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MACAULAY'S THEOREM FOR SOME PROJECTIVE MONOMIAL CURVES

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1. Introduction. Throughout this paper S stands for the polynomial ring $k[x_1, \dots, x_n]$ over a field k with the standard grading $\deg(x_i) = 1$ for $1 \leq i \leq n$. For any graded ideal J of S, the size of J is measured by the Hilbert function

$$\begin{aligned} h : \mathbf{N} &\longrightarrow \mathbf{N} \\ i &\longmapsto \dim_k J_i, \end{aligned}$$

where $\mathbf{N} = \{0, 1, 2, \dots\}$ and J_i is the vector space of all homogeneous polynomials in J of degree i. In 1927, Macaulay [**9**] proved that, for every graded ideal in S, there exists a lex ideal with the same Hilbert function. Since then, lex ideals have played a key role in the study of Hilbert functions: in 1966, Hartshorne [**5**] proved that the Hilbert scheme is connected, namely, every graded ideal in S is connected by a sequence of deformations to the lex ideal with the same Hilbert function; then in the 1990s, Bigatti [**1**], Hulett [**6**] and Pardue [**11**] proved that every lex ideal in S attains maximal Betti numbers among all graded ideals with the same Hilbert function.

It is interesting to know if similar results hold for graded quotient rings of the polynomial ring S. One important class of graded quotient rings over which Macaulay's theorem holds is the Clements-Lindström ring $S/(x_1^{c_1}, \dots, x_n^{c_n})$, where $c_1 \leq \cdots \leq c_n \leq \infty$. In 1969, Clements and Lindström [**2**] proved that Macaulay's theorem holds over the ring $S/(x_1^{c_1}, \dots, x_n^{c_n})$, that is, for every graded ideal in $S/(x_1^{c_1}, \dots, x_n^{c_n})$, there exists a lex ideal with the same Hilbert function. In the case $c_1 = \cdots = c_n = 2$, the result was obtained earlier by Katona [**7**] and Kruskal [**8**]. Recently, Mermin and Peeva [**10**] raised the problem to find other graded quotient rings over which Macaulay's theorem holds.

Toric varieties, cf. [**3**], have been extensively studied in algebraic geometry. They are very interesting because they can be studied with

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methods and ideas from algebraic geometry, combinatorics, commutative algebra and computational algebra. In [**4**], Gasharov, Horwitz and Peeva introduced the notion of a lex ideal in the toric ring and raised the question [**4**, 4.1] to find projective toric rings over which Macaulay's theorem holds. They proved in [**4**, Theorem 5.1] that Macaulay's theorem holds for the rational normal curves. The goal of this paper is to study whether Macaulay's theorem holds for other projective monomial curves.

Let $\mathcal{A} = \{\binom{a_1}{1}, \dots, \binom{a_n}{1}\}$ be a subset of $\mathbf{N}^2 \backslash \{\vec{0}\}$. We set $A = \begin{pmatrix} a_1 & \cdots & a_n \\ 1 & \cdots & 1 \end{pmatrix}$ to be the matrix associated to $\mathcal{A}$, and assume $\operatorname{rank} A = 2$. The *toric ideal* associated to $\mathcal{A}$ is the kernel $I_{\mathcal{A}}$ of the homomorphism:

$$\begin{aligned} \varphi: \quad k[x_1, \dots, x_n] &\longrightarrow k[u, v] \\ x_i &\longmapsto u^{a_i} v. \end{aligned}$$

The ideal $I_{\mathcal{A}}$ is graded and prime. Set $R = S/I_{\mathcal{A}} \cong k[u^{a_1}v, \dots, u^{a_n}v]$. Then R is a graded ring with $\deg(x_i) = 1$ for $1 \le i \le n$. We call $R = S/I_{\mathcal{A}}$ the *toric ring* associated to $\mathcal{A}$. Every projective monomial curve in $\mathbf{P}^{n-1}$ can be defined by $I_{\mathcal{A}}$ for some $\mathcal{A}$. For example, the rational normal curves are defined by the toric ideals associated to matrices of the form $A = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 1 & \cdots & 1 \end{pmatrix}$. We say that Macaulay's theorem holds for a projective monomial curve defined by $I_{\mathcal{A}}$, or that Macaulay's theorem holds over the toric ring $R = S/I_{\mathcal{A}}$ if, for any homogeneous ideal J in R, there exists a lex ideal L with the same Hilbert function. Throughout, we assume that $x_1 > \cdots > x_n$.

In Theorem 4.1 we prove that Macaulay's theorem holds for projective monomial curves defined by the toric ideals associated to matrices of the form

$$A = \begin{pmatrix} 0 & 1 & \cdots & n-2 & n-1+h \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \quad \text{where } n \ge 3,\ h \in \mathbf{Z}^+.$$

In Theorem 5.1 we consider matrices of the form

$$A = \begin{pmatrix} 0 & 1+h & 2+h & \cdots & n-1+h \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}, \quad \text{where } n \ge 3, h \in \mathbf{Z}^+,$$

and prove that if $h = 1$ or $n = 3$, Macaulay's theorem holds; otherwise, Macaulay's theorem does not hold.

Finally, in Theorem 5.5 we prove that Macaulay's theorem does not hold if

$$A = \begin{pmatrix} 0 & 1 & \cdots & m-1 & m+h & \cdots & n-1+h \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $n \geq 4$, $2 \leq m \leq n-2$ and $h \in \mathbf{Z}^+$.

2. Preliminaries. Throughout this paper, we fix the order of the variables in S to be $x_1 > \cdots > x_n$ and consider the induced lex order $>_{\text{lex}}$ on S.

To define the lex ideals in the toric ring $R = S/I_{\mathcal{A}}$, we need the following definition introduced in Section 3 in [**4**]:

Definition 2.1. An element $m \in R$ is a *monomial* if there exists a monomial preimage $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ of m in S. For simplicity, by writing $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in R, we mean $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n} + I_{\mathcal{A}}$ in R. An ideal in R is a *monomial ideal* if it can be generated by monomials in R. Let $m \in R$ be a monomial; the set of all monomial preimages of m in S is called the *fiber* of m. The lex-greatest monomial in a fiber is called the *top-representative* of the fiber.

Let $m, m' \in R_d$ be two monomials of degree d in R. Let p, p' be the top-representatives of the fibers of m and m', respectively. We say that $m \succ_{\text{lex}} m'$ in R_d if $p >_{\text{lex}} p'$ in S.

A *d-monomial space* W is a vector subspace of R_d spanned by some monomials of degree d. A d-monomial space W is *lex* if the following property holds: for monomials $m \in W$ and $q \in R_d$, if $q \succ_{\text{lex}} m$ then $q \in W$. A monomial ideal L in R is *lex* if, for every $d \geq 0$, the d-monomial space L_d is lex.

By [**4**, Theorem 2.5], we know that for any homogeneous ideal J in R, there exists a monomial ideal M in R such that M has the same Hilbert function as J. So, to show that Macaulay's theorem holds over R, we only need to prove that, given any monomial ideal M in R, there exists a lex ideal L in R with the same Hilbert function. Furthermore, we will use [**4**, Lemma 4.2], which states:

Lemma 2.2 (Gasharov-Horwitz-Peeva). *Macaulay's theorem holds over R if and only if, for every $d \geq 0$ and for every d-monomial space*

W, we have the inequality:

$$\dim_k R_1 L_W \leq \dim_k R_1 W,$$

where L_W is the lex d-monomial space in R_d such that $\dim_k L_W = \dim_k W$.

Remark 2.3. Let W be a d-monomial space spanned by monomials $w_1, \dots, w_s \in R_d$; then we have that

$$\dim_k W = |\{w_1, \dots, w_s\}|$$

and

$$\dim_k R_1 W = |\{x_i w_j \in R_{d+1} \mid 1 \leq i \leq n, 1 \leq j \leq s\}|.$$

If W' is another d-monomial space spanned by monomials $w'_1, \dots, w'_t \in R_d$, then we have

$$\dim_k W \cap W' = |\{w_1, \dots, w_s\} \cap \{w'_1, \dots, w'_t\}|.$$

Remark 2.4. Let m be a monomial in R. Pick a representative $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ from the fiber of m. Then

$$\varphi(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = u^{\alpha_1 a_1 + \cdots + \alpha_n a_n} v^{\alpha_1 + \cdots + \alpha_n},$$

which is independent of the choice of the representative. Define

$$u(m) = u(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) := \alpha_1 a_1 + \cdots + \alpha_n a_n.$$

Note that $\deg m = \alpha_1 + \cdots + \alpha_n$. Then, for monomials m, $m' \in R$,

$$m = m' \iff u(m) = u(m') \quad \text{and} \quad \deg m = \deg m'.$$

Hence, for any $d \geq 1$, we have a natural order $>_u$ on the monomials in R_d: for monomials $m, m' \in R_d$, we say that $m >_u m'$ if $u(m) < u(m')$.

Note that the lex order $\succ_{\text{lex}}$ may not coincide with the natural order $>_u$. This is illustrated in the following example.

Example 2.5. Let $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$. Then, in R_2, $x_1x_3 \succ_{\text{lex}} x_2^2$, but $x_2^2 >_u x_1x_3$.

We use lex order $\succ_{\text{lex}}$ instead of $>_u$ to define lex ideals in R because we want to have the following crucial property: *If L_d is a lex d-monomial space in R_d, then R_1L_d is a lex $(d+1)$-monomial space in R_{d+1}.* By [**4**, Theorem 3.4], we know that this property holds for the lex order $\succ_{\text{lex}}$. However, by the above example, it is easy to see that this property does not hold for the natural order $>_u$. Indeed, let $L_1 = \text{span}\,\{x_1\} \subseteq R_1$. Then L_1 is lex with respect to the natural order $>_u$ and $R_1L_1 = \text{span}\,\{x_1^2, x_1x_2, x_1x_3\} \subseteq R_2$; but in R_2, since $x_1^2 >_u x_1x_2 >_u x_2^2 >_u x_1x_3$, one sees that R_1L_1 is not lex with respect to the natural order $>_u$.

