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STABILITY OF QUASI-SOCLE IDEALS

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ABSTRACT. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $\dim A > 0$. Let $\mathrm{G}(\mathfrak{m}) = \oplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the associated graded ring of $\mathfrak{m}$. This paper explores quasi-socle ideals in A, i.e., ideals of the form $I = Q : \mathfrak{m}^q$ $(q \geq 1)$ where Q is a parameter ideal. Goto, Sakurai, and the author have shown that the methods developed by Wang also work in the non Cohen-Macaulay case with some modification. The purpose of this paper is to solve a problem that has remained open. We will show that, if A is a generalized Cohen-Macaulay ring with $\operatorname{depth}\mathrm{G}(\mathfrak{m}) \geq 2$, then for each integer $q \geq 1$ one can find an integer $t = t(q) \gg 0$, depending upon q, such that $I^2 = QI$ for every parameter ideal Q contained in $\mathfrak{m}^t$, where $I = Q : \mathfrak{m}^q$. Therefore, the associated graded ring $\mathrm{G}(I) = \oplus_{n\geq 0} I^n/I^{n+1}$ of I is a Buchsbaum ring whenever A is Buchsbaum.

1. Introduction. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. This paper studies quasi-socle ideals, i.e., ideals of the form $I = Q : \mathfrak{m}^q$ $(q \geq 1)$ where Q is a parameter ideal in A. We are interested in determining when $I^2 = QI$, in which case we call I stable. To state the results, we need to first fix some notation and terminology.

For each $\mathfrak{m}$-primary ideal I in A, we denote by $\{\mathrm{e}_I^i(A)\}_{0\leq i\leq d}$ the Hilbert coefficients of A with respect to I. The Hilbert function of I is then given by the formula

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}_I^d(A)$$

for all $n \gg 0$, where $\ell_A(M)$ denotes the length of the A-module M.

Let Q be a parameter ideal in A. We set $\mathbf{I}(Q) = \ell_A(A/Q) - e_Q^0(A)$. Then A is a Cohen-Macaulay ring if and only if $\mathbf{I}(Q) = 0$ for some (and

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hence every) parameter ideal Q. We say that A is a *Buchsbaum* ring if $\mathbf{I}(Q)$ is constant and independent of the choice of parameter ideals Q in A.

We say that A is a *generalized Cohen-Macaulay* ring if $\sup_Q \mathbf{I}(Q) < \infty$, where Q runs through parameter ideals in A. This definition is equivalent to saying that all the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(A)$ $(i \neq d)$ of A with respect to $\mathfrak{m}$ are finitely generated. When this is the case, one has the equality $\sup_Q \mathbf{I}(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$. A good reference for generalized Cohen-Macaulay rings is [**18**].

Let $Q = (a_1, a_2, \dots, a_d)$ be a parameter ideal in a generalized Cohen-Macaulay ring A. Then we say that Q is *standard* if $\mathbf{I}(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$. This condition is equivalent to saying that, for all integers $n_i > 0$, the sequence $a_1^{n_1}, a_2^{n_2}, \dots, a_d^{n_d}$ forms a d-sequence in any order ([**18**, Proposition 3.2]). It is known that, for a given generalized Cohen-Macaulay ring A, one can find an integer $\ell \gg 0$ such that every parameter ideal Q contained in $\mathfrak{m}^\ell$ is standard ([**18**, Section 3]).

For each ideal I in A, we set

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n, \qquad \mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1} \quad \text{and} \quad \mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$$

and call them, respectively, the Rees algebra, the associated graded ring, and the fiber cone of I.

With this notation and terminology our purpose is to prove the following.

Theorem 1.1. *Let A be a generalized Cohen-Macaulay ring, and suppose that* $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$. *Let $\ell \geq 1$ be an integer such that every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Then, for each integer $q \geq 1$, one can find an integer $t = t(q) \geq q + \ell + 1$ such that I is stable for every parameter ideal Q of A contained in $\mathfrak{m}^t$, where $I = Q : \mathfrak{m}^q$.*

Applying the results in [**12**, Section 5] and [**13**, Section 2] to our ideals $I = Q : \mathfrak{m}^q$, we readily get the following, which is the most important consequence of Theorem 1.1. Notice that, in both Theorem 1.1 and Corollary 1.2, one can choose $\ell = 1$ when A is a Buchsbaum ring.

