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GENERALIZING SPERNER'S LEMMA TO A FREE MODULE OVER A SPECIAL PRINCIPAL IDEAL RING

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ABSTRACT. Sperner's lemma states that if $\mathcal{A}$ is an anti-chain from the power set of an n-element set, then $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Rota and Harper provide the following q-analogue to a number of classical generalizations of Sperner's lemma: *if* $\mathcal{A}$ *is an* l*-chain-free family of subspaces of a finite vector space* $\mathbf{F}_q^n$, *then* $\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{\dim(A)}_q} \leq l$ *and* $|\mathcal{A}|$ *is bounded by the sum of the* l *largest Gaussian coefficients* $\binom{n}{k}_q$. In this work, the original Sperner's lemma as well as Rota and Harper's result are extended to multiple generalizations in the setting of a finitely-generated free module over a finite special principal ideal ring.

1. Introduction. As given in [**10, 11**], Sperner's lemma (regarding finite sets) states that the cardinality of any anti-chain (i.e., collection of incomparable subsets) from the power set $\mathcal{P}([n])$ of an n-element set $[n]$ does not exceed the combination $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Some well-known n-set generalizations of Sperner's lemma are stated in the next theorem. (As in [**1**], we use the convention that an *l-chain free family* is a collection of subsets that contain no chain, $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_l$, of length l.)

Theorem 1.1 ([**2, 6–8, 12**]). *Let* $\mathcal{A}$ *be an* l*-chain-free family of subsets from* $\mathcal{P}([n])$. *Then*: (a) $\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq l$.

(b) $|\mathcal{A}|$ *is bounded by the sum of the* l *largest values* $\binom{n}{k}$, $0 \leq k \leq n$.

(c) *Letting* $\mathcal{S}_k$ *denote the set of all* k*-element subsets in* $\mathcal{P}([n])$, *there is equality in* (a) *and* (b) *when* $\mathcal{A}$ *consists of the* l *largest sets* $\mathcal{S}_k$, $0 \leq k \leq n$.

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Part (a) of Theorem 1.1 when $l = 1$ is independently credited to Lubell [**7**], Yamamoto [**12**], Meshalkin [**8**], and Bollobás [**2**], and part (b) is Erdös's theorem on l-chain-free families [**6**]. Observe that Sperner's lemma readily follows from both (a) and (b).

The following theorem, given by Rota and Harper, is a q-analogue of Theorem 1.1. Recall that the Gaussian coefficient $\binom{n}{k}_q$ gives the number of k-subspaces of an n-dimensional vector space $\mathbf{F}_q^n$ over a finite field $\mathbf{F}$ with q elements (cf., [**11**, page 291]). Explicitly, one can calculate that $\binom{n}{k}_q = [(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1)]/[(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)]$. Along with the following results, Gaussian coefficients will play a prominent role in what is to come.

Theorem 1.2 [**9**, page 200]. *Let $\mathcal{A}$ be an l-chain-free family of subspaces of $\mathbf{F}_q^n$. Then*: (a) $\sum_{A\in\mathcal{A}} 1/\binom{n}{\dim(A)}_q \leq l$.

(b) $|\mathcal{A}|$ *is bounded by the sum of the l largest Gaussian coefficients $\binom{n}{k}_q$, $0 \leq k \leq n$.*

(c) *Letting $\mathcal{S}_k$ denotes the set of all k-dimensional subspaces of $\mathbf{F}_q^n$, there is equality in* (a) *and* (b) *when $\mathcal{A}$ consists of the l largest sets $\mathcal{S}_k$, $0 \leq k \leq n$.*

Recent work by Beck and Zaslavsky in [**1**] has yielded generalizations of Theorem 1.2 where $\mathcal{A}$ is replaced by certain families of tuples of subspaces called Meshalkin sequences.

In another direction, our interest is in finding analogues of Rota and Harper's result above in the setting of a finitely-generated free module over a finite special principal ideal ring. This context seems appropriate since principal ideal rings (or PIRs) are precisely those rings R with the field-like property that no R-submodule of an R-module A has a minimal generating set with cardinality greater than the minimum number of R-generators of A. We restrict our attention to the case where the PIR is local or "special" in order to capture more field-like behavior and avoid additional structure technicalities induced by the non-local case (cf., [**4**]).

Following [**12**], a principal ideal ring R is called a *special principal ideal ring* (or SPIR) if it has only one prime ideal M and M is nilpotent, that is, $M^i = 0$ for some $i > 0$. By the index of nilpotency of M, we mean the least positive integer e such that $M^e = 0$. We use V to denote a finitely-generated free module over a finite SPIR with index of nilpotency e. Observe that a (finite) SPIR is a (finite) field and V is a (finite) vector space when $e = 1$ (cf. Section 2 for all conventions and notation of the paper). To create an example of R and V, notice that if S is a principal ideal domain and u is an irreducible element in S, then S/Su^e is an SPIR with unique prime ideal $M = Su/Su^e$ having index of nilpotency e. Therefore, if p is a prime number, $S = \mathbf{Z}_p[X]$ and $u = aX + b$ with $a \neq 0$ and b in $\mathbf{Z}_p$, then $R = S/Su^e$ is a finite SPIR and $V = R^n$ is a finitely-generated free R-module.

Referring to V as above, our main results are Theorems 3.1, 3.4, 3.6 and 3.7. Theorems 3.6 and 3.7 and their proofs are generalizations of Rota and Harper's q-analogue of Sperner's lemma (Theorem 1.2); Theorem 3.6 gives upper bounds concerning an l-free chain family of R-submodules of V when all the "dimensions" (in the form of e-tuples) of submodules of V are comparable, and Theorem 3.7 provides a similar bound when the comparability restriction is removed. Theorems 3.1 and 3.4 both generalize ancillary results often used in proofs of Sperner's lemma (cf. [**11**, Theorem 24.1]), and both are useful resources when investigating the ramifications of Theorems 3.6 and 3.7; Theorem 3.1 gives a generalization of the inequality $\binom{n}{k}_q \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}_q$ in the setting of V, and Theorem 3.4 provides a formula, albeit recursive, for the number of maximal chains of R-submodules of V.

The necessary machinery for this paper was introduced largely in [**3**]. In moving from the vector space context to modules, it was shown there that a submodule of V can be studied by a certain associated ascending e-tuple of natural numbers. This e-tuple plays the role of dimension and is just that when R is a field (i.e., $e = 1$). In order to make the paper self-contained, we give a summary in Section 2 of the parts of [**3**] that we need for our proofs. The reader may wish to consult Section 2 as needed. The main results (Section 3) are presented in the spirit of Rota and Harper's work [**9**] but with the machinery of [**3**]. The technical proofs are given in the Appendix (Section 4) and are needed only for the proofs of Theorems 3.1, 3.4 and 3.6. A first reading of the main results is feasible by augmenting Section 3 with the statements of results in Section 4 and background from Section 2 as needed.

2. Preliminaries. As explained in the introduction, this section establishes the setting of the paper and gives an updated summary of the results from [**3**] needed for our proofs. Of central importance is Definition 2.1 which establishes e-tuples that act as dimension for our submodules and Proposition 2.5 which is used to count submodules of the ambient module V.

We pause to establish *running hypotheses and notation for the rest of the paper.* All rings here are commutative with $1 \neq 0$. Empty sums are 0, and empty products are 1. Let R be a finite special principal ideal ring (SPIR) with maximal ideal $M = R\pi$ and e the index of nilpotency of M. Let V be a nonzero finitely-generated R-module that is R-free on a basis of n elements $(1 \leq n < \infty)$. Since the case in which R is a field has been handled in [**9**], we often implicitly assume that $\pi \neq 0$ and $e \geq 2$, that is, R is not a field. Also, we let q denote the cardinality of the field $\mathbf{F} = R/M$.

The following definition establishes vector space-theoretic data used to study R-submodules of V.

Definition 2.1 [**3**, Definition 2.4]. Let W be an R-submodule of V. For each $i \in \{1, \ldots, e\}$, let $p_{W,i}$ denote the R/M-dimension of $((W \cap M^{i-1}V) + M^iV)/M^iV$. When there is no ambiguity, we denote this dimension by p_i. Also, by convention, we let $p_i = 0$ for any integer $i \notin \{1, \ldots, e\}$.