Remark 2.6. In the polynomial ring S we have the following property: if L_d is a lex d-monomial space in S_d and m is the first monomial in $S_d \backslash L_d$, then

$$(*) \qquad \dim_k S_1(L_d + km) > \dim_k S_1 L_d,$$

and, in particular, $x_n m \notin S_1L_d$. However, this may not be true in R, and we have the following example.

Example 2.7. Let $A = \begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $L_2 = \text{span}\,\{x_1^2, x_1x_2, x_1x_3, x_1x_4\}$ and $m = x_2^2$. Then L_2 is lex in R_2 and m is the first monomial after x_1x_4. Since

$$u(x_1x_2^2) = u(x_2x_1x_2), \qquad u(x_2x_2^2) = u(x_1x_1x_3),$$
$$u(x_3x_2^2) = u(x_2x_1x_4), \qquad u(x_4x_2^2) = u(x_3x_1x_3),$$

it follows that $R_1(L_2 + km) = R_1L_2$ and $x_4m \in R_1L_2$. Thus, $\dim_k R_1(L_2 + km) = \dim_k R_1L_2$ and $(*)$ fails.

3. Lemmas for general projective monomial curves. In this section, we prove three lemmas which hold for projective monomial curves. These lemmas will be used later in Sections 4 and 5.

First we make the following observation. Let $I_{\mathcal{A}}$ be the toric ideal associated to $\mathcal{A} = \left\{ \binom{a_1}{1}, \dots, \binom{a_n}{1} \right\}$; then, without loss of generality, we can assume that $a_i \neq a_j$ for $i \neq j$. By changing the order of the variables in S, we can assume $a_1 < \cdots < a_n$. Let $B = \begin{pmatrix} 1 & -a_1 \\ 0 & 1 \end{pmatrix}$ and $p = \gcd(a_2 - a_1, \dots, a_n - a_1)$. Then we have

$$\frac{1}{p} BA = \begin{pmatrix} 0 & (a_2 - a_1)/p & \cdots & (a_n - a_1)/p \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Since A and $(BA)/p$ have the same kernel, they define the same toric ideal, so that we can always assume that $0 = a_1 < a_2 < \cdots < a_n$ and $\gcd(a_2, \dots, a_n) = 1$.

Given a d-monomial space W, in order to calculate $\dim_k R_1 W$ efficiently, we have the following lemma.

Lemma 3.1. *Let W be a d-monomial space spanned by monomials $w_1, \dots, w_s \in R_d$ with $u(w_1) < \cdots < u(w_s)$. Then*

$$\dim_k R_1 W = sn - \sum_{1 \le i < j \le s} \lambda(w_i, w_j),$$

where

$$\begin{aligned} \lambda(w_i, w_j) = |\{(p, q) \mid 1 \le p < q \le n, u(x_q) - u(x_p) = u(w_j) - u(w_i), \\ \textit{and there exist no } p < r < q,\ i < k < j \\ \textit{such that } u(x_r) - u(x_p) = u(w_j) - u(w_k)\}|. \end{aligned}$$

Proof. By induction on s. If $s = 1$, then the assertion is clear. If $s > 1$, then setting $W' = \operatorname{span}\{w_1, \dots, w_{s-1}\}$, we get

$$\begin{aligned} \dim_k R_1 W &= \dim_k R_1 (W' + k w_s) \\ &= \dim_k (R_1 W' + R_1 (k w_s)) \\ &= \dim_k R_1 W' + \dim_k R_1 (k w_s) - \dim_k R_1 W' \cap R_1 (k w_s). \end{aligned}$$

By the induction hypothesis, we have that

$$\dim_k R_1 W' = (s-1)n - \sum_{1 \le i < j \le s-1} \lambda(w_i, w_j),$$

and

$$\dim_k R_1(kw_s) = n.$$

Note that

$$\begin{aligned}
&\dim_k R_1 W' \cap R_1(kw_s)\\
&\quad = |\{1 \le p \le n \mid x_p w_s = x_q w_i \text{ in } R_{d+1},\\
&\qquad \text{for some } 1 \le i \le s-1, q > p\}|\\
&\quad = \sum_{1 \le i \le s-1} |\{1 \le p \le n \mid x_p w_s = x_q w_i \text{ in } R_{d+1},\\
&\qquad\qquad \text{for some } q > p, \text{ and there exists no}\\
&\qquad\qquad i < k < s \text{ such that } x_p w_s = x_r w_k \text{ for some } r > p\}|\\
&\quad = \sum_{1 \le i \le s-1} \lambda(w_i, w_s).
\end{aligned}$$

So we have

$$\begin{aligned}
\dim_k R_1 W &= (s-1)n - \sum_{1 \le i < j \le s-1} \lambda(w_i, w_j)\\
&\qquad + n - \sum_{1 \le i \le s-1} \lambda(w_i, w_s)\\
&= sn - \sum_{1 \le i < j \le s} \lambda(w_i, w_j). \qquad \square
\end{aligned}$$

The following two lemmas will be helpful when we prove Theorem 5.1.

Lemma 3.2. *Let* $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ *and* $A' = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ *be such that* $0 = a_1 < a_2 < \cdots < a_n$, $0 = b_1 < b_2 < \cdots < b_n$ *and* $a_i + b_{n+1-i} = a_n$ *for* $i = 1, \ldots, n$. *Set* $S = k[x_1, \ldots, x_n]$ *and* $S' = k[y_1, \ldots, y_n]$. *Then we have an isomorphism* $\widehat{f} : S \to S'$ *with* $\widehat{f}(x_i) = y_{n+1-i}$. *Let* $R = S/I_{\mathcal{A}}$ *be the toric ring associated to* A *and* $R' = S'/I_{\mathcal{A}'}$ *the toric ring associated to* A'; *then* $\widehat{f}$ *induces an isomorphism* $f : R \to R'$ *such that* $f(x_i + I_{\mathcal{A}}) = y_{n+1-i} + I_{\mathcal{A}'}$.

Proof. Given a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in S, we have

$$\begin{aligned} u(m) + u(\widehat{f}(m)) &= u(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) + u(y_n^{\alpha_1} \cdots y_1^{\alpha_n}) \\ &= \alpha_1 a_1 + \cdots + \alpha_n a_n + \alpha_1 b_n + \cdots + \alpha_n b_1 \\ &= \alpha_1(a_1 + b_n) + \cdots + \alpha_n(a_n + b_1) \\ &= (\alpha_1 + \cdots + \alpha_n)a_n \\ &= \deg(m)a_n. \end{aligned}$$

If $m - m' \in I_{\mathcal{A}}$ for some monomials $m, m' \in S$, then by Remark 2.4 we have that $u(m) = u(m')$ and $\deg(m) = \deg(m')$. Hence $u(\widehat{f}(m)) = u(\widehat{f}(m'))$ and $\deg(\widehat{f}(m)) = \deg(\widehat{f}(m'))$, so that $\widehat{f}(m) - \widehat{f}(m') = \widehat{f}(m - m') \in I_{\mathcal{A}'}$. Similarly, if $m - m' \in I_{\mathcal{A}'}$, then $\widehat{f}^{-1}(m - m') \in I_{\mathcal{A}}$. Thus, $\widehat{f}(I_{\mathcal{A}}) = I_{\mathcal{A}'}$, and therefore, $\widehat{f}$ induces an isomorphism f from R to R' such that $f(x_i + I_{\mathcal{A}}) = y_{n+1-i} + I_{\mathcal{A}'}$. □

Lemma 3.3. *Under the assumption of Lemma* 3.2, *we have the following two properties.*

(1) *If $W \subseteq R_d$ is a d-monomial space spanned by monomials $m_1, \ldots, m_r \in R_d$ with $u(w_1) < \cdots < u(w_r)$, then $f(W) \subseteq R'_d$ is a d-monomial space spanned by monomials $f(w_1), \ldots, f(w_r) \in R'_d$ with $u(f(w_1)) > \cdots > u(f(w_r))$, and* $\dim_k R_1 W = \dim_k R'_1 f(W)$.

(2) *Note that we have defined a* lex *order $\succ_{\text{lex}}$ in R_d. Now setting $y_n > \cdots > y_1$, we have a* lex *order $>_{\text{lex}'}$ in S' which induces a* lex *order $\succ_{\text{lex}'}$ in R'_d. Let m be a monomial in R_d with top representative $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Then $f(m)$ is a monomial in R'_d with top representative $\widehat{f}(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = y_n^{\alpha_1} \cdots y_1^{\alpha_n}$. Furthermore, if monomials $m, m' \in R_d$ are such that $m \succ_{\text{lex}} m'$, then $f(m) \succ_{\text{lex}'} f(m')$ in R'_d; if L_d is a lex d-monomial space in R_d, then $f(L_d)$ is a* lex *d-monomial space in R'_d; if Macaulay's theorem holds over R, then Macaulay's theorem holds over R'.*

Proof. (1) It is clear that $f(W)$ is a d-monomial space in R'_d. By the proof of Lemma 3.2, we see that $u(w_i) + u(f(w_i)) = da_n$, which implies that $u(f(w_i)) > u(f(w_j))$ for $i < j$. Note that $a_p - a_q = b_q - b_p$ for any $p \neq q$ and $u(w_i) - u(w_j) = u(f(w_j)) - u(f(w_i))$, for any $i \neq j$, so that the last part of the assertion follows directly from Lemma 3.1.

