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SEMILOCAL FORMAL FIBERS OF PRINCIPAL PRIME IDEALS

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ABSTRACT. Let $(T, \mathfrak{m})$ be a complete local (Noetherian) ring, C a finite set of pairwise incomparable nonmaximal prime ideals of T, and $p \in T$ a nonzero element. We provide necessary and sufficient conditions for T to be the completion of an integral domain A containing the prime ideal pA whose formal fiber is semilocal with maximal ideals the elements of C.

1. Introduction. One way to better understand the relationship between a commutative local ring and its completion is to examine the formal fibers of the ring. Given a local ring A with maximal ideal $\mathfrak{m}$ and $\mathfrak{m}$-adic completion $\widehat{A}$, the formal fiber of a prime ideal $P \in \operatorname{Spec} A$ is defined to be $\operatorname{Spec}(\widehat{A} \otimes_A k(P))$, where $k(P) := A_P/PA_P$. Since there is a one-to-one correspondence between the elements in the formal fiber of P and the prime ideals in the inverse image of P under the map from $\operatorname{Spec} \widehat{A}$ to $\operatorname{Spec} A$ given by $Q \to Q \cap A$, we can think of $Q \in \operatorname{Spec} \widehat{A}$ as being in the formal fiber of P if and only if $Q \cap A = P$.

One fruitful way of researching formal fibers has been, instead of directly computing the formal fibers of rings, to investigate "inverse" formal fiber questions–that is, given a complete local ring T, when does there exist a local ring A such that $\widehat{A} = T$ and both A and the formal fibers of prime ideals in A meet certain prespecified conditions? One important result of this type is due to Charters and Loepp, who show in [**1**] that, given a complete local ring T with maximal ideal $\mathfrak{m}$ and $G \subset \operatorname{Spec} T$ where G is a finite set of prime ideals which are pairwise incomparable by inclusion, a local domain A exists such that $\widehat{A} = T$ and the formal fiber of the zero ideal of A is semilocal with maximal

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ideals exactly the elements of G if and only if certain relatively weak conditions are satisfied.

In this paper we address a similar question: what are the necessary and sufficient conditions for T to be the completion of a local domain A possessing a principal prime ideal with a specified semilocal formal fiber?

Partial results on this subject were achieved by Dundon et al. in [**2**], under the constraint that the specified set $C = \{Q_1, Q_2, \dots, Q_k\}$ of nonmaximal ideals in the formal fiber is such that $\cap_{i=1}^k Q_i$ contains a nonzero regular *prime* element p of T. In particular, suppose this holds and with Π denoting the prime subring of T, we have that either $\Pi \cap Q_i = (0)$ for every i or $\Pi \cap Q_i = p\Pi$ for every i. In [**2**] it is shown that, if $\dim T > 1$, then a local domain A exists such that $\widehat{A} = T$, $p \in A$, $pA \in \operatorname{Spec} A$, and the formal fiber of pA is semilocal with maximal ideals the elements of C if and only if $p \in Q_i$ for every i and $\mathfrak{m} \notin C$ (here $\mathfrak{m}$ is the maximal ideal of T). The authors in [**2**] also consider the case $\dim T = 1$, but the subject of formal fibers is not a rich one when considering rings of dimension 1 or smaller. Therefore, we will always assume throughout this paper that all of our rings have dimension strictly larger than 1.

The main theorem in this paper is an improvement on the results in [**2**]. We eliminate the assumption that p is a prime element in T. Theorem 1.1 in this paper provides necessary and sufficient conditions for a complete local ring to be the completion of an integral domain containing a height one principal prime ideal with specified semilocal formal fiber. If, for an integral domain R, we define F_R to be the quotient field of R then we can state our main result as follows:

Theorem 1.1. *Let $(T, \mathfrak{m})$ be a complete local ring, Π the prime subring of T, and $C = \{Q_1, Q_2, \dots, Q_k\}$ a finite set of non-maximal incomparable prime ideals of T. Let $p \in \cap_{i=1}^k Q_i$ with $p \neq 0$. Then a local domain $A \subseteq T$ exists with $p \in A$ such that $\widehat{A} = T$ and pA is a prime ideal whose formal fiber is semilocal with maximal ideals the elements of C if and only if*:

(1) *$P \cap \Pi[p] = (0)$ for all $P \in \operatorname{Ass} T$,*

(2) *for every $P' \in \operatorname{Ass}(T/pT)$, we have $P' \subseteq Q_i$ for some $i \in \{1, 2, \dots, k\}$,*

(3) $F_{\Pi[p]} \cap Q_i \subseteq pT$ *for all* $i \in \{1, 2, \ldots, k\}$.

Remark. Notice that, if $p \in \cap_{i=1}^{k} Q_i$ is a prime element in T, then condition (2) is trivially satisfied (see [**6**, 9.41]). Therefore, under the hypotheses used in [**2**], the statement of our theorem can be simplified.

The proof that the above conditions are necessary is relatively short. Therefore, most of this paper is devoted to showing that they are sufficient by constructing an integral domain A with the desired properties. The general strategy behind our construction, which is similar to constructions in both [**1, 2**], is to start with the prime subring of T localized at its maximal ideal and recursively build up an ascending chain of subrings maintaining some specific properties. Our final ring A will be the union of all the subrings in the chain. Most of the work in the construction goes toward ensuring that A simultaneously meets three conditions: the map $A \to T/J$ is onto for every ideal J such that $J \nsubseteq Q_i$ for all $i \in \{1, 2, \ldots, k\}$; $IT \cap A = I$ for every finitely generated ideal I of A; and $F_A \cap Q_i \subseteq pT$ for all $i \in \{1, 2, \ldots, k\}$. These conditions will ensure that $\widehat{A} = T$ and that $pA \in \operatorname{Spec} A$ has a semilocal formal fiber with maximal ideals precisely the elements of C.

Throughout this paper, all rings will be commutative with unity. When we say a ring is "quasilocal" we mean that it has one maximal ideal. A "local" ring will be a Noetherian quasilocal ring.