The collection of p_i-values acts in much the same manner as dimension does for vector spaces. For example, it was shown in [**3**, Remark 2.5] that an R-submodule W of V equals 0 if and only if each $p_{W,i}$-value is 0, and $W = V$ if and only if each p_i-value equals $n(= \text{rank}\,(V))$. Also. from [**3**, Remark 2.5], using $\nu_R(W)$ to denote the cardinality of a minimal R-generating set of W, $0 \leq p_1 \leq p_2 \leq \cdots \leq p_e = \nu_R(W) \leq n$ for any R-submodule W of V. We now give the terminology that will be used to reference these p_i-values throughout the rest of the paper.

Definition 2.2 [**3**, Definition 2.8 and 2.9 (1)]. Call $d = (d_1, d_2, \ldots, d_e)$ an *ascending e-tuple* if $d_1, d_2, \ldots, d_e \in \{0, \ldots, n\}$ and $d_1 \leq d_2 \leq \cdots \leq d_e$. By convention, let $d_0 = 0$. Also, say that an R-submodule W of V *realizes* the ascending e-tuple d if $p_{W,i} = d_i$ for each $i \in \{1, \ldots, e\}$. In such a case, W may be written as W_d.

The next definition is key to our work: it involves ordered lists of elements from the established free module V. The collections of elements in the respective lists serve as minimal generating sets of R-submodules of V and possess some of the properties of vector space bases. Theorem 2.4 rigorously establishes the connection between these lists and submodules of V.

Definition 2.3 [**3**, Definition 2.9 (2)]. Let $d = (d_1, d_2, \ldots, d_e)$ be a nonzero ascending e-tuple. Let $\lambda_0 = 0$, and let $\{\lambda_1, \lambda_2, \ldots, \lambda_\beta\}$ be the set $\{i \in \{1, \ldots, e\} : d_i - d_{i-1} \neq 0\}$ labeled such that $\lambda_1 < \lambda_2 < \cdots < \lambda_\beta$. For each $k \in \{1, \ldots, \beta\}$, let ψ_k denote $d_{\lambda_k} - d_{\lambda_{k-1}}$. After labeling the coordinates of a d_e-tuple w so that

$$w = (w_{1,1}, w_{1,2}, \ldots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \ldots, w_{2,\psi_2}, \ldots, \\ w_{\beta,1}, w_{\beta,2}, \ldots, w_{\beta,\psi_\beta}),$$

say that the tuple w *realizes* the ascending e-tuple d if

$$w_{k,\gamma} \in M^{\lambda_k - 1}V \Bigg\backslash \left(\left(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} w_{i,j} \right) \right. \\ \left. + \left(\sum_{j=1}^{\gamma-1} R w_{k,j} \right) + M^{\lambda_k} V \right)$$

for each $k \in \{1, \ldots, \beta\}$ and $\gamma \in \{1, \ldots, \psi_k\}$. As a convention, we set $d_{\lambda_0} = 0$, $d_{\lambda_{\beta+1}} = n$ and $\lambda_{\beta+1} = e + 1$, giving that $\psi_{\beta+1} = n - d_{\lambda_\beta} = n - d_e$. (Recall $d_0 = 0$. Notice that $d_{\lambda_\beta} = d_e$ and telescoping sum $\sum_{i=1}^{\beta} \psi_i = d_{\lambda_\beta}$. Since $w_{i,j} \in M^{\lambda_i - 1}V$ for $i < k$, $\pi^{\lambda_k - \lambda_i} w_{i,j} \in M^{\lambda_k - 1}V$. An inductive argument gives that $(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} w_{i,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + M^{\lambda_k} V$ is an R-submodule of $M^{\lambda_k - 1}V$ for all k, γ as above.)

Theorem 2.4 [**3**, Theorem 2.17]. *For any nonzero R-submodule W of V and nonzero ascending e-tuple d, W realizes d if and only if W is R-generated by the set of coordinates of some tuple that realizes d. Moreover, if these equivalent conditions hold, then each of the generating sets in question is a minimal generating set of W.*

Given an ascending e-tuple $d = (d_1, \dots, d_e)$, let $N(d) = N(d_1, \dots, d_e)$ denote the number of R-submodules in V that realize the ascending e-tuple d. Also, when an R-submodule W of V realizes the ascending e-tuple d, we use the notation $N(d_W)$ to represent $N(d)$. The next result generalizes Gaussian coefficients by providing an explicit formula for $N(d)$ as a rational function in q.

Proposition 2.5 [**3**, Proposition 2.27 (c)]. *Given a nonzero ascending e-tuple $d = (d_1, d_2, \dots, d_e)$, define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta$, $d_{\lambda_0}, d_{\lambda_{\beta+1}}$ as in Definition* 2.5. *Then, the number of submodules of V that realize d is*

$$N(d) = q^{\varepsilon}\binom{d_{\lambda_1}}{d_{\lambda_0}}_q\binom{d_{\lambda_2}}{d_{\lambda_1}}_q \cdots \binom{d_{\lambda_\beta}}{d_{\lambda_{\beta-1}}}_q\binom{d_{\lambda_{\beta+1}}}{d_{\lambda_\beta}}_q \quad \textit{where}$$

$$\begin{aligned}\varepsilon &= \sum_{j=1}^{e} d_j(n-d_j) - \sum_{j=1}^{\beta} d_{\lambda_j}(d_{\lambda_{j+1}} - d_{\lambda_j}) \\ &= \sum_{j=1}^{\beta} d_{\lambda_j}(n-d_{\lambda_j})(\lambda_{j+1}-\lambda_j) \\ &\quad - \sum_{j=1}^{\beta} d_{\lambda_j}(d_{\lambda_{j+1}} - d_j).\end{aligned}$$

Further, $N(d)$ is the evaluation at q of a monic polynomial in $\mathbf{Z}[X]$ of degree $\sum_{i=1}^{e} d_i(n-d_i)$; it can be arranged that this polynomial is independent of q.

As further evidence of the analogy between $N(d)$ in Proposition 2.5 and the Gaussian coefficient $\binom{n}{k}_q$, we give the following result which generalizes the symmetric property $\binom{n}{k}_q = \binom{n}{n-k}_q$. Its proof can be deduced from Proposition 2.5.

Proposition 2.6. *Given a nonzero ascending e-tuple $d = (d_1, d_2, \dots, d_e)$, $N(d_1, \dots, d_e) = N(n-d_e, n-d_{e-1}, \dots, n-d_1)$.*

3. Results concerning Sperner's lemma. In this section, we generalize well-known q-analogues to setting the ambient module

V. Theorem 3.1 gives an upper bound for $N(d)$, the number of submodules which realizes ascending e-tuple d (cf. Proposition 2.5), and Theorem 3.4 provides a recursive formula for the number of maximal chains of submodules of V. Theorems 3.6 and 3.7 both give generalizations of Rota and Harper's q-analogue of Sperner's lemma (cf. Theorem 1.2) in the setting of V.

The inequality $\binom{n}{k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for n-element sets as well as its q-analogue $\binom{n}{k}_q \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}_q$ appears in many proofs of Sperner's lemma (cf. [**11**, Theorem 24.1 page 293], [**9**, page 100]). The following result gives an SPIR generalization of these inequalities.

Theorem 3.1. *For any ascending e-tuple* $d = (d_1, d_2, \dots, d_e)$*,*

$$N(d) \leq q^{(n/2)^2(e-1)} \binom{n}{n/2}_q \text{ when } n \text{ is even}$$

and

$$N(d) \leq q^{\lfloor n/2 \rfloor \lceil n/2 \rceil (e-1) - \lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{1}_q \binom{n}{\lfloor n/2 \rfloor}_q \text{ when } n \text{ is odd.}$$

Further, equality is reached only if $d_1 = \lfloor n/2 \rfloor$ *and* $d_e = \lceil n/2 \rceil$*. (Notice that if n is even then there is one such ascending e-tuple that achieves equality, and if n is odd then there are* $e - 1$*.)*

Proof. Given a nonzero ascending e-tuple $d = (d_1, \dots, d_e)$, define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta$, $d_{\lambda_0}, d_{\lambda_{\beta+1}}$ as in Definition 2.3. A $k \in \{0, \dots, \beta\}$ exists such that $0 = d_{\lambda_0} < d_{\lambda_1} < \cdots < d_{\lambda_k} \leq \lfloor n/2 \rfloor \leq \lceil n/2 \rceil \leq d_{\lambda_{k+1}} < d_{\lambda_{k+2}} < \cdots < d_{\lambda_{\beta+1}} = n$. There are three cases: (i) $k = 0$, (ii) $k = \beta$ and (iii) $1 \leq k \leq \beta - 1$.