Corollary 1.2. *Let A be a generalized Cohen-Macaulay ring with* $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$, *and choose an integer $\ell \geq 1$ so that every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Then, for each integer $q \geq \ell$, there exists an integer $t = t(q) \geq q + \ell + 1$ such that the following assertions hold true for every parameter ideal Q of A contained in $\mathfrak{m}^t$, where $I = Q : \mathfrak{m}^q$.*

(1) $\mathrm{e}_I^1(A) = \mathrm{e}_I^0(A) + \mathrm{e}_Q^1(A) - \ell_A(A/I)$.

(2) *The Hilbert function of I is given by* $\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \sum_{i=2}^{d}(-1)^i[\mathrm{e}_Q^{i-1}(A) + \mathrm{e}_Q^i(A)]\binom{n+d-i}{d-i}$ *for all $n \geq 0$.*

(3) $\mathrm{H}_{\mathcal{M}}^i(\mathrm{G}(I)) = [\mathrm{H}_{\mathcal{M}}^i(\mathrm{G}(I))]_{1-i} \cong \mathrm{H}_{\mathfrak{m}}^i(A)$ *as an A-module for all $i < d$ and*

$$\max\left\{n \in \mathbf{Z} \mid [\mathrm{H}_{\mathcal{M}}^d(\mathrm{G}(I))]_n \neq (0)\right\} \leq 1 - d.$$

(4) *The associated graded ring $\mathrm{G}(I) = \oplus_{n \geq 0} I^n/I^{n+1}$ of I is a Buchsbaum ring whenever A is Buchsbaum.*

Here $\mathcal{M} = \mathfrak{m}\mathrm{G}(I) + \mathrm{G}(I)_+$ and $[\mathrm{H}_{\mathcal{M}}^i(\mathrm{G}(I))]_n$ ($i, n \in \mathbf{Z}$) denotes the homogeneous component with degree n in the ith graded local cohomology module $\mathrm{H}_{\mathcal{M}}^i(\mathrm{G}(I))$ of $\mathrm{G}(I)$ with respect to $\mathcal{M}$.

In [**7**] Goto, Sakurai and the author proved Theorem 1.1 and Corollary 1.2, assuming the extra condition on systems $a_1, a_2, \ldots, a_d$ of parameters that $a_d = ab$ for some $a \in \mathfrak{m}^q$ and $b \in \mathfrak{m}$. This is a technical but crucial condition in order to use the result of Goto and Sakurai [**16**, Lemma2.3], and thanks to the condition, they were able to get the equality $I^2 = QI$ by induction on dimension d, where $I = Q : \mathfrak{m}^q$ and $Q \subseteq \mathfrak{m}^{q+\ell+1}$. The present proof of Theorem 1.1 and Corollary 1.2 is substantially different from the one in [**7**]. It is based on Proposition 2.2 and valid for every parameter ideal Q contained in $\mathfrak{m}^t$, choosing an integer t such that $t \geq q + \ell + 1$.

Our research dates back to the works of Corso, Polini, Huneke, Vasconcelos and Goto, where they explored the socle ideals $Q : \mathfrak{m}$ for parameter ideals Q in Cohen-Macaulay rings and proved the following.

Theorem 1.3 [**1–4, 6**]. *Let Q be a parameter ideal in a Cohen-Macaulay ring A, and let $I = Q : \mathfrak{m}$. Then the following conditions are equivalent.*

(1) $I^2 \neq QI$.

(2) *Q is integrally closed in A.*

(3) *A is a regular local ring and the A-module $\mathfrak{m}/Q$ is cyclic.*

Therefore, if A is a Cohen-Macaulay ring which is not regular, then $I^2 = QI$ for every parameter ideal Q in A, so that $\mathrm{G}(I)$ *and* $\mathrm{F}(I)$ *are both Cohen-Macaulay rings, where $I = Q : \mathfrak{m}$. The Rees algebra $\mathcal{R}(I)$ is also Cohen-Macaulay if* $\dim A \geq 2$.