(2) By contradiction, we assume that $y_n^{\beta_1} \cdots y_1^{\beta_n}$ is in the fiber of $f(m)$ and $y_n^{\beta_1} \cdots y_1^{\beta_n} >_{\text{lex}'} y_n^{\alpha_1} \cdots y_1^{\alpha_n}$ in S'. Then $\widehat{f}^{-1}(y_n^{\beta_1} \cdots y_1^{\beta_n}) = x_1^{\beta_1} \cdots x_n^{\beta_n}$ is also in the fiber of m and $x_1^{\beta_1} \cdots x_n^{\beta_n} >_{\text{lex}} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in S, which is a contradiction. So we have proved the first part of the assertion, and the rest of the assertion follows easily. □

Remark 3.4. If we set $y_1 > \cdots > y_n$ in Lemma 3.3 (2), then the assertion may not hold. Indeed, considering Example 2.7, we have that $A = A'$; let $m = x_1 x_3^2$ in R. Then $x_1 x_3^2$ is the top-representative of the fiber of m, but $\widehat{f}(x_1 x_3^2) = y_4 y_2^2$ is not the top-representative of the fiber of $f(m)$. Also, by Theorems 4.1 and 5.1, we will see that even if Macaulay's theorem holds over R, it may not hold over R'.

4. A class of projective monomial curves. Throughout this section,

$$A = \begin{pmatrix} 0 & 1 & \cdots & n-2 & n-1+h \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \quad \text{where } n \geq 3, h \in \mathbf{Z}^+,$$

and R is the toric ring associated to A. We prove:

Theorem 4.1. *Macaulay's theorem holds over R.*

For the proof of Theorem 4.1, we need Lemmas 4.2, 4.3, 4.5, 4.7–4.11.

Lemma 4.2. *Let m be a monomial in R. Suppose that*

$$u(m) = \alpha(n-1+h) + \beta(n-2) + \gamma,$$

where α, β and γ are nonnegative integers such that $\beta(n-2) + \gamma < n-1+h$ and $\gamma < n-2$. If $\gamma \neq 0$, then $x_1^{\deg(m)-\alpha-\beta-1} x_{r+1} x_{n-1}^{\beta} x_n^{\alpha}$ is the top-representative of the fiber of m. If $\gamma = 0$, then $x_1^{\deg(m)-\alpha-\beta} x_{n-1}^{\beta} x_n^{\alpha}$ is the top-representative of the fiber of m.

Proof. Pick a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ from the fiber of m, and run the following algorithm.

Input: $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

Step 1: If $\sum_{i=1}^{n-1} \alpha_i(i-1) < n-1+h$, go to Step 2. Otherwise, choose $\beta_2, \ldots, \beta_{n-1} \in \mathbf{Z}$ such that $0 \leq \beta_2 \leq \alpha_2, \ldots, 0 \leq \beta_{n-1} \leq \alpha_{n-1}$, $\sum_{i=2}^{n-1} \beta_i(i-1) \geq n-1+h$ and $\sum_{i=2}^{n-1} \beta_i(i-1)$ is minimal with respect to this property. Running the division algorithm, we get $\sum_{i=2}^{n-1} \beta_i(i-1) = \beta_n(n-1+h)+\delta$, for some $\beta_n \geq 1$ and $0 \leq \delta < n-1+h$. Let $j = \min\{i \mid \beta_i \neq 0\}$. Then $\delta < j-1$; otherwise, it contradicts the minimality of $\sum_{i=1}^{n-1} \beta_i(i-1)$. Setting

$$\begin{aligned}
\alpha_j &:= \alpha_j - \beta_j, \\
&\cdots\cdots, \\
\alpha_{n-1} &:= \alpha_{n-1} - \beta_{n-1}, \\
\alpha_n &:= \alpha_n + \beta_n, \\
\alpha_{\delta+1} &:= \alpha_{\delta+1} + 1, \\
\alpha_1 &:= \alpha_1 + (\beta_j + \cdots + \beta_{n-1}) - \beta_n - 1,
\end{aligned}$$

we get a new monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ which is still in the fiber of m and is strictly bigger with respect to $>_{\text{lex}}$ in S. Go back to Step 1.

Step 2: If $\sum_{i=1}^{n-2} \alpha_i(i-1) < n-2$, stop. Otherwise, choose $\beta_2, \ldots, \beta_{n-2} \in \mathbf{Z}$ such that $0 \leq \beta_2 \leq \alpha_2, \ldots, 0 \leq \beta_{n-2} \leq \alpha_{n-2}$, $\sum_{i=2}^{n-2} \beta_i(i-1) \geq n-2$ and $\sum_{i=2}^{n-2} \beta_i(i-1)$ is minimal with respect to this property. Running the division algorithm, we get $\sum_{i=2}^{n-2} \beta_i(i-1) = \beta_{n-1}(n-2)+\delta$, for some $\beta_{n-1} \geq 1$ and $0 \leq \delta < n-2$. Let $j = \min\{i \mid \beta_i \neq 0\}$. Then $\delta < j-1$; otherwise, it contradicts the minimality of $\sum_{i=2}^{n-2} \beta_i(i-1)$. Setting

$$\begin{aligned}
\alpha_j &:= \alpha_j - \beta_j, \\
&\cdots\cdots, \\
\alpha_{n-2} &:= \alpha_{n-2} - \beta_{n-2}, \\
\alpha_{n-1} &:= \alpha_{n-1} + \beta_{n-1}, \\
\alpha_{\delta+1} &:= \alpha_{\delta+1} + 1, \\
\alpha_1 &:= \alpha_1 + (\beta_j + \cdots + \beta_{n-2}) - \beta_{n-1} - 1,
\end{aligned}$$

we get a new monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ which is still in the fiber of m and is strictly bigger with respect to $>_{\text{lex}}$ in S. Go back to Step 2.

The algorithm stops after finitely many steps, and the output of the algorithm is the monomial described in the lemma. If the top-representative of the fiber of m is different from the monomial given in the lemma, then we can run the algorithm on the top-representative to get a bigger monomial in the fiber, which is a contradiction. So the monomial given in the lemma is the top-representative of the fiber of m. $\square$

Lemma 4.3. *R has the following two properties.*

(1) *Let m be a monomial in R_d. If $w \in S$ is the top-representative of the fiber of m, then $x_n w \in S$ is the top-representative of the fiber of $x_n m \in R_{d+1}$.*

(2) *If L_d is a* lex *d-monomial space in R_d and m is the first monomial in $R_d \backslash L_d$, then $\dim_k R_1(L_d + km) > \dim_k R_1 L_d$ and $x_n m \notin R_1 L_d$.*

Proof. (1) Let $\widehat{m} \in S$ be the top-representative of the fiber of $x_n m$. Since $u(x_n m) \geq n-1+h$, by Lemma 4.2 we have $x_n | \widehat{m}$. Suppose that $\widehat{m} = x_n w'$ for some monomial $w' \in S$. Then it is easy to see that w' is the top-representative of the fiber of m, so that $w' = w$ and $\widehat{m} = x_n w$. So $x_n w$ is the top-representative of the fiber of $x_n m$.

(2) It suffices to prove that $x_n m \notin R_1 L_d$. By contradiction, we assume $x_n m \in R_1 L_d$. Then there exist x_i, $1 \leq i < n$ and $m' \in L_d$ such that $x_n m = x_i m'$ in R_{d+1}. Let w, w' be the top-representatives of the fibers of m and m', respectively; then, by (1), $x_n w$ is the top-representative of the fiber of $x_n m$. Since $m' \succ_{\text{lex}} m$ in R_d, we have $w' >_{\text{lex}} w$ in S, and then $x_i w'$ is in the fiber of $x_n m$ such that $x_i w' >_{\text{lex}} x_n w$, which is a contradiction. So, $x_n m \notin R_1 L_d$. $\square$

Definition 4.4. Let W be a d-monomial space spanned by monomials $w_1, \dots, w_s \in R_d$ with $0 = u(w_1) < \cdots < u(w_s)$. For $i \geq 0$, set

$$W(i) = \{w_j \mid \text{the top representative of } w_j \text{ can be divided by } x_n^i \text{ but not by } x_n^{i+1}\}.$$

The set $W(i)$ is called *n-compressed* if $W(i) = \varnothing$ or $W(i) = \{w_{k_i}, w_{k_i+1}, \dots, w_{k_i+t}\}$, for some $t \geq 0$ and $1 \leq k_i \leq s$, such that

$$u(w_{k_i}) = i(n-1+h),$$

$$u(w_{k_i+1}) = i(n-1+h)+1,$$
$$\cdots,\cdots,$$
$$u(w_{k_i+t}) = i(n-1+h)+t.$$

We say that a d-monomial space C is *n-compressed* if $C(i)$ is n-compressed for every $i \geq 0$.

Lemma 4.5. *Let m_1 and m_2 be two monomials in R_d with $u(m_1) < u(m_2)$. Suppose that $u(m_1) = \alpha_1(n-1+h)+\beta_1$ and $u(m_2) = \alpha_2(n-1+h)+\beta_2$, where α_1, α_2, β_1, β_2 are nonnegative integers and $\beta_1, \beta_2 < n-1+h$.*

(1) *If $\alpha_1 = \alpha_2$, then $m_1 \succ_{\text{lex}} m_2$.*

(2) *If $\alpha_1 < \alpha_2$ and $\beta_1 - \beta_2 \leq (\alpha_2-\alpha_1)(n-2)$, then $m_1 \succ_{\text{lex}} m_2$.*

(3) *If $\alpha_1 < \alpha_2$ and $\beta_1 - \beta_2 > (\alpha_2-\alpha_1)(n-2)$, then $m_2 \succ_{\text{lex}} m_1$.*

Proof. By Lemma 4.2, we can assume that $\alpha_1 = 0$.