Suppose that n is odd and $1 \leq k \leq \beta - 1$ (case iii). By repeated use of Lemma 4.1 (in the Appendix), one can find an ascending e-tuple $d' = (d'_1, d'_2, \dots, d'_e)$ with $N(d) \leq N(d')$ and (using the notation of Definition 2.3 applied to d')

$$d'_{\lambda'_1} < d'_{\lambda'_2} < \cdots < d'_{\lambda'_k} = \left\lfloor \frac{n}{2} \right\rfloor < \left\lceil \frac{n}{2} \right\rceil = d'_{\lambda'_{k+1}} < \cdots < d'_{\lambda'_{\beta'}},$$

where $d'_{\lambda'_1}, d'_{\lambda'_2}, \ldots, d'_{\lambda'_{\beta'}}$ is a list of consecutive whole numbers. From this point, one can use Lemmas 4.1 and 4.2 repeatedly to update d' and its notation to the point that $\beta' = 2$, $d'_{\lambda'_1} = \lfloor n/2 \rfloor$, $d'_{\lambda'_{\beta'}} = \lceil n/2 \rceil$, and $N(d) \leq N(d')$. With this, the assertion follows in this case.

In cases (i) and (ii) when n is odd, we can similarly use Lemmas 4.1 and 4.2 to show that $N(d) \leq N(\lfloor n/2 \rfloor, \ldots, \lfloor n/2 \rfloor) = N(\lceil n/2 \rceil, \ldots, \lceil n/2 \rceil)$. Using Proposition 2.5, note that $N(\lfloor n/2 \rfloor, \ldots, \lfloor n/2 \rfloor) = q^{\lfloor n/2 \rfloor \lceil n/2 \rceil (e-1)} \binom{n}{\lfloor n/2 \rfloor}_q$ and

$$\begin{aligned} N(f) &= q^{\lfloor n/2 \rfloor \lceil n/2 \rceil (e-1) - \lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{\lfloor n/2 \rfloor}_q \binom{n}{\lceil n/2 \rceil}_q \\ &= q^{\lfloor n/2 \rfloor \lceil n/2 \rceil (e-1) - \lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{1}_q \binom{n}{\lfloor n/2 \rfloor}_q \end{aligned}$$

for any ascending e-tuple $f = (f_1, \ldots, f_e)$ with $f_1 = \lfloor n/2 \rfloor$ and $f_e = \lceil n/2 \rceil$. Since $q^{-\lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{1}_q > 1$, it follows that $N(\lfloor n/2 \rfloor, \ldots, \lfloor n/2 \rfloor) < N(f)$, and so

$$N(d) \leq q^{\lfloor n/2 \rfloor \lceil n/2 \rceil (e-1) - \lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{1}_q \binom{n}{\lfloor n/2 \rfloor}_q$$

for any ascending e-tuple d when n is odd.

When n is even, it can similarly be shown that $N(d) \leq N(n/2, \ldots, n/2) = q^{(e-1)(n/2)^2} \binom{n}{n/2}_q$ for any ascending e-tuple d. In all cases, the equality in the statement of Theorem 3.1 is reached only if $d_1 = \lfloor n/2 \rfloor$ and $d_e = \lceil n/2 \rceil$ since any evocation of Lemma 4.1 or 4.2 involves a strict inequality. □

The number of maximal chains of subspaces is classically important in the study of Gaussian numbers and q-analogues (cf. [**11**, Section 24]). In what follows until Theorem 3.6, we develop and explore the tools necessary to count maximal chains of R-submodules of V.

Definition 3.2. Given R-submodules $U_1, \ldots, U_i$ of V and ascending e-tuples $d^{(1)}, \ldots, d^{(j)}$ that satisfy the coordinate-wise inequality $d^{(1)} \leq d^{(2)} \leq \cdots \leq d^{(j)}$, let $\mathcal{M}_W(U_1, \ldots, U_i, d^{(1)}, \ldots, d^{(j)})$ denote the set of maximal chains of R-submodules of a fixed R-submodule W in V that contain the submodules $U_1,\dots,U_i$ as well as R-submodules that realize the ascending e-tuples $d^{(1)},\dots,d^{(j)}$. The cardinality of $\mathcal{M}_W(U_1,\dots,U_i,d^{(1)},\dots,d^{(j)})$ is denoted by $\mathcal{M}_W(U_1,\dots,U_i,d^{(1)},\dots,d^{(j)})$. We use $\mathcal{M}_W$ to denote the set, $\mathcal{M}_W(-)$, of maximal chains of R-submodules in W and use $\mathcal{M}_W$ to denote the cardinality of $\mathcal{M}_W$.

Remark 3.3. (a) Recall that each maximal chain of subspaces of a finite vector space contains subspaces of all possible dimensions. When, $e > 1$ the analogy is not true. That is, given a maximal chain of submodules in V when $e \geq 2$, there will be some ascending e-tuples that are not realized by any submodule. The typical maximal chain in $\mathcal{M}_V$ contains $ne+1$ R-submodules $0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{ne} = V$ realizing ascending e-tuples $d^{(0)}, d^{(1)}, \dots, d^{(ne)}$, respectively, such that the consecutive e-tuples differ by 1 in exactly one coordinate. For example, if $n = 2$ and $e = 3$, it can happen that $d^{(0)} = (0,0,0)$, $d^{(1)} = (0,0,1)$, $d^{(2)} = (0,0,2)$, $d^{(3)} = (0,1,2)$, $d^{(4)} = (0,2,2)$, $d^{(5)} = (1,2,2)$, $d^{(6)} = (2,2,2)$. Notice that $(1,1,1)$ and $(1,1,2)$ are not represented in this case. So, the inclusion of $d^{(1)}, \dots, d^{(j)}$ in the notation $\mathcal{M}_W(U_1, \dots, U_i, d^{(1)}, \dots, d^{(j)})$ is a non-trivial restriction.

(b) When $e = 1$ (i.e., V is a vector space) it is well known that

$$N(k) = \binom{n}{k}_q, \qquad \mathcal{M}_V = \binom{n}{1}_q \binom{n-1}{1}_q \cdots \binom{1}{1}_q,$$

and

$$\begin{aligned}\mathcal{M}_V(W) &= \frac{\binom{n}{1}_q\binom{n-1}{1}_q\cdots\binom{1}{1}_q}{\binom{n}{k}_q} \\ &= \binom{k}{1}_q\binom{k-1}{1}_q\cdots\binom{1}{1}_q\binom{n-k}{1}_q\binom{n-k-1}{1}_q\cdots\binom{1}{1}_q\end{aligned}$$

for any k-dimensional R-subspace W of V (cf., [**11**, Section 24]). For $e \geq 1$, one sees that $\mathcal{M}_V(W) = \mathcal{M}_V(d_W)/N(d_W)$ and, more generally,

$$\mathcal{M}_V(W, d^{(1)}, \dots, d^{(j)}) = \mathcal{M}_V(d_W, d^{(1)}, \dots, d^{(j)})/N(d_W)$$

for any R-submodule W of V and ascending e-tuples $d^{(1)}, \dots, d^{(j)}$. The latter equation paves the way for finding $\mathcal{M}_V(W, d^{(1)}, \dots, d^{(j)})$ explicitly in q since the righthand side can be found using Proposition 2.5 and the next theorem (cf. Example 3.5).

We now give a recursive formula for counting maximal chains of R-submodules of V.

Theorem 3.4. *Given a nonzero ascending e-tuple* $d = (d_1, d_2, \ldots, d_e)$*, define* $\beta, \lambda_0, \lambda_1, \ldots, \lambda_\beta, \psi_1, \ldots, \psi_\beta, \psi_{\beta+1}, d_{\lambda_0}, d_{\lambda_{\beta+1}}$ *as in Definition* 2.3. *Suppose that* W *is an* R*-submodule that realizes the ascending e-tuple* d*, and* f *is an ascending e-tuple that satisfies the coordinate-wise inequality* $f \leq d$*. Then the number of maximal chains of* R*-submodules in* W *that contain an* R*-submodule that realizes* f *is given by the recursive formula*

$$\mathcal{M}_W(f) = \sum_{i \in \Lambda} q^{d_{\lambda_{i-1}}} \binom{\psi_i}{1}_q \mathcal{M}_{W_{\delta(i)}}(f)$$

where $\delta(i)$ *is the ascending e-tuple given by decreasing the* λ_i*th entry of* d *by* 1*,* $W_{\delta(i)}$ *is any fixed* R*-submodule that realizes* $\delta(i)$ *and* $\Lambda = \{j \in \{1, \ldots, \beta\} : f \leq \delta(j)\}$.

