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COMULTIPLICATION MODULES OVER COMMUTATIVE RINGS II

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ABSTRACT. Let R be a commutative ring with identity. A unital R-module M is a comultiplication module provided that, for each submodule N of M, there exists an ideal A of R such that N is the set of elements m in M such that $Am = 0$. It is proved that every comultiplication module with zero radical is semisimple. Moreover, for any comultiplication module M, every submodule has a unique complement and a unique closure in M. Every Noetherian comultiplication module is an Artinian quasi-injective module. In case R is a semilocal ring containing precisely n distinct maximal ideals, for some positive integer n, every comultiplication R-module has Goldie dimension at most n. On the other hand, if R is a ring with finite Goldie dimension n, for some positive integer n, then it is proved that certain faithful comultiplication R-modules have hollow dimension at most n.

1. Introduction. This paper is a continuation of [**1**]. Throughout R is a ring with identity and M is a unitary right R-module. Moreover, unless stated otherwise, *R will always denote a commutative ring.* Given submodules N and L of M, we denote by $(N :_R L)$ the set of elements r in R such that $rL \subseteq N$. Note that $(N :_R L)$ is the annihilator in R of the R-module $(L + N)/N$ and is an ideal of R. In particular, if N is a submodule of M and $m \in M$, then $(N :_R Rm)$ will be denoted simply by $(N :_R m)$, so that $(N :_R m) = \{r \in R : rm \in N\}$. On the other hand, if N is again a submodule of M and A is an ideal of R, then $(N :_M A)$ is the set of elements m in M such that $Am \subseteq N$, and it is clear that $(N :_M A)$ is a submodule of M. Recall that M is a *comultiplication module* if, for each submodule N of M, there exists an ideal A of R such that $N = (0 :_M A)$. The first result is taken from [**2**, Theorem 3.17 (d)].

Lemma 1.1. *Every submodule of a comultiplication module is also a comultiplication module.*

Proof. Clear. $\square$

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Let A be any ideal of a ring R, and let M be an R-module. Then we define $T_A(M)$ to be the set of elements m in M such that $(1-a)m = 0$ for some $a \in A$. Note that $T_A(M)$ is a submodule of M. Module M is called A-*torsion* provided $M = T_A(M)$. On the other hand, module M will be called A-*cocyclic* provided there exists a completely irreducible submodule L of M such that $(1-b)L = 0$ for some $b \in A$.

Lemma 1.2. *Let B be any ideal of R, and let M be a comultiplication R-module such that $(0 :_M B) = 0$. Then $Rm = Bm$ for every element $m \in M$. In particular, M is B-torsion.*

Proof. See [**3**, Theorem 3.2 (a)]. □

Lemma 1.3. *Every finitely generated comultiplication module is finitely cogenerated.*

Proof. See [**3**, Theorem 3.5 (b)]. □

Lemma 1.4. *Let L_i $(i \in I)$ be any collection of submodules of a comultiplication module M with $\cap_{i\in I} L_i = 0$. Then $N = \cap_{i\in I} (N + L_i)$ for every submodule N of M.*

Proof. See [**2**, Proposition 3.14 (a)]. □

Recall that a module is called *cocyclic* if it is nonzero and the intersection of all its nonzero submodules is nonzero. A submodule N of M is called *completely irreducible* provided $N \neq M$ and M/N is a cocyclic module.

Corollary 1.5. *Let L_i $(i \in I)$ be any collection of submodules of a comultiplication module M with $\cap_{i\in I} L_i = 0$, and let N be any completely irreducible submodule of M. Then there exists a $j \in I$ such that $L_j \subseteq N$.*

Proof. By Lemma 1.4,

$$N = \cap_{i\in I} (N + L_i),$$

and hence $N = N + L_j$ for some $j \in I$. It follows that $L_j \subseteq N$. □

Given any R-module M, the *radical* of M, denoted by $\operatorname{Rad} M$, is the intersection of all maximal submodules of M and, if M has no maximal submodule, $\operatorname{Rad} M$ is defined to be M. An R-module M is called a *V-module* provided every simple R-module is M-injective. Note that a module is a V-module if and only if $\operatorname{Rad}(M/N) = 0$ for every submodule N of M (see, for example, [**6**, 2.13]). We now give another application of Lemma 1.4.

Theorem 1.6. *Every comultiplication module with zero radical is semisimple.*

Proof. Let R be a ring, and let M be a comultiplication R-module such that $\operatorname{Rad} M = 0$. If M does not contain a maximal submodule, then $M = 0$, and hence M is semisimple trivially. Now suppose that M does contain a maximal submodule, and let P_i $(i \in I)$ denote the maximal submodules of M. By hypothesis, $\cap_{i\in I} P_i = 0$. Let N be any proper submodule of M. By Lemma 1.3, $N = \cap_{i\in I}(N + P_i)$. Note that, for each $i \in I$, either $N + P_i = M$ or $N + P_i = P_i$. Thus, N is an intersection of maximal submodules of M, and hence $\operatorname{Rad}(M/N) = 0$. By [**6**, 2.13], M is a V-module.

Let $m \in M$. By Lemma 1.1, Rm is a comultiplication module, and by Lemma 1.3, Rm is finitely cogenerated. There exist a positive integer n and independent simple submodules U_i $(1 \le i \le n)$ of Rm such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of Rm. But U_i is M-injective for all $1 \le i \le n$, and hence $U_1 \oplus \cdots \oplus U_n$ is M-injective. But this implies that $U_1 \oplus \cdots \oplus U_n$ is a direct summand of Rm, and hence $Rm = U_1 \oplus \cdots \oplus U_n$. Thus Rm is semisimple for each $m \in M$. Therefore M is semisimple. □

The next result can be found in [**4**, Theorem 2.3].