This result has led people to explore quasi-socle ideals in arbitrary local rings. In [**14–16**], Goto and Sakurai explored the socle ideals $I = Q : \mathfrak{m}$ inside Buchsbaum rings. They showed that I is stable and $\mathrm{G}(I)$ is a Buchsbaum ring whenever $\mathrm{e}^0_{\mathfrak{m}}(A) \geq 2$ and Q is contained in a sufficiently high power of the maximal ideal $\mathfrak{m}$. Wang [**19**] and Goto, Matsuoka, Takahashi, Kimura, Phuong and Truong [**8–11**] explored quasi-socle ideals in both Cohen-Macaulay and Gorenstein rings with ample examples. In [**11**] the quasi-socle ideals $Q : \mathfrak{m}^2$ in Gorenstein rings A with $\dim A > 0$ and $\mathrm{e}^0_{\mathfrak{m}}(A) \geq 3$ are explored, and in [**8–10**] the quasi-socle ideals $Q : \mathfrak{m}^q$ $(q \geq 1)$ in Cohen-Macaulay local rings of dimension 1 are closely studied.

Perhaps Wang has provided the greatest achievement so far by affirmatively answering a conjecture posed by Polini and Ulrich that is rooted in linkage theory. We state his result in the following way.

Theorem 1.4 [**19**]. *Suppose that A is a Cohen-Macaulay ring, and let $q \geq 1$ be an integer. Let Q be a parameter ideal in A such that $Q \subseteq \mathfrak{m}^{q+1}$, and put $I = Q : \mathfrak{m}^q$. Then*

$$\mathfrak{m}^q I = \mathfrak{m}^q Q, \qquad I \subseteq \mathfrak{m}^{q+1} \quad \text{and} \quad I^2 = QI,$$

provided $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$.

It seems, however, natural to ask what we can expect when the base local ring is not necessarily Cohen-Macaulay. Goto, Sakurai and the author [**7**, Theorem 1.1] gave an answer in the case where the base ring A is Buchsbaum, showing the assumption that $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$ is sufficient in order for Wang's methods to work. Generalizing the results in [**7, 14–16**] our Theorem 1.1 answers the question with substantial generality in the case where A is a generalized Cohen-Macaulay ring; although, the author does not know sharp estimations of integers $t = t(q)$ given in Theorem 1.1 even in the case where A is a Buchsbaum ring.

2. Proof of Theorem 1.1. In what follows, unless otherwise specified, we denote by A a Noetherian local ring with maximal ideal $\mathfrak{m}$ and dimension $d > 0$. Let $\mathrm{H}^i_{\mathfrak{m}}(*)$ $(i \in \mathbf{Z})$ be the local cohomology functors of A with respect to $\mathfrak{m}$. The purpose of this section is to prove Theorem 1.1.

Our proof is based on the following result of Cuong and Truong [**5**, Theorem 3.3, Corollary 4.1]. They deal with the case when $q = 1$, but this can be generalized to when $q \geq 1$ in a straightforward manner.

Theorem 2.1 ([**5**, Theorem 3.3, Corollary 4.1]). *Suppose that A is a generalized Cohen-Macaulay ring, and let $q \geq 1$ be an integer. Then*

$$\sup_Q \ell_A([Q : \mathfrak{m}^q]/Q) = \sum_{i=0}^{d} \binom{d}{i} \ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q)$$

where Q runs through standard parameter ideals in A. Furthermore, one can find an integer $k = k(q) \geq 1$ such that every parameter ideal Q of A contained in $\mathfrak{m}^k$ is standard with

$$\ell_A([Q : \mathfrak{m}^q]/Q) = \sum_{i=0}^{d} \binom{d}{i} \ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q).$$

We begin with the following.

Proposition 2.2. *Suppose that A is a generalized Cohen-Macaulay ring, and let $q \geq 1$ be an integer. Let Q be a standard parameter ideal in A, and assume that*

$$\ell_A([Q : \mathfrak{m}^q]/Q) = \sum_{i=0}^{d} \binom{d}{i} \ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q).$$

Then

$$[Q+W]:\mathfrak{m}^q = [Q:\mathfrak{m}^q] + W,$$

where $W = \mathrm{H}^0_{\mathfrak{m}}(A)$.