Proof. Given a nonzero ascending e-tuple $d = (d_1, \ldots, d_e)$, define $\beta, \lambda_0, \lambda_1, \ldots, \lambda_\beta, \psi_1, \ldots, \psi_\beta, \psi_{\beta+1}, d_{\lambda_0}, d_{\lambda_{\beta+1}}$ as in Definition 2.3. Let W be an R-submodule of V that realizes d. Any R-submodule of W that is maximal with respect to being a proper submodule of W must realize an ascending e-tuple $\delta(i)$ given by decreasing the λ_ith entry of d for some $i \in \{1, \ldots, \beta\}$. Any maximal chain of R-submodules of W that contains an R-submodule that realizes f must also contain an R-submodule that realizes $\delta(i)$ for some i such that $f \leq \delta(i)$. By Lemma 4.4, the number of R-submodules of W that realize $\delta(i)$ is given by $q^{d_{\lambda_{i-1}}} \binom{\psi_i}{1}_q$, so multiplication gives the assertion. ◻

Example 3.5. (a) Suppose that $e = 2$ and $n = 3$. Then

$$\mathcal{M}_V = \mathcal{M}_{W_{(3,3)}} = \left[\binom{3}{1}_q \binom{2}{1}_q \right]^2 + q \binom{3}{1}_q \binom{2}{1}_q \left[\binom{2}{1}_q + q \right]^2$$

since the recursive formula from Theorem 3.4 (using $f = (0,0)$) gives

$$\mathcal{M}_V = \mathcal{M}_{W_{(3,3)}} = \binom{3}{1}_q \mathcal{M}_{W_{(2,3)}},$$

$$\mathcal{M}_{W_{(2,3)}} = \binom{2}{1}_q \mathcal{M}_{W_{(1,3)}} + q^2 \mathcal{M}_{W_{(2,2)}},$$
$$\mathcal{M}_{W_{(2,2)}} = \binom{2}{1}_q \mathcal{M}_{W_{(1,2)}},$$
$$\mathcal{M}_{W_{(1,3)}} = \mathcal{M}_{W_{(0,3)}} + q\binom{2}{1}_q \mathcal{M}_{W_{(1,2)}},$$
$$\mathcal{M}_{W_{(1,2)}} = \mathcal{M}_{W_{(0,2)}} + q\mathcal{M}_{W_{(1,1)}},$$
$$\mathcal{M}_{W_{(0,3)}} = \binom{3}{1}_q \mathcal{M}_{W_{(0,2)}},$$
$$\mathcal{M}_{W_{(0,2)}} = \binom{2}{1}_q \mathcal{M}_{W_{(0,1)}},$$

and

$$\mathcal{M}_{W_{(1,1)}} = \mathcal{M}_{W_{(0,1)}} = \mathcal{M}_{W_{(0,0)}} = 1.$$

The number of these maximal chains that include an R-submodule that realizes the ascending e-tuple $(1,2)$ is $\mathcal{M}_V((1,2)) = q\binom{3}{1}_q\binom{2}{1}_q[\binom{2}{1}_q+q]^2$ since

$$\mathcal{M}_{W_{(3,3)}}((1,2)) = \binom{3}{1}_q \mathcal{M}_{W_{(2,3)}}((1,2)),$$
$$\mathcal{M}_{W_{(2,3)}}((1,2)) = \binom{2}{1}_q \mathcal{M}_{W_{(1,3)}}((1,2)) + q^2\mathcal{M}_{W_{(2,2)}}((1,2)),$$
$$\mathcal{M}_{W_{(2,2)}}((1,2)) = \binom{2}{1}_q \mathcal{M}_{W_{(1,2)}},$$

and

$$\mathcal{M}_{W_{(1,3)}}((1,2)) = q\binom{2}{1}_q \mathcal{M}_{W_{(1,2)}}.$$

(b) Again, suppose that $e = 2$ and $n = 3$. To find the number of maximal chains from $\mathcal{M}_V$ that include a fixed R-submodule $W_{(1,2)}$

of V (that realizes the ascending e-tuple $(1,2)$), use (a) above and Proposition 2.5 to observe that

$$\mathcal{M}_V(W_{(1,2)}) = \frac{\mathcal{M}_V((1,2))}{N((1,2))} = \frac{q\binom{3}{1}_q\binom{2}{1}_q\left[\binom{2}{1}_q + q\right]^2}{q\binom{3}{1}_q\binom{2}{1}_q} = \left[\binom{2}{1}_q + q\right]^2.$$

(c) In regards to finding $\mathcal{M}_W(d^{(1)},\dots,d^{(j)})$ as defined in Definition 3.2, notice that $\mathcal{M}_W(d^{(1)},\dots,d^{(j)}) = 0$ unless the ascending e-tuples $d^{(i)}$ can be relabeled such that $d^{(1)} \le \cdots \le d^{(j)}$. If this is the case, then Theorem 3.4 can initially be used to find $\mathcal{M}_W(d^{(1)},\dots,d^{(j)})$ in terms of $\mathcal{M}_{W_{d^{(j)}}}(d^{(1)},\dots,d^{(j-1)})$. Iterating in this manner, $\mathcal{M}_W(d^{(1)},\dots,d^{(j)})$ can be found explicitly in q. For example, when $e=2$ and $n=3$, $\mathcal{M}_V((1,1),(1,3)) = q^2\binom{3}{1}_q\binom{2}{1}_q^2$ since $\mathcal{M}_V((1,1),(1,3)) = \binom{3}{1}_q\mathcal{M}_{W_{(2,3)}}((1,1),(1,3))$, $\mathcal{M}_{W_{(2,3)}}((1,1),(1,3)) = \binom{2}{1}_q\mathcal{M}_{W_{(1,3)}}((1,1))$, $\mathcal{M}_{W_{(1,3)}}((1,1)) = q\binom{2}{1}_q\mathcal{M}_{W_{(1,2)}}((1,1))$, $\mathcal{M}_{W_{(1,2)}}((1,1)) = q\mathcal{M}_{W_{(1,1)}}$ and $\mathcal{M}_{W_{(1,1)}} = \mathcal{M}_{W_{(0,1)}} = \mathcal{M}_{W_{(0,0)}} = 1$.

The next two theorems generalize Rota and Harper's Theorem 1.2. The premise of Theorem 3.6 requires that the associated e-tuples of all submodules in the l-chain free family $\mathcal{A}$ of submodules of V be pairwise comparable. Theorem 3.7 has no such restriction.

Theorem 3.6. *Suppose that $d^{(0)} < d^{(1)} < \cdots < d^{(ne)}$ is a maximal chain of ascending e-tuples (with respect to a fixed n). Also, suppose that $\mathcal{A}$ is an l-chain free family of submodules of V such that each $W \in \mathcal{A}$ realizes an ascending e-tuple from within the set $\{d^{(1)}, d^{(2)}, \dots, d^{(ne)}\}$. Then:*

(a) $\sum_{W\in\mathcal{A}}(1/N(d_W)) \le l$.

(b) $|\mathcal{A}|$ *is at most the sum of the l largest terms $N(d^{(i)})$.*

(c) *Letting $\mathcal{S}_{d^{(k)}}$ denote the set of all R-submodules of V that realize the ascending e-tuple $d^{(k)}$, there is equality in* (a) *and* (b) *when $\mathcal{A}$ consists of the l largest classes $\mathcal{S}_{d^{(k)}}$, $0 \le k \le ne$.*

Proof. (a) Each of the $\mathcal{M}_V(d^{(0)}, d^{(1)}, \dots, d^{(ne)})$ maximal chains of R-submodules in V contains at most l members of $\mathcal{A}$. On the other hand, for any $W \in \mathcal{A}$, there are $\mathcal{M}_V(W, d^{(0)}, d^{(1)}, \ldots, d^{(ne)})$ of these maximal chains that contain W. Therefore,

$$\sum_{W \in \mathcal{A}} \mathcal{M}_V(W, d^{(0)}, d^{(1)}, \ldots, d^{(ne)}) \leq \mathcal{M}_V(d^{(0)}, d^{(1)}, \ldots, d^{(ne)})l.$$

The result follows by division, Proposition 4.3, and the observation that

$$\begin{aligned}\mathcal{M}_V(W, d^{(0)}, d^{(1)}, \ldots, d^{(ne)}) &= \frac{\mathcal{M}_V(d_W, d^{(0)}, d^{(1)}, \ldots, d^{(ne)})}{N(d_W)} \\ &= \frac{\mathcal{M}_V(d^{(0)}, d^{(1)}, \ldots, d^{(ne)})}{N(d_W)}\end{aligned}$$

for any submodule $W \in \mathcal{A}$.