Corollary 1.7. *Let R be a von Neumann regular ring. Then every comultiplication R-module is semisimple.*

Proof. Since R is a von Neumann regular ring it follows that every simple R-module is injective (see, for example, [**8**, 23.5]), and hence every R-module is a V-module and has zero radical. Now apply Theorem 1.6. □

Lemma 1.8. *Let R be a ring, let P be any maximal ideal of R and let M be any comultiplication R-module. Then M is P-torsion or P-cocyclic.*

Proof. See [**4**, Lemma 2.10]. □

It is natural to ask whether the converse of Lemma 1.8 is true. It is true if M is a Noetherian module as we shall show below. First we prove the following result.

Proposition 1.9. *Let R be a ring, and let M be an R-module such that every cocyclic homomorphic image of M is a comultiplication module. Suppose further that, for every maximal ideal P of R, the module M is P-torsion or P-cocyclic. Then M is a comultiplication module.*

Proof. Let N be a submodule of M, let $m \in (0 :_M (0 :_R N))$ and let $A = \{r \in R : rm \in N\}$. Suppose that $m \notin N$. Then $A \neq R$, and there exists a maximal ideal P of R such that $A \subseteq P$. Clearly M is not P-torsion. By hypothesis, M is P-cocyclic. Let L be a completely irreducible submodule of M such that $(1-p)L = 0$ for some $p \in P$. By hypothesis, M/L is a comultiplication module. Thus, $(N + L :_M 1 - p)/L = (0 :_{M/L} B) = (L :_M B)/L$ for some ideal B of R. Note that

$$N \subseteq (N :_M 1-p) \subseteq (N + L :_M 1-p) = (L :_M B) \subseteq (0 :_M B(1-p)),$$

and it follows that $B(1-p) \subseteq (0 :_R N)$. Thus, $B(1-p)m = 0$. This implies that $(1-p)(m+L) \subseteq (0 :_{M/L} B)$, and hence $(1-p)^2 m \in N+L$. Thus, $(1-p)^3 m \in N$ and $(1-p)^3 \in A \subseteq P$, a contradiction. Thus, $m \in N$. It follows that $N = (0 :_M (0 :_R N))$ for every submodule N of M. This proves that M is a comultiplication module by [**1**, Theorem 1.5]. □

Proposition 1.9 has the following corollary which is essentially [**4**, Theorem 2.10].

Corollary 1.10. *A Noetherian R-module M is a comultiplication module if and only if for each maximal ideal P of R the module M is P-torsion or P-cocyclic.*

Proof. Every Noetherian cocyclic module is a comultiplication module by [**4**, Theorem 2.6] (or [**1**, Corollary 2.11 and Theorem 3.11]). The result follows by Lemma 1.8 and Proposition 1.9. □

Lemma 1.11. *A module M is a comultiplication module if, and only if, given an element $m \in M$ and a completely irreducible submodule L of M, $(0 :_R L) \subseteq (0 :_R m)$ implies that $m \in L$.*

Proof. By [**1**, Proposition 1.3 and Theorem 1.5]. □

Let R be any ring, and let M be an R-module. Let L_i $(i \in I)$ be the collection of all completely irreducible submodules of M. Then we define

$$\alpha(M) = \sum_{i \in I} (0 :_R L_i).$$

Note that $\alpha(M)$ is an ideal of R. By a *minimal completely irreducible submodule* of M we mean a completely irreducible submodule L of M such that there does not exist a completely irreducible submodule K of M with $K \subset L$.

Proposition 1.12. *Let R be a ring and M an R-module.*

(i) *If M is a comultiplication module then $R = \alpha(M) + (0 :_R m)$ for all $m \in M$.*

(ii) *If $R = \alpha(M) + (0 :_R m)$ for all $m \in M$ then $L = (0 :_M (0 :_R L))$ for every minimal completely irreducible submodule L of M.*

Proof. Let L_i $(i \in I)$ be the collection of all completely irreducible submodules of M.

(i) Note that

$$\begin{aligned} 0 = \cap_{i \in I} L_i &= \cap_{i \in I} (0 :_M (0 :_R L_i)) \\ &= \left(0 :_M \sum_{i \in I} (0 :_R L_i)\right) = (0 :_R \alpha(M)). \end{aligned}$$

Then $R = \alpha(M) + (0 :_R m)$ for all $m \in M$ by Lemma 1.2.

(ii) Suppose that $R = \alpha(M) + (0 :_R m)$ for all $m \in M$. Let L be any minimal completely irreducible submodule of M. Suppose that $i \in I$ with $L = L_i$. Suppose further that $(0 :_R L) \subseteq (0 :_R m)$ for some $m \in M$. By hypothesis,

$$R = \sum_{j \neq i} (0 :_R L_j) + (0 :_R m).$$

Because M/L is cocyclic, there exists an element $x \in M \setminus L$ with $(L :_R x) = P$, a maximal ideal of R, such that $Rx + L \subseteq Ry + L$ for all $y \in M \setminus L$. For each $y \in M \setminus L$,

$$(0 :_R y)x \subseteq (0 :_R y)(Ry + L) \subseteq L.$$

Thus, $(0 :_R y) \subseteq (L :_R x) = P$ for all $y \in M \setminus L$. Because L is a minimal completely irreducible submodule of M we have $L_j \nsubseteq L \, (j \neq i)$, and hence there exists $z_j \in L_j \setminus L$ $(j \neq i)$. Note that

$$(0 :_R L_j) \subseteq (0 :_R z_j) \subseteq P \, (j \neq i).$$

Next observe that

$$R = \sum_{j \neq i} (0 :_R L_j) + (0 :_R m) \subseteq P + (0 :_R m).$$

Thus, $(0 :_R m) \nsubseteq P$, and hence $m \in L$ by the above argument. It follows that $L = (0 :_M (0 :_R L))$, as required. □

Now let R be a (not necessarily commutative) ring, and let M be a right R-module. Let N and L be submodules of M such that $N \cap L = 0$. By Zorn's lemma, the collection of submodules H of M with $L \subseteq H$ and $N \cap H = 0$ has a maximal member. In particular, this shows that there exists a submodule K of M maximal with respect to the property $N \cap K = 0$, and such a submodule K is called a *complement of* N (*in* M). Thus, given $N \cap L = 0$, there exists a complement Q of N such that $L \subseteq Q$.