Proof. We set $\overline{A} = A/W$. Then $Q \cap W = (0)$ [**18**, Corollary 2.3], we have the exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(A) \longrightarrow A/Q \xrightarrow{\varepsilon} \overline{A}/Q\overline{A} \longrightarrow 0,$$

since $Q\overline{A}$ is also a standard parameter ideal of $\overline{A}$. By applying $\mathrm{Hom}_A(A/\mathfrak{m}^q, *)$ and using Theorem 2.1, we get

$$\begin{aligned}
\ell_A([Q:\mathfrak{m}^q]/Q) &\leq \ell_A((0):_{\mathrm{H}^0_{\mathfrak{m}}(A)} \mathfrak{m}^q) + \ell_A([Q\overline{A}:_{\overline{A}} \mathfrak{m}^q]/Q\overline{A}) \\
&\leq \ell_A((0):_{\mathrm{H}^0_{\mathfrak{m}}(A)} \mathfrak{m}^q) + \sum_{i=0}^{d} \binom{d}{i} \ell_A((0):_{\mathrm{H}^i_{\mathfrak{m}}(\overline{A})} \mathfrak{m}^q) \\
&= \ell_A((0):_{\mathrm{H}^0_{\mathfrak{m}}(A)} \mathfrak{m}^q) + \sum_{i=1}^{d} \binom{d}{i} \ell_A((0):_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q) \\
&\leq \sum_{i=0}^{d} \binom{d}{i} \ell_A((0):_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q),
\end{aligned}$$

since $\mathrm{H}^0_{\mathfrak{m}}(\overline{A}) = (0)$ and $\mathrm{H}^i_{\mathfrak{m}}(\overline{A}) = \mathrm{H}^i_{\mathfrak{m}}(A)$ for all $i \geq 1$. Therefore, because

$$\ell_A([Q:\mathfrak{m}^q]/Q) = \sum_{i=0}^{d} \binom{d}{i} \ell_A((0):_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q),$$

we have

$$\ell_A([Q:\mathfrak{m}^q]/Q) = \ell_A((0):_{\mathrm{H}^0_{\mathfrak{m}}(A)} \mathfrak{m}^q) + \ell_A([Q\overline{A}:_{\overline{A}} \mathfrak{m}^q]/Q\overline{A}).$$

This shows that homomorphism $A/Q \xrightarrow{\varepsilon} \overline{A}/Q\overline{A}$ gives rise to an epimorphism

$$\mathrm{Hom}_A(A/\mathfrak{m}^q, \varepsilon) : \mathrm{Hom}_A(A/\mathfrak{m}^q, A/Q) \longrightarrow \mathrm{Hom}_A(A/\mathfrak{m}^q, \overline{A}/\overline{Q}).$$

Hence,

$$[Q+W]:\mathfrak{m}^q = [Q:\mathfrak{m}^q] + W. \qquad \square$$

The following is the key for our proof of Theorem 1.1. This is a generalization of the result of Goto and Sakurai [**14**, Theorem 3.9].

Theorem 2.3. *Suppose that A is a generalized Cohen-Macaulay ring, and let $q \geq 1$ be an integer. Let Q be a standard parameter ideal in A, and set $I = Q : \mathfrak{m}^q$. Assume that the following three conditions are satisfied.*

(1) $\ell_A(I/Q) = \sum_{i=0}^{d} \binom{d}{i} \ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q)$.

(2) $\mathfrak{m}^q I = \mathfrak{m}^q Q$.

(3) $I^2 \subseteq Q$.

Then I is stable.

Proof. We have

$$[Q + W] : \mathfrak{m}^q = [Q : \mathfrak{m}^q] + W = I + W$$

by Proposition 2.2, where $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. Let $Q = (a_1, a_2, \dots , a_d)$.

Suppose that $d = 1$. We put $\overline{A} = A/W$, $\overline{\mathfrak{m}} = \mathfrak{m}/W$, $\overline{I} = I\overline{A}$ and $\overline{Q} = Q\overline{A}$. Then $\overline{\mathfrak{m}}^q{\cdot}\overline{I} = \overline{\mathfrak{m}}^q{\cdot}\overline{Q}$; hence, $\overline{\mathfrak{m}}^q{\cdot}\overline{I}^n = \overline{\mathfrak{m}}^q{\cdot}\overline{Q}^n$ for all $n \in \mathbf{Z}$. By the equality $[Q + W] : \mathfrak{m}^q = I + W$, we have $\overline{I} = \overline{Q} : \overline{\mathfrak{m}}^q$. Let $x \in \overline{I}^2$. Then, since $\overline{I}^2 \subseteq \overline{Q}$, we have $x = a_1 y$ with $y \in \overline{A}$. Let $\alpha \in \overline{\mathfrak{m}}^q$. Then, $a_1(\alpha y) = \alpha x \in \overline{\mathfrak{m}}^q{\cdot}\overline{I}^2 = \overline{\mathfrak{m}}^q{\cdot}\overline{Q}^2$, and we get $a_1(\alpha y) = a_1^2 z$ for some $z \in \overline{A}$. Hence, $\alpha y \in \overline{Q}$ (notice that $\overline{A}$ is Cohen-Macaulay so that a_1 is $\overline{A}$-regular); hence, we have $x = a_1 y \in \overline{Q} \cdot \overline{I}$, because $y \in \overline{Q} : \overline{\mathfrak{m}}^q = \overline{I}$. Thus, we have $\overline{I}^2 = \overline{Q}{\cdot}\overline{I}$, so that $I^2 \subseteq QI + W$. Therefore, since $W \cap Q = (0)$ and $I^2 \subseteq Q$, we get $I^2 \subseteq (QI + W) \cap Q = QI$ as claimed.

