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ON THE PRIME IDEAL STRUCTURE OF SYMBOLIC REES ALGEBRAS

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ABSTRACT. This paper contributes to the study of the prime spectrum and dimension theory of symbolic Rees algebra over Noetherian domains. We first establish some general results on the prime ideal structure of subalgebras of affine domains, which actually arise, in the Noetherian context, as domains between a domain A and $A[a^{-1}]$. We then examine closely the special context of symbolic Rees algebras (which yielded the first counterexample to the Zariski-Hilbert problem). One of the results states that if A is a Noetherian domain and p a maximal ideal of A, then the Rees algebra of p inherits the Noetherian-like behavior of being a stably strong S-domain. We also investigate graded rings associated with symbolic Rees algebras of prime ideals p such that A_p is a rank-one DVR and close with an application related to Hochster's result on the coincidence of the ordinary and symbolic powers of a prime ideal.

1. Introduction. All rings considered in this paper are integral domains and all ring homomorphisms are unital. Examples of finite dimensional non-Noetherian Krull (or factorial) domains are scarce in the literature. One of these stems from the generalized fourteenth problem of Hilbert (also called Zariski-Hilbert problem). Let k be a field of characteristic zero, and let T be a normal affine domain over k. Let F be a subfield of the field of fractions of T. Set $R := F \cap T$. The Hilbert-Zariski problem asks whether R is an affine domain over k. Counterexamples on this problem were constructed by Rees [**27**], Nagata [**24**] and Roberts [**28, 29**]. In 1958, Rees constructed the first counter-example giving rise to (what is now called) Rees algebras. In

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1970, based on Rees's work, Eakin and Heinzer constructed in [**11**] a first example of a three dimensional non-Noetherian Krull domain which arose as a symbolic Rees algebra. In 1973, Hochster studied in [**17**] criteria for the ordinary and symbolic powers of a prime ideal to coincide (i.e., the Rees and symbolic Rees algebras are equal) within Noetherian contexts. Since then, these special graded algebras has been capturing the interest of many commutative algebraists and geometers.

In this line, Anderson, Dobbs, Eakin, and Heinzer [**2**] asked whether R and its localizations inherit from T the Noetherian-like main behavior of having Krull and valuative dimensions coincide (i.e., R is a Jaffard domain). This can be viewed in the larger context of Bouvier's conjecture about whether finite dimensional non-Noetherian Krull domains are Jaffard [**7, 14**]. In [**5**], we showed that while most examples existing in the literature are (locally) Jaffard, the question about those arising as symbolic Rees algebras is still open. This lies behind our motivation to contribute to the study of the prime ideal structure of this construction. We examine contexts where it inherits the (locally) Jaffard property and hence compute its Krull and valuative dimensions.

A finite-dimensional domain R is said to be Jaffard if $\dim(R[X_1,\dots, X_n]) = n+\dim(R)$ for all $n \geq 1$ or, equivalently, if $\dim(R) = \dim_v(R)$, where $\dim(R)$ denotes the (Krull) dimension of R and $\dim_v(R)$ its valuative dimension (i.e., the supremum of dimensions of the valuation overrings of R). As this notion does not carry over to localizations, R is said to be locally Jaffard if R_p is a Jaffard domain for each prime ideal p of R (equivalently, $S^{-1}R$ is a Jaffard domain for each multiplicative subset S of R).

In order to study Noetherian domains and Prüfer domains in a unified manner, Kaplansky [**21**] introduced the notions of S-domain and strong S-ring. A domain R is called an S-domain if, for each height-one prime ideal p of R, the extension $p[X]$ to the polynomial ring in one variable also has height 1. A ring R is said to be a strong S-ring if R/p is an S-domain for each $p \in \operatorname{Spec}(R)$. While $R[X]$ is always an S-domain for any domain R [**13**], $R[X]$ need not be a strong S-ring even when R is a strong S-ring. Thus, R is said to be a stably (or universally) strong S-ring if the polynomial ring $R[X_1,\dots,X_n]$ is a strong S-ring for each positive integer n [**19, 20, 22**]. A stably strong S-domain is locally Jaffard [**1, 19**]. An example of a strong S-domain which is not a stably strong S-domain was constructed in [**8**].

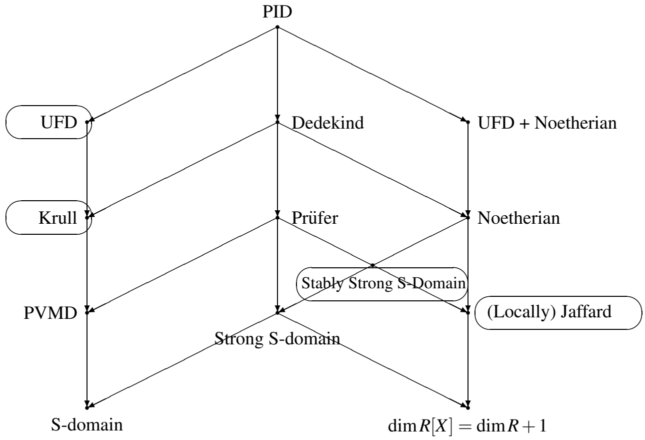

FIGURE 1. Diagram of Implications.