Theorem 1.13. *Let R be any ring, and let M be a comultiplication R-module. Then there exist minimal completely irreducible submodules L_i $(i \in I)$ of M such that the following hold.*

(i) $\cap_{i\in I} L_i = 0$.

(ii) $\cap_{i\neq j} L_i \neq 0$ *for all* $j \in I$.

(iii) *For each completely irreducible submodule* L *of* M *there exists an* $i \in I$ *such that* $L_i \subseteq L$.

(iv) $\alpha(M) = \sum_{i\in I} (0 :_R L_i)$.

Proof. (i) By Lemma 1.3, M has essential socle S. There exist an index set I and independent simple submodules U_i $(i \in I)$ such that $S = \oplus_{i\in I} U_i$. For each $j \in I$ there exists a complement L_j of U_j such that $\oplus_{i\neq j} U_i \subseteq L_j$. Suppose that $\cap_{i\in I} L_i \neq 0$. Then $S \cap (\cap_{i\in I} L_i) \neq 0$, because S is essential in M. There exists a finite subset J of I and nonzero elements $u_j \in U_j$ $(j \in J)$ such that $\sum_{j\in J} u_j \in \cap_{i\in I} L_i$. Let $j \in J$. Note that $u_k \in L_j$ for all $k \in J\backslash\{j\}$, and hence $u_j \in U_j \cap L_j = 0$, a contradiction. It follows that $\cap_{i\in I} L_i = 0$. Let $i \in I$. Suppose that N is a completely irreducible submodule of M such that $N \subseteq L_i$. By Corollary 1.5 there exists a $j \in I$ such that $L_j \subseteq N$. Suppose that $j \neq i$. Then $U_i \subseteq L_j \subseteq N \subseteq L_i$, and hence $U_i = 0$, a contradiction. Thus, $j = i$ and hence $L_i \subseteq N$ and $L_i = N$. Thus L_i is a minimal completely irreducible submodule of M for each $i \in I$.

(ii) For each $j \in I$, $U_j \subseteq \cap_{i\neq j} L_i$, and hence $\cap_{i\neq j} L_i \neq 0$.

(iii) By Corollary 1.5.

(iv) By (iii). □

2. Modules with unique complements. In the first part of this section we shall assume that R is a (not necessarily commutative) ring and M is a right R-module. Given a submodule N of M, we know that there exists at least one complement K of N. However, K need not be unique. For example, if F is a field, V a two-dimensional vector space over F and U a one-dimensional subspace of V, then every one-dimensional subspace X of V other than U is a complement of U. In particular, if F is an infinite field, then there are an infinite number of one-dimensional subspaces of V, and hence there are an infinite number of complements of U in V. We shall say that an R-module M has *unique complements* provided, for each submodule N of M, there exists a unique complement of N in M. For example, simple modules have unique complements. More generally, uniform modules have unique complements. Recall that a module U is called *uniform* provided $U \neq 0$ and $X \cap Y \neq 0$ for all nonzero submodules X and Y of U. If U is a uniform module, then 0 is the unique complement of every nonzero submodule and U is the unique complement of 0. Thus, U has unique complements. In this section we shall investigate modules with unique complements over arbitrary rings and then show that comultiplication modules over commutative rings have unique complements.

Theorem 2.1. *The following statements are equivalent for a submodule N of a right R-module M.*

(i) *N has a unique complement in M.*

(ii) *$\{m \in M : mR \cap N = 0\}$ is a submodule of M.*

(iii) *Given elements x and y in M with $xR \cap N = yR \cap N = 0$ then $(x+y)R \cap N = 0$.*

(iv) *Given submodules K and L of M such that $K \cap N = L \cap N = 0$, then $(K+L) \cap N = 0$.*

(v) *Given submodules L_i $(i \in I)$ such that $N \cap L_i = 0$ $(i \in I)$, then $N \cap (\sum_{i \in I} L_i) = 0$.*

Moreover, in this case, $\{m \in M : mR \cap N = 0\}$ is the unique complement of N in M.

Proof. Let $Q = \{m \in M : mR \cap N = 0\}$.

(i) $\Rightarrow$ (ii). Let K denote the unique complement of N in M. Since every submodule H of M with $H \cap N = 0$ is contained in a complement of N (Zorn's lemma) it follows that $Q \subseteq K$. Clearly $0 \in Q$. Let $m_1, m_2 \in Q$, $r \in R$. Then $m_1 - m_2 \in K$ implies that $(m_1 - m_2)R \cap N \subseteq K \cap N = 0$, so that $m_1 - m_2 \in Q$. Moreover, $m_1 rR \cap N \subseteq m_1 R \cap N = 0$ gives that $m_1 r \in Q$. It follows that Q is a submodule of M.

(ii) $\Rightarrow$ (iii). If x and y belong to M with $xR \cap N = yR \cap N = 0$, then $x \in Q$ and $y \in Q$ so that $x + y \in Q$, and hence $(x+y)R \cap N = 0$.

(iii) $\Rightarrow$ (iv). Let K and L be submodules of M such that $K \cap N = L \cap N = 0$. Let $u \in (K+L) \cap N$. Then $u = x + y$ for some $x \in K$ and $y \in L$. But $xR \cap N \subseteq K \cap N = 0$ gives that $xR \cap N = 0$. Similarly, $yR \cap N = 0$. By (iii), $uR \cap N = 0$, and hence $u = 0$. It follows that $(K+L) \cap N = 0$.

(iv) $\Rightarrow$ (v). Let L_i $(i \in I)$ be submodules of M such that $N \cap L_i = 0$ $(i \in I)$. Let $v \in N \cap (\sum_{i \in I} L_i)$. Then there exists a finite subset J of I such that $v \in N \cap (\sum_{i \in J} L_i)$. But by (iv) and induction, $N \cap (\sum_{i \in J} L_i) = 0$ so that $v = 0$. It follows that $N \cap (\sum_{i \in I} L_i) = 0$.