(b) Deny. Suppose that $N(d^{(1)}) \geq N(d^{(2)}) \geq \cdots \geq N(d^{(l)})$ are the l largest values of $N(d)$, and assume that $|\mathcal{A}| > \sum_{i=1}^{l} N(d^{(i)})$. Enumerate the elements W of $\mathcal{A}$ in a list $W_1, \ldots, W_{|\mathcal{A}|}$ such that $1/N(d_{W_1}) \leq 1/N(d_{W_2}) \leq \cdots \leq 1/N(d_{W_{|\mathcal{A}|}})$.

On the left-hand side of the inequality $1/N(d_{W_1}) + \cdots + 1/N(d_{W_{|\mathcal{A}|}}) \leq l$ from (a), replace each of the first $N(d^{(1)})$ terms with $1/N(d^{(1)})$; replace each of the next $N(d^{(2)})$ terms with $1/N(d^{(2)})$; replace each of the next $N(d^{(3)})$ terms with $1/N(d^{(3)})$; and so on, till the left-hand side is of the form

$$N(d^{(1)})\left(\frac{1}{N(d^{(1)})}\right) + N(d^{(2)})\left(\frac{1}{N(d^{(2)})}\right) + \cdots + N(d^{(l)})\left(\frac{1}{N(d^{(l)})}\right) + r$$

where r is the sum of unaltered remaining terms of the form $1/N(d_W)$. Notice that the assumption $|\mathcal{A}| > \sum_{i=1}^{l} N(d^{(i)})$ guarantees that $r > 0$, and it is clear that each replacement term does not exceed the original term. Therefore, $l + r \leq l$. The assertion then follows since r cannot be both zero and nonzero.

(c) Referring to $S_{d^{(k)}}$ in the assertion, (c) follows from the observations that $\sum_{W \in S_{d^{(k)}}} \mathcal{M}_V(W, d^{(0)}, d^{(1)}, \ldots, d^{(ne)}) = \mathcal{M}_V(d^{(0)}, d^{(1)}, \ldots, d^{(ne)})$ and $|S_{d^{(k)}}| = N(d^{(k)})$ for any $d^{(k)}$. □

Theorem 3.7. *For an l-chain free family $\mathcal{A}$ of R-submodules of V,*

$$\sum_{W\in\mathcal{A}} \frac{1}{[\mathcal{M}_V/\mathcal{M}_V(W)]} \leq l.$$

Proof. Each of the $\mathcal{M}_V$ maximal chains of R-submodules in V contains at most l members of $\mathcal{A}$. On the other hand, there are $\mathcal{M}_V(W)$ maximal chains containing any R-submodule $W \in \mathcal{A}$. Therefore,

$$\sum_{W\in\mathcal{A}} \mathcal{M}_V(W) \leq l\mathcal{M}_V.$$

The result follows by division. □

Remark 3.8. (a) In the context of Theorem 3.7, $\mathcal{M}_V\mathcal{M}_V(d_W) \geq 1$ and $\mathcal{M}_V\mathcal{M}_V(W) = N(d_W)\mathcal{M}_V\mathcal{M}_V(d_W)$ for all $W \in V$, and so $|\mathcal{A}| \leq \max_{d\in\Delta}\{N(d)\frac{\mathcal{M}_V}{\mathcal{M}_V(d)}\}$ where Δ is the collection of all ascending e-tuples of V. For example, using Theorems 3.1 and 3.4 when $e = 2$, $n = 3$ and $q = 5$, it can be shown that

$$\begin{aligned}\max_{d\in\Delta}\left\{N(d)\frac{\mathcal{M}_V}{\mathcal{M}_V(d)}\right\} &= N((1,1))\frac{\mathcal{M}_V}{\mathcal{M}_V((1,1))}\\ &= N((2,2))\frac{\mathcal{M}_V}{\mathcal{M}_V((2,2))}\\ &= 2229.\overline{18}.\end{aligned}$$

(b) One can show, using an argument similar to the proof of Theorem 3.6 (b), that $|\mathcal{A}|$ is bounded above by the sum of the largest l values of $\lceil N(d)\mathcal{M}_V\mathcal{M}_V(d)\rceil$ as d ranges over the set of all ascending e-tuples of V.

APPENDIX

4. This Appendix contains the majority of the new, technical information in the paper. All of the results presented here are inevitably used in the proofs of Theorems 3.1, 3.4 and 3.6. The statements in this section are arranged in the order that they are invoked in the preceding sections. We begin with two lemmas that are the heart of the proof of Theorem 3.1.

Lemma 4.1. *Let $d = (d_1, \dots, d_e)$ be a nonzero ascending e-tuple, and define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, d_{\lambda_0}, d_{\lambda_{\beta+1}}$ as in Definition 2.3. For some $k \in \{1, \dots, \beta\}$, suppose that $d_{\lambda_k} + 1 < d_{\lambda_{k+1}}$ and $d' = (d'_1, \dots, d'_e)$ is the ascending e-tuple d with the modification that each instance of d_{λ_k} is replaced with $d_{\lambda_k} + 1$.*

(1) *If $d_{\lambda_k} < \lfloor n/2 \rfloor$, then $N(d) < N(d')$.*

(2) *If $\lceil n/2 \rceil \le d_{\lambda_k}$, then $N(d') < N(d)$.*

Proof. We only prove (1) since (2) follows by similar techniques. Assume the premise of (1), and define the parameters β', $\lambda'_0, \lambda'_1, \dots, \lambda'_{\beta'}$, $d'_{\lambda'_0}, d'_{\lambda'_{\beta'+1}}$ with respect to the tuple d' (as in Definition 2.3).

Given the relationship between d and d', notice that

$$\begin{aligned} \beta' = \beta, \lambda'_0 &= \lambda_0, \lambda'_1 = \lambda_1, \dots, \lambda'_{\beta'} = \lambda_\beta, \\ d'_{\lambda'_0} &= d_{\lambda_0}, \\ d'_{\lambda'_1} &= d_{\lambda_1}, \dots, d'_{\lambda'_{k-1}} = d_{\lambda_{k-1}}, \\ d'_{\lambda'_k} &= d_{\lambda_k} + 1, \\ d'_{\lambda'_{k+1}} &= d_{\lambda_{k+1}}, \dots, d'_{\lambda'_{\beta'+1}} = d_{\lambda_{\beta+1}}. \end{aligned}$$

With this and Proposition 2.5, observe that

$$N(d') = q^{\varepsilon'} \binom{d_{\lambda_1}}{d_{\lambda_0}}_q \binom{d_{\lambda_2}}{d_{\lambda_1}}_q \cdots \binom{d_{\lambda_{k-1}}}{d_{\lambda_{k-2}}}_q \binom{d_{\lambda_k} + 1}{d_{\lambda_{k-1}}}_q \binom{d_{\lambda_{k+1}}}{d_{\lambda_k} + 1}_q \binom{d_{\lambda_{k+2}}}{d_{\lambda_{k+1}}}_q \cdots \binom{d_{\lambda_{\beta+1}}}{d_{\lambda_\beta}}_q$$

where $\varepsilon' = \sum_{j=1, j \ne k}^{\beta} d_{\lambda_j}(n - d_{\lambda_j})(\lambda_{j+1} - \lambda_j) - \sum_{j=1, j \notin \{k,k-1\}}^{\beta} d_{\lambda_j}(d_{\lambda_{j+1}} - d_{\lambda_j}) + (d_{\lambda_k} + 1)(n - (d_{\lambda_k} + 1))(\lambda_{k+1} - \lambda_k) - (d_{\lambda_k} + 1)(d_{\lambda_{k+1}} - (d_{\lambda_k} + 1)) - (d_{\lambda_{k-1}}((d_{\lambda_k} + 1) - d_{\lambda_{k-1}}) = \varepsilon + (n - 2d_{\lambda_k} - 1)(\lambda_{k+1} - \lambda_k) + 2d_{\lambda_k} - (d_{\lambda_{k-1}} + d_{\lambda_{k+1}}) + 1$ where $\varepsilon = \sum_{j=1}^{\beta} d_{\lambda_j}(n - d_{\lambda_j})(\lambda_{j+1} - \lambda_j) - \sum_{j=1}^{\beta} d_{\lambda_j}(d_{\lambda_{j+1}} - d_{\lambda_j})$.