2. Semilocal formal fibers of principal prime ideals of a domain. Suppose we are given a complete local ring $(T, \mathfrak{m})$, and a finite set $C = \{Q_1, Q_2, \ldots, Q_k\} \subseteq \operatorname{Spec} T$ of pairwise incomparable (that is, $Q_i \subseteq Q_j$ if and only if $Q_i = Q_j$) nonmaximal prime ideals. In this section we answer the following question. When is it true that there is a local domain A such that $\widehat{A} = T$ and there is some principal prime $P \in \operatorname{Spec} A$ such that the formal fiber of P is semilocal with maximal ideals $\{Q_1, Q_2, \ldots, Q_k\}$?

As mentioned in the introduction, the proof of necessity of the conditions in Theorem 1.1 is relatively short and we give it now.

Proof of necessity in Theorem 1.1. Suppose we have an $A \subseteq T$ with $\widehat{A} = T$ and that pA is a prime ideal with a semilocal formal fiber with maximal ideals exactly the elements of C. Since the extension $A \subseteq \widehat{A} = T$ is faithfully flat, any zero divisor of T which is in A must be a zero divisor of A. Since we assume A is a domain, A can contain no such nonzero zero divisor, and in particular, since certainly $\Pi[p] \subseteq A$, we must have $P \cap \Pi[p] = (0)$ for all $P \in \operatorname{Ass} T$. Furthermore, since the completion of $A/(pT \cap A) = A/pA$ is T/pT, we can say that all zero divisors of T/pT (that is, all elements in the image of $\cup \operatorname{Ass} T/pT$ under the canonical map $T \to T/pT$) contained in A/pA are zero divisors of A/pA. But A/pA is a domain since pA is prime; thus, A/pA cannot contain any nonzero zero divisor of T/pT and so A does not contain any element of $\cup \operatorname{Ass}(T/pT)$ which is not in pT. Let $P \in \operatorname{Ass}(T/pT)$. The argument above shows that $P \cap A \subseteq pT \cap A = pA$, and since $p \in P$, we also have $pA \subseteq A \cap P$ giving us $P \cap A = pA$. Thus, P is in the formal fiber of pA, and since we have assumed this formal fiber is semilocal with maximal ideals $\{Q_1, Q_2, \dots, Q_k\}$, we know $P \subseteq Q_i$ for some i.

Finally, suppose that for some i there is a $q \in F_{\Pi[p]} \cap (Q_i \setminus pT)$. We know $qg = h$ for some $g, h \in \Pi[p] \subseteq A$ with $g \neq 0$. Since we showed above it is necessary that $P \cap \Pi[p] = (0)$ for all $P \in \operatorname{Ass} T$, we know that g is not a zero divisor of T. Since $A \subseteq T$ is a faithfully flat extension, we know $gT \cap A = gA$ (see [**5**, Chapter 8]) and so $gq \in gA$ which implies $q \in A$. Therefore $Q_i \cap A \nsubseteq pT$, contradicting the assumption that Q_i is in the formal fiber of pA. Thus, it is necessary that $F_{\Pi[p]} \cap (Q_i \setminus pT) = \varnothing$ for all i. □

Now we proceed with the construction that will guarantee the sufficiency in Theorem 1.1.

Definition 2.1. Let S be a set. Define $\Gamma(S) = \sup(|S|, \aleph_0)$.

Note that, clearly, if T and S are sets, $\Gamma(S)\Gamma(T) = \sup(\Gamma(S), \Gamma(T))$. This definition simplifies the statement of some of our lemmas.

Definition 2.2. Let $(T, \mathfrak{m})$ be a complete local ring, and suppose we have a finite, pairwise incomparable set $C = \{Q_1, Q_2, \dots, Q_k\} \subseteq \operatorname{Spec} T$. Let $p \in \cap_{i=1}^{k} Q_i$ be a nonzero regular element of T. Suppose that $(R, R \cap \mathfrak{m})$ is a quasilocal subring of T containing p with the following properties:

(1) $\Gamma(R) < |T|$;

(2) If P is an associated prime ideal of T then $R \cap P = (0)$;

(3) For all $i \in \{1, 2, \ldots, k\}$, $F_R \cap Q_i \subseteq pT$.

Then we call R a *pT-complement avoiding subring of T*, which we shorten to $p$ca *subring*.

Remark. Observe that condition (2) in Definition 2.2 implies every $p$ca subring is an integral domain.

To show the existence of our local domain A, we construct a chain of intermediate $p$ca subrings and then let A be the union of these subrings. The following two lemmas give us ways in which we can enlarge $p$ca subrings to obtain new $p$ca subrings.

Lemma 2.3. *If R is a $p$ca subring, then $F_R \cap T$ is also a $p$ca subring.*

Proof. We begin by checking that $F_R \cap T$ is in fact quasi-local and, for this, it suffices to show that $x \in F_R \cap T$ is a non-unit if and only if $x \in \mathfrak{m} \cap F_R \cap T$. Clearly, if $x \in \mathfrak{m}$, then x is not a unit. Now, suppose for contradiction, that x is not a unit, but $x \notin \mathfrak{m}$. Since T is local, x must be a unit in T, so we write $xt = 1$ in T. However, if $rx = r'$ with $r, r' \in R$, then we get $r't = r$ in T, and hence $t \in F_R \cap T$ so x is a unit, which is a contradiction. It remains to check the three conditions in Definition 2.2.

Condition (1) is obvious. For condition (2), we suppose $q \in P \cap (F_R \cap T)$ where $P \in \operatorname{Ass} T$. Since $q \in F_R$, we write $qr_2 = r_1$ with $r_1, r_2 \in R$ so $r_1 \in R \cap P = (0)$ so $q = 0$ since r_2 is not a zero divisor in T. To check condition (3), suppose $q \in F_{F_R \cap T} \cap Q_i$. Then $qs_2 = s_1$ where $r_3 s_1 = r_4$ and $r_5 s_2 = r_6$ with $r_i \in R$ for all i. This implies $qr_3 r_6 = r_4 r_5$ so $q \in F_R \cap Q_i$, and hence $q \in pT$ as desired. □

We will eventually need the following lemma, which will allow us to retain the property of being a $p$ca subring after adjoining an element and localizing.