(v) $\Rightarrow$ (i). Let L_i $(i \in I)$ denote the collection of all submodules H of M with the property that $H \cap N = 0$. Clearly, (v) gives that $\sum_{i \in I} L_i$ is the unique complement of N in M.

Finally, if K is the unique complement of N, then K is contained in the submodule Q. Let $w \in Q \cap N$. Then $w \in wR \cap N = 0$ so that $w = 0$. It follows that $Q \cap N = 0$, and hence $K = Q$. □

Corollary 2.2. *Let R be any ring. Then a right R-module M has unique complements if and only if for every submodule N of M the set $\{m \in M : mR \cap N = 0\}$ is a submodule of M.*

Proof. By Theorem 2.1. □

Corollary 2.3. *Let R be any ring, and let M be a right R-module with unique complements. Then every submodule of M has unique complements.*

Proof. Let L be any submodule of M. Let N be a submodule of L. Let G and H be submodules of L such that $G \cap N = H \cap N = 0$. By Theorem 2.1, $(G + H) \cap N = 0$. Again using Theorem 2.1, we see that L has unique complements. □

Recall that a submodule L of a general module M is called *essential* provided $L \cap Q \neq 0$ for every nonzero submodule Q of M. Let N be a submodule of a module M. By Zorn's lemma there exists a submodule K containing N such that K is maximal in the collection of submodules H of M such that N is essential in H. We call K a *closure of N (in M)*. In general, a submodule N will have many closures in M. Following [**7**] we shall call the module M a *UC-module* provided every submodule N has a unique closure in M. In [**7**], necessary and sufficient conditions are given for a module to be a UC-module.

Proposition 2.4. *Let R be any ring, and let M be a right R-module with unique complements. Then M is a UC-module.*

Proof. Let N be any submodule of M. Let K be the unique complement of N in M, and let L be the unique complement of K in M. Note that $N \cap K = 0$, and hence $N \subseteq L$. Moreover, it is well known (and easily checked) that $N \oplus K$ is an essential submodule of M. This implies that $N = L \cap (N \oplus K)$ is an essential submodule of L. Let H be any closure of N in M. Then N is essential in H, and hence $H \cap K = 0$. It follows that $H \subseteq L$, and hence $H = L$. Thus, L is the unique closure of N in M. □

If F is a field and V is a nonzero vector space over F with dimension at least 2, then V is a UC-module because every submodule is its own closure but V does not have unique complements as we have already observed. Thus, the converse of Proposition 2.4 is false in general. Next we aim to show that, if R is a commutative ring, then every comultiplication R-module has unique complements. We shall suppose from now on that the ring R is a commutative ring. First we prove a result which is well known but which is included for completeness.

Lemma 2.5. *Let N be a submodule of an R-module M, and let $m \in M$. Then $\mathrm{Hom}_R(Rm, N) \cong (0 :_N (0 :_R m))$.*

Proof. Define a mapping $\lambda : \mathrm{Hom}_R(Rm, N) \to (0 :_N (0 :_R m))$ by $\lambda(\varphi) = \varphi(m)$ for all $\varphi \in \mathrm{Hom}_R(Rm, N)$. Note that, if $\varphi \in \mathrm{Hom}_R(Rm, N)$, then $\varphi(m) \in N$ and $(0 :_R m)\varphi(m) = \varphi((0 :_R m)m) = \varphi(0) = 0$ so that $\varphi(m) \in (0 :_N (0 :_R m))$. It is easy to check that λ is an R-homomorphism. Suppose that $\lambda(\theta) = 0$ for some $\theta \in \mathrm{Hom}_R(Rm, N)$. Then $\theta(m) = 0$, and hence $\theta = 0$. Thus, λ is a monomorphism. Finally, let $x \in (0 :_N (0 :_R m))$. Define a mapping $\alpha : Rm \to N$ by $\alpha(rm) = rx$ for all $r \in R$. Note that, if $rm = 0$ for some $r \in R$, then $r \in (0 :_R m)$, and hence $rx = 0$. Thus, α is well defined. It is easy to check that $\alpha \in \mathrm{Hom}_R(Rm, N)$ and that $\lambda(\alpha) = x$. Thus, λ is an epimorphism and hence an isomorphism. □

Lemma 2.6. *Let N be a submodule of an R-module M, and let $m \in M$. Then $\mathrm{Hom}_R(Rm, N) = 0$ implies that $Rm \cap N = 0$. Moreover, the converse holds in case $Rm = (0 :_M (0 :_R m))$.*

Proof. Suppose first that $\mathrm{Hom}_R(Rm, N) = 0$. Let $x \in Rm \cap N$. Then there exists an $r \in R$ such that $x = rm$. Define a mapping $\varphi : Rm \to N$ by $\varphi(sm) = rsm$ $(s \in R)$. Clearly φ is a homomorphism. By hypothesis, $\varphi = 0$, and hence $x = \varphi(m) = 0$. Thus, $Rm \cap N = 0$.

Now suppose that $Rm = (0 :_M (0 :_R m))$ and that $Rm \cap N = 0$. Let $\theta : Rm \to N$ be any homomorphism. By [**1**, Lemma 4.2], there exists an $a \in R$ such that $\theta(m) = am$. Now $am \in Rm \cap N$ so that $am = 0$. It follows that $\theta = 0$. $\square$

Lemma 2.7. *Let N be a submodule of a comultiplication R-module M, let $A = (0 :_R N)$ and let $m \in M$. Then*

$$\begin{aligned} T_A(M) &= \{m \in M \,:\, Rm \cap N = 0\} \\ &= \{m \in M : \mathrm{Hom}_R(Rm, N) = 0\} \\ &= \{m \in M : (0 :_N (0 :_R m)) = 0\} \\ &= \{m \in M \,:\, Rz = (0 :_R m)z \textit{ for all } z \in N\}. \end{aligned}$$

Proof. Let $x \in T_A(M)$. Then $(1 - a)x = 0$ for some $a \in A$. Now let $u \in Rx \cap N$. There exists an $r \in R$ such that $u = rx \in N$, and hence $u = (1 - a)u = (1 - a)rx = 0$. Thus, $Rx \cap N = 0$. Now suppose that $Rm \cap N = 0$ for some $m \in M$. Then $Rm \cap (0 :_M A) = 0$ so that $(0 :_{Rm} A) = 0$. But Rm is a comultiplication module (Lemma 1.1) and Lemma 1.2 gives that $Rm = Am$ and $(1 - b)m = 0$ for some $b \in A$. Thus, $m \in T_A(M)$. We have proved that $T_A(M) = \{m \in M : Rm \cap N = 0\}$.