We assume familiarity with these concepts, as in [**1, 3, 6, 7, 10, 18–20, 22**].

In Figure 1, a diagram of implications indicates how the classes of Noetherian domains, Prüfer domains, UFDs, Krull domains and PVMDs [**16**] interact with the notion of Jaffard domain as well as with the S-properties.

Let A be a domain, $a \neq 0 \in A$, and $(I_n)_{n\geq 0}$ an a-filtration of A. Section 2 of this paper provides some general results on the prime ideal structure of the graded ring $R := \sum_{n\geq 0} a^{-n} I_n$. In particular, we prove that $p \in \operatorname{Spec}(A)$ with $a \in p$ is the contraction of a prime ideal of R if and only if $a^{-n}pI_n \cap A = p$ for each n. Moreover, if any one condition holds, then $p \supseteq I_1 \supseteq I_2, \cdots$ (Corollary 2.4). Section 3 closely examines the special construction of symbolic Rees algebra. One of the main results (Theorem 3.2) reveals the fact that three or more prime ideals of the symbolic Rees algebra may contract on the same prime ideal in the base ring. Also, we show that if A is a Noetherian domain and p a maximal ideal of A, then the Rees algebra of p is a stably strong S-domain, hence locally Jaffard (Theorem 3.5). Section 4 investigates the dimension of graded rings associated with symbolic Rees algebras of prime ideals p such that A_p is a rank-one DVR and closes with an application related to Hochster's study of criteria that force the coincidence of the ordinary and symbolic powers of a prime ideal.

2. The general context. Recall that an affine domain over a ring A is a finitely generated A-algebra that is a domain [**25**, page 127]. In light of the developments described in [**5**, Section 3], in order to investigate the prime ideal structure of subalgebras of affine domains over a Noetherian domain, we are reduced to those domains R between a Noetherian domain A and its localization $A[a^{-1}]$ for a nonzero element a of A. For this purpose, we use the language of filtrations.

From [**4, 23**], a filtration of a ring A is a descending chain $(I_n)_n$ of ideals of A such that $A = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$ and $I_n I_m \subseteq I_{n+m}$ for all n, m. The associated graded ring of A with respect to the filtration $(I_n)_n$ is given by $\mathrm{gr}\,(A) := \oplus_n I_n/I_{n+1}$. The filtration $(I_n)_n$ is said to be an I-filtration, for a given ideal I of A, if $II_n \subseteq I_{n+1}$ for each integer $n \geq 0$.

Let A be a domain and a a nonzero element of A. Let $(I_n)_n$ be an a-filtration of A and $R := A + a^{-1}I_1 + a^{-2}I_2 + \cdots = \sum_{n\geq 0} a^{-n}I_n$. Clearly, R is a domain, which is an ascending union of the fractional ideals $(a^{-n}I_n)_n$, such that $A \subseteq R \subseteq A[a^{-1}]$. The converse is also true as shown below.

Lemma 2.1. *Let A be a domain and $a \neq 0 \in A$. Then R is a domain such that $A \subseteq R \subseteq A[a^{-1}]$ if and only if $R := A + a^{-1}I_1 + a^{-2}I_2 + \cdots = \sum_{n\geq 0} a^{-n}I_n$ for some a-filtration $(I_n)_{n\geq 0}$ of A.*

Proof. We only need to prove necessity. Let $I_n := \{x \in A \mid x/a^n \in R\}$ for each positive integer n. It is fairly easy to see that I_n is an ideal of A for each integer n. Now, let $x \in I_{n+1}$. Then $x/a^{n+1} \in R$, so that $a(x/a^{n+1}) = x/a^n \in R$. Thus, $x \in I_n$. Also, observe that $aI_n \subseteq I_{n+1}$ for each n. It follows that $(I_n)_n$ is an a-filtration of A, as desired. $\square$

Lemma 2.2. *Let A be a domain, $a \neq 0 \in A$, and $(I_n)_{n\geq 0}$ an a-filtration of A. Let $R := \sum_{n\geq 0} a^{-n}I_n$. Then the prime ideals of R which don't contain a are in one-to-one correspondence with the prime ideals of A which don't contain a.*

Proof. This follows from the fact that $S^{-1}A = S^{-1}R$, where S is the multiplicatively closed subset of A defined by $S := \{a^n \mid n \in \mathbf{N}\}$.

Moreover, if $P \in \mathrm{Spec}\,(R)$ with $a \notin P$ and $p := P \cap A$, then

$$P = S^{-1}p \cap R = p[a^{-1}] \cap R = \sum_{n \geq 0} a^{-n}(p \cap I_n). \qquad \square$$

The question which naturally arises is under what conditions a chain $q \subset p$ in $\mathrm{Spec}\,(A)$ with $a \in p \setminus q$ lifts to a chain in $\mathrm{Spec}\,(R)$. This is handled by the main result of this section.

Theorem 2.3. *Let A be a domain, $a \neq 0 \in A$, $(I_n)_{n \geq 0}$ an a-filtration of A and $R := \sum_{n \geq 0} a^{-n} I_n$. Let $q \subset p \in \mathrm{Spec}\,(A)$ such that $a \in p \setminus q$, and let $Q := q[a^{-1}] \cap R$. Then the following assertions are equivalent*:

(1) *A $P \in \mathrm{Spec}\,(R)$ exists such that $Q \subset P$ and $P \cap A = p$;*

(2) *$a^{-n}(pI_n + q \cap I_n) \cap A = p$, for each $n \geq 0$.*

Proof. Recall first that $Q := q[a^{-1}] \cap R = \sum_{n \geq 0} a^{-n}(q \cap I_n)$ is the unique prime ideal of R lying over q in A (Lemma 2.2).