(1) Now $u(m_1) = \beta_1$, $u(m_2) = \beta_2$, $0 \leq \beta_1 < \beta_2 < n-1+h$, and we only need to prove the case $\beta_2 = \beta_1 + 1$. Suppose that $\beta_1 = \beta(n-2)+\gamma$, where β, γ are nonnegative integers and $\gamma < n-2$. If $\gamma = 0$, then $\beta_2 = \beta(n-2)+1$, so that by Lemma 4.2, $x_1^{d-\beta}x_{n-1}^{\beta}$ and $x_1^{d-\beta-1}x_2x_{n-1}^{\beta}$ are the top-representatives of the fibers of m_1 and m_2, respectively; thus, $m_1 \succ_{\text{lex}} m_2$. If $\gamma > 0$, then $\beta_2 = \beta(n-2)+\gamma+1$, so that by Lemma 4.2, $x_1^{d-\beta-1}x_{\gamma+1}x_{n-1}^{\beta}$ and $x_1^{d-\beta-1}x_{\gamma+2}x_{n-1}^{\beta}$ are the top-representatives of the fibers of m_1 and m_2, respectively; thus, $m_1 \succ_{\text{lex}} m_2$.

(2) Suppose that $\beta_1 = \beta(n-2)+\gamma$ and $\beta_2 = \beta'(n-2)+\gamma'$, where β, β', γ, γ' are nonnegative integers and $\gamma, \gamma' < n-2$. Then

$$\beta_1 - \beta_2 = (\beta-\beta')(n-2)+\gamma-\gamma' \leq \alpha_2(n-2),$$

that is,

$$(*) \qquad (\beta-(\beta'+\alpha_2))(n-2) \leq \gamma'-\gamma.$$

If $\gamma = \gamma' = 0$, then by $(*)$, we have $\beta \leq \beta'+\alpha_2$ and, by Lemma 4.2, we see that $x_1^{d-\beta}x_{n-1}^{\beta}$ and $x_1^{d-(\beta'+\alpha_2)}x_{n-1}^{\beta'}x_n^{\alpha_2}$ are the top-representatives of the fibers of m_1 and m_2, respectively, so that $m_1 \succ_{\text{lex}} m_2$. If $\gamma = 0$ and $\gamma' > 0$, then $\gamma' - \gamma < n-2$; hence, by $(*)$, we have $\beta \le \beta' + \alpha_2$ and, by Lemma 4.2, we see that $x_1^{d-\beta} x_{n-1}^{\beta}$ and $x_1^{d-(\beta'+\alpha_2)-1} x_{\gamma'+1} x_{n-1}^{\beta'} x_n^{\alpha_2}$ are the top-representatives of the fibers of m_1 and m_2, respectively, so that $m_1 \succ_{\text{lex}} m_2$. If $\gamma > 0$ and $\gamma' = 0$, then $\gamma' - \gamma < 0$; hence, by $(*)$, we have $\beta < \beta' + \alpha_2$. By Lemma 4.2, we see that $x_1^{d-\beta-1} x_{\gamma+1} x_{n-1}^{\beta}$ and $x_1^{d-(\beta'+\alpha_2)} x_{n-1}^{\beta'} x_n^{\alpha_2}$ are the top-representatives of the fibers of m_1 and m_2, respectively, so that $m_1 \succ_{\text{lex}} m_2$. If $\gamma > 0$ and $\gamma' > 0$, then by Lemma 4.2, we see that $x_1^{d-\beta-1} x_{\gamma+1} x_{n-1}^{\beta}$ and $x_1^{d-(\beta'+\alpha_2)-1} x_{\gamma'+1} x_{n-1}^{\beta'} x_n^{\alpha_2}$ are the top-representatives of the fibers of m_1 and m_2, respectively. And, by $(*)$, we have either $\gamma' \ge \gamma$, $\beta \le \beta' + \alpha_2$ or $\gamma' < \gamma$, $\beta < \beta' + \alpha_2$; then, it follows that $m_1 \succ_{\text{lex}} m_2$.

(3) We use the notations in the proof of (2). Now $(\beta - (\beta' + \alpha_2))(n - 2) > \gamma' - \gamma$. If $\gamma' \ge \gamma$, then $\beta > \beta' + \alpha_2$, and, similar to the proof of (2), it is easy to check that $m_2 \succ_{\text{lex}} m_1$. If $\gamma' < \gamma$, then $\gamma' - \gamma > -(n-2)$; hence, $\beta \ge \beta' + \alpha_2$, so that, similar to the proof of (2), we get $m_2 \succ_{\text{lex}} m_1$. □

Remark 4.6. By Lemma 4.5, we make the following remarks.

(1) By Lemma 4.5, we see that the lex order $\succ_{\text{lex}}$ induces a total order on the set of nonnegative integers.

(2) If L_d is a lex d-monomial space, then by Lemma 4.5, it is easy to see that L_d is n-compressed and $|L_d(0)| \ge |L_d(1)| \ge |L_d(2)| \ge \cdots$.

(3) If L_d is a lex d-monomial space and $|L_d(i)| < n - 1 + h$ for some $i \ge 0$, then by Lemma 4.5, one easily sees that $|L_d(i+1)| \le \max\{0, |L_d(i)| - (n-2)\}$.

(4) If L_d is a lex d-monomial space, then $|L_d(i+j)| \ge (|L_d(i)| - 1) - j(n-2)$ for $i, j \ge 0$. Indeed, if $|L_d(i)| - (|L_d(i+j)| + 1) > j(n-2)$, then by Lemma 4.5 (3), it is easy to see that L_d is not lex, which is a contradiction.

(5) Let L_d be a lex d-monomial space spanned by monomials $m_1, \dots, m_s \in R_d$ with $0 = u(m_1) < \cdots < u(m_s)$, and $L'_{d'}$ a lex d'-monomial space spanned by monomials $m'_1, \dots, m'_s \in R_{d'}$ with $0 = u(m'_1) < \cdots < u(m'_s)$. Then, by Lemma 4.5, we have $u(m_i) = u(m'_i)$ for $1 \le i \le s$. In particular, by Lemma 3.1, we have $\dim_k R_1 L_d = \dim_k R_1 L'_{d'}$.

(6) Let W be a d-monomial space spanned by monomials $w_1, \dots, w_s \in R_d$ with $u(w_1) < \cdots < u(w_s)$. If $u(w_s) > d$, setting $\alpha = u(w_s) - d$ and $W' = \operatorname{span}\{x_1^{\alpha} w_1, \dots, x_1^{\alpha} w_s\} \subseteq R_{d+\alpha}$, we have that $u(x_1^{\alpha} w_i) = u(w_i)$, $u(x_1^{\alpha} w_s) = d+\alpha$, and Lemma 3.1 implies that $\dim_k R_1 W = \dim_k R_1 W'$. So, by (5) and the above observation, to prove Lemma 2.2, we can always assume that $u(w_s) \le d$, and then, for any $0 \le j \le u(w_s)$, there exists an $m = x_1^{d-j} x_2^{j}$ in R_d such that $u(m) = j$. Furthermore, there exists a $\widehat{w_i} \in R_d$ such that $u(\widehat{w_i}) = u(w_i) - u(w_1)$. Let $\widehat{W} = \operatorname{span}\{\widehat{w_1}, \dots, \widehat{w_s}\} \subseteq R_d$; then, by Lemma 3.1, we have $\dim_k R_1 W = \dim_k R_1 \widehat{W}$, so that, to prove Lemma 2.2, we can also assume that $u(w_1) = 0$.

Lemma 4.7. *Let L_d be a* lex *d-monomial space in R_d such that $L_d \ne R_d$, and let m be the first monomial in $R_d \backslash L_d$. Then*

$$\dim_k R_1(L_d + km) - \dim_k R_1 L_d = \begin{cases} n & \text{if } u(m) = 0 \\ 2 & \text{if } 1 \le u(m) \le h \\ 1 & \text{if } u(m) > h. \end{cases}$$

Proof. Let $a_m = \dim_k R_1(L_d + km) - \dim_k R_1 L_d$; by Lemma 3.1 and Remark 4.6 (5), we see that a_m depends only upon $u(m)$ and does not depend upon d. If $u(m) = 0$, then it is clear that $a_m = n$. If $u(m) > h$, then by Lemma 4.3 (2), we see that $a_m \ge 1$.

If $1 \le u(m) \le h$, then $a_m \ge 2$. Indeed, if $x_{n-1} m \in R_1 L_d$, then $x_{n-1} m = x_j m'$ in R_d for some $j \ne n-1$ and $m' \in L_d$. Since $u(x_{n-1} m) = u(x_{n-1}) + u(m) \le n - 2 + h$, it follows that $u(m') \le n - 2 + h$. Note that $m' \succ_{\text{lex}} m$. Then, by Lemma 4.5 (1), we see that $u(m') < u(m)$; hence, $x_j = x_n$, and then $u(x_{n-1} m) = u(x_n m') \ge n - 1 + h$, which is a contradiction. Thus, $x_{n-1} m \notin R_1 L_d$. By Lemma 4.3 (2), we see that $x_n m$ is also not in $R_1 L_d$, so $a_m \ge 2$.