Next, let $v \in M$ such that $Rv \cap N = 0$. Let $B = (0 :_R v)$. Then $(0 :_M B) \cap N = Rv \cap N = 0$, and thus $(0 :_N B) = 0$. By Lemmas 1.1 and 1.2, for all $z \in N$, $Rz = Bz$. Conversely, suppose that $w \in M$ satisfies $Rz = (0 :_R w)z$ for all $z \in N$. Let $g \in Rw \cap N$. Then $Rg = (0 :_R w)g = 0$. It follows that $Rw \cap N = 0$. Thus, we have proved that $\{m \in M : Rm \cap N = 0\} = \{m \in M : Rz = (0 :_R m)z$ for all $z \in N\}$. The result now follows by Lemmas 2.5 and 2.6. $\square$

Theorem 2.8. *Every comultiplication module has unique complements. Moreover, if N is any submodule of a comultiplication R-module M, then the unique complement of N in M is $T_A(M)$ where A is the ideal $(0 :_R N)$ of R.*

Proof. Let M be a comultiplication module over a ring R, and let N be any submodule of M. By Lemma 2.7, the set $\{m \in M : Rm \cap N = 0\}$ is a submodule of M and, by Theorem 2.1, N has a unique complement in M. Moreover, if $A = (0 :_R N)$, then $T_A(M)$ is the unique complement of N by Lemma 2.7 again. $\square$

It is not too difficult to give an example of a module which has unique complements but which is not a comultiplication module. Let F be any field of nonzero characteristic p, and let G denote the Prüfer p-group $\mathbf{Z}(p^\infty)$. Then the group algebra $R = FG$ is a commutative ring with unique maximal ideal $A = \sum_{x \in G} R(x-1)$, the augmentation ideal of R. For every finitely generated, i.e., proper, subgroup H of G, H is a cyclic p-group so that the subring FH of R is an Artinian local ring whose unique maximal ideal is the augmentation ideal of FH which equals $A \cap FH$ and which is principal. Thus, the subring FH is a chain ring; in other words, the lattice of ideals of R is totally ordered for every finitely generated subgroup H of G. It follows that ring R is a chain ring, and hence the R-module R is uniform. Therefore, the R-module R has unique complements. However, $(0 :_R A) = 0$, so that the R-module R is not a comultiplication module.

3. Quasi-injective modules. Let R be a commutative ring. The next result is concerned with an arbitrary module. It is well known but we give a proof for completeness.

Lemma 3.1. *Let M be any R-module, and let A and B be ideals of R. Then*

(i) $(0 :_M AB) = ((0 :_M A) :_M B)$, *and*

(ii) $(0 :_R AM) = ((0 :_M A) :_R M)$.

Proof. (i) Let $m \in M$. Then

$$m \in (0 :_M AB) \Longleftrightarrow ABm = 0 \Longleftrightarrow Bm \subseteq (0 :_M A) \Longleftrightarrow m \in ((0 :_M A) :_M B).$$

(ii) Let $r \in R$. Then

$$r \in (0 :_R AM) \Longleftrightarrow ArM = 0 \Longleftrightarrow rM \subseteq (0 :_M A) \Longleftrightarrow r \in ((0 :_M A) :_R M). \qquad \square$$

Lemma 3.2. *Let K and L be submodules of a comultiplication R-module M. Then $(0 :_M (K :_R L)) = (0 :_R K)L$.*

Proof. Note first that

$$(K :_R L)(0 :_R K)L = (0 :_R K)(K :_R L)L \subseteq (0 :_R K)K = 0,$$

and hence $(0 :_R K)L \subseteq (0 :_M (K :_R L))$. Because M is a comultiplication module, $(0 :_R K)L = (0 :_M A)$ for some ideal A of R. It follows that $(0 :_R K)AL = 0$ so that $AL \subseteq (0 :_M (0 :_R K)) = K$, again using the fact that M is a comultiplication module. Thus, $A \subseteq (K :_R L)$, and hence

$$(0 :_M (K :_R L)) \subseteq (0 :_M A) = (0 :_R K)L.$$

Therefore, $(0 :_R K)L = (0 :_M (K :_R L))$, as required. $\square$

Lemma 3.2 has many consequences, and we next give a selection.

Corollary 3.3. *Let M be a comultiplication R-module. Then $BM = (0 :_R (0 :_M B))M$ for every ideal B of R.*

Proof. Let B be any ideal of R. By Lemma 3.1,

$$BM = (0 :_M (0 :_R BM)) = (0 :_M ((0 :_M B) :_R M)),$$

and hence by Lemma 3.2,

$$BM = (0 :_R (0 :_M B))M. \qquad \square$$

Corollary 3.4. *Let M be a comultiplication R-module, and let B and C be ideals of R such that $(0 :_M B) = (0 :_M C)$. Then $BM = CM$.*

Proof. By Corollary 3.3. $\square$

Corollary 3.5. *Let M be a comultiplication module. Then $(0 :_R L)K = (0 :_R L \cap K)K$ for all submodules K and L of M.*

Proof. By Lemma 3.2,

$$\begin{aligned}(0 :_R L)K &= (0 :_M (L :_R K))\\ &= (0 :_M (L \cap K :_R K))\\ &= (0 :_R L \cap K)K. \qquad \square\end{aligned}$$