(1) $\Rightarrow$ (2) Suppose that a $P \in \mathrm{Spec}\,(R)$ exists such that $Q \subset P$ and $P \cap A = p$. It is worth noting that $R_p = \sum_{n \geq 0} a^{-n} I_n A_p$ is associated with the a-filtration $(I_n A_p)_n$ of A_p and $Q_p = \sum_{n \geq 0} a^{-n}(qA_p \cap I_n A_p)$ is the unique prime ideal of R_p lying over qA_p in A_p. Also $pR + Q = \sum_{n \geq 0} a^{-n}(pI_n + q \cap I_n)$, and hence $pR_p + Q_p = \sum_{n \geq 0} a^{-n}(pI_n A_p + qA_p \cap I_n A_p) = \cup_{n \geq 0} a^{-n}(pI_n A_p + qA_p \cap I_n A_p)$, an ascending union of fractional ideals of R_p. Now $pR_p + Q_p$ is a proper ideal of R_p. Therefore, for each n, $a^n \notin pI_n A_p + qA_p \cap I_n A_p$. Hence, $sa^n \notin pI_n + q \cap I_n$ for every $s \in A \backslash p$, whence $(a^{-n}(pI_n + q \cap I_n) \cap A) \cap (A \setminus p) = \varnothing$. It follows that $a^{-n}(pI_n + q \cap I_n) \cap A = p$ for each $n \geq 0$.

(2) $\Rightarrow$ (1) Suppose $a^{-n}(pI_n + q \cap I_n) \cap A = p$ for each $n \geq 0$. Then $(pR + Q) \cap A = \cup_{n \geq 0}(a^{-n}(pI_n + q \cap I_n) \cap A) = p$. Therefore, [**4**, Proposition 3.16] applied to the ring homomorphism $A/q \hookrightarrow R/Q$ leads to the conclusion. $\square$

The special case where $q = 0$ yields a necessary and sufficient condition for a prime ideal of A containing a to lift to a prime ideal of R.

Corollary 2.4. *Let A be a domain, $a \neq 0 \in A$, $(I_n)_{n \geq 0}$ an a-filtration of A, and $R := \sum_{n \geq 0} a^{-n} I_n$. Let $p \in \mathrm{Spec}\,(A)$ such that $a \in p$. Then p is the contraction of a prime ideal of R if and only if*

$a^{-n}pI_n \cap A = p$ for each n. Moreover, if any one condition holds, then $p \supseteq I_1$.

Proof. Equivalence is ensured by the above theorem with $q = 0$. Moreover, $a \in p$ yields $I_1 \subseteq a^{-1}pI_1 \cap A = p$, as desired. $\square$

Now it is legitimate to ask whether a chain of prime ideals may exist of R of length ≥ 2 lying over a given prime ideal p of A containing a. Ahead, Corollary 3.3 gives an affirmative answer to this question.

3. The case of symbolic Rees algebras. Here we will focus on the special case of symbolic Rees algebras. In 1958, Rees constructed in [**27**] a first counter-example to the Zariski-Hilbert problem (initially posed at the Second International Congress of Mathematicians at Paris in 1900). His construction gave rise to (what is now called) Rees algebras. Since then, these special graded algebras have been capturing the interest of many mathematicians, particularly in the fields of commutative algebra and algebraic geometry.

Let A be a domain, t an indeterminate over A, and $p \in \operatorname{Spec}(A)$. For each $n \in \mathbf{Z}$, set $p^{(n)} := p^n A_p \cap A$, the nth symbolic power of p, with $p^{(n)} = A$ for each $n \leq 0$. Notice that $p = p^{(1)}$ and $p^n \subseteq p^{(n)}$ for all $n \geq 2$. We recall the following definitions:

- $\oplus_{n \in \mathbf{Z}} p^n t^n = A[t^{-1}, pt, \ldots, p^n t^n, \ldots]$ is the Rees algebra of p.
- $\oplus_{n \in \mathbf{Z}} p^{(n)} t^n = A[t^{-1}, p^{(1)}t, \ldots, p^{(n)}t^n, \ldots]$ is the symbolic Rees algebra of p.

In 1970, based on Rees's work, Eakin and Heinzer constructed in [**11**] the first example of a three dimensional non-Noetherian Krull domain. It arose as a symbolic Rees algebra. This enhances our interest for these constructions. In this section, we wish to push further the analysis of the prime ideal structure of symbolic Rees algebras. Precisely, we plan to investigate the lifting of prime ideals of $A[t^{-1}]$ in the symbolic Rees algebra R. We prove that any prime ideal of $A[t^{-1}]$ lifts to a prime ideal in R. We also examine the length of chains of prime ideals of R lying over a prime ideal of $A[t^{-1}]$.