Suppose that $d \geq 2$, and our assertion holds true for $d - 1$. Let $B = A/(a_1)$. Then conditions (1), (2) and (3) are satisfied for the parameter ideal QB in B. This is clear for conditions (2) and (3). As for condition (1), for all $0 \leq i \leq d-2$ we have the short exact sequence

$$0 \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(B) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \longrightarrow 0$$

of local cohomology modules, since $a_1\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ $(0 \leq i \leq d-1)$ and $\ell_A((0) : a_1) = \ell_A(W) < \infty$ [**18**, Theorem 2.5]. Hence, by Theorem 2.1, we get

$$\begin{aligned}\ell_A(I/Q) &= \ell_A([QB :_B \mathfrak{m}^q]/QB)\\ &\leq \sum_{i=0}^{d-1}\binom{d-1}{i}\ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(B)} \mathfrak{m}^q)\\ &\leq \sum_{i=0}^{d-1}\binom{d-1}{i}\left[\ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q) + \ell_A((0) :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(A)} \mathfrak{m}^q)\right]\\ &= \sum_{i=0}^{d}\binom{d}{i}\ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(A)} \mathfrak{m}^q)\\ &= \ell_A(I/Q),\end{aligned}$$

so that

$$\ell_A([QB :_B \mathfrak{m}^q]/QB) = \sum_{i=0}^{d-1}\binom{d-1}{i}\ell_A((0) :_{\mathrm{H}^i_{\mathfrak{m}}(B)} \mathfrak{m}^q).$$

Therefore, condition (1) is satisfied also for QB. Thus, we have $I^2 \subseteq QI + (a_1)$ by the hypothesis of induction on d. Let us now choose $x \in I^2$ and write $x = y + a_1 z$ with $y \in QI$ and $z \in A$. Also, let $\alpha \in \mathfrak{m}^q$. We then have

$$\alpha x = \alpha y + a_1(\alpha z) \in Q^2,$$

because $x \in I^2$ and $\mathfrak{m}^q I = \mathfrak{m}^q Q$. Consequently, $a_1(\alpha z) \in Q^2$ (notice that $\alpha y \in Q^2$), since $a_1, a_2, \ldots, a_d$ form a d-sequence in A [**18**, Proposition 3.1], we have $a_1(\alpha z) \in (a_1) \cap Q^2 = a_1 Q$. Hence, $\alpha z - v \in (0) : a_1 \subseteq W$ ([**18**, Theorem 2.5]) for some $v \in Q$, which guarantees $z \in (Q + W) : \mathfrak{m}^q = I + W$. Since $a_1 W = (0)$, we get $x = y + a_1 z \in QI$. Hence, $I^2 = QI$. □

To prove Theorem 1.1 we need the following result of [**7**], in which we make use of the assumption that $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$. In [**7**, Proposition 2.2] the integer q is assumed to be $q \geq 2$. The assertion also holds for $q = 1$. Since the proof of the case $q = 1$ is quite different from that of the case $q \geq 2$, we have included it here.

Proposition 2.4 [**7**, Proposition 2.2]. *Let A be a generalized Cohen-Macaulay ring with* $\operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$. *Choose an integer $\ell \geq 1$ so that* *every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Let $q \geq 1$ be an integer, and let Q be a parameter ideal of A such that $Q \subseteq \mathfrak{m}^{q+\ell+1}$. We then have*

$$\mathfrak{m}^q I = \mathfrak{m}^q Q, \qquad I \subseteq \mathfrak{m}^{q+\ell+1}, \quad \textit{and} \quad I^2 \subseteq Q,$$

where $I = Q : \mathfrak{m}^q$.