Corollary 3.6. *Let φ be any endomorphism of a comultiplication R-module M. Then $\varphi(M) = (0 :_R \ker \varphi)M$.*

Proof. Because $\varphi(M) \cong M/\ker\varphi$, we have $(0 :_R \varphi(M)) = (\ker\varphi :_R M)$. Now

$$\varphi(M) = (0 :_M (0 :_R \varphi(M))) = (0 :_M (\ker\varphi :_R M)) = (0 :_R \ker\varphi)M,$$

by Lemma 3.2. $\square$

Lemma 3.7. *Let K and L be submodules of a comultiplication R-module M. Then $K \cap L = (0 :_K (0 :_R L))$.*

Proof. Note that

$$K \cap L = K \cap (0 :_M (0 :_R L)) = (0 :_K (0 :_R L)),$$

as required. $\square$

Lemma 3.8. *Let K and L be submodules of a comultiplication R-module M such that, for all ideals A and B of R, $AK \subseteq BK$ implies that $A \subseteq B + (0 :_R K)$. Then*

$$(0 :_R L) + (0 :_R K) = (0 :_R L \cap K).$$

Proof. By Corollary 3.5, we have $(0 :_R L)K = (0 :_R L \cap K)K$, and hence

$$(0 :_R L \cap K) \subseteq (0 :_R L) + (0 :_R K).$$

But $(0 :_R L) \subseteq (0 :_R L \cap K)$ and $(0 :_R K) \subseteq (0 :_R L \cap K)$. Thus,

$$(0 :_R L) + (0 :_R K) = (0 :_R L \cap K). \qquad \square$$

Corollary 3.9. *Let L be any submodule, and let m be any element of a comultiplication R-module M. Then $(0 :_R L) + (0 :_R m) = (0 :_R Rm \cap L)$.*

Proof. By Lemma 3.8. $\square$

Theorem 3.10. *Let R be any ring. Then an R-module M is a comultiplication module if and only if*

(i) $(0 :_R L) + (0 :_R m) = (0 :_R Rm \cap L)$, *and*

(ii) *Rm is a comultiplication module,*

for each $m \in M$ and submodule L of M.

Proof. The necessity follows by Lemma 1.1 and Corollary 3.9. Conversely, suppose module M satisfies (i) and (ii). Let L be a submodule of M, and let $m \in M$ be such that $(0 :_R L) \subseteq (0 :_R m)$. By (i), we have

$$(0 :_R m) = (0 :_R L) + (0 :_R m) = (0 :_R Rm \cap L).$$

But Rm is a comultiplication module by (ii), and hence

$$Rm = (0 :_{Rm} (0 :_R Rm)) = (0 :_{Rm} (0 :_R Rm \cap L)) = Rm \cap L,$$

and thus $m \in L$. By [**1**, Theorem 1.5], M is a comultiplication module. $\square$

Let L be a submodule of an R-module M. Then a homomorphism $\varphi : L \to M$ will be called *trivial* provided there exists an $r \in R$ such that $\varphi(x) = rx$ $(x \in L)$.

Corollary 3.11. *Let L be any finitely generated submodule of a comultiplication module M. Then every homomorphism $\varphi : L \to M$ is trivial.*

Proof. By Theorem 3.10 and the proof of (ii) $\Rightarrow$ (i) in [**1**, Lemma 4.3]. $\square$

Corollary 3.12. *Every Noetherian comultiplication module over a commutative ring is an Artinian quasi-injective module.*

Proof. By Corollary 3.11 and [**1**, Corollary 2.11]. ◻

4. Goldie dimension. Let R be a commutative ring and M any nonzero R-module. A nonempty collection of submodules L_i $(i \in I)$ of M is called *independent* provided the sum $\sum_{i\in J} L_i$ is direct for every nonempty finite subset J of I. The module M has finite *Goldie dimension* provided M does not contain an infinite independent family of nonzero submodules. In this case there exists a unique positive integer n, called the *Goldie dimension* of M, denoted here by $\operatorname{Gdim} M$, such that n is the supremum of the cardinalities of independent families of nonzero submodules (see, for example, [**6**, Section 5]). Following [**5**, page 8], a nonempty family of submodules N_i $(i \in I)$ of a module M is called *coindependent* provided for each nonempty finite subset J of I and element $i \in I \setminus J$,

$$N_i + \cap_{j\in J} N_j = M.$$

We shall say that module M has finite *dual Goldie dimension* provided M does not contain an infinite coindependent family of proper submodules. In this case, there exists a unique positive integer k, called the *dual Goldie dimension* of M, denoted here by $\operatorname{dGdim} M$, such that k is the supremum of the cardinalities of coindependent families of nonzero submodules (see, for example, [**5**, 5.2]).

Recall that a submodule L of a module M is called *small* (*in* M) provided $M \neq L + K$ for any proper submodule K of M. Following [**5**, page 15], a module H is called *hollow* if $H \neq 0$ and every proper submodule of H is small in H. If R is a ring, then the R-module R is hollow if and only if R is a local ring, that is, R contains a unique maximal ideal. It is proved in [**5**, 5.2] that a module M has finite dual Goldie dimension if and only if there exist a positive integer n, hollow modules H_i $(1 \leq i \leq n)$ and an epimorphism $\varphi : M \to H_1 \oplus \cdots \oplus H_n$ such that $\ker\varphi$ is a small submodule of M, and in this case $\operatorname{dGdim} M = n$.