Let us fix the notation for the rest of this section. Let A be a domain and t an indeterminate over A. Let $p \in \operatorname{Spec}(A)$, and let

$$R := A[t^{-1}, pt, p^{(2)}t^2, \ldots, p^{(n)}t^n, \ldots]$$

be the symbolic Rees algebra of p. Consider the t^{-1}-filtration $(I_n)_{n\geq 0}$ of $A[t^{-1}]$, where $I_0 = A[t^{-1}]$, $I_1 = p[t^{-1}] + t^{-1}A[t^{-1}]$, and for $n \geq 2$

$$I_n := p^{(n)}[t^{-1}] + t^{-1}p^{(n-1)}[t^{-1}] + \cdots + t^{-(n-1)}p[t^{-1}] + t^{-n}A[t^{-1}].$$

One can easily check that

$$A[t^{-1}] \subseteq R \subseteq A[t^{-1}, t] \quad \text{and} \quad R = \sum_{n\geq 0} I_n t^n = \bigcup_{n\geq 0} I_n t^n.$$

Finally, for $q \supseteq p$ in $\mathrm{Spec}\,(A)$, set

$$\mathrm{G}(A) := \bigoplus_{n\geq 0} \frac{p^{(n)}}{p^{(n+1)}} \quad \text{and} \quad \mathrm{G}(A_q) := \bigoplus_{n\geq 0} \frac{pA_q^{(n)}}{pA_q^{(n+1)}}.$$

The first result examines the transfer of the Jaffard property.

Proposition 3.1. *Assume A to be a Jaffard domain. Then R is a Jaffard domain with* $\dim(R) = 1 + \dim(A)$.

Proof. Notice that $A[t^{-1}] \subseteq R \subseteq A[t^{-1}, t]$. By [**1**, Lemma 1.15], $\dim_v(R) = \dim_v(A[t^{-1}])$. On the other hand, the equality $R[t] = A[t^{-1}, t]$ combined with [**1**, Proposition 1.14] yields $\dim(A[t^{-1}]) \leq \dim(R)$. Now A is Jaffard and then so is $A[t^{-1}]$ [**1**, Proposition 1.2]. Consequently, $\dim(R) = \dim_v(R) = 1 + \dim(A)$, as desired. □

Next, we investigate the prime ideals of $A[t^{-1}]$ that lift in the symbolic Rees algebra R. In view of Lemma 2.2, one has to narrow the focus to the prime ideals which contain t^{-1}. Moreover, by Corollary 2.4, these primes must necessarily contain $I_1 = (p, t^{-1})$. Consequently, we reduce the study to the prime ideals of $A[t^{-1}]$ of the form (q, t^{-1}) where $q \supseteq p \in \mathrm{Spec}\,(A)$.

Theorem 3.2. *Let q be a prime ideal of A containing p. Then the following lattice isomorphisms hold*:

(1)

$$\{Q \in \mathrm{Spec}\,(R) \mid Q \cap A[t^{-1}] = (q, t^{-1})\} \simeq \mathrm{Spec}\left(\frac{\mathrm{G}(A_q)}{(qA_q/pA_q)\mathrm{G}(A_q)}\right).$$

(2)
$$\{P \in \mathrm{Spec}\,(R) \mid P \cap A[t^{-1}] = (p, t^{-1})\} \simeq \mathrm{Spec}\,(\mathrm{G}(A_p)).$$

Proof. (1) Let
$$\mathrm{G}(q) := \frac{A}{q} \oplus \frac{p}{p^{(2)} + qp} \oplus \frac{p^{(2)}}{p^{(3)} + qp^{(2)}} \oplus \cdots .$$
We claim that
$$\frac{R}{(q, t^{-1})R} \cong \mathrm{G}(q) \cong \frac{\mathrm{G}(A)}{(q/p)\mathrm{G}(A)}.$$
Indeed, notice the following:
$$\begin{cases} R = A[t^{-1}] \oplus pt \oplus p^{(2)}t^2 \oplus \cdots \oplus p^{(n)}t^n \oplus \cdots \\ t^{-1}R = (p, t^{-1}) \oplus p^{(2)}t \oplus \cdots \oplus p^{(n+1)}t^n \oplus \cdots \\ qR = q[t^{-1}] \oplus qpt \oplus qp^{(2)}t^2 \oplus \cdots \oplus qp^{(n)}t^n \oplus \cdots \\ (q, t^{-1})R = (q, t^{-1}) \oplus (p^{(2)} + qp)t \oplus \cdots \oplus (p^{(n+1)} + qp^{(n)})t^n \oplus \cdots . \end{cases}$$
Then it is easily seen that
$$\frac{R}{t^{-1}R} \cong \mathrm{G}(A)$$
and
$$\frac{R}{(q, t^{-1})R} \cong \mathrm{G}(q).$$
Moreover,
$$\mathrm{G}(q) \cong \frac{A/p}{q/p} \oplus \frac{p/p^{(2)}}{(q/p)(p/p^{(2)})} \oplus \cdots \oplus \frac{p^{(n)}/p^{(n+1)}}{(q/p)(p^{(n)}/p^{(n+1)})} \oplus \cdots$$
$$\cong \frac{\mathrm{G}(A)}{(q/p)\mathrm{G}(A)}.$$

Now, observe that $R_q = A_q[t^{-1}, pA_q^{(1)}t, \ldots, pA_q^{(n)}t^n, \ldots]$ is the symbolic Rees algebra of pA_q. This is due to the fact that $p^{(n)}A_q = p^nA_p \cap A_q = pA_q^{(n)}$ for each $n \geq 0$. We obtain
$$\frac{R_q}{(q, t^{-1})R_q} = \frac{R_q}{(qA_q, t^{-1})R_q} \cong \frac{\mathrm{G}(A_q)}{(qA_q/pA_q)\mathrm{G}(A_q)}.$$

Hence, the set of prime ideals of R lying over (q, t^{-1}) in $A[t^{-1}]$ is lattice isomorphic to the spectrum of $\mathrm{G}(A_q)/[(qA_q/pA_q)\mathrm{G}(A_q)]$.