Lemma 2.4. *Let $(T, \mathfrak{m})$, C and p be as in Definition 2.2, and suppose $\widetilde{R}$ is a subring of $(T, \mathfrak{m})$. If $\widetilde{R}$ satisfies any of conditions (1), (2) or (3)* *in Definition* 2.2 (*with* $\widetilde{R}$ *in place of* R), *then* $\widetilde{S} := \widetilde{R}_{\widetilde{R}\cap\mathfrak{m}}$ *satisfies those same conditions. In particular, if all three conditions are satisfied by* $\widetilde{R}$ *and* $p \in \widetilde{R}$, *then* $\widetilde{S}$ *is a* $p$ca *subring.*

Remark. When we say $\widetilde{R}$ satisfies condition (3), this implicitly assumes $\widetilde{R}$ is an integral domain so that we may form the ring $F_{\widetilde{R}}$.

Proof. The cardinality condition is clearly preserved under localization. To see that property (2) is preserved under localization, let $P \in \operatorname{Ass} T$ and take $x \in P \cap \widetilde{S}$. Then $u_2 x = u_1$ with $u_1, u_2 \in \widetilde{R}$ and u_2 a unit in T. Since $x \in P$, we must have $u_2 x \in \widetilde{R} \cap P$. Therefore, if $\widetilde{R}$ satisfies condition (2), then $x = 0$ as desired.

To see that property (3) is preserved under localization, let us assume $\widetilde{R}$ satisfies condition (3), and suppose for contradiction that we can find an element $q \in (Q_i \setminus pT) \cap F_{\widetilde{S}}$ where $s_2 q = s_1$ with $s_1, s_2 \in \widetilde{S}$. We can then write $s_1 = fg^{-1} = qf'(g')^{-1} = qs_2$ with $f, g, f', g' \in \widetilde{R}$ with g and g' units in T. Then we have $fg' = qf'g$, and so clearly $q \in (Q_i \setminus pT) \cap F_{\widetilde{R}}$, which is a contradiction. $\square$

In our later constructions, we will often need to take unions of $p$ca subrings at intermediate steps. The purpose of Lemma 2.5 is to avoid repeating the arguments checking that the union is still a $p$ca subring.

Lemma 2.5. *Let* $(T, \mathfrak{m})$ *be a complete local ring, and suppose we have a finite, pairwise incomparable set* $C = \{Q_1, Q_2, \ldots, Q_k\} \subseteq \operatorname{Spec} T$. *Let* $p \in \cap_{i=1}^k Q_i$ *be a nonzero regular element of* T. *Let* Ω *be a well-ordered set, and let* $\{R_\alpha \mid \alpha \in \Omega\}$ *be a set of* $p$ca *subrings indexed by* Ω *with the property* $R_\alpha \subseteq R_\beta$ *for all* α *and* β *such that* $\alpha < \beta$. *Let* $S = \cup_{\alpha\in\Omega} R_\alpha$. *Then* $S \cap P = (0)$ *for all associated primes* P *of* T, $F_S \cap Q_i \subseteq pT$ *for each* $i \in \{1, 2, \ldots, k\}$, *and* S *is quasi-local. Furthermore, if* $\Gamma(R_\alpha) \leq \lambda$ *for all* $\alpha \in \Omega$ *we have* $\Gamma(S) \leq \lambda\Gamma(\Omega)$ *and so, if* $\Gamma(\Omega) \leq \lambda$ *and* $\Gamma(R_\alpha) = \lambda$ *for some* α, *we have* $\Gamma(S) = \lambda$.

Proof. No further explanation is necessary for the cardinality conditions. Clearly $S \cap P = (0)$ for all $P \in \operatorname{Ass} T$ because the R_α are $p$ca subrings and so none contain a nonzero element of any associated prime ideal of T. Next, suppose we have $q \in F_S \cap (Q_i \setminus pT)$. Then

$qs_1 = s_2$ for some $s_1, s_2 \in S$. If we choose $\alpha \in \Omega$ such that $s_1, s_2 \in R_\alpha$, then $q \in F_{R_\alpha} \cap (Q_i \setminus pT)$, contradicting the hypothesis that R_α is a $p$ca subring. To see that S is quasi-local, let $x, y \in S$ be non-units, and suppose for contradiction that $z(x+y) = 1$ for some $z \in S$. Choose α large enough so $x, y, z \in R_\alpha$. Then x and y are non-units in R_α and hence are both contained in $R_\alpha \cap \mathfrak{m}$. Therefore, $x + y \in R_\alpha \cap \mathfrak{m}$ so $x+y$ is not a unit in R_α, contradicting the existence of z. This contradiction shows $x + y$ is a non-unit in S, and it follows easily that the collection of non-units in S is an ideal, so S is quasi-local. □

For some steps of the construction we need the additional condition that $pT \cap R = pR$ for our subring R. The following lemma shows that, given a $p$ca subring R, we can find a larger $p$ca subring S with this property. This lemma will allow us to present a much simpler construction than in [**2**].

Lemma 2.6. *Suppose we have $(T, \mathfrak{m})$, C and p as in the hypotheses of Lemma 2.5. Let $(R, R \cap \mathfrak{m})$ be a $p$ca subring of $(T, \mathfrak{m})$. Then a $p$ca subring S of T exists with $\Gamma(S) = \Gamma(R)$ such that $R \subseteq S \subseteq T$ and $pT \cap S = pS$.*

Proof. We set $S = F_R \cap T$. By Lemma 2.3, we know S is a $p$ca subring. Take any $x \in pT \cap S$. We can write $pt = x$ and $xr_2 = r_1$ with $r_1, r_2 \in R$. This implies $ptr_2 = r_1$ so that $t \in S$ implying $pT \cap S \subseteq pS$. The reverse inclusion is obvious. □

The following is Proposition 1 from [**4**]. It helps us to ensure that the final ring we create has T as its completion.