Next we set $d = n + h$ and consider R_{n+h}. By Lemma 4.2, it is easy to see that, for any monomial $m \in R_{n+h}$, $u(m) \ge n - 1 + h$ if and only if $m = x_n m'$ for some monomial $m' \in R_{n-1+h}$, so that

$$R_{n+h} = x_n R_{n-1+h} \bigoplus \Bigg(\bigoplus_{i=0}^{n-2+h} k m_i \Bigg),$$

where $m_i = x_1^{n+h-i} x_2^i$ in R_{n+h} is such that $u(m_i) = i$; thus, we have

$$\dim_k R_{n+h} - \dim_k R_{n-1+h} = n - 1 + h.$$

On the other hand, since R_{n-1+h} is a lex $(n-1+h)$-monomial space and $R_{n+h} = R_1 R_{n-1+h}$, it follows that

$$\begin{aligned} &\dim_k R_{n+h} - \dim_k R_{n-1+h} \\ &\qquad = (n-1) + \sum_{1 \le u(m) \le h} (a_m - 1) + \sum_{u(m) > h} (a_m - 1) \\ &\qquad \ge n - 1 + h. \end{aligned}$$

Since the equality holds, we must have that $a_m = 2$ if $1 \le u(m) \le h$ and $a_m = 1$ if $u(m) > h$. □

Lemma 4.8. *Let C be an n-compressed d-monomial space.*

(1) *$R_1 C$ is an n-compressed $(d+1)$-monomial space.*

(2) *If C is spanned by monomials $c_1, \dots, c_s \in R_d$ with $u(c_i) = i - 1$ and $s \le h + 1$, then $|R_1 C(0)| = n - 2 + s$, $|R_1 C(1)| = s$, $|R_1 C(j)| = 0$ for $j \ge 2$, and $\dim_k R_1 C = n + 2(s-1)$.*

(3) *If C is spanned by monomials $c_1, \dots, c_s \in R_d$ with $u(c_i) = i - 1$ and $h + 2 \le s \le n - 1 + h$, then $|R_1 C(0)| = n - 1 + h$, $|R_1 C(1)| = s$, $|R_1 C(j)| = 0$ for $j \ge 2$, and $\dim_k R_1 C = n - 1 + h + s$.*

Proof. (1) Let m be a monomial in $R_1 C$ such that $u(m) = p(n - 1 + h) + q$ for some $p \ge 0$ and $1 \le q < n - 1 + h$; then $m = x_j m'$ for some j and $m' \in C$. If $n - 1 + h$ divides $u(m')$, then $j \ne 1$ or n, so that $x_{j-1} m' \in R_1 C$ and $u(x_{j-1} m') = u(x_j m') - 1 = u(m) - 1$; if $n - 1 + h$ does not divide $u(m')$, then since C is n-compressed, we have a monomial $m'' \in C$ such that $u(m'') = u(m') - 1$, so that $x_j m'' \in R_1 C$ and $u(x_j m'') = u(x_j m') - 1 = u(m) - 1$. So $R_1 C$ is an n-compressed $(d+1)$-monomial space.

(2) It is clear that $|R_1 C(j)| = 0$ for $j \ge 2$. By Lemma 3.1, we have

$$\begin{aligned} \dim_k R_1 C &= sn - \sum_{1 \le i \le s-1} \lambda(c_i, c_{i+1}) \\ &= sn - (s-1)(n-2) \\ &= n + 2(s-1). \end{aligned}$$

Thus, $|R_1C(0)| + |R_1C(1)| = n + 2(s-1)$. By (1), we know that R_1C is n-compressed, so that $u(x_{n-1}c_s) = n-2+s-1 < n-1+h$ and $u(x_nc_s) = n-1+h+s-1$ imply that $|R_1C(0)| \geq n-2+s$ and $|R_1C(1)| \geq s$. Thus, $|R_1C(0)| = n-2+s$ and $|R_1C(1)| = s$.

(3) It is clear that $|R_1C(j)| = 0$ for $j \geq 2$. By Lemma 3.1, we have

$$\begin{aligned}\dim_k R_1C &= sn - \sum_{1\leq i\leq s-1} \lambda(c_i, c_{i+1}) \\ &\quad - \sum_{1\leq i\leq s-h-1} \lambda(c_i, c_{i+h+1}) \\ &= sn - (s-1)(n-2) - (s-h-1) \\ &= n-1+h+s.\end{aligned}$$

Thus, $|R_1C(0)| + |R_1C(1)| = n-1+h+s$. By (1), we know that R_1C is n-compressed, so that $u(x_{n+h-s}c_s) = n-2+h < n-1+h$ and $u(x_nc_s) = n-1+h+s-1$ imply that $|R_1C(0)| \geq n-1+h$ and $|R_1C(1)| \geq s$. Thus, $|R_1C(0)| = n-1+h$ and $|R_1C(1)| = s$. □

Lemma 4.9. *Let W be a d-monomial space spanned by monomials $w_1, \ldots, w_s \in R_d$ with $u(w_1) < \cdots < u(w_s) \leq d$, and $u(w_s) - u(w_1) < n-1+h$. Let C be the n-compressed d-monomial space spanned by monomials $c_1, \ldots, c_s \in R_d$ with $u(c_i) = i-1$ for $1 \leq i \leq s$, and set $\widehat{W} = \{\text{monomial } m \in R_1W \mid u(w_1) \leq u(m) < u(w_1)+n-1+h\}$. Then $|\widehat{W}| \geq |R_1C(0)|$ and $\dim_k R_1W \geq \dim_k R_1C$.*

Proof. By Remark 4.6 (6), we can assume that $u(w_1) = 0$. Then $u(w_s) < n-1+h$, and $\widehat{W} = R_1W(0)$. By Lemma 4.8, we see that $|R_1C(1)| = s$; hence, $|R_1W(1)| \geq s = |R_1C(1)|$. Note that $\dim_k R_1W = |R_1W(0)| + |R_1W(1)|$ and $\dim_k R_1C = |R_1C(0)| + |R_1C(1)|$; thus, we only need to prove that $|R_1W(0)| \geq |R_1C(0)|$.

First we suppose $s \leq h+1$; then, by Lemma 4.8, we have $|R_1C(0)| = n-2+s$. If there exist w_i and w_{i+1} such that $u(w_{i+1}) - u(w_i) > n-2$, then $0 = u(x_1w_1) < u(x_1w_2) < \cdots < u(x_1w_i) < u(x_2w_i) < \cdots < u(x_{n-1}w_i) < u(x_1w_{i+1}) < \cdots < u(x_1w_s) < n-1+h$, which implies that $|R_1W(0)| \geq s+n-2 = |R_1C(0)|$. So we can assume that $u(w_{i+1}) - u(w_i) \leq n-2$ for $1 \leq i \leq s-1$. For any nonnegative integer $l \leq u(x_{n-1}w_s)$, there exists a w_i such that $u(w_i)$ is maximal with respect to the property that $u(w_i) \le l$. Then it is easy to see that $0 \le l - u(w_i) \le n-3$ and $u(x_{l-u(w_i)+1}w_i) = l$. Therefore, if $u(x_{n-1}w_s) \ge n-1+h$, then

$$|R_1W(0)| = n-1+h \ge n-2+s = |R_1C(0)|;$$

if $u(x_{n-1}w_s) < n-1+h$, then

$$\begin{aligned}|R_1W(0)| &= u(x_{n-1}w_s)+1 \ge (n-2)+(s-1)+1\\ &= |R_1C(0)|.\end{aligned}$$

Next we suppose $h+2 \le s \le n-1+h$. Then, by Lemma 4.8, we have $|R_1C(0)| = n-1+h$, and it is easy to see that $u(w_{i+1})-u(w_i) \le n-2$ for $1 \le i \le s-1$ and $u(x_{n-1}w_s) \ge n-1+h$. Therefore, similar to the above argument, we have $|R_1W(0)| = n-1+h = |R_1C(0)|$. □

Lemma 4.10. *Let W be a d-monomial space spanned by monomials $w_1, \dots, w_s \in R_d$ with $u(w_1) < \cdots < u(w_s) \le d$. If there exists $1 \le i < j \le s$ such that $j-i \ge h$ and $u(w_j)-u(w_i) < n-1+h$, then*

$$\dim_k R_1L_W \le \dim_k R_1W,$$

where L_W is the lex *d-monomial space in R_d such that* $\dim_k L_W = \dim_k W$.

Proof. By Lemma 4.7, we have that $\dim_k R_1L_W \le \dim_k L_W + (n-1) + h = \dim_k W + n-1+h = s+n-1+h$. On the other hand, it is easy to check that, if $1 \le p < i$, then $x_1w_p \notin R_1\mathrm{span}\{w_{p+1}, \dots, w_i, \dots, w_j\}$; if $j < q \le s$, then $x_nw_q \notin R_1\mathrm{span}\{w_1, \dots, w_j, \dots, w_{q-1}\}$. Thus, we have

$$\dim_k R_1W \ge \dim_k R_1\mathrm{span}\{w_i, \dots, w_j\} + (i-1) + (s-j).$$

By Lemmas 4.8 and 4.9, it is easy to see that

$$\dim_k R_1\mathrm{span}\{w_i, \dots, w_j\} \ge n-1+h+(j-i+1).$$

Therefore, we have

$$\begin{aligned}\dim_k R_1W &\ge n-1+h+(j-i+1)+(i-1)+(s-j)\\ &= n-1+h+s\\ &\ge \dim_k R_1L_W. \quad \square\end{aligned}$$

Lemma 4.11. *Let C be an n-compressed d-monomial space in R_d, and suppose that there exists a $t \geq 0$ such that $0 < |C(i)| \leq h$ for $i = 0, \ldots, t$ and $|C(i)| = 0$ for $i > t$. Then*

$$\dim_k R_1 L_C \leq \dim_k R_1 C,$$

where L_C is the lex *d-monomial space in R_d such that $\dim_k L_C = \dim_k C$.*

Proof. If $|C(j)| < |C(j+1)| + (n-2)$ for some $0 \leq j \leq t-1$, then we consider the n-compressed d-monomial space C' such that

$$\begin{aligned} |C'(j)| &= |C(j)| + 1, \\ |C'(t)| &= |C(t)| - 1, \\ |C'(i)| &= |C(i)| \text{ if } i \neq j, t. \end{aligned}$$

By Lemma 4.8, one easily sees that

$$\begin{aligned} |R_1 C(0)| &= |C(0)| + (n-2), \\ |R_1 C(i)| &= \max\{|C(i)| + (n-2), |C(i-1)|\} \text{ for } 1 \leq i \leq t, \\ |R_1 C(t+1)| &= |C(t)|, \\ |R_1 C(i)| &= 0 \text{ for } i > t+1, \end{aligned}$$

and we have similar formulas for C'. Then it is easy to check that

$$\begin{aligned} |R_1 C'(j)| &\leq |R_1 C(j)| + 1, \\ |R_1 C'(t)| &\leq |R_1 C(t)|, \\ |R_1 C'(t+1)| &= |R_1 C(t+1)| - 1, \\ |R_1 C'(i)| &= |R_1 C(i)| \text{ for } i \neq j, t, t+1. \end{aligned}$$

Therefore, we have that $\dim_k C' = \dim_k C$ and $\dim_k R_1 C' \leq \dim_k R_1 C$. If $|C'(j)| = h+1$, then by Lemma 4.10, $\dim_k R_1 L_C \leq \dim_k R_1 C'$, and then $\dim_k R_1 L_C \leq \dim_k R_1 C$. So we can assume that $|C'(j)| \leq h$, that is, C' satisfies the assumption of the Lemma.