(2) Take $q := p$ in (1), completing the proof of the theorem. □

We deduce the following result in the Noetherian case. It shows, in particular, that a chain of prime ideals of R may exist of length ≥ 2 lying over (p, t^{-1}) in $A[t^{-1}]$.

Corollary 3.3. *Assume that A is Noetherian, and let $n := \mathrm{ht}\,(p)$. Then the set $\{P \in \mathrm{Spec}\,(R) \mid P \cap A[t^{-1}] = (p, t^{-1})\}$ is lattice isomorphic to the spectrum of an n-dimensional finitely generated algebra over the field A_p/pA_p.*

Proof. Let $y_1, \dots, y_r \in A$ be such that $pA_p = (y_1, \dots, y_r)A_p$, and let $e_1, \dots, e_r$ denote their respective images in pA_p/p^2A_p. By Theorem 3.2, $R_p/[(p, t^{-1})R_p] \cong \mathrm{G}(A_p)$. Note that R_p coincides with the Rees algebra of pA_p (since $p^nA_p = p^{(n)}A_p$ for all $n \geq 1$). It follows that

$$\mathrm{G}(A_p) = \mathrm{gr}\,(A_p) = \frac{A_p}{pA_p}[e_1, \dots, e_r]$$

(cf. [**23**, page 93]). On the other hand, by [**23**, Theorem 15.7], $\dim\,(\mathrm{gr}\,(A_p)) = \dim\,(A_p) = n$, completing the proof. □

Notice at this point that $t^{-1}R = (p, t^{-1})R$ (see the proof of Theorem 3.2). This translates into the fact that prime ideals of R containing t^{-1} contain necessarily $p[t^{-1}]$ (stated in Corollary 2.4). Given a prime ideal q of A, we next exhibit particular prime ideals of R that lie over q.

Proposition 3.4. *Let $q \in \mathrm{Spec}\,(A)$. The following hold*:

(1) *Assume $p \subseteq q$. Then $Q := (q, t^{-1}) \oplus pt \oplus p^{(2)}t^2 \oplus \cdots$ is a prime ideal of R lying over (q, t^{-1}) in $A[t^{-1}]$ and Q is maximal with this property.*

(2) *$q[t^{-1}, t] \cap R = q[t^{-1}] \oplus (p \cap q)t \oplus (p^{(2)} \cap q)t^2 \oplus \cdots$ is the unique prime ideal of R lying over $q[t^{-1}]$ in $A[t^{-1}]$.*

Proof. (1) Assume $p \subseteq q$. It is easily seen that $R/Q \cong A/q$. It follows that Q is a prime ideal of R and $Q \cap A[t^{-1}] = (q, t^{-1})$. Now R_q is the symbolic Rees algebra of pA_q with $(R_q/QR_q) \cong (A_q/qA_q)$, a field. Therefore, Q is maximal among the prime ideals of R lying over (q, t^{-1}).

(2) By Lemma 2.2, the unique prime ideal of R lying over $q[t^{-1}]$ is $q[t^{-1},t]\cap R$. Further, observe that $q[t^{-1},t]=q[t^{-1}]\oplus qt\oplus qt^2\oplus\cdots$. So that $q[t^{-1},t]\cap R=q[t^{-1}]\oplus(p\cap q)t\oplus(p^{(2)}\cap q)t^2\oplus\cdots$, as claimed. ◻

Theorem 3.5. *Assume that A is Noetherian and $p\in\operatorname{Max}(A)$. Then R is a stably strong* S*-domain* (*hence locally Jaffard*).

Proof. Let $T:=A[t^{-1},pt,p^2t^2,\dots,p^nt^n,\dots]$ be the Rees algebra of p. Let n be a positive integer. Consider the natural injective ring homomorphism:

$$T[X_1,\dots,X_n]\hookrightarrow R[X_1,\dots,X_n].$$

This induces the following map

$$f:\operatorname{Spec}(R[X_1,\dots,X_n])\longrightarrow\operatorname{Spec}(T[X_1,\dots,X_n])$$

defined by $f(P)=P\cap T[X_1,\dots,X_n]$. We claim that f is an order-preserving bijection. Indeed, let Q be a prime ideal of $T[X_1,\dots,X_n]$. If $t^{-1}\notin Q$, then Q survives in