Proposition 2.7 [**4**]. *If $(R, \mathfrak{m} \cap R)$ is a quasilocal subring of a complete local ring $(T, \mathfrak{m})$, the map $R \to T/\mathfrak{m}^2$ is onto, and $IT \cap R = I$ for every finitely generated ideal I of R, then R is Noetherian and the natural homomorphism $\widehat{R} \to T$ is an isomorphism.*

We will construct A so that the map $A \to T/\mathfrak{m}^2$ is onto. To do this, we will need Lemma 2.8, which lets us adjoin an element of a coset of T/J to a $p$ca subring R where J is an ideal of T such that $J \not\subseteq Q_i$ for every $i \in \{1, 2, \ldots, k\}$ to get a new $p$ca subring. With $J = \mathfrak{m}^2$, we will get that $A \to T/\mathfrak{m}^2$ is onto as desired. Note that Lemma 2.8 is similar in purpose to Lemma 3.9 of [**2**].

Lemma 2.8. *Let $(T, \mathfrak{m})$ be a complete local ring, and suppose we have a finite, pairwise incomparable set of nonmaximal ideals $C = \{Q_1, Q_2, \dots, Q_k\} \subseteq \operatorname{Spec} T$. Let $p \in \cap_{i=1}^k Q_i$ be a nonzero regular element of T such that for every $P \in \operatorname{Ass}(T/pT)$ we have $P \subseteq \cup_{i=1}^k Q_i$.*

Let $(R, R \cap \mathfrak{m})$ be a pca *subring of T such that $pT \cap R = pR$, and let $u + J \in T/J$ where J is an ideal of T with $J \not\subseteq Q_i$ for all $i \in \{1, 2, \dots, k\}$. Then there exists a* pca *subring S of T meeting the following conditions*:

(1) $R \subseteq S \subseteq T$,

(2) $\Gamma(S) = \Gamma(R)$,

(3) *$u + J$ is in the image of the map $S \to T/J$,*

(4) *if $u \in J$, then $S \cap J \not\subseteq Q_i$ for each $i \in \{1, 2, \dots, k\}$,*

(5) $pT \cap S = pS$.

Proof. For each $i \in \{1, 2, \dots, k\}$, let $D_{(Q_i)}$ be a full set of coset representatives of the cosets $t + Q_i \in T/Q_i$ with $t \in T$ that make $(u+t)+Q_i$ algebraic over $R/R \cap Q_i$. Let $D := \cup_{i=1}^k D_{(Q_i)}$. By Lemma 2.3 of [**1**], we know that $|T| \geq |\mathbf{R}|$. Thus, because $\Gamma(R) < |T|$, we have $|R| < |T|$, and so $|D_{(Q_i)}| < |T|$ for all $i \in \{1, \dots, k\}$, and thus we have that $|D| < |T|$.

We can now employ Lemma 2.4 of [**1**] with $I = J$ to find an $x \in J$ such that $x \notin \cup\{r + P \mid r \in D, P \in C\}$ since the set C is finite. We claim that $S' = R[u+x]_{(R[u+x] \cap \mathfrak{m})}$ is a $p$ca subring. It is clear that S' satisfies $\Gamma(S') = \Gamma(R)$.

Now consider any $P \in \operatorname{Ass} T$. We claim that $P \subseteq Q_i$ for some $Q_i \in C$. To see this, let $z \in P$ be arbitrary. Since P contains only zero divisors in T, there must be some nonzero $y \in T$ so that $zy = 0$. Let $\ell \geq 0$ be the largest integer so that $y \in p^\ell T$, and write $y = p^\ell t$ with $t \notin pT$. Since p is regular, it must be that $zt = 0$, and so z annihilates the element $t + pT \neq 0 + pT$ in T/pT. Therefore, z is a zerodivisor on T/pT and so is contained in some $\widetilde{P} \in \operatorname{Ass}(T/pT)$. It follows that $P \subseteq \cup_{P' \in \operatorname{Ass}(T/pT)} P'$ and so, by the Prime Avoidance theorem, is contained in one particular $P^* \in \operatorname{Ass}(T/pT)$. By hypothesis P^* is contained in the union of the elements of C and again by the Prime Avoidance theorem must be contained in one of them, proving our claim.

Now suppose we have $0 \neq f = r_n(u+x)^n + \cdots + r_1(u+x) + r_0 \in R[u+x] \cap P \subseteq R[u+x] \cap Q_i$. Let $m \geq 0$ be the largest integer such that $r_j \in (pT)^m$ for all $0 \leq j \leq n$. Since $pT \cap R = pR$, we have $(pT)^m \cap R = p^m R$, so we write $f = p^m(r'_n(u+x)^n + \cdots + r'_1(u+x) + r'_0)$. Since $p \notin P$ because p is regular, we must have $r'_n(u+x)^n + \cdots + r'_1(u+x) + r'_0 \in P$ and at least one of the coefficients r'_j is not in $pT \supseteq R \cap Q_i$ (by the maximality of m). This contradicts the fact that $(u+x) + Q_i$ is transcendental over $R/(R \cap Q_i)$. We thus have $R[u+x] \cap P = (0)$ for every $P \in \operatorname{Ass} T$ and Lemma 2.4 shows that the same is true for S'.

Finally, we claim that, for each $i \in \{1, 2, \ldots, k\}$, $(Q_i \setminus pT) \cap F_{S'} = \varnothing$. First, suppose for contradiction, we have a $q \in (Q_i \setminus pT) \cap F_{R[u+x]}$ for some i. Then we have $r_n(u+x)^n + \cdots + r_1(u+x) + r_0 = q(s_{n'}(u+x)^{n'} + \cdots + s_1(u+x) + s_0)$ for some $r_0, r_1, \ldots, r_n, s_0, s_1, \ldots, s_{n'} \in R$ with $r_k \neq 0$ for some $0 \leq k \leq n$. Let m be the largest integer such that $r_i \in (pT)^m$ for all $0 \leq i \leq n$, and let m' be the largest integer such that $s_j \in (pT)^{m'}$ for all $0 \leq j \leq n'$. As above, we have $(pT)^m \cap R = p^m R$ (and similarly for m'), and we can write $f = p^m(r'_n(u+x)^n + \cdots + r'_1(u+x) + r'_0) = qp^{m'}(s'_{n'}(u+x)^{n'} + \cdots + s'_1(u+x) + s'_0)$ for some $r'_0, r'_1, \ldots, r'_n, s'_0, s'_1, \ldots, s'_{n'} \in R$.