Proof of the case where $q = 1$. Let $I = Q : \mathfrak{m}$. Firstly we will show that $\mathfrak{m}I = \mathfrak{m}Q$. Assume on the contrary that, by [**14**, Lemma 2.2] we have $e^0_{\mathfrak{m}}(A) = 1$. Hence, A is a regular local ring, since it is unmixed (notice that A is a generalized Cohen-Macaulay ring and $\dim A \geq \operatorname{depth} A \geq \operatorname{depth} \mathrm{G}(\mathfrak{m}) \geq 2$). Since $\mathfrak{m}I \subseteq Q$ but $\mathfrak{m}I \neq \mathfrak{m}Q$, the parameter ideal Q cannot be a reduction of I, so that $I = Q : \mathfrak{m} \not\subseteq \overline{Q}$. Hence, $Q = \overline{Q}$, because $\ell_A([Q : \mathfrak{m}]/Q) = 1$ (recall that A is a Gorenstein ring). Therefore, $\mathfrak{m}/Q$ is a cyclic A-module by Theorem 1.3, which forces $\dim A \leq 1$, because $Q \subseteq \mathfrak{m}^{\ell+2} \subseteq \mathfrak{m}^2$. This is impossible, since $\dim A \geq 2$. It now follows from [**7**, Lemma 2.1] that $I \subseteq \mathfrak{m}^{\ell+2}$; in fact, $I = Q : \mathfrak{m} \subseteq \mathfrak{m}^{\ell+3} : \mathfrak{m} = \mathfrak{m}^{\ell+2}$. Thus, $I \subseteq \mathfrak{m}$, so that we have $I^2 \subseteq \mathfrak{m}I \subseteq Q$. □

We are now ready to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. Let $\ell \geq 1$ be an integer such that every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Take $t(q) = \max\{k(q), q+\ell+1\}$, where $k(q)$ is the integer obtained by Theorem 2.1. Then, by Theorem 2.3 and Proposition 2.4, we readily get that I is stable for every parameter ideal $Q = (a_1, a_2, \ldots, a_d)$ of A contained in $\mathfrak{m}^t$, where $I = Q : \mathfrak{m}^q$. □

Before entering into the proof of Corollary 1.2, let us give the notion introduced by [**13**]. Let I be an $\mathfrak{m}$-primary ideal of A, and let $\underline{a} = a_1, a_2, \ldots, a_d$ be a system of parameters in A. We assume that $Q = (a_1, a_2, \ldots, a_d)$ is a reduction of I. Then we say that condition (C_2) is satisfied for $\underline{a}$ and I, if

$$(a_1, \ldots, \check{a}_i, \ldots, a_d) : a_i \subseteq I$$

for all $1 \leq i \leq d$.

Proof of Corollary 1.2. We have $I^2 = QI$ by Theorem 1.1. We notice that, if $q \geq \ell$, condition (C_2) is satisfied for our system $\underline{a}$ of parameters and the ideal $I = Q : \mathfrak{m}^q$. In fact,

$$(a_1, \dots, \check{a}_i, \dots, a_d) : a_i = (a_1, \dots, \check{a}_i, \dots, a_d) : \mathfrak{m}^\ell$$

for each $1 \leq i \leq d$ ([**18**, Lemma 1.1]), because $Q \subseteq \mathfrak{m}^\ell$ and every parameter ideal of A contained in $\mathfrak{m}^\ell$ is standard. Therefore, since $q \geq \ell$, we get

$$(a_1, \dots, \check{a}_i, \dots, a_d) : a_i = (a_1, \dots, \check{a}_i, \dots, a_d) : \mathfrak{m}^\ell \subseteq (a_1, \dots, \check{a}_i, \dots, a_d) : \mathfrak{m}^q \subseteq I,$$

as wanted. Hence, the detailed description of the Hilbert function of our ideal $I = Q : \mathfrak{m}^q$ follows from [**13**]. By [**12**, Section 5] the associated graded ring $G(I)$ of I is Buchsbaum, if A is Buchsbaum. Assertions (1) and (2) (respectively (3) and (4)) of Corollary 1.2 readily follow from [**13**, Propositions 2.4, 2.5] (respectively [**12**, Theorem 1.3, Section 5]). □

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