$$A[t^{-1},t,X_1,\dots,X_n]=R[t,X_1,\dots,X_n]=T[t,X_1,\dots,X_n].$$

Therefore, $P:=QA[t^{-1},t,X_1,\dots,X_n]\cap R[X_1,\dots,X_n]$. Hence, P is the unique prime ideal of $R[X_1,\dots,X_n]$ such that $f(P)=Q$. Now, let $t^{-1}\in Q$. Then $(p,t^{-1})\subseteq Q\cap A[t^{-1}]$ by Corollary 2.4, whence $p=Q\cap A$ as p is maximal in A. Moreover, recall that $R_p=T_p$. Therefore, Q survives in $T_p[X_1,\dots,X_n]=R_p[X_1,\dots,X_n]$, and hence $P:=QR_p[X_1,\dots,X_n]\cap R[X_1,\dots,X_n]$ is the unique prime ideal of $R[X_1,\dots,X_n]$ such that $f(P)=Q$. It follows that f is bijective. Obviously, it also preserves the inclusion order. Now assume that $p=(a_1,\dots,a_r)$. One can easily check that $T=A[t^{-1},a_1t,\dots,a_rt]$, so that T is Noetherian and thus a stably strong S-domain. It follows that $T[X_1,\dots,X_n]$ is a strong S-domain and so is $R[X_1,\dots,X_n]$, as desired. ◻

Remark 3.6. It is worth noting that the proof of the above theorem is still valid if we weaken the assumption “A is Noetherian” to “A is a stably strong S-domain and p is finitely generated" since the concept of strong S-domain is stable under quotient ring.

4. Associated graded rings and applications. This section investigates the dimension theory of graded rings associated with special symbolic Rees algebras. Recall that the Krull dimension of the graded ring associated with the (ordinary) Rees algebra of an ideal I of a Noetherian domain A is given by the formula (cf. [**12**, Exercise 13.8]):

$$\dim(\mathrm{gr}_I(A)) = \max\{\mathrm{ht}\,(q) \mid q \in \mathrm{Spec}\,(A) \quad \text{and} \quad I \subseteq q\}.$$

Let us fix the notation for this section. Throughout, A will denote a Noetherian domain and p a prime ideal of A such that A_p is a rank-one DVR. Thus, any height-one prime ideal of an integrally closed Noetherian domain falls within the scope of this study. Let R, $\mathrm{G}(A)$, and $\mathrm{gr}\,(A)$ denote the symbolic Rees algebra of p, the associated graded ring of A with respect to the filtration $(p^{(n)})_n$, and the associated graded ring of p, respectively. That is,

$$\begin{aligned} R &:= A[t^{-1}, pt, p^{(2)}t^2, \dots, p^{(n)}t^n, \dots], \\ \mathrm{G}(A) &:= \bigoplus_{n\geq 0} \frac{p^{(n)}}{p^{(n+1)}}, \\ \mathrm{gr}\,(A) &:= \bigoplus_{n\geq 0} \frac{p^n}{p^{n+1}}. \end{aligned}$$

Finally, let $u \in p$ such that $pA_p = uA_p$ and $v := \overline{u}$ is the image of u in $p/p^{(2)}$.

Lemma 4.1. *For each $n \geq 0$, let $E_n := (A :_{A_p} u^n) = \{x \in A_p \mid xu^n \in A\}$ and $F_n := \overline{E_n}$ be the image of E_n in $K := A_p/pA_p$. Then*:

(1) *$(E_n)_{n\geq 0}$ is an ascending sequence of fractional ideals of A such that $A \subseteq E_n \subseteq A_p$ and $p^{(n)} = E_n u^n$, for each n.*

(2) *$(F_n)_{n\geq 0}$ is an ascending sequence of fractional ideals of A/p such that $A/p \subseteq F_n \subseteq K$ and $p^{(n)}/p^{(n+1)} = F_n v^n$, for each n.*

(3) $\mathrm{G}(A) = \oplus_{n\geq 0} F_n v^n$.

Proof. Clearly, $(E_n)_n$ is an ascending sequence of fractional ideals of A. Fix $n \geq 0$. We have $x \in p^{(n)}$ if and only if $x \in u^n A_p$ and $x \in A$ if and only if a $y \in A_p$ exists such that $x = yu^n \in A$ if and only if $x \in E_n u^n$. This proves (1). Assertion (2) is a consequence of (1), and the proof is left to the reader. Also, (3) is trivial from (2). □

Next, we announce the main result of this section.

Theorem 4.2. *Let $D := \cup_{n\geq 0} F_n$, and let X be an indeterminate over D. Then*:

(1) $\mathrm{G}(A)$ *is a Jaffard domain and* $\dim(\mathrm{G}(A)) = 1 + \dim(A/p)$.

(2) $\dim(\mathrm{G}(A)/v\mathrm{G}(A)) = \dim(A/p)$ *and* $\dim(\mathrm{G}(A)[v^{-1}]) = \dim(D[X])$.

Proof. We first prove the following claims.

Claim 1. *D is an overring of A/p, and v is transcendental over D.*

It is fairly easy to see that $F_n F_m \subseteq F_{n+m}$ for any n and m. It follows that D is an overring of A/p contained in K. Let $P = b_0 + b_1 X + \cdots + b_n X^n \in (A/p)[X]$ be such that $P(v) = 0 = b_0 + b_1 v + \cdots + b_n v^n$. Let $i \in \{0, 1, \ldots, n\}$. Since $b_i v^i \in F_i v^i$, $b_i v^i = 0$ by Lemma 4.1. So $b_i = \overline{a_i} \pmod{pA_p}$, for some $a_i \in E_i$, and $a_i u^i \in p^i A_p \cap A = p^{(i)}$. Therefore, $\overline{a_i u^i} = 0$ in $p^{(i)}/p^{(i+1)}$, that is, $a_i u^i \in p^{(i+1)}$. Hence, $a_i \in pAp$, whence $b_i = 0$. Consequently, $P = 0$, proving that v is transcendental over D.

