\mathbf{k}} W^*.$$

Note that the dimension of T equals the rank of $V \otimes_{\mathbf{k}} \mathcal{Q}$.

Proposition 4.3. *Suppose that* $\mathbf{k}$ *is an infinite field. The locus where the composition*

$$\alpha : T \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}(\vec{W})} \hookrightarrow V \otimes_{\mathbf{k}} W^* \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}(\vec{W})} \longrightarrow V \otimes_{\mathbf{k}} \mathcal{Q}$$

degenerates to rank $\dim_{\mathbf{k}} T - i$*, has codimension* $\geq i$.

Proof. The map α is the direct sum of maps

$$\alpha_i : E_i \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}(\vec{W})} \longrightarrow \mathcal{Q}.$$

The rank of α is then the sum of the ranks of these maps. Now fix a subset K of $[n]$, and let $E_K = \cap_{i \in K} E_i$. For each $i \in K$, also fix a number $q_i \geq 1$. Let X be the locus of points in $\mathbf{P}(\vec{W})$ where the image of $\alpha_{i,\mathbf{k}(x)}$ has corank $\geq q_i$ for $i \in K$. Let X' be the locus of points in $\mathbf{P}(\vec{W})$ where $E_K \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}(\vec{W})} \stackrel{\alpha_K}{\to} \mathcal{Q}$ degenerates to corank $\geq \sum_{i \in K} q_i$. We will show that: i) either X is empty, or $X \subseteq X'$, and ii) $\operatorname{codim} X' \geq \sum_{i \in K} q_i$. This will show the proposition.

i) Suppose X is nonempty, and let $x \in \mathbf{P}(\vec{W})$ be a point where the image of $\alpha_{i,\mathbf{k}(x)}$ has corank $\geq q_i$. Clearly, for any $I \subseteq [n]$, the image of $\alpha_{I,\mathbf{k}(x)}$ is contained in $\cap_{i \in I} \operatorname{im} \alpha_{i,\mathbf{k}(x)}$. Suppose there is an $I \subseteq K$ such that this intersection in $\mathcal{Q}_{\mathbf{k}(x)}$ does not have corank $\geq \sum_{i \in I} q_i$, and let I be minimal in K. Then, clearly, $|I| \leq \operatorname{rk} \mathcal{Q} + 1$ since all $q_i \geq 1$. The image of $E_{I \setminus \{j\}}$ is contained in the intersection $\cap_{i \in I \setminus \{j\}} \operatorname{im} \alpha_{i,\mathbf{k}(x)}$, and these do not generate $\mathcal{Q}_{\mathbf{k}(x)}$ by Lemma 4.2. By Lemma 4.1, this is not possible since the $E_{I \setminus \{j\}}$ generate W^*, and the map $W^* \otimes \mathcal{O}_{\mathbf{P}(\vec{W})} \to \mathcal{Q}$ is surjective. Hence, the image of $E_K \stackrel{\alpha_{K,\mathbf{k}(x)}}{\to} \mathcal{Q}_{\mathbf{k}(x)}$ must be of corank $\geq \sum_{i \in K} q_i$. This proves part i).

ii) The image of $E_K \otimes_{\mathbf{k}} \mathcal{O}_{\mathbf{P}(\vec{W})} \stackrel{\alpha_K}{\to} \mathcal{Q}$ is a sheaf of corank $\geq \operatorname{rk} \mathcal{Q} - \dim_{\mathbf{k}} E_K = |K| - 1$ at all points x in $\mathbf{P}(\vec{W})$. Since $\mathcal{Q}$ is generated by its global sections, the locus of points where this map has corank $\geq c + |K| - 1$, for some $c \geq 1$, by [**13**, Example 14.3.2 (d)], has codimension in $\mathbf{P}(\vec{W})$ greater than or equal to

$$\begin{aligned} c(c + \operatorname{rk} \mathcal{Q} - \dim_{\mathbf{k}} E_K) &= c(c + |K| - 1) \\ &\geq c + |K| - 1. \end{aligned} \tag{22}$$

Hence, the locus of points where the corank is $\geq \sum_{i \in K} q_i = c + |K| - 1$ has codimension $\geq \sum_{i \in K} q_i$. □

Corollary 4.4. *Let T be the sections of $\mathcal{E} = V \otimes_{\mathbf{k}} \mathcal{Q}$ given by the composition α. The subscheme $\mathcal{X} = \operatorname{Spec}_{\mathcal{O}_{\mathbf{P}(\vec{W})}} \mathcal{R}$ of $\mathbf{V}_{\mathbf{P}(\vec{W})}(\mathcal{E})$ defined by the vanishing of T, see* (20), *has dimension less than or equal to the dimension of $\mathbf{P}(\vec{W})$.*

Proof. This follows by the above Proposition 4.3 and the expression for the dimension given by (21). □

4.4. Construction of pure free squarefree resolutions from tensor complexes. Recall that $x_1, \ldots, x_n$ is a basis for V. Consider the map

$$V \otimes W^* = \bigoplus_{i=1}^{n} x_i \otimes W^* \longrightarrow \bigoplus_{i=1}^{n} x_i \otimes (W^*/E_i) \cong V. \tag{23}$$

This identifies V as the quotient space of $V \otimes W^*$ by the subspace T. It induces a homomorphism of algebras

$$S(V \otimes W^*) \longrightarrow S(V).$$

Recall the pure resolution $F_\bullet(V; \vec{W})$ of subsection 4.2 whose degree sequence is given by the set A.