Lemma 4.1. *Let R be a local ring. Then every nonzero comultiplication R-module M is uniform.*

Proof. Let K and L be submodules of M such that $K \cap L = 0$. Suppose that $K \neq 0$ and that x is a nonzero element of K. Let $y \in L$. Then $Rx \cap Ry = 0$. By Corollary 3.9,

$$(0 :_R x) + (0 :_R y) = (0 :_R Rx \cap Ry) = R.$$

Because R is a local ring, either $R = (0 :_R x)$ and $x = 0$, a contradiction, or $R = (0 :_R y)$ and $y = 0$. Thus, $y = 0$ for all $y \in L$. In other words, $L = 0$. It follows that M is a uniform module. □

Theorem 4.2. *Let R be a commutative ring, and let M be a comultiplication R-module such that the submodules Rm_i $(1 \leq i \leq n)$ are independent, for some positive integer n and nonzero elements $m_i \in M$ $(1 \leq i \leq n)$. Then $(0 :_R m_i)$ $(1 \leq i \leq n)$ is a coindependent family of proper ideals of R.*

Proof. The result is trivially true if $n = 1$. Suppose that $n \geq 2$. Let $L = Rm_1 \oplus \cdots \oplus Rm_{n-1}$. By hypothesis, $L \cap Rm_n = 0$. Then Corollary 3.9 gives that

$$(0 :_R L) + (0 :_R m_n) = (0 :_R L \cap Rm_n) = R.$$

But $(0 :_R L) = \cap_{1 \leq i \leq n-1} (0 :_R m_i)$. Thus,

$$R = [\cap_{1 \leq i \leq n-1} (0 :_R m_i)] + (0 :_R m_n).$$

It follows that the ideals $(0 :_R m_i)$ $(1 \leq i \leq n)$ are coindependent. □

A ring R is called *semilocal* if it contains only a finite number of maximal ideals, say P_i $(1 \leq i \leq n)$. In this case, if k is a positive integer and A_i $(1 \leq i \leq k)$ any coindependent collection of proper ideals of R, then $R = A_i + A_j$ for all $1 \leq i < j \leq k$. Thus, for each $1 \leq i \leq n$, there exists a unique integer j with $1 \leq j \leq k$ and $A_j \subseteq P_i$. Thus, $k \leq n$. It follows that the R-module R has finite dual Goldie dimension. On the other hand, if R is any ring and Q_i $(1 \leq i \leq t)$ any collection of distinct maximal ideals of R, for some positive integer t, then clearly Q_i $(1 \leq i \leq t)$ are coindependent submodules of the R-module R. Thus, a ring R is semilocal if and only if the R-module R

has finite dual Goldie dimension and, in this case, dGdim R is precisely the number of distinct maximal ideals of R.

It is proved in Lemma 1.3 that every finitely generated comultiplication module is finitely cogenerated and hence has finite Goldie dimension. On the other hand, Lemma 1.3 also shows that every comultiplication module has essential socle, so that if it has finite Goldie dimension then it is finitely cogenerated. Moreover, in [**1**, Corollary 2.2] it is proved that, if M is a nonzero finitely generated comultiplication module over a ring R, then the ring $R/(0:_R M)$ is semilocal. Now note the following corollary of Theorem 4.2.

Corollary 4.3. *Let R be a semilocal ring, and let M be a comultiplication R-module. Then M has finite Goldie dimension. Moreover,* $\operatorname{Gdim} M \leq \operatorname{dGdim} R$.

Proof. Let n be a positive integer such that L_i $(1 \leq i \leq n)$ is an independent collection of nonzero submodules of M. For each $1 \leq i \leq n$, let $0 \neq m_i \in L_i$. Then Rm_i $(1 \leq i \leq n)$ is an independent collection of nonzero submodules of M. By Theorem 4.2, $n \leq \operatorname{dGdim} R$. The result follows. ◻

Corollary 4.4. *Let R be a ring, and let M be a nonzero finitely generated comultiplication R-module. Then M has finite Goldie dimension if and only if the ring $R/(0:_R M)$ is semilocal.*

Proof. By Corollary 4.3 and [**1**, Corollary 2.2]. ◻

If M is a Noetherian comultiplication module, then M is Artinian (see [**1**, Corollary 2.11]) and hence M has finite hollow dimension (see [**5**, 5.2]). We next investigate when comultiplication modules have finite hollow dimension. In particular, we would like to know whether Corollary 4.3 has an analogue for hollow dimension. Recall that an ideal A of a ring R is called (*meet*) *irreducible* provided A is a proper ideal of R and $A \neq B \cap C$ for any ideals B and C, both properly containing A. Clearly, A is an irreducible ideal of R if and only if the R-module R/A is uniform.

Lemma 4.5. *Let R be a ring and M a comultiplication R-module such that $(0:_R M)$ is an irreducible ideal of R. Then M is a hollow module.*

Proof. Let K and L be submodules of M with $M = K + L$. Then

$$(0:_R M) = (0:_R K + L) = (0:_R K) \cap (0:_R L).$$

Without loss of generality, we can suppose that $(0:_R M) = (0:_R K)$. Because M is a comultiplication module, we see that $M = (0:_M (0:_R M)) = (0:_M (0:_R K)) = K$. It follows that M is a hollow module. $\square$

We would like to extend Lemma 4.5 to comultiplication R-modules such that $R/(0:_R M)$ has finite Goldie dimension but can only do so in certain cases, as we show next.

Proposition 4.6. *Let R be a ring, and let M be a comultiplication R-module such that $R/(0:_R M)$ has finite Goldie dimension and $(0:_R K \cap L) = (0:_R K) + (0:_R L)$ for all submodules K and L of M. Then M has finite hollow dimension and moreover* $\mathrm{dGdim}\, M \leq \mathrm{Gdim}\,(R/(0:_R M))$.