Claim 2. $A/p[v] \subseteq \mathrm{G}(A) \subseteq D[v]$.

This follows from the facts that $A/p \subseteq F_n \subseteq D$ for each $n \geq 0$ and $\mathrm{G}(A) = \oplus_{n\geq 0} F_n v^n$ by Lemma 4.1.

Claim 3. $S^{-1}\mathrm{G}(A) = D[v, v^{-1}]$, *where* $S := \{v^n \mid n \geq 0\}$.

Clearly, $S^{-1}\mathrm{G}(A) \subseteq D[v, v^{-1}]$. Also note that $D \subseteq S^{-1}\mathrm{G}(A)$ since $F_n = (F_n v^n) v^{-n} \subseteq S^{-1}\mathrm{G}(A)$ for each positive integer n. Hence, $D[v, v^{-1}] \subseteq S^{-1}\mathrm{G}(A) \subseteq D[v, v^{-1}]$ establishing the desired equality.

(1) In view of Claim 2, we get $\dim_v(\mathrm{G}(A)) \leq 1 + \dim_v(A/p) = 1 + \dim(A/p)$. On the other hand, notice that, for each prime ideal q of A containing p, the ideal $Q := (q/p) \oplus p/p^{(2)} \oplus p^{(2)}/p^{(3)} \oplus \cdots \in \mathrm{Spec}(\mathrm{G}(A))$

with $\mathrm{G}(A)/Q \cong A/q$. So $\dim(\mathrm{G}(A)) \geq 1 + \dim(A/p)$ as $\mathrm{G}(A)$ is a domain. Thus $\dim(\mathrm{G}(A)) = \dim_v(\mathrm{G}(A)) = 1 + \dim(A/p)$.

(2) First notice that

$$\mathrm{ht}\,(v\mathrm{G}(A)) + \dim(\mathrm{G}(A)/v\mathrm{G}(A)) \leq \dim(\mathrm{G}(A)) = 1 + \dim(A/p).$$

Then $\dim(\mathrm{G}(A)/v\mathrm{G}(A)) \leq \dim(A/p)$. Consider the prime ideal of $\mathrm{G}(A)$ given by $P := F_1 v \oplus F_2 v^2 \oplus \cdots$ and $p \subset p_1 \subset p_2 \subset \cdots \subset p_h \in \mathrm{Spec}\,(A)$ with $h := \dim(A/p)$. We get the following chain of prime ideals of $\mathrm{G}(A)$ containing the ideal $v\mathrm{G}(A)$

$$v\mathrm{G}(A) \subset P \subset \frac{p_1}{p} \oplus P \subset \frac{p_2}{p} \oplus P \subset \cdots \subset \frac{p_h}{p} \oplus P.$$

It follows that $\dim(\mathrm{G}(A)/v\mathrm{G}(A)) = \dim(A/p)$. Moreover, Claims 1 and 3 yield

$$\dim(S^{-1}\mathrm{G}(A)) = \dim(D[v, v^{-1}]) = \dim(D[X]),$$

completing the proof of the theorem. □

Let $B := \cup_{n\geq 0} E_n = \cup_{n\geq 0} u^{-n} p^{(n)}$. Notice that B is an overring of A contained in A_p and $\overline{B} := B/(pA_p \cap B) = D$. The next result investigates some properties of B and its relation with D, in view of the fact that an essential part of the spectrum of $\mathrm{G}(A)$ (and hence that of R) is strongly linked to D.

Proposition 4.3. *Let* $B := \cup_{n\geq 0} u^{-n} p^{(n)}$. *Then*:

(1) $pB = uB$ *is a height-one prime ideal of* B *and it is the unique prime ideal of* B *lying over* p *in* A.

(2) $B/uB = D$.

(3) $B = A_p \cap A[u^{-1}]$. *Then, if* A *is a Krull domain, so is* B.

(4) B *is locally Jaffard if and only if so is* D.

Proof. (1) Let $z \in pA_p \cap B$. Then $z = a/s = x/u^n$ for some positive integer n, with $x \in p^{(n)}$, $a \in p$ and $s \in A \setminus p$. Then $sx = au^n \in p^{n+1}$ which means that $x \in p^{(n+1)}$. Hence, $z =$

$u(x/u^{n+1}) \in u(p^{(n+1)}/u^{n+1}) \subseteq uB$. Therefore, $pA_p \cap B = uB = pB$, whence $pB \in \mathrm{Spec}\,(B)$ with $pB \cap A = p$. Moreover, observe that $B_p := \cup_{n \geq 0} u^{-n} p^{(n)} A_p = \cup_{n \geq 0} u^{-n} p^n A_p = A_p$ is a rank-one DVR. Then $\mathrm{ht}\,(pB) = \mathrm{ht}\,(pB_p) = \mathrm{ht}\,(pA_p) = 1$ and pB is the unique prime ideal of B lying over p in A.