Proposition 4.5. *Let* $\mathbf{k}$ *be an infinite field. The complex*

$$F_\bullet = F_\bullet(V; \vec{W}) \otimes_{S(V \otimes W^*)} S(V)$$

is a pure free squarefree resolution of a Cohen-Macaulay squarefree $S(V)$*-module. Its degree sequence is* A*.*

Remark 4.6. The essential thing about the tensor complex $F_\bullet(V; \vec{W})$ that makes this construction work is that, in the generators of its free modules in (19), the only representations of V that occur are the exterior forms $\wedge^d V$. Choosing a basis $x_1, \ldots, x_n$ for V, this is generated by (squarefree) exterior monomials. This is why the tensor complexes are "tailor made" for our construction.

Remark 4.7. In our construction we could equally well have used the $GL(F) \times GL(G)$-equivariant complex of [**7**, Section 4]. Again, in the generators of the free modules, the $\wedge^d F$ are the only representations of F that occur. In contrast, all kinds of irreducible representations of G are involved, and it also only works when char. $\mathbf{k} = 0$, which is why we focus on the tensor complexes of [**1**].

From this we obtain as a corollary the following.

Theorem 4.8. *For a balanced degree triplet (A, B, C) of type n, where B and C are intervals, i.e., the only internal nondegrees are in the interval associated to A, there exists a triplet of pure free squarefree modules over the polynomial ring in n variables over an infinite field, whose degree sequences are given by A, B and C.*

Proof of Theorem 4.8. Let the endpoints of A be c and $n-b$. All nondegrees of this triplet are in $[c, n-b]$, so the number e of such is the cardinality of $[c, n-b]\backslash A$. Since $a+b+c+e=n$, we see that a is determined by A. Hence, B and C are determined by A. Starting with the complex $F_\bullet$ in Proposition 4.5, by Lemma 1.9, $(\mathbf{A}\circ\mathbf{D})(F_\bullet)$ and $(\mathbf{A}\circ\mathbf{D})^2(F_\bullet)$ are both linear and so have degrees given by B and C. $\square$

Remark 4.9. In the forthcoming paper [**12**] we consider the construction of triplets of pure complexes in general. We transfer Conjecture 2.11 to a conjecture on the existence of certain complexes of coherent sheaves on projective spaces. In the case of the above theorem, these complexes reduce to a single coherent sheaf, the line bundle $\mathcal{O}_{\mathbf{P}(\vec{W})}(u_0, \ldots, u_{r+1})$ on the Segre embedding of $\mathbf{P}(\vec{W})$ in the projective space $\mathbf{P}(W)$.

Proof of Proposition 4.5. Let Z' be the pullback in the diagram

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d, hook] & \mathbf{V}(V)\times\mathbf{P}(\vec{W}) \arrow[d, hook] \\ Z \arrow[r, hook] & \mathbf{V}(V\otimes W^*)\times\mathbf{P}(\vec{W}). \end{tikzcd}$$

The subscheme $\mathbf{V}(V)$ of $\mathbf{V}(V\otimes W^*)$ is defined by the vanishing of the subspace T of $V\otimes W^*$. Since $Z=\mathbf{V}_{\mathbf{P}(\vec{W})}(V\otimes_{\mathbf{k}}\mathcal{Q})$, we see that Z' is the subscheme of Z defined by the vanishing of the sections T of $V\otimes\mathcal{Q}$ given by the composition α in Proposition 4.3. By Corollary 4.4 the dimension of Z' is less than or equal to $\dim\mathbf{P}(\vec{W})$. Since $\dim_{\mathbf{k}}T$ equals the rank of $V\otimes\mathcal{Q}$, the dimension of Z is $\dim\mathbf{P}(\vec{W})+\dim_{\mathbf{k}}T$ and so $\dim Z'\le\dim Z-\dim_{\mathbf{k}}T$.

Let Y' be the pullback in the diagram

$$\begin{tikzcd} Y' \arrow[r, hook] \arrow[d, hook] & \mathbf{V}(V) \arrow[d, hook] \\ Y \arrow[r, hook] & \mathbf{V}(V \otimes W^*). \end{tikzcd}$$

Since the image of Z is Y, the image of Z' is Y'. This gives

$$\dim Y' \leq \dim Z' \leq \dim Z - \dim_k T = \dim Y - \dim_k T.$$

The complex $F_\bullet(V;\vec{W})$ is a resolution of a Cohen-Macaulay module M supported on Y. The module $M' = M \otimes_{S(V\otimes W^*)} S(V)$ where $S(V) = S(V \otimes W^*)/(T)$ is supported on Y' and so

$$\dim M' \leq \dim Y' \leq \dim Y - \dim_k T = \dim M - \dim_k T.$$

Since $M' = M/(T \cdot M)$, a basis for T must form a regular sequence, and so M' is a Cohen-Macaulay module with resolution given by $F_\bullet$. Therefore, $F_\bullet$ becomes a pure resolution of a Cohen-Macaulay where the term with generators of degree $d \in A$ is
(24)

$$\bigwedge^d V \bigotimes (\otimes_{d \leq u_i} S^{u_i - d}(W_i)) \bigotimes (\otimes_{d \geq u_i + w_i} \widetilde{D}^{d-u_i-w_i}(W_i)) \bigotimes S(V).$$

The basis $x_1, \dots, x_n$ of V induces a maximal torus D of $GL(V)$, the diagonal matrices. The quotient map (23) is equivariant for the torus action where $t = (t_1, \dots, t_n) \in D$ acts on $w = \sum x_i \otimes w_i^*$ as $t.w = \sum (t_i.x_i) \otimes w_i^*$. Thus the complex above is equivariant for the torus action and so is $\mathbf{Z}^n$-graded. The action on term (24) in the complex is given by the natural actions on $\wedge^d V$ and $S(V)$ and the trivial action on the rest of the tensor factors. Hence, the multidegrees of the generators of the terms above are of squarefree degree, and so the resolution is squarefree. $\square$

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