Also, observe that

$$\binom{d_{\lambda_k}+1}{d_{\lambda_{k-1}}}_q \binom{d_{\lambda_{k+1}}}{d_{\lambda_k}+1}_q = \binom{d_{\lambda_k}}{d_{\lambda_{k-1}}}_q \binom{d_{\lambda_{k+1}}}{d_{\lambda_k}}_q \left[\frac{q^{d_{\lambda_{k+1}}-d_{\lambda_k}}-1}{q^{d_{\lambda_k}-d_{\lambda_{k-1}}+1}-1}\right].$$

Since Proposition 2.5 also gives that $N(d) = q^{\varepsilon}\binom{d_{\lambda_1}}{d_{\lambda_0}}_q \binom{d_{\lambda_2}}{d_{\lambda_1}}_q \binom{d_{\lambda_3}}{d_{\lambda_2}}_q \cdots \binom{d_{\lambda_{\beta+1}}}{d_{\lambda_\beta}}_q$, we have that

$$N(d') = N(d) q^{\delta}\left[\frac{q^{d_{\lambda_{k+1}}-d_{\lambda_k}}-1}{q^{d_{\lambda_k}-d_{\lambda_{k-1}}+1}-1}\right]$$

where $\delta = (n-2d_{\lambda_k}-1)(\lambda_{k+1}-\lambda_k)+2d_{\lambda_k}-(d_{\lambda_{k-1}}+d_{\lambda_{k+1}})+1$. Now,

$$\begin{aligned}
q^{\delta}\left[\frac{q^{d_{\lambda_{k+1}}-d_{\lambda_k}}-1}{q^{d_{\lambda_k}-d_{\lambda_{k-1}}+1}-1}\right] &\geq q^{\delta}\left[\frac{q^{d_{\lambda_{k+1}}-d_{\lambda_k}}-1}{q^{d_{\lambda_k}-d_{\lambda_{k-1}}+1}}\right] \\
&= q^{(n-2d_{\lambda_k}-1)(\lambda_{k+1}-\lambda_k)-d_{\lambda_{k+1}}+d_{\lambda_k}}\left[q^{d_{\lambda_{k+1}}-d_{\lambda_k}}-1\right] \\
&= q^{(n-2d_{\lambda_k}-1)(\lambda_{k+1}-\lambda_k)}\left(1-\frac{1}{q^{d_{\lambda_{k+1}}-d_{\lambda_k}}}\right) \\
&\geq q^{(n-2d_{\lambda_k}-1)(\lambda_{k+1}-\lambda_k)}\left(1-\frac{1}{q^2}\right) \\
&\geq 2^{(n-2d_{\lambda_k}-1)(\lambda_{k+1}-\lambda_k)}\left(1-\frac{1}{2^2}\right) \\
&\geq 2^{n-2d_{\lambda_k}-1}(3/4) > 2^{n-2d_{\lambda_k}-2} \geq 1.
\end{aligned}$$

(The last inequality is given by the fact that $n-2d_{\lambda_k}-2 \geq n-2(n/2-1)-2 = 0$ since $d_{\lambda_k} < \lfloor n/2 \rfloor$.) As

$$q^{\delta}\left[\frac{q^{d_{\lambda_{k+1}}-d_{\lambda_k}}-1}{q^{d_{\lambda_k}-d_{\lambda_{k-1}}+1}-1}\right] > 1$$

whenever $d_{\lambda_k} < \lfloor n/2 \rfloor$, we have that $N(d) < N(d')$ which is the assertion of (1). □

We omit the proof of the Lemma 4.2 since it is analogous to that of Lemma 4.1.

Lemma 4.2. *Let $d = (d_1, \dots, d_e)$ be a nonzero ascending e-tuple, and define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, d_{\lambda_0}, d_{\lambda_{\beta+1}}$ with respect to d.*

(1) *Suppose that $d_{\lambda_{k-1}} = \lfloor n/2 \rfloor - 1$ and $d_{\lambda_k} = \lfloor n/2 \rfloor$. Then $N(d) < N(d')$ where $d' = (d'_1, \dots, d'_e)$ is the ascending e-tuple d with the modification that each instance of $d_{\lambda_{k-1}}$ is replaced with $\lfloor n/2 \rfloor$.*

(2) *Suppose that $d_{\lambda_k} = \lceil n/2 \rceil$ and $d_{\lambda_{k+1}} = \lceil n/2 \rceil + 1$. Then $N(d) < N(d')$ where d' is d with the modification that each instance of $d_{\lambda_{k+1}}$ is replaced with $\lceil n/2 \rceil$.*

When $e = 1$ (i.e., V is a finite vector space), the number of maximal chains $\mathcal{M}_V(W)$ (of subspaces of V containing subspace W) does not change as W ranges over the set of subspaces of the same dimension. For any e, the next proposition establishes the invariance of $\mathcal{M}_V(W)$ as W ranges over the set of R-submodules of V that realize the same ascending e-tuple. In addition, Proposition 4.3 allows for restrictions on the types of ascending e-tuples that must be realized by the R-submodules in the maximal chains.

Proposition 4.3. *If U and W are both R-submodules of V which realize the same nonzero ascending e−tuple d and $d^{(1)}, d^{(2)}, \dots, d^{(j)}$ are ascending e-tuples, then $\mathcal{M}_V(U, d^{(1)}, \dots, d^{(j)}) = \mathcal{M}_V(W, d^{(1)}, \dots, d^{(j)})$.*

Proof. Observe that it suffices to display an R-automorphism Γ on V with $\Gamma(U) = W$. This will give the assertion since Γ would induce a one-to-one correspondence between $\mathcal{M}_V(U, d^{(1)}, \dots, d^{(j)})$ and $\mathcal{M}_V(W, d^{(1)}, \dots, d^{(j)})$, and any submodule realizing $d^{(i)}$ would be sent to a submodule of the same type.

Assume the premise of Proposition 4.3. For the ascending e-tuple $d = (d_1, \dots, d_e)$, define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta, \ \psi_1, \dots, \psi_\beta, \psi_{\beta+1}, d_{\lambda_0}, d_{\lambda_{\beta+1}}$ as in Definition 2.3. As U and W realize d, they both have minimal R-generating sets which are both sets of coordinates of tuples which realize d (cf., Definition 2.3). That is,

$$U = \langle u_{1,1}, u_{1,2}, \dots, u_{1,\psi_1}, u_{2,1}, u_{2,2}, \dots, u_{2,\psi_2}, \dots, u_{\beta,1}, u_{\beta,2}, \dots, u_{\beta,\psi_\beta} \rangle$$

and

$$W = \langle w_{1,1}, w_{1,2}, \dots, w_{1,\psi_1}, w_{2,1}, w_{2,2}, \dots, w_{2,\psi_2}, \dots, w_{\beta,1}, w_{\beta,2}, \dots, w_{\beta,\psi_\beta} \rangle$$

for some

$$u_{k,\gamma} \in M^{\lambda_k - 1}V \Big\backslash \left(\left(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} u_{i,j} \right) + \left(\sum_{j=1}^{\gamma-1} Ru_{k,j} \right) + M^{\lambda_k} V \right)$$

and

$$w_{k,\gamma} \in M^{\lambda_k - 1}V \Big\backslash \left(\left(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} w_{i,j} \right) + \left(\sum_{j=1}^{\gamma-1} Rw_{k,j} \right) + M^{\lambda_k} V \right)$$

for each $k \in \{1, \dots, \beta\}$ and $\gamma \in \{1, \dots, \psi_k\}$.