(2) It is straightforward from (1) and the fact that

$$\overline{B} := B/(pA_p \cap B) = D.$$

(3) It is clear that $B \subseteq A_p \cap A[u^{-1}]$. Let $z \in A_p \cap A[u^{-1}]$. Then $z = x/u^n = a/s$ for some positive integer n, and $x, a \in A$ and $s \in A \setminus p$. So $xs = au^n \in p^n$. Hence, $x \in p^{(n)}$ which means that $z \in p^{(n)}/u^n \subseteq B$. Then the desired equality holds.

(4) Applying (1), one can check that the following diagram is cartesian:

$$\begin{array}{ccc} B & \longrightarrow & D \\ \downarrow & & \downarrow \\ A_p & \longrightarrow & K, \end{array}$$

which allows the transfer of the locally Jaffard property between B and D (recall that A_p is a rank-one DVR). $\square$

In [**17**], Hochster investigated when the symbolic power $p^{(n)}$ of a given prime ideal p of a Noetherian domain A coincides with the ordinary power p^n for any $n \geq 0$. His main theorem gives sufficient conditions guaranteeing this equality. Applying this theorem, he proves that the Cohen-Macaulayness of A/p has nothing to do with the coincidence of the symbolic and ordinary powers, by providing a polynomial ring in four indeterminates A such that A/p is not Cohen-Macaulay while $p^{(n)} = p^n$ for any positive integer n. In this vein, we recall Northcott's example [**26**, Example 3, page 29] in which p is the defining ideal of a curve (so that its residue class ring is Cohen-Macaulay) while $p^{(2)} \neq p^2$. Other examples appeared in the literature of Noetherian domains A for which a prime ideal p of A exists such that $p^{(n)} \neq p^n$ for some positive integer n, especially in works dealing with the Noetherian property of symbolic Rees algebras.

From Theorem 4.2 we deduce a necessary condition for the symbolic and ordinary powers to coincide for a height-one prime ideal p of a

Noetherian domain A. This will allow us to provide new and original examples of Noetherian domains for which a prime ideal p exists such that $p^{(n)} \neq p^n$ for some positive integer n.

Corollary 4.4. *Let A be a local Noetherian domain and p a prime ideal of A such that A_p is a rank-one DVR. Then*:

$$p^{(n)} = p^n, \quad \text{for all } n \geq 0 \Longrightarrow \dim(A) = 1 + \dim(A/p).$$

Proof. Assume $p^{(n)} = p^n$ for all $n \geq 0$. Then $\mathrm{G}(A) = \mathrm{gr}(A)$. So a combination of [**23**, Theorem 15.7] and Theorem 4.2 leads to the conclusion. □

Corollary 4.5. *Let A be an integrally closed local Noetherian domain which is not catenarian. Then a prime ideal p of A exists such that $p^{(n)} \neq p^n$ for some positive integer n.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of A. Since A is not catenarian, a height-one prime ideal $p \subsetneqq \mathfrak{m}$ of A exists such that

$$1 + \dim(A/p) \lneqq \mathrm{ht}(\mathfrak{m}) = \dim(A).$$

By Corollary 4.4, an $n \geq 2$ exists such that $p^{(n)} \neq p^n$. □

Next, we exhibit an explicit example of a local Noetherian domain A containing a prime ideal p such that $p^{(n)} \neq p^n$ for some positive integer n. For this purpose, we'll use Nagata's well-known example of a Noetherian domain which is catenarian but not universally catenarian [**25**].

Example 4.6. Let k be a field and X, Y, Z, t indeterminates over k. Consider the k-algebra homomorphism $\varphi : k[X, Y] \to k[[t]]$ defined by $\varphi(X) = t$ and $\varphi(Y) = s := \sum_{n \geq 1} t^{n!}$. Since s is known to be transcendental over $k(t)$, φ is injective. This induces an embedding $\overline{\varphi} : k(X, Y) \to k((t))$ of fields. So $B_1 := \overline{\varphi}^{-1}(k[[t]])$ is a rank-one discrete valuation overring of $k[X, Y]$ of the form $B_1 = k + XB_1$. Let $B_2 := k[X, Y]_{(X-1,Y)}$ and $B := B_1 \cap B_2$. Then $\mathrm{Max}(B) = \{M, N\}$ with $M = XB_1 \cap B$ and $N = (X - 1, Y)B_2 \cap B$, and B is Noetherian.

Let $C := k + \mathfrak{m}$ with $\mathfrak{m} := M \cap N$. It turns out that C is a two-dimensional Noetherian domain such that the polynomial ring $C[Z]$ is not catenarian. So there is an upper Q to $\mathfrak{m}$ which contains an upper P to zero such that the chain $(0) \subsetneqq P \subsetneqq Q$ is saturated with $\mathrm{ht}\,(Q) = 3$. Now, let $A := C[Z]_Q$ and $p := PC[Z]_Q$. Then A is a local Noetherian domain and $A_p \cong C[Z]_P$ is a rank-one DVR with $1 + \dim (A/p) = 2 \lneqq \dim (A) = 3$. Consequently, by Corollary 4.4, an $n \geq 2$ exists such that $p^{(n)} \neq p^n$. □

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