By the above observation, we can assume that C is an n-compressed d-monomial space in R_d and there exists $t \geq 0$, such that $0 < |C(i)| \leq h$

for $0 \le i \le t$, $|C(i)| \ge |C(i+1)|+(n-2)$ for $0 \le i \le t-1$, and $|C(i)| = 0$ for $i > t$. Then by Lemma 4.8, it is easy to see that

$$\begin{aligned}\dim_k R_1 C &= |C(0)| + (n-2) + |C(0)| + |C(1)| + \cdots + |C(t)| \\ &= |C(0)| + n - 2 + \dim_k C.\end{aligned}$$

If $|L_C(0)| > |C(0)|$, then by Remark 4.6 (4), we have that, for $1 \le i \le t$,

$$|L_C(i)| \ge |L_C(0)| - 1 - i(n-2) \ge |C(0)| - i(n-2) \ge |C(i)|,$$

and then

$$\begin{aligned}\dim_k L_C &\ge |L_C(0)| + |L_C(1)| + \cdots + |L_C(t)| \\ &> |C(0)| + |C(1)| + \cdots + |C(t)| \\ &= \dim_k C,\end{aligned}$$

which is a contradiction. So we have $|L_C(0)| \le |C(0)| \le h$. By Remark 4.6 (2), we see that $|L_C(i)| \le h$ for $i \ge 0$. Thus, by Remark 4.6 (3), one easily sees that there exists a $t' \ge 0$ such that $|L_C(i)| \ge |L_C(i+1)| + (n-2)$ for $0 \le i \le t'-1$, and $|L_C(i)| = 0$ for $i > t'$. Therefore, by Lemma 4.8, it is easy to see that

$$\begin{aligned}\dim_k R_1 L_C &= |L_C(0)| + (n-2) + |L_C(0)| + |L_C(1)| + \cdots + |L_C(t')| \\ &= |L_C(0)| + (n-2) + \dim_k L_C \\ &\le |C(0)| + n - 2 + \dim_k C \\ &= \dim_k R_1 C. \qquad \square\end{aligned}$$

Proof of Theorem 4.1. Let W be a d-monomial space spanned by monomials $w_1, \ldots, w_s$ in R_d with $u(w_1) < \cdots < u(w_s)$; by Lemma 2.2, we only need to prove that

$$\dim_k R_1 L_W \le \dim_k R_1 W,$$

where L_W is the lex d-monomial space in R_d such that $\dim_k L_W = \dim_k W$.

By Remark 4.6 (6), we can assume that $u(w_1) = 0$ and $u(w_s) \le d$. Note that there exist $1 = i_0 < i_1 < \cdots < i_t \le s$ for some $t \ge 0$ such that $u(w_s)-u(w_{i_t}) < n-1+h$, and for $1 \leq j \leq t$, $u(w_{i_j-1})-u(w_{i_{j-1}}) < n-1+h$ and $u(w_{i_j}) - u(w_{i_{j-1}}) \geq n-1+h$. Set

$$\begin{aligned} W[0] &= \{w_{i_0}, \dots, w_{i_1-1}\}, \\ W[1] &= \{w_{i_1}, \dots, w_{i_2-1}\}, \\ &\cdots\cdots, \\ W[t] &= \{w_{i_t}, \dots, w_s\}. \end{aligned}$$

Then, by Lemma 4.10, we can assume that $|W[j]| \leq h$ for $0 \leq j \leq t$.

Let C be the n-compressed d-monomial space such that $|C(j)| = |W[j]|$ for $0 \leq j \leq t$ and $|C(j)| = 0$ for $j \geq t+1$. Then $\dim_k C = \dim_k W$, and it is easy to see that

$$\begin{aligned} \dim_k R_1 C &= |R_1C(0)| + |R_1C(1)| + \cdots \\ &\quad + |R_1C(t)| + |R_1C(t+1)|, \\ \dim_k R_1 W &= |(R_1W)[0]| + |(R_1W)[1]| + \cdots \\ &\quad + |(R_1W)[t]| + |(R_1W)[t+1]|, \end{aligned}$$

where $(R_1W)[0] = R_1W(0)$, $(R_1W)[t+1]$ is the set of monomials $m \in R_1W$ such that $u(m) \geq u(w_{i_t}) + n - 1 + h$, and for $1 \leq j \leq t$, $(R_1W)[j]$ is the set of monomials $m \in R_1W$ such that $u(w_{i_{j-1}}) + n - 1 + h \leq u(m) < u(w_{i_j}) + n - 1 + h$. First, it is easy to see that

$$|(R_1W)[t+1]| \geq |W[t]| = |C(t)| = |R_1C(t+1)|.$$

Then By Lemma 4.9, we get

$$|R_1W(0)| \geq |R_1C(0)|.$$

Finally, by Lemma 4.8 it is easy to see that, for $1 \leq j \leq t$,

$$|R_1C(j)| = \max\{|C(j-1)|, |C(j)| + (n-2)\};$$

if $|R_1C(j)| = |C(j-1)|$, then we have

$$|(R_1W)[j]| \geq |W[j-1]| = |C(j-1)| = |R_1C(j)|;$$

if $|R_1C(j)| = |C(j)| + (n-2)$, then by Lemma 4.9, we also have

$$|(R_1W)[j]| \geq |R_1C(j)|.$$

So, we get $\dim_k R_1W \geq \dim_k R_1C$. By Lemma 4.11, we know that $\dim_k R_1C \geq \dim_k R_1L_C$, where L_C is the lex d-monomial space such that $\dim_k L_C = \dim_k C$. Note that $L_C = L_W$, so $\dim_k R_1W \geq \dim_k R_1L_W$. □

5. Two other classes of projective monomial curves. The main results of this section are Theorems 5.1 and 5.5.

Theorem 5.1. *Let*

$$A = \begin{pmatrix} 0 & 1+h & 2+h & \cdots & n-1+h \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix},$$
$$\text{where } n \geq 3, \quad h \in \mathbf{Z}^+.$$

Let R be the toric ring associated to A.

(1) *If $h = 1$, then Macaulay's theorem holds over R.*

(2) *If $n = 3$, then Macaulay's theorem holds over R.*

(3) *If $h \geq 2$ and $n \geq 4$, then Macaulay's theorem does not hold over R.*

In order to prove Theorem 5.1, we need Lemmas 5.2, 5.3 and 5.4.

Lemma 5.2. *Let R be the toric ring defined in Theorem* 5.1 *and R' the toric ring defined in Section* 4 *such that R and R' satisfy the assumptions of Lemma* 3.2. *Then we have an isomorphism $\widehat{f}$: $S = k[x_1, \ldots, x_n] \to S' = k[y_1, \ldots, y_n]$ with $\widehat{f}(x_i) = y_{n+1-i}$, which induces an isomorphism f from R to R'. Setting $x_1 > \cdots > x_n$ and $y_1 > \cdots > y_n$ as usual, by Definition* 2.1, *we have the* lex *orders $\succ_{\text{lex}}$, $\succ_{\text{lex}'}$ in R and R'.*

(1) *Let m be a monomial in R_d such that $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ is the top representative of the fiber of the monomial $f(m) \in R'_d$. Then $\widehat{f}^{-1}(y_1^{\alpha_1} \cdots y_n^{\alpha_n}) = x_1^{\alpha_n} \cdots x_n^{\alpha_1}$ is the top-representative of the fiber of m.*

(2) *Let m and m' be two monomials in R_d such that $u(m) < u(m')$. Then $m \succ_{\text{lex}} m'$ in R_d, so that the* lex *order $\succ_{\text{lex}}$ in R_d is the same as the natural order $>_u$ defined in Remark* 2.4.

Proof. (1) Suppose that $x_1^{\beta_n} \cdots x_n^{\beta_1}$ is the top representative of the fiber of m. Then $\beta_n \geq \alpha_n$ and $\widehat{f}(x_1^{\beta_n} \cdots x_n^{\beta_1}) = y_1^{\beta_1} \cdots y_n^{\beta_n}$ is a monomial in the fiber of $f(m)$. Since $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ is the top representative of the fiber of $f(m)$, by Lemma 4.2 we have $\beta_n \leq \alpha_n$, so that $\beta_n = \alpha_n$, and then $\beta_{n-1} \geq \alpha_{n-1}$. But, by Lemma 4.2, we have $\beta_{n-1} \leq \alpha_{n-1}$, so that $\beta_{n-1} = \alpha_{n-1}$. If there exists $2 \leq i \leq n-2$ such that $\beta_i > \alpha_i$ and $\beta_j = \alpha_j$ for $j > i$, then the monomial $y_1^{\beta_1} \cdots y_i^{\beta_i} y_{i+1}^{\alpha_{i+1}} \cdots y_n^{\alpha_n}$ is in the fiber of $f(m)$. By Lemma 4.2, one easily sees that $\beta_i \leq \alpha_i$, which is a contradiction, so we have $\beta_i = \alpha_i$ for $i = 2, \ldots, n-2$. Since $\deg(m) = \beta_1 + \cdots + \beta_n = \alpha_1 + \cdots + \alpha_n$, it follows that $\beta_1 = \alpha_1$, and then $x_1^{\alpha_n} \cdots x_n^{\alpha_1} = x_1^{\beta_n} \cdots x_n^{\beta_1}$ is the top-representative of the fiber of m.