By the maximality of m and m', we know that there is an l such that $r'_l \notin pT$ and a j such that $s'_j \notin pT$. Since $(Q \setminus pT) \cap F_R = \varnothing$ for all $Q \in C$, we know $Q \cap R \subseteq pT$ and thus $r'_l, s'_j \notin Q \cap R$ for all $Q \in C$. Since $(u+x)+Q$ is transcendental over $R/R \cap Q$ for all $Q \in C$, we therefore know that $r'_n(u+x)^n + \cdots + r'_1(u+x) + r'_0 \notin \cup_{i=1}^k Q_i$ and $s'_{n'}(u+x)^{n'} + \cdots + s'_1(u+x) + s'_0 \notin \cup_{i=1}^k Q_i$. Now suppose that $m \leq m'$. Since p is not a zero divisor, we may cancel it on both sides of our equation to get $r'_n(u+x)^n + \cdots + r'_1(u+x) + r'_0 = qp^{m'-m}(s'_{n'}(u+x)^{n'} + \cdots + s'_1(u+x) + s'_0)$. The left-hand side is not in $\cup_{i=1}^k Q_i$ while the right-hand side is clearly in Q_i, which is a contradiction. On the other hand, suppose $m > m'$. Then, canceling, we have $p^{m-m'}(r'_n(u+x)^n + \cdots + r'_1(u+x) + r'_0) = q(s'_{n'}(u+x)^{n'} + \cdots + s'_1(u+x) + s'_0)$. The left-hand side is clearly in pT but, since $s'_{n'}(u+x)^{n'} + \cdots + s'_1(u+x) + s'_0$ is not in $\cup_{i=1}^k Q_i$, it is not in any associated prime of pT and so is not a zero divisor of T/pT.

Since $q \notin pT$, we have that the right-hand side is not in pT, which is a contradiction. Thus, we have $(Q_i \setminus pT) \cap F_{R[u+x]} = \varnothing$. By Lemma 2.4, we know that localizing preserves this property and so $(Q_i \setminus pT) \cap F_{S'} = \varnothing$ for all $Q_i \in C$. We have now shown that S' is a $p$ca subring of T.

We now employ Lemma 2.6 to find a $p$ca subring S with $S' \subseteq S \subseteq T$ and $\Gamma(S) = \Gamma(S') = \Gamma(R)$ such that $pT \cap S = pS$. Since $S' \subseteq S$, the image of S in T/J contains $u + x + J = u + J$. Furthermore, if $u \in J$, then $u + x \in J \cap S$, but since $(u+x) + Q_i$ is transcendental over $R/R \cap Q_i$ for each $i \in \{1, 2, \dots, k\}$, we have $u + x \notin Q_i$ so $J \cap S \nsubseteq Q_i$ for all i. □

The following two lemmas, which are similar to Lemmas 3.10 and 3.11 of [**2**], allow us to construct A such that $IT \cap A = I$ for every finitely generated ideal I of A. Recall that this is one of the conditions from Proposition 2.7 needed to show that $\widehat{A} = T$.

Lemma 2.9. *Suppose we have $(T, \mathfrak{m})$, C and p as in the hypotheses of Lemma* 2.8. *Let $(R, R \cap \mathfrak{m})$ be a* $p$ca *subring of $(T, \mathfrak{m})$ such that $pT \cap R = pR$, let I be a finitely generated ideal of R and let $c \in IT \cap R$. Then a* $p$ca *subring S of T exists meeting the following conditions*:

(1) $R \subseteq S \subseteq T$,

(2) $\Gamma(S) = \Gamma(R)$,

(3) $c \in IS$,

(4) $pT \cap S = pS$.

Proof. We first show that a $p$ca subring S' of T exists satisfying the first three conditions. Induct on the number of generators of I. Suppose $I = aR$. If $a = 0$, then $c = 0$ so $S' = R$ is the desired $p$ca subring. If $a \neq 0$, then $c = au$ for some $u \in T$. We will show that $S' = R[u]_{(R[u] \cap \mathfrak{m})}$ is a $p$ca subring satisfying the first three conditions and then apply Lemma 2.6 and set $S = F_{S'} \cap T$ to get a $p$ca subring satisfying all four conditions.

To verify that S' is a $p$ca subring, first note that the cardinality condition is clearly satisfied. To prove condition (2), consider an arbitrary $f \in R[u]$ with $f \neq 0$. We can write $f = r_n u^n + \cdots + r_1 u + r_0$

for some $r_0, r_1, \ldots, r_n \in R$. Then

$$\begin{aligned} a^n f &= r_n(au)^n + ar_{n-1}(au)^{n-1} + \cdots + a^{n-1}r_1(au) + a^n r_0 \\ &= r_n c^n + ar_{n-1}c^{n-1} + \cdots + a^{n-1}r_1 c + a^n r_0, \end{aligned}$$

and thus we see $a^n f \in R$. Now, let $P \in \operatorname{Ass} T$, and let $f \in P \cap R[u]$. Choose an n such that $a^n f \in R$. Then $a^n f \in R \cap P$ and so $a^n f = 0$ since R is a $p$ca subring. Since a is not a zerodivisor, $f = 0$ and so we have that $P \cap R[u] = (0)$. Lemma 2.4 then implies $P \cap S' = 0$. For condition (3), suppose for contradiction that we have an element $q \in (Q_i \setminus pT) \cap F_{R[u]}$ where $u_2 q = u_1$ with $u_1, u_2 \in R[u]$. By our above calculation, we can find $m \in \mathbf{N}$ so that $a^m u_i \in R$ for $i = 1, 2$. This means $a^m u_1 = a^m u_2 q$, so $q \in F_R \cap Q_i \subseteq pT$ giving the desired contradiction. Lemma 2.4 now shows S' is a $p$ca subring as claimed. This completes the base case of the induction.

Now let I be an ideal of R that is generated by $m > 1$ elements, and assume that the lemma holds for all ideals with $m-1$ generators. Let $I = (y_1, \ldots, y_m)R$. Since $c \in IT$, we can choose $t_1, t_2, \ldots, t_m \in T$ such that $c = y_1 t_1 + y_2 t_2 + \cdots + y_m t_m$.