In regards to U, each $u_{k,\gamma} = \pi^{\lambda_k - 1} f_{k,\gamma}$ for some $f_{k,\gamma} \in V$. Observe that

$$f_{k,\gamma} \notin \sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} Rf_{i,j} + \sum_{j=1}^{\gamma-1} Rf_{k,j} + MV$$

since

$$\begin{aligned} &\pi^{\lambda_k - 1} \left(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} Rf_{i,j} + \sum_{j=1}^{\gamma-1} Rf_{k,j} + MV \right) \\ &\qquad = \sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} R\pi^{\lambda_k - \lambda_i} u_{i,j} + \sum_{j=1}^{\gamma-1} Ru_{k,j} + M^{\lambda_k} V. \end{aligned}$$

The observations that $\psi_1 + \cdots + \psi_\beta = d_{\lambda_\beta} = d_e$ and

$$f_{k,\gamma} \in V \Big\backslash \left(\sum_{i=1}^{k-1} \sum_{j=1}^{\psi_i} Rf_{i,j} + \sum_{j=1}^{\gamma-1} Rf_{k,j} + MV \right)$$

for each $k \in \{1, \dots, \beta\}$ and $\gamma \in \{1, \dots, \psi_k\}$ give that

$$(f_{1,1}, f_{1,2}, \dots, f_{1,\psi_1}, f_{2,1}, f_{2,2}, \dots, f_{2,\psi_2}, \dots, f_{\beta,1}, f_{\beta,2}, \dots, f_{\beta,\psi_\beta})$$

is a d_e-tuple of elements of V which realizes the constant e-tuple $(d_e, d_e, \dots, d_e)$. Choose $f_{\beta+1,\gamma}$ from $V \backslash (\sum_{i=1}^{\beta} \sum_{j=1}^{\psi_i} Rf_{i,j} + \sum_{j=1}^{\gamma-1}$

$Rf_{\beta+1,j} + MV)$ for each $\gamma \in \{1, \dots, \psi_{\beta+1} = n - d_e\}$. (Note that any such $f_{\beta+1,\gamma}$ is available to be chosen since

$$\dim_{R/M}\left(\frac{\sum_{i=1}^{\beta}\sum_{j=1}^{\psi_i} Rf_{i,j} + \sum_{j=1}^{\gamma-1} Rf_{\beta+1,j} + MV}{MV}\right) = d_e + \gamma - 1 < n = \dim_{R/M}(V/MV).$$

Under this construction,

$$(f_{1,1}, f_{1,2}, \dots, f_{1,\psi_1}, f_{2,1}, f_{2,2}, \dots, f_{2,\psi_2}, \dots, f_{\beta+1,1}, \dots, f_{\beta+1,\psi_{\beta+1}})$$

is an ascending e-tuple which realizes the constant e-tuple $(n, n, \dots, n)$. By [**3**, Remark 2.5 (a) and Proposition 2.19],

$$\{f_{1,1}, f_{1,2}, \dots, f_{1,\psi_1}, f_{2,1}, f_{2,2}, \dots, f_{2,\psi_2}, \dots, f_{\beta+1,1}, f_{\beta+1,2}, \dots, f_{\beta+1,\psi_{\beta+1}}\}$$

is an R-basis of V. Similarly, we can find $g_{k,\gamma}$ such that $w_{k,\gamma} = \pi^{\lambda_k - 1} g_{k,\gamma}$ for each $k \in \{1, \dots, \beta\}$ and $\gamma \in \{1, \dots, \psi_k\}$. Choosing $g_{\beta+1,\gamma}$ for each $\gamma \in \{1, \dots, \psi_{\beta+1} = n - d_e\}$ in a manner analogous to how the $f_{\beta+1,\gamma}$ were selected above gives another R-basis $\{g_{1,1}, g_{1,2}, \dots, g_{1,\psi_1}, g_{2,1}, g_{2,2}, \dots, g_{2,\psi_2}, \dots, g_{\beta+1,1}, \dots, g_{\beta+1,\psi_{\beta+1}}\}$ of V. Consider the automorphism Γ induced by sending $f_{k,\gamma}$ to $g_{k,\gamma}$ for each $k \in \{1, \dots, \beta + 1\}$ and $\gamma \in \{1, \dots, \psi_k\}$. Since $\Gamma(u_{k,\gamma}) = \pi^{\lambda_k - 1}\Gamma(f_{k,\gamma}) = \pi^{\lambda_k - 1} g_{k,\gamma} = w_{k,\gamma}$ for each $k \in \{1, \dots, \beta\}$ and $\gamma \in \{1, \dots, \psi_k\}$, this automorphism satisfies $\Gamma(U) = W$. $\square$

The next lemma gives the crucial step needed for the proof of the recursive formula in Theorem 3.4.

Lemma 4.4. *Suppose that W is an R-submodule of V that realizes the nonzero ascending e-tuple $d = (d_1, \dots, d_e)$. Define $\beta, \lambda_0, \lambda_1, \dots, \lambda_\beta$, $\psi_1, \dots, \psi_\beta, \psi_{\beta+1}, d_{\lambda_0}, d_{\lambda_{\beta+1}}$ with respect to d as in Definition* 2.3. *For $i \in \{1, \dots, \beta\}$, let $\delta(i)$ denote the ascending e-tuple given by decreasing the λ_ith entry of d by* 1. *Then, the number of R-submodules of W which realize the ascending e-tuple $\delta(i)$ is given by $q^{d_{\lambda_{i-1}}}\binom{\psi_i}{1}_q$.*

Proof. Assume the premise of the assertion. Fix $i \in \{1, \dots, \beta\}$, and define $\beta', \lambda'_0, \lambda'_1, \dots, \lambda'_{\beta'}, \psi'_1, \dots, \psi'_{\beta'}, \psi'_{\beta'+1}, d'_{\lambda'_0}, d'_{\lambda'_{\beta'+1}}$ with respect to the ascending e-tuple $\delta(i)$.

To find the cardinality of the set of all R-submodules of W which realize $\delta(i)$, we begin by finding the cardinality of the set of all tuples

$$w = (w_{1,1}, w_{1,2}, \dots, w_{1,\psi'_1}, w_{2,1}, \dots, w_{2,\psi'_2}, \dots, w_{\beta',1}, \dots, w_{\beta',\psi'_{\beta'}})$$

that realize $\delta(i)$ and have coordinates in W. For this purpose, fix $k \in \{1, \dots, \beta'\}$ and $\gamma \in \{1, \dots, \psi'_k\}$. Also, assume that any coordinates preceding $w_{k,\gamma}$ in the tuple w have been chosen "properly" according to Definition 2.3.

In other words, assume that

$$w_{k',\gamma'} \in (W \cap M^{\lambda'_{k'}-1}V) \Big\backslash \left(\left(\sum_{\alpha=1}^{k'-1} \sum_{j=1}^{\psi'_\alpha} R\pi^{\lambda'_{k'}-\lambda'_\alpha} w_{\alpha,j} \right) + \left(\sum_{j=1}^{\gamma'-1} R w_{k',j} \right) + M^{\lambda'_{k'}} V \right)$$

for any $k' \in \{1, \dots, k-1\}$ and $\gamma' \in \{1, \dots, \psi'_{k'}\}$, and that

$$w_{k,\gamma'} \in (W \cap M^{\lambda'_k-1}V) \Big\backslash \left(\left(\sum_{\alpha=1}^{k-1} \sum_{j=1}^{\psi'_\alpha} R\pi^{\lambda'_k-\lambda'_\alpha} w_{\alpha,j} \right) + \left(\sum_{j=1}^{\gamma'-1} R w_{k,j} \right) + M^{\lambda'_k} V \right)$$

for any $\gamma' \in \{1, \dots, \gamma-1\}$. As a consequence, observe that

$$\left(\sum_{\alpha=1}^{k-1} \sum_{j=1}^{\psi'_\alpha} R\pi^{\lambda'_k-\lambda'_\alpha} w_{\alpha,j} \right) + \left(\sum_{j=1}^{\gamma-1} R w_{k,j} \right) + (W \cap M^{\lambda'_k} V) \subseteq W \cap M^{\lambda'_k-1} V.$$