(2) Let $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, $y_1^{\beta_1} \cdots y_n^{\beta_n}$ be the top-representatives of the fibers of $f(m)$ and $f(m')$. Then (1) implies that $x_1^{\alpha_n} \cdots x_n^{\alpha_1}$, $x_1^{\beta_n} \cdots x_n^{\beta_1}$ are the top-representatives of the fibers of m and m'. Since $u(m) < u(m')$, by Lemma 3.3 (1), we have $u(f(m)) > u(f(m'))$, so that Lemma 4.2 implies $\alpha_n \geq \beta_n$. If $\alpha_n > \beta_n$, then $m \succ_{\text{lex}} m'$ and we are done. So we may assume $\alpha_n = \beta_n$. Then similarly, by Lemma 4.2, we have $\alpha_{n-1} \geq \beta_{n-1}$, and if $\alpha_{n-1} > \beta_{n-1}$, we are done. So we can also assume that $\alpha_{n-1} = \beta_{n-1}$. Then, applying Lemma 4.2 again, we see that there exist $2 \leq r \leq n-2$ and $1 \leq r' \leq r-1$ such that

$$y_1^{\alpha_1} \cdots y_n^{\alpha_n} = y_1^{d-1-\alpha_{n-1}-\alpha_n} y_r y_{n-1}^{\alpha_{n-1}} y_n^{\alpha_n},$$
$$y_1^{\beta_1} \cdots y_n^{\beta_n} = y_1^{d-1-\alpha_{n-1}-\alpha_n} y_{r'} y_{n-1}^{\alpha_{n-1}} y_n^{\alpha_n},$$

and then we have that

$$\begin{aligned} x_1^{\alpha_n} \cdots x_n^{\alpha_1} &= x_1^{\alpha_n} x_2^{\alpha_{n-1}} x_{n+1-r} x_n^{d-1-\alpha_{n-1}-\alpha_n} \\ &>_{\text{lex}} x_1^{\alpha_n} x_2^{\alpha_{n-1}} x_{n+1-r'} x_n^{d-1-\alpha_{n-1}-\alpha_n} \\ &= x_1^{\beta_n} \cdots x_n^{\beta_1}, \end{aligned}$$

which implies $m \succ_{\text{lex}} m'$. □

Lemma 5.3. *Let R be the toric ring defined in Theorem* 5.1*, and suppose $h = 1$. Let L_d be an r-dimensional* lex *d-monomial space in R_d with $0 \leq r < \dim_k R_d$ and m the first monomial in $R_d \backslash L_d$. If we set*

$$a_r = \dim_k R_1(L_d + km) - \dim_k R_1 L_d,$$

then $a_0 = n$, $a_1 = 2$ *and* $a_r = 1$ *for* $1 < r < \dim_k R_d$.

Proof. Without loss of generality, we can assume $d \geq 1$. It is clear that $a_0 = n$. If $r = 1$, then it is easy to see that $L_d = \operatorname{span}\{x_1^d\}$ and $m = x_1^{d-1}x_2$ in R_d, so that by Lemma 3.1,

$$\dim_k R_1(L_d + km) = 2n - \lambda(x_1^d, x_1^{d-1}x_2) = 2n - (n-2) = n+2;$$

hence, $a_0 + a_1 = n + 2$, and then $a_1 = 2$. If $1 < r < \dim_k R_d$, by Lemma 5.2, we see that $u(x_n m) > u(x_j m')$ for any $1 \leq j \leq n$ and any monomial $m' \in L_d$; hence, $x_n m \notin R_1 L_d$, and then $a_r \geq 1$ for $1 < r < \dim_k R_d$. Note that $\dim_k R_1 R_d = \dim_k R_{d+1}$, and it is easy to see that

$$\dim_k R_{d+1} - \dim_k R_d = \dim_k R'_{d+1} - \dim_k R'_d = n - 1 + h = n,$$

where R' is the toric ring defined in Lemma 5.2. Thus,

$$(a_0 - 1) + (a_1 - 1) + \sum_{1<r<\dim_k R_d} (a_r - 1) = n,$$

so that $\sum_{1<r<\dim_k R_d}(a_r - 1) = 0$, which implies $a_r = 1$ for $1 < r < \dim_k R_d$. □

Lemma 5.4. *Let* R *and* R' *be the toric rings defined in Lemma* 5.2, *and suppose* $n = 3$. *If* L_d, L'_d *are* lex d*-monomial spaces in* R_d *and* R'_d *such that* $\dim_k L_d = \dim_k L'_d$, *then* $\dim_k R_1 L_d = \dim_k R'_1 L'_d$.

Proof. Since the toric ring R is defined by the matrix $A = \begin{pmatrix} 0 & 1+h & 2+h \\ 1 & 1 & 1 \end{pmatrix}$ and $\operatorname{Ker} A$ has dimension 1, one easily sees that the toric ideal $I_{\mathcal{A}}$ is generated by the binomial $x_2^{2+h} - x_1 x_3^{1+h}$, so that we have $R = k[x_1, x_2, x_3]/(x_2^{2+h} - x_1 x_3^{1+h})$, and similarly, $R' = k[y_1, y_2, y_3]/(y_2^{2+h} - y_1^{1+h} y_3)$.

Let T_d be the set of monomials in $k[x_1, x_2, x_3]_d$ which cannot be divided by x_2^{2+h}, and let T'_d be the set of monomials in $k[y_1, y_2, y_3]_d$ which cannot be divided by y_2^{2+h}. It is easy to see that, for any monomial

$m \in R_d$, there is one and only one monomial in the fiber of m that cannot be divided by x_2^{2+h}. Then it follows that the monomials in R_d are in one-to-one correspondence with the monomials in T_d. Furthermore, if $\dim_k L_d = r$ and L_d is spanned by the monomials $m_1, \dots, m_r \in R_d$ with $u(m_1) < \cdots < u(m_r)$, then $m_1, \dots, m_r$ have top-representatives $w_1, \dots, w_r \in T_d$ that are the first r monomials in T_d. Similarly, if $\dim_k L'_d = r$ and L'_d is spanned by monomials $m'_1, \dots, m'_r \in R'_d$, then $m'_1, \dots, m'_r$ have top-representatives $w'_1, \dots, w'_r \in T'_d$ that are the first r monomials in T'_d.

Note that the natural isomorphism $g : S = k[x_1, x_2, x_3] \to S' = k[y_1, y_2, y_3]$ with $g(x_j) = y_j$ for $j = 1, 2, 3$ induces an order-preserving bijection between T_d and T'_d. Then $g(w_i) = w'_i$ for $1 \le i \le r$. Setting $W = \operatorname{span}\{w_1, \dots, w_r\} \subseteq S_d$ and $W' = \operatorname{span}\{w'_1, \dots, w'_r\} \subseteq S'_d$, one easily sees that $\dim_k S_1 W = \dim_k S'_1 W'$. Let p be the number of monomials in $S_1 W$ that can be divided by x_2^{2+h}, and let p' be the number of monomials in $S'_1 W'$ that can be divided by y_2^{2+h}; then we have $p = p'$. Note that if $x_2 w_i$ can be divided by x_2^{2+h} for some i, then $x_2 w_i = x_3(x_1 x_3^h w_i / x_2^{1+h})$ in R_{d+1} and $x_1 x_3^h w_i / x_2^{1+h} = w_j$ for some $j < i$. Therefore, the monomials in the lex $(d+1)$-monomial space $R_1 L_d$ are in one-to-one correspondence with the monomials in $S_1 W$ that cannot be divided by x_2^{2+h}, so that we have

$$\dim_k R_1 L_d = \dim_k S_1 W - p.$$

Similarly, we have

$$\dim_k R'_1 L'_d = \dim_k S'_1 W - p',$$

and so $\dim_k R_1 L_d = \dim_k R'_1 L'_d$. □

Proof of Theorem 5.1. (1) Let W be a d-monomial space spanned by monomials $w_1, \dots, w_r \in R_d$ with $u(w_1) < \cdots < u(w_r)$. By Lemma 2.2, it suffices to prove that $\dim_k R_1 L_W \le \dim_k R_1 W$, where L_W is the lex d-monomial space in R_d such that $\dim_k L_W = \dim_k W = r$.

We prove by induction on r. If $r = 1$, then $\dim_k R_1 L_W = \dim_k R_1 W = n$. If $r = 2$, then by Lemma 5.3, $\dim_k R_1 L_W = a_0 + a_1 = n + 2$, and by Lemma 3.1, $\dim_k R_1 W = 2n - \lambda(w_1, w_2)$. It is easy to see that $\lambda(w_1, w_2) \le n - 2$. Thus, we have

$$\dim_k R_1 W \ge 2n - (n-2) = n + 2 = \dim_k R_1 L_W.$$

If $r > 2$,let $\widehat{W}$ be the d-monomial space spanned by monomials $w_1, \ldots, w_{r-1} \in R_d$ and $L_{\widehat{W}}$ the lex d-monomial space in R_d such that $\dim_k L_{\widehat{W}} = \dim_k \widehat{W} = r-1$. Then, by induction we have $\dim_k R_1 L_{\widehat{W}} \le \dim_k R_1 \widehat{W}$. By Lemma 5.3, we see that $\dim_k R_1 L_W = \dim_k R_l L_{\widehat{W}} + 1$. On the other hand, since $u(x_n w_r) > u(x_j w_i)$ for any $1 \le j \le n$ and any $1 \le i \le r-1$, we have $x_n w_r \notin R_1 \widehat{W}$, and then $\dim_k R_1 W \ge \dim_k R_1 \widehat{W} + 1$. Therefore,

$$\dim_k R_1 W \ge \dim_k R_1 \widehat{W} + 1 \ge \dim_k R_1 L_{\widehat{W}} + 1 = \dim_k R_1 L_W,$$

and we are done.

(2) Let W be an r-dimensional d-monomial space in R_d. By Lemma 2.2, it suffices to prove that $\dim_k R_1 L_W \le \dim_k R_1 W$ where L_W is the lex d-monomial space in R_d such that $\dim_k L_W = r$.