First suppose that $y_j \notin pT \cap R = pR$ for some $j = 1, 2, \ldots, m$. Without loss of generality, reorder the y_i's so that $y_2 \notin pT \cap R$. Our goal is now to find a $t \in T$ such that we may adjoin $t_1 + y_2 t$ to our subring R without disturbing the $p$ca properties. First note that if $(t_1 + y_2 t) + Q_i = (t_1 + y_2 t') + Q_i$ for any i, then we have that $y_2(t - t') \in Q_i$. However, by the assumption that $y_2 \notin pR$ and the fact that $Q_i \cap R = pT \cap R = pR$, we know that $y_2 \notin Q_i$. Since Q_i is prime, we must have $(t - t') \in Q_i$; thus, $t + Q_i = t' + Q_i$. Therefore, if $t + Q_i \neq t' + Q_i$, then $(t_1 + y_2 t) + Q_i \neq (t_1 + y_2 t') + Q_i$.

For each i, let $D_{(Q_i)}$ be a full set of coset representatives of the cosets $t + Q_i$ that make $t_1 + y_2 t + Q_i$ algebraic over $R/R \cap Q_i$. Let $D = \cup_{i=1}^k D_{(Q_i)}$. Using the fact from the previous paragraph that $(t_1 + y_2 t) + Q_i \neq (t_1 + y_2 t') + Q_i$ whenever $t + Q_i \neq t' + Q_i$, it can be easily checked that $|D| < |T|$, and thus we use [**1**, Lemma 2.4] with $I = T$ to find an element $t \in T$ such that $t \notin \cup\{r + P \mid r \in D, P \in C\}$. We will let $x = t_1 + y_2 t$ so that $x + Q_i$ is transcendental over $R/R \cap Q_i$ for all i. We now know that $R' := R[x]_{(R[x] \cap \mathfrak{m})}$ is a $p$ca subring of T by the argument in the proof of Lemma 2.8.

We now both add and subtract $y_1 y_2 t$ to see that $c = y_1 t_1 + y_1 y_2 t - y_1 y_2 t + y_2 t_2 + \cdots + y_m t_m = y_1 x + y_2(t_2 - y_1 t) + y_3 t_3 + \cdots + y_m t_m$.

Let $J = (y_2, \ldots, y_m)R'$ and $c^* = c - y_1 x$. Then $c^* \in JT \cap R'$ and so we use the induction assumption to find a $p$ca subring S' of T with $\Gamma(S') = \Gamma(R)$ such that $R' \subseteq S' \subseteq T$ and $c^* \in JS'$. Then $c = y_1 x + c^* \in IS'$, and S' is a $p$ca subring satisfying the first three conditions of the lemma.

Now suppose that $y_j \in pT \cap R$ for all j. Then let k be the largest integer such that $y_j \in (pT)^k \cap R$ for all j. Since $pT \cap R = pR$, we know $(pT)^k \cap R = p^k R$ and we can write $c = p^k(y_1' t_1 + \cdots + y_m' t_m)$ for some $y_1', y_2', \ldots, y_m' \in R$ such that $y_i' \notin pT$ for some i. Now let $I' = (y_1', \ldots, y_m')R$ so that we have $c' := y_1' t_1 + \cdots + y_m' t_m \in I'T$. We can now apply the argument above to find a $p$ca subring S' such that $c' \in I'S'$ and so $c' = y_1' s_1 + \cdots + y_m' s_m$ for some $s_1, \ldots, s_m \in S'$. Then we have $c = p^k c' = p^k y_1' s_1' + \cdots + p^k y_m' s_m = y_1 s_1 + \cdots + y_m s_m$ and so $c \in IS'$ showing that S' is a $p$ca subring satisfying the first three conditions of the lemma.

Now we apply Lemma 2.6 to find a $p$ca subring S with $R \subseteq S' \subseteq S \subseteq T$ and $\Gamma(S) = \Gamma(S') = \Gamma(R)$ such that $pT \cap S = pS$. We know $c \in IS$ since $c \in IS'$ and $S' \subseteq S$. Thus S is a $p$ca subring meeting the conditions stated in the lemma. $\square$

Lemma 2.11 allows us to create a subring S of T that satisfies many of the conditions we want to be true for our final ring A. First we require some additional notation.

Definition 2.10. Let Ω be a well-ordered set and $\alpha \in \Omega$. We define $\gamma(\alpha) = \sup\{\beta \in \Omega \mid \beta < \alpha\}$.

Lemma 2.11. *Suppose we have $(T, \mathfrak{m})$, C and p as in the hypotheses of Lemma 2.8. Let $(R, R \cap \mathfrak{m})$ be a* $p$ca *subring of T such that $pT \cap R = pR$, let J be an ideal of T with $J \nsubseteq Q_i$ for all $i \in \{1, 2, \ldots, k\}$ and let $u + J \in T/J$. Then a* $p$ca *subring S of T exists such that*:

(1) $R \subseteq S \subseteq T$,

(2) $\Gamma(S) = \Gamma(R)$,

(3) *$u + J$ is in the image of the map $S \to T/J$,*

(4) *If $u \in J$, then $S \cap J \nsubseteq Q_i$ for each $i \in \{1, 2, \ldots, k\}$,*