Therefore, heeding Definition 2.3, the number of ways to choose $w_{k,\gamma}$ from W is

$$\begin{aligned}
&\Bigg|(W \cap M^{\lambda'_k - 1}V)\Bigg\backslash\Bigg(\Bigg(\sum_{\alpha=1}^{k-1}\sum_{j=1}^{\psi'_\alpha} R\pi^{\lambda'_k-\lambda'_\alpha} w_{\alpha,j}\Bigg) \\
&\qquad + \Bigg(\sum_{j=1}^{\gamma-1} R w_{k,j}\Bigg) + M^{\lambda'_k}V\Bigg)\Bigg| \\
&= |W \cap M^{\lambda'_k-1}V| - \Bigg|\Bigg(\sum_{\alpha=1}^{k-1}\sum_{j=1}^{\psi'_\alpha} R\pi^{\lambda'_k-\lambda'_\alpha} w_{\alpha,j}\Bigg) \\
&\qquad + \Bigg(\sum_{j=1}^{\gamma-1} R w_{k,j}\Bigg) + (W \cap M^{\lambda'_k}V)\Bigg| \\
&= |W \cap M^{\lambda'_k-1}V| \\
&\quad - \left|\frac{(\sum_{\alpha=1}^{k-1}\sum_{j=1}^{\psi'_\alpha} R\pi^{\lambda'_k-\lambda'_\alpha} w_{\alpha,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda'_k}V)}{W \cap M^{\lambda'_k}V}\right| \\
&\quad \times |W \cap M^{\lambda'_k}V|.
\end{aligned}$$

As

$$\frac{(\sum_{\alpha=1}^{k-1}\sum_{j=1}^{\psi'_\alpha} R\pi^{\lambda'_k-\lambda'_\alpha} w_{\alpha,j}) + (\sum_{j=1}^{\gamma-1} R w_{k,j}) + (W \cap M^{\lambda'_k}V)}{W \cap M^{\lambda'_k}V}$$

is an R/M-vector space on a basis of $\psi'_1 + \cdots + \psi'_{k-1} + \gamma - 1$ elements, it has cardinality $q^{\psi'_1+\cdots+\psi'_{k-1}+\gamma-1} = q^{d'_{\lambda'_{k-1}}+\gamma-1}$. By [**3**, Corollary 2.25 (2)], we have that $|W \cap M^{\lambda'_k-1}V| = q^{\sum_{\alpha=\lambda'_k}^{e} d_\alpha}$ and $|W \cap M^{\lambda'_k}V| = q^{\sum_{\alpha=\lambda'_k+1}^{e} d_\alpha}$. Therefore,

$$\begin{aligned}
&\Bigg|(W \cap M^{\lambda'_k - 1}V)\Bigg\backslash\Bigg(\Bigg(\sum_{\alpha=1}^{k-1}\sum_{j=1}^{\psi'_\alpha} R\pi^{\lambda'_k-\lambda'_\alpha} w_{\alpha,j}\Bigg) \\
&\qquad + \Bigg(\sum_{j=1}^{\gamma'-1} R w_{k,j}\Bigg) + M^{\lambda'_k}V\Bigg)\Bigg| \\
&\qquad = \Big(q^{\sum_{\alpha=\lambda'_k+1}^{e} d_\alpha}\Big)\Big(q^{d_{\lambda'_k}} - q^{d'_{\lambda'_{k-1}}+\gamma-1}\Big).
\end{aligned}$$

Therefore, the number of tuples which realize $\delta(i)$ and have coordinates in W is given by

$$\prod_{k=1}^{\beta'}\prod_{\gamma=1}^{\psi'_k}\left[\left(q^{\sum_{\alpha=\lambda'_k+1}^{e} d_\alpha}\right)\left(q^{d_{\lambda'_k}}-q^{d'_{\lambda'_{k-1}}+\gamma-1}\right)\right]$$
$$=\left(q^{\sum_{k=1}^{\beta'}\psi'_k(\sum_{\alpha=\lambda'_k+1}^{e} d_\alpha)}\right)\prod_{k=1}^{\beta'}\prod_{\gamma=1}^{\psi'_k}\left[\left(q^{d_{\lambda'_k}}-q^{d'_{\lambda'_{k-1}}+\gamma-1}\right)\right].$$

By [**3**, Proposition 2.27 (b)] and the last part of its proof, any R-submodule of W which realizes $\delta(i)$ is R-generated by the coordinates of exactly

$$q^{\sum_{k=1}^{\beta'}\psi'_k(\sum_{\alpha=\lambda'_k+1}^{e} d'_\alpha)}\prod_{k=1}^{\beta'}\prod_{\gamma=1}^{\psi'_k}\left[\left(q^{d'_{\lambda'_k}}-q^{d'_{\lambda'_{k-1}}+\gamma-1}\right)\right]$$

tuples which realize the ascending e-tuple $\delta(i)$. So, the number of R-submodules in W which realize $\delta(i)$ is given by

$$q^{\varepsilon}\left[\prod_{k=1}^{\beta'}\frac{\prod_{\gamma=1}^{\psi'_k}\left(q^{d_{\lambda'_k}}-q^{d'_{\lambda'_{k-1}}+\gamma-1}\right)}{\prod_{\gamma=1}^{\psi'_k}\left(q^{d'_{\lambda'_k}}-q^{d'_{\lambda'_{k-1}}+\gamma-1}\right)}\right]$$

where $\varepsilon=\sum_{k=1}^{\beta'}\psi'_k(\sum_{\alpha=\lambda'_k+1}^{e}(d_\alpha-d'_\alpha))$. Since d_α and d'_α differ (by one) only if $\alpha=i$ and $\lambda'_{i-1}=\lambda_{i-1}$, we have that $\varepsilon=\sum_{k=1}^{i-1}\psi'_k=d_{\lambda'_{i-1}}=d_{\lambda_{i-1}}$. Also, observe that $d_{\lambda'_k}=d'_{\lambda'_k}$ for all $k\neq i$. Therefore,

$$\left[\prod_{k=1}^{\beta'}\frac{\prod_{\gamma=1}^{\psi'_k}\left(q^{d_{\lambda'_k}}-q^{d'_{\lambda'_{k-1}}+\gamma-1}\right)}{\prod_{\gamma=1}^{\psi'_k}\left(q^{d'_{\lambda'_k}}-q^{d'_{\lambda'_{k-1}}+\gamma-1}\right)}\right]=\frac{\prod_{\gamma=1}^{\psi'_i}\left(q^{d_{\lambda'_i}}-q^{d'_{\lambda'_{i-1}}+\gamma-1}\right)}{\prod_{\gamma=1}^{\psi'_i}\left(q^{d'_{\lambda'_i}}-q^{d'_{\lambda'_{i-1}}+\gamma-1}\right)}$$
$$=\frac{\prod_{\gamma=1}^{\psi'_i}\left(q^{d_{\lambda'_i}-d'_{\lambda'_{i-1}}-(\gamma-1)}-1\right)}{\prod_{\gamma=1}^{\psi'_i}\left(q^{d'_{\lambda'_i}-d'_{\lambda'_{i-1}}-(\gamma-1)}-1\right)}$$

$$= \frac{\prod_{\gamma=1}^{\psi_i'}\left(q^{(d_{\lambda_i'}-d'_{\lambda_i'})+\psi_i'-(\gamma-1)}-1\right)}{\prod_{\gamma=1}^{\psi_i'}\left(q^{\psi_i'-(\gamma-1)}-1\right)}$$

$$= \binom{d_{\lambda_i'}-d'_{\lambda_i'}+\psi_i'}{\psi_i'}_q.$$

In the case where $\beta' = \beta - 1$, it happens that $d_{\lambda_i'} = d'_{\lambda_i'} (= d_{\lambda_{i+1}})$ and $\psi_i = 1$, so $\binom{d_{\lambda_i'}-d'_{\lambda_i'}+\psi_i'}{\psi_i'}_q = 1 = \binom{\psi_i}{1}_q$. Otherwise, β' is β or $\beta+1$. In either of these cases, $\lambda_i' = \lambda_i$, $d'_{\lambda_i'} = d_{\lambda_i'} - 1$, and $\psi_i' = d'_{\lambda_i'} - d'_{\lambda_{i-1}'} = (d_{\lambda_i} - 1) - d_{\lambda_{i-1}} = \psi_i - 1$, giving $\binom{d_{\lambda_i'}-d'_{\lambda_i'}+\psi_i'}{\psi_i'}_q = \binom{1+\psi_i-1}{\psi_i-1}_q = \binom{\psi_i}{1}_q$. So, the assertion follows. □

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