Proof. Let A denote the ideal $(0:_R M)$ of R. Let n be a positive integer, and let L_i $(1 \leq i \leq n)$ be coindependent proper submodules of M. If $n = 1$, then there is nothing to prove. Suppose that $n \geq 2$. Let $N = L_1 \cap \cdots \cap L_{n-1}$. Then $M = N + L_n$, and it follows that

$$\begin{aligned} A &= (0:_R N) \cap (0:_R L_n) \\ &= [(0:_R L_1) + \cdots + (0:_R L_{n-1})] \cap (0:_R L_n), \end{aligned}$$

by hypothesis and induction. Thus the nonzero ideals $(0:_R L_i)/A$ $(1 \leq i \leq n)$ of the ring R/A are independent. Thus $n \leq \mathrm{Gdim}\,(R/A)$. The result follows. $\square$

5. Final remarks. Let R be a ring, and let N be a submodule of an R-module M. A submodule L of M is called a *supplement of* N (*in* M) provided L is minimal in the collection of submodules K of M such that $M = N + K$. The module M is called *supplemented* if every submodule has a supplement. Moreover, module M is called *amply supplemented* if for all submodules N and K with $M = N+K$, K contains a supplement of N in M. Clearly Artinian modules are amply supplemented. Thus, [**1**, Corollary 2.11] gives that Noetherian comultiplication modules are amply supplemented. We question whether all comultiplication modules are amply supplemented. We have the following partial result.

Proposition 5.1. *Let R be a ring, and let M be a comultiplication R-module such that $M = (0 :_M C) + (0 :_M D)$ for all ideals C and D of R with $C \cap D = (0 :_R M)$. Then M is amply supplemented.*

Proof. Suppose that $M = N + K$ for some submodules N and K of M. If $A = (0 :_R M)$, then $A = (0 :_R N) \cap (0 :_R K)$. By Zorn's lemma, there exists an ideal B of R maximal with respect to the properties $(0 :_R K) \subseteq B$ and $(0 :_R N) \cap B = A$. By hypothesis,

$$M = (0 :_M (0 :_R N)) + (0 :_M B) = N + (0 :_M B).$$

Suppose that $M = N+H$ for some submodule H of M with $H \subseteq (0 :_M B)$. Then $B \subseteq (0 :_R H)$ and $A = (0 :_R N) \cap (0 :_R H)$. The choice of B gives that $B = (0 :_R H)$, and hence $(0 :_M B) = (0 :_M (0 :_R H)) = H$. Thus, $(0 :_M B)$ is a supplement of N in M with the property that $(0 :_M B) \subseteq (0 :_M (0 :_R K)) = K$. It follows that M is amply supplemented. $\square$

Let R be any ring, and let M be any R-module. It is well known that the collection of submodules of M forms a modular lattice $\mathcal{L}(_RM)$ with least element the zero submodule and greatest element M. Given two submodules N and L of M, the least upper bound of N and L in $\mathcal{L}(_RM)$ is $N+L$ and the greatest lower bound in $\mathcal{L}(_RM)$ is $N \cap L$. Now let $\mathcal{L}_M(_RR)$ denote the collection of ideals in R of the form $(0 :_R K)$ for some submodule K of M. Note that $\mathcal{L}_M(_RR)$ is a subset of $\mathcal{L}(_RR)$ but it need not be a sublattice even if M is a comultiplication module. Note the following result.

Theorem 5.2. *Let R be a commutative ring, and let M be a comultiplication R-module. Then $\mathcal{L}_M(_RR)$ is a sublattice of $\mathcal{L}(_RR)$ if and only if $(0 :_R N \cap L) = (0 :_R N) + (0 :_R L)$ for all submodules N and L of M. Moreover, in this case, the mapping $\varphi : \mathcal{L}(_RM) \to \mathcal{L}_M(_RR)$, defined by $\varphi(K) = (0 :_R K)$ for every submodule K of M, is an anti-isomorphism from the lattice $\mathcal{L}(_RM)$ to the lattice $\mathcal{L}_M(_RR)$.*

Proof. Suppose first that $(0 :_R N \cap L) = (0 :_R N) + (0 :_R L)$ for all submodules N, L of M. Let G and H be any submodules of M. Clearly, $(0 :_R G) \cap (0 :_R H) = (0 :_R G + H) \in \mathcal{L}_M({}_R R)$ and, by hypothesis, $(0 :_R G) + (0 :_R H) = (0 :_R G \cap H) \in \mathcal{L}_M({}_R R)$. It follows that $\mathcal{L}_M({}_R R)$ is a sublattice of $\mathcal{L}({}_R R)$. Conversely, suppose that $\mathcal{L}_M({}_R R)$ is a sublattice of $\mathcal{L}({}_R R)$. Let N and L be any submodules of M. Then $(0 :_R N) + (0 :_R L) = (0 :_R K)$ for some submodule K of M. It follows that $(0 :_R N) \subseteq (0 :_R K)$, and hence $K \subseteq N$ because M is a comultiplication module (see [**1**, Theorem 1.5 (i) $\Rightarrow$ (ii)]). Similarly, $K \subseteq L$, and we have $K \subseteq N \cap L$. Thus,

$$(0 :_R N \cap L) \subseteq (0 :_R K) = (0 :_R N) + (0 :_R L) \subseteq (0 :_R N \cap L),$$

so that $(0 :_R N \cap L) = (0 :_R N) + (0 :_R L)$, as required. $\square$

Let R be a ring and M a comultiplication R-module such that $(0 :_R N \cap L) = (0 :_R N) + (0 :_R L)$ for all submodules N and L of M. By Theorem 5.2, $\mathcal{L}_M({}_R R)$ is a lattice with least element $(0 :_R M)$. By standard arguments using the anti-isomorphism φ in Theorem 5.2, M is amply supplemented (compare Proposition 5.1). Moreover, also by Theorem 5.2 we have at once that if R has finite hollow dimension, i.e., R is semilocal, then the R-module M has finite Goldie dimension and $\operatorname{Gdim} M \leq \operatorname{dGdim} R$ (Corollary 4.3). On the other hand, if the ring $R/(0 :_R M)$ has finite Goldie dimension, then the R-module M has finite hollow dimension and $\operatorname{dGdim} M \leq \operatorname{Gdim}(R/(0 :_R M))$, which is precisely Proposition 4.6. However, note finally that there exist comultiplication modules which are not of this type and therefore Theorem 5.2 does not apply. For example, in [**1**, Example 3.4] an example is given of a ring R and a semisimple comultiplication R-module M such that $(0 :_R N \cap L) \neq (0 :_R N) + (0 :_R L)$ for some submodules N and L of M.

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