(5) *For every finitely generated ideal I of S, we have $IT \cap S = I$.*

Proof. We first apply Lemma 2.8 to find a $p$ca subring R' of T satisfying conditions (1), (2), (3) and (4) and such that $pT \cap R' = pR'$. We will now construct the desired S such that S satisfies conditions (2) and (5) and $R' \subseteq S \subseteq T$ which will ensure that the first, third, and fourth conditions of the lemma hold true. Let $\Omega = \{(I, c) \mid I \text{ is a finitely generated ideal of } R' \text{ and } c \in IT \cap R'\}$. Letting $I = R'$, we see that $|\Omega| \geq |R'|$. Since R' is infinite, the number of finitely generated ideals of R' is $|R'|$, and therefore $|R'| \geq |\Omega|$, giving us the equality $|R'| = |\Omega|$ and thus $\Gamma(\Omega) = \Gamma(R)$. Well order Ω so that it does not have a maximal element, and let 0 denote its first element. We will now inductively define a family of $p$ca subrings of T, one for each element of Ω. Let $R_0 = R'$, and let $\alpha \in \Omega$. Assume that R_β has been defined for all $\beta < \alpha$ and that $pT \cap R_\beta = pR_\beta$ and $\Gamma(R_\beta) = \Gamma(R)$ hold for all $\beta < \alpha$. If $\gamma(\alpha) < \alpha$ and $\gamma(\alpha) = (I, c)$, then define R_α to be the $p$ca subring obtained from Lemma 2.9. Note that, clearly, $pT \cap R_\alpha = pR_\alpha$ and $\Gamma(R_\alpha) = \Gamma(R_{\gamma(\alpha)}) = \Gamma(R)$. If, on the other hand, $\gamma(\alpha) = \alpha$, define $R_\alpha = \cup_{\beta<\alpha} R_\beta$. By Lemma 2.5, R_α is a $p$ca subring with $\Gamma(R_\alpha) = \Gamma(R)$. Furthermore, if $t \in pT \cap R_\alpha$, then $t \in R_\beta$ for some $\beta < \alpha$ and so $t \in pT \cap R_\beta = pR_\beta \subseteq pR_\alpha$. Thus, $pT \cap R_\alpha = pR_\alpha$.

Now let $R_1 = \cup_{\alpha \in \Omega} R_\alpha$. We see from Lemma 2.5 that R_1 is a $p$ca subring and $\Gamma(R_1) = \Gamma(R_0) = \Gamma(R)$. Also, since we know by induction that $pT \cap R_\alpha = pR_\alpha$ for all $\alpha \in \Omega$, we see by the same argument made at the end of the last paragraph that $pT \cap R_1 = pR_1$. Furthermore, notice that if I is a finitely generated ideal of R_0 and $c \in IT \cap R_0$, then $(I, c) = \gamma(\alpha)$ for some $\alpha \in \Omega$ with $\gamma(\alpha) < \alpha$. It follows from the construction that $c \in IR_\alpha \subseteq IR_1$. Thus, $IT \cap R_0 \subseteq IR_1$ for every finitely generated ideal I of R_0.

Following this same pattern, build a $p$ca subring R_2 of T with $\Gamma(R_2) = \Gamma(R_1) = \Gamma(R)$ and $pT \cap R_2 = pR_2$ such that $R_1 \subseteq R_2 \subseteq T$ and $IT \cap R_1 \subseteq IR_2$ for every finitely generated ideal I of R_1. Continue by induction, forming a chain $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ of $p$ca subrings of T such that $IT \cap R_n \subseteq IR_{n+1}$ for every finitely generated ideal I of R_n and $\Gamma(R_i) = \Gamma(R_0)$ for all i.

We now claim that $S = \cup_{i=1}^{\infty} R_i$ is the desired $p$ca subring. To see this, first note $R \subseteq S \subseteq T$ and that we know from Lemma 2.5 that S is indeed a $p$ca subring and $\Gamma(S) = \Gamma(R)$. Now set $I = (y_1, y_2, \ldots, y_k)S$, and let $c \in IT \cap S$. Then an $N \in \mathbf{N}$ exists such that $c, y_1, \dots, y_k \in R_N$. Thus, $c \in (y_1, \dots, y_k)T \cap R_N \subseteq (y_1, \dots, y_k)R_{N+1} \subseteq IS$. From this, it follows that $IT \cap S = I$, so the fifth condition of the statement of the lemma holds. □

In Lemma 2.12 we construct a domain A that has the desired completion and the formal fiber of pA is semilocal with maximal ideals the elements of C.

Lemma 2.12. *Suppose we have $(T, \mathfrak{m})$, C and p as in the hypotheses of Lemma 2.8. Let Π denote the prime subring of T. Suppose $F_{\Pi[p]} \cap Q_i \subseteq pT$ for all $Q_i \in C$ and that $P \cap \Pi[p] = (0)$ for all $P \in \operatorname{Ass} T$. Then a local domain $A \subseteq T$ exists such that*

(1) *$p \in A$,*

(2) *$\widehat{A} = T$,*

(3) *pA is a prime ideal in A and has a semilocal formal fiber with maximal ideals the elements of C,*

(4) *If J is an ideal of T satisfying $J \not\subseteq Q_i$ for all $i \in \{1, 2, \dots, k\}$, then the map $A \to T/J$ is onto and $J \cap A \not\subseteq Q_i$ for all $i \in \{1, 2, \dots, k\}$.*

Proof. Let

$$\Omega = \{u + J \in T/J : J \text{ is an ideal of } T \text{ with } J \not\subseteq Q_i \text{ for all } i \in \{1, \dots, k\}\},$$

and for each $\alpha \in \Omega$, define $\Omega_\alpha := \{\beta \in \Omega \mid \beta \leq \alpha\}$. Since T is infinite and Noetherian, $|\{J \text{ is an ideal of } T \text{ with } J \not\subseteq Q \text{ for all } Q \in C\}| \leq |T|$. Also, if J is an ideal of T, then $|T/J| \leq |T|$. It follows that $|\Omega| \leq |T|$. Well order Ω so that each element has fewer than $|\Omega|$ predecessors. Let 0 denote the first element of Ω. Define R_0' to be $\Pi[p]$ localized at $\Pi[p] \cap \mathfrak{m}$. We know $\Gamma(R_0') = \aleph_0$, and by [**1**, Lemma 2.3], we know that $|T| \geq |\mathbf{R}|$ and thus $\Gamma(R_0') < |T|$. Our hypotheses and Lemma 2.4 imply that R_0' is a $p$ca subring of T. We now apply Lemma 2.6 to find a $p$ca subring R_0'' with $R_0' \subseteq R_0''$ such that $pT \cap R_0'' = pR_0''$ and $\Gamma(R_0'') = \Gamma(R_0') = \aleph_0$. Next apply Lemma 2.11 with $J = T$ to find a $p$ca subring R_0 with $R_0'' \subseteq R_0$ such that $IT \cap R_0 = I$ for every finitely generated ideal I of R_0 and $\Gamma(R_0) = \Gamma(R_0'') = \aleph_0$.