Let f and R' be as in Lemma 5.2. Then, by Lemma 3.3 (1), we see that $f(W)$ is an r-dimensional d-monomial space in R'_d and $\dim_k R_1 W = \dim_k R'_1 f(W)$. Let $L'_{f(W)}$ be the lex d-monomial space in R'_d such that $\dim_k L'_{f(W)} = r$. Then, by Lemma 5.4, we have $\dim_k R_1 L_W = \dim_k R'_1 L'_{f(W)}$. By Theorem 4.1, we see that R' satisfies Macaulay's theorem; hence, $\dim_k R'_1 L'_{f(W)} \le \dim_k R'_1 f(W)$. So, $\dim_k R_1 L_W \le \dim_k R_1 W$, and we are done.

(3) Considering the 1-monomial space $W = \operatorname{span}\{x_2, x_3\}$ and the lex 1-monomial space $L_W = \operatorname{span}\{x_1, x_2\}$ in R_1, we have $\dim_k W = \dim_k L_W = 2$. However, by Lemma 3.1, it is easy to see that

$$\dim_k R_1 W = 2n - \lambda(x_2, x_3) = 2n - (n-2) = n+2,$$

and

$$\begin{aligned}&\dim_k R_1 L_W \\ &= 2n - \lambda(x_1, x_2) = \begin{cases} 2n-1 & \text{if } n \le h+2 \\ 2n - (1+n-h-2) = n+h+1 & \text{if } n \ge h+3. \end{cases}\end{aligned}$$

Since $h \ge 2$ and $n \ge 4$, one can easily check that $\dim_k R_1 L_W > \dim_k R_1 W$. So, by Lemma 2.2, Macaulay's theorem does not hold over R. □

Theorem 5.5. *Let*

$$A = \begin{pmatrix} 0 & 1 & \cdots & m-1 & m+h & \cdots & n-1+h \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $n \geq 4$, $2 \leq m \leq n-2$ and $h \in \mathbf{Z}^+$. Let R be the toric ring associated to A. Then Macaulay's theorem does not hold over R.

Proof. We have three cases.

Case 1. $h \leq m-1$. Let $W = \text{span}\,\{x_1^2, x_1x_2, \dots, x_1x_m, x_2x_m\} \subseteq R_2$ and $L_W = \text{span}\,\{x_1^2, x_1x_2, \dots, x_1x_m, x_1x_{m+1}\} \subseteq R_2$. Then W is a 2-monomial space in R_2 and L_W is a lex 2-monomial space in R_2 such that $\dim_k W = \dim_k L_W = m+1$. By Lemma 3.1, we have

$$\begin{aligned} \dim_k R_1 W &= (m+1)n - \sum_{1\leq i<j\leq m} \lambda(x_1x_i, x_1x_j) \\ &\quad - \sum_{1\leq i\leq m} \lambda(x_1x_i, x_2x_m), \\ \dim_k R_1 L_W &= (m+1)n - \sum_{1\leq i<j\leq m} \lambda(x_1x_i, x_1x_j) \\ &\quad - \sum_{1\leq i\leq m} \lambda(x_1x_i, x_1x_{m+1}), \end{aligned}$$

so that we get

$$\begin{aligned} &\dim_k R_1 L_W - \dim_k R_1 W \\ &\qquad = \sum_{1\leq i\leq m} \lambda(x_1x_i, x_2x_m) - \sum_{1\leq i\leq m} \lambda(x_1x_i, x_1x_{m+1}). \end{aligned}$$

It is easy to see that

$$\lambda(x_1x_m, x_2x_m) = n-2, \qquad \lambda(x_1x_{m-h}, x_2x_m) = 1,$$

and

$$\lambda(x_1x_i, x_2x_m) = 0 \quad \text{for } 1 \leq i \leq m-1 \text{ and } i \neq m-h.$$

Thus, we have

$$\sum_{1\leq i\leq m} \lambda(x_1x_i, x_2x_m) = n-2+1 = n-1.$$

On the other hand, one easily sees that

$$\lambda(x_1x_i, x_1x_{m+1}) = \begin{cases} 1 & \text{if } m-h \leq i \leq m-1; \\ 0 & \text{if } i < m-h. \end{cases}$$

If $n-m-1 \geq h+1$, then it is easy to check that

$$\begin{aligned} \lambda(x_1x_m, x_1x_{m+1}) &= 1 + ((m-1) - (h+1) + 1) \\ &\quad + ((n-m-1) - (h+1) + 1) \\ &= n - 2h - 1, \end{aligned}$$

so that we have

$$\sum_{1 \leq i \leq m} \lambda(x_1x_i, x_1x_{m+1}) = h + n - 2h - 1 = n - h - 1,$$

and then

$$\dim_k R_1 L_W - \dim_k R_1 W = n - 1 - (n - h - 1) = h \geq 1 > 0;$$

therefore, by Lemma 2.2, we see that Macaulay's theorem does not hold over R. If $n-m-1 < h+1$, then it is easy to check that

$$\lambda(x_1x_m, x_1x_{m+1}) = 1 + ((m-1) - (h+1) + 1) = m - h,$$

so that we have

$$\sum_{1 \leq i \leq m} \lambda(x_1x_i, x_1x_{m+1}) = h + m - h = m,$$

and then

$$\dim_k R_1 L_W - \dim_k R_1 W = n - 1 - m \geq n - 1 - (n-2) = 1 > 0;$$

therefore, by Lemma 2.2, we see that Macaulay's theorem does not hold over R.

Case 2. $h \geq m$ and $m < n-2$. Let W and L_W be the same 2-monomial spaces as in Case 1. Then

$$\begin{aligned} &\dim_k R_1 L_W - \dim_k R_1 W \\ &\qquad = \sum_{1 \leq i \leq m} \lambda(x_1x_i, x_2x_m) - \sum_{1 \leq i \leq m} \lambda(x_1x_i, x_1x_{m+1}). \end{aligned}$$

It is easy to see that

$$\lambda(x_1x_m, x_2x_m) = n-2, \text{ and } \lambda(x_1x_i, x_2x_m) = 0 \quad \text{for } 1 \le i \le m-1.$$

Thus, we have

$$\sum_{1\le i\le m} \lambda(x_1x_i, x_2x_m) = n-2.$$

On the other hand, one easily sees that

$$\lambda(x_1x_i, x_1x_{m+1}) = 1 \text{ for } 1 \le i \le m-1.$$

If $n-m-1 \ge h+1$, then it is easy to check that

$$\begin{aligned}\lambda(x_1x_m, x_1x_{m+1}) &= 1 + ((n-m-1) - (h+1) + 1)\\ &= n-m-h,\end{aligned}$$

so that we have

$$\sum_{1\le i\le m} \lambda(x_1x_i, x_1x_{m+1}) = m-1+n-m-h = n-h-1,$$

and then

$$\begin{aligned}\dim_k R_1 L_W - \dim_k R_1 W &= n-2-(n-h-1)\\ &= h-1 \ge m-1 \ge 1 > 0.\end{aligned}$$

Therefore, by Lemma 2.2, we see that Macaulay's theorem does not hold over R. If $n-m-1 < h+1$, then it is easy to check that $\lambda(x_1x_m, x_1x_{m+1}) = 1$, so that we have

$$\sum_{1\le i\le m} \lambda(x_1x_i, x_1x_{m+1}) = m-1+1 = m,$$

and then

$$\dim_k R_1 L_W - \dim_k R_1 W = n-2-m > n-2-(n-2) = 0.$$

Therefore, by Lemma 2.2, we see that Macaulay's theorem does not hold over R.

Case 3. $h \geq m$ and $m = n - 2$. Let p be the maximal integer such that $p \leq (h-1)/(m-1)$; then $p \geq 1$. Considering R_{p+1}, we see that, for any monomial $w \in R_{p+1}$, $0 \leq u(w) \leq (p+1)(n-1+h)$. More precisely, one can easily check that there are $(n-1)+(p-i)(m-1)+i$ monomials $w \in R_{p+1}$ such that $i(n-1+h) \leq u(w) < (i+1)(n-1+h)$ for $0 \leq i \leq p$, so that

$$\begin{aligned}\dim_k R_{p+1} &= 1 + \sum_{i=0}^{p}(n-1)+(p-i)(m-1)+i \\ &= 1+(p+1)\left(n+\frac{pm}{2}-1\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\dim_k R_{p+2} &= (n-1+h)+1 \\ &\quad + \sum_{i=0}^{p}(n-1)+(p-i)(m-1)+(i+1) \\ &= n+h+p+1+(p+1)\left(n+\frac{pm}{2}-1\right).\end{aligned}$$

Setting $l = 1+(p+1)(n+(pm/2)-1)$, we have that

$$\dim_k R_{p+1} = l$$

and

$$\dim_k R_1 R_{p+1} = \dim_k R_{p+2} = n+h+p+l.$$

Let W be the l-monomial space spanned by the monomials $w_1, \dots, w_l \in R_l$ such that $u(w_i) = i-1$ for $1 \leq i \leq l$. Let monomials $w'_1, \dots, w'_l$ be a basis of R_{p+1}, and let L_W be the l-monomial space spanned by the monomials $x_1^{l-p-1}w'_1, \dots, x_1^{l-p-1}w'_l \in R_l$. Then it is easy to see that L_W is a lex l-monomial space such that

$$\dim_k L_W = \dim_k W = l$$

and

$$\dim_k R_1 L_W = \dim_k R_1 R_{p+1} = n+h+p+l.$$

However, by Lemma 3.1, one can easily check that

$$\begin{aligned}\dim_k R_1 W &= ln - (l-1)(n-2) - ((l-1) - (h+1) + 1)\\ &= n + h - 1 + l,\end{aligned}$$

so that

$$\begin{aligned}\dim_k R_l L_W - \dim_k R_1 W &= (n+h+p+l) - (n+h-1+l)\\ &= p + 1 \geq 2 > 0;\end{aligned}$$

therefore, by Lemma 2.2, we see that Macaulay's theorem does not hold over R. □

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