Starting with R_0, recursively define a family of $p$ca subrings as follows. Let $\alpha \in \Omega$ and assume that R_β has already been defined to be a $p$ca subring for all $\beta < \alpha$ with $IT \cap R_\beta = IR_\beta$ for every finitely generated ideal I of R_β and $\Gamma(R_\beta) \leq \Gamma(\Omega_\beta)$ (note that this condition holds for R_0 since $\Gamma(R_0) = \Gamma(\Omega_0) = \aleph_0$). Then $\gamma(\alpha) = u + J$ for some ideal J of T with $J \not\subseteq Q_i$ for every $i \in \{1, 2, \ldots, k\}$. If $\gamma(\alpha) < \alpha$, use Lemma 2.11 to obtain a $p$ca subring R_α with $\Gamma(R_\alpha) = \Gamma(R_{\gamma(\alpha)})$ such that $R_{\gamma(\alpha)} \subseteq R_\alpha \subseteq T$, $u + J$ is in the image of the map $R_\alpha \to T/J$ and $IT \cap R_\alpha = I$ for every finitely generated ideal I of R_α. Moreover, this gives us that $R_\alpha \cap J \not\subseteq Q_i$ for every $i \in \{1, 2, \ldots, k\}$ if $u \in J$. Also, since $\Gamma(R_\alpha) = \Gamma(R_{\gamma(\alpha)})$ and $\Gamma(\Omega_\alpha) = \Gamma(\Omega_{\gamma(\alpha)})$ we have that $\Gamma(R_\alpha) \leq \Gamma(\Omega_\alpha)$.

If $\gamma(\alpha) = \alpha$, define $R_\alpha = \cup_{\beta<\alpha} R_\beta$. Then, by Lemma 2.5, we see that R_α is a $p$ca subring of T. Furthermore, we have $\Gamma(R_\beta) \leq \Gamma(\Omega_\beta) \leq \Gamma(\Omega_\alpha)$ for all $\beta < \alpha$, so by Lemma 2.5 we see that $\Gamma(R_\alpha) \leq \Gamma(\Omega_\alpha)$. Now, let $I = (y_1, \ldots, y_k)$ be a finitely generated ideal of R_α, and let $c \in IT \cap R_\alpha$. Then $\{c, y_1, \ldots, y_k\} \subseteq R_\beta$ for some $\beta < \alpha$. By the inductive hypothesis, $(y_1, \ldots, y_k)T \cap R_\beta = (y_1, \ldots, y_k)R_\beta$. As $c \in (y_1, \ldots, y_k)T \cap R_\beta$, we have that $c \in (y_1, \ldots, y_k)R_\beta \subseteq I$. Hence, $IT \cap R_\alpha = I$.

We now know by induction that, for each $\alpha \in \Omega$, R_α is a $p$ca subring with $\Gamma(R_\alpha) \leq \Gamma(\Omega_\alpha)$ and $IT \cap R_\alpha = I$ for all finitely generated ideals I of R_α. We claim that $A = \cup_{\lambda \in \Omega} R_\lambda$ is the desired domain.

First note that by construction, condition (4) of the lemma is satisfied and by Lemma 2.5 A is a domain and is quasi-local. We now show that the completion of A is T. Note that as Q_i is nonmaximal in T for all i, we have that $\mathfrak{m}^2 \not\subseteq Q_i$ for all i. Thus, by the construction, the map $A \to T/\mathfrak{m}^2$ is onto. Furthermore, by an argument identical to the one used to show that $IT \cap R_\alpha = I$ for all finitely generated ideals I of R_α in the case $\gamma(\alpha) = \alpha$, we know $I'T \cap A = I'$ for all finitely generated ideals I' of A. It follows from Proposition 2.7 that A is Noetherian and $\widehat{A} = T$.

Now we show that the formal fiber of pA is semilocal with maximal ideals exactly the ideals in C. We know that if $P \in \operatorname{Spec} T$ with $P \not\subseteq Q_i$ for all i, then $P \cap A \not\subseteq Q_i$ for all i, and so $P \cap A \neq pA$ which shows that P is not in the formal fiber of pA. Furthermore, since each R_α is $p$ca, the argument in Lemma 2.5 tells us that $(Q_i \setminus pT) \cap F_A = \varnothing$, and so in particular $(Q_i \setminus pT) \cap A = \varnothing$ for all i. Thus, $Q_i \cap A = pT \cap A = pA$ for each i and so pA is prime and Q_i is in its formal fiber for every $i \in \{1, 2, \dots, k\}$. We have now shown the formal fiber of pA is semilocal with maximal ideals exactly the members of C. $\square$

We are now ready to complete the proof of Theorem 1.1; our main result.

Proof of Theorem 1.1. The condition that $P \cap \Pi[p] = (0)$ for all $P \in \operatorname{Ass} T$ ensures that p is regular. Since every $P' \in \operatorname{Ass}(T/pT)$ is contained in some Q_i, we know $P' \subseteq \cup_{i=1}^{k} Q_i$. With these observations, Lemma 2.12 now shows the conditions are sufficient. We have already shown they are necessary. $\square$

We conclude with an example showing where our result can be applied.

Example 2.13. Let T be the complete local ring $\mathbf{R}[[x, y, z, w]]/(x^2 - yz)$ and Q the non-maximal prime ideal (x, y, z). T is a domain as $(x^2 - yz)$ is a prime ideal in $\mathbf{R}[[x, y, z, w]]$. Note that if $P \in \operatorname{Ass}(T/xT) = \{(x, y), (x, z)\}$, then $P \subseteq Q$. It is also the case that $Q \cap F_{\Pi[x]} \subseteq xT$. Thus, the conditions of Theorem 1.1 are satisfied, and a domain A exists such that $\widehat{A} = \mathbf{R}[[x, y, z, w]]/(x^2 - yz)$, xA is a prime ideal in A, and the formal fiber of xA is local with maximal ideal (x, y, z).

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