

P -PARTITIONS REVISITED

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ABSTRACT. We compare a traditional and non-traditional view on the subject of P -partitions, leading to formulas counting linear extensions of certain posets.

1. Introduction. Our goal is to re-examine Stanley's theory of P -partitions from a non-traditional viewpoint, one that arose originally from ring-theoretic considerations in [2]. Comparing viewpoints, for example, gives an application to counting linear extensions of certain posets. We describe these viewpoints here, followed by this enumerative application, and then give an indication of the further ring-theoretic results.

1.1. Traditional viewpoint. Given a partial order P on the set $\{1, 2, \dots, n\}$ a *weak P -partition* [21, subsection 4.5] is a map $f : P \rightarrow \mathbf{N} := \{0, 1, 2, \dots\}$ satisfying $f(i) \geq f(j)$ whenever $i <_P j$.

In Stanley's original work [20] and that of Garsia [11], it was important that one can express a weak P -partition f uniquely as a sum $f = \chi_{I_1} + \chi_{I_2} + \dots + \chi_{I_{\max(f)}}$ of indicator functions χ_{I_i} for a multiset of nonempty, nested order ideals I_i in P ; specifically, $I_i := \{j \in P : f(j) \geq i\}$. An important special case occurs when f takes on each value in $\{1, 2, \dots, n\}$ exactly once, so that the nested sequence of order ideals $I_1 \supset \dots \supset I_n \supset I_{n+1} := \emptyset$ corresponds to a permutation $w = (w(1), \dots, w(n))$ of $\{1, 2, \dots, n\}$ defined by $w(i) = I_i \setminus I_{i+1}$. Such permutations w are called *linear extensions* of P because the order $<_w$ given by $w(1) <_w \dots <_w w(n)$ strengthens the partial order P to a linear order.

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This has a geometric interpretation: the weak P -partitions f are the integral points inside a rational polyhedral cone in \mathbf{R}^n defined by the inequalities $f_i \geq f_j \geq 0$ for $i <_P j$, and the set $\mathcal{L}(P)$ of all linear extensions of P indexes the maximal simplicial subcones in a unimodular triangulation of this P -partition cone. The simplicial complex underlying this triangulation is the order complex for the finite distributive lattice structure on the set $\mathcal{J}(P)$ of all order ideals in P ; see [22].

1.2. New viewpoint. Here a much larger role is played by the subset $\mathcal{J}_{\text{conn}}(P) \subset \mathcal{J}(P)$ consisting of all nonempty *connected* order ideals J in P , that is, those order ideals J whose Hasse diagram is a connected graph. Say that two connected order ideals J_1, J_2 *intersect trivially* if either they are disjoint or they are nested, that is, comparable under inclusion; otherwise, say that they *intersect nontrivially*.

It will be important that one can express a P -partition f uniquely as a sum

$$(1.1) \quad f = \chi_{J_1} + \chi_{J_2} + \cdots + \chi_{J_{\nu(f)}}$$

of the indicator functions χ_{J_i} where $\{J_1, J_2, \dots, J_{\nu(f)}\}$ is a multiset of nonempty connected order ideals in P that pairwise intersect trivially; specifically, one takes the $\{J_\ell\}_{\ell=1}^{\nu(f)}$ to be the multiset of connected components of the Hasse diagrams for the order ideals $I_i = \{j \in P : f(j) \geq i\}$ mentioned earlier.

Geometrically, this corresponds to a different (non-unimodular) triangulation of the P -partition cone. This triangulation is intimately related to the refinement of the normal fan of a *graphic zonotope* by the normal fan of one of Carr and Devadoss's *graph-associahedra* [5]; see Section 11.

1.3. Counting linear extensions. Computing the number $|\mathcal{L}(P)|$ of linear extensions of P for general posets P is known to be a $\#P$ -hard problem by work of Brightwell and Winkler [3]. However, for the class of posets which we are about to define, a formula for $|\mathcal{L}(P)|$ will follow easily from the above considerations.

Say that a finite poset P is a *forest with duplications* if it can be constructed from one-element posets by iterating the following three operations:

Disjoint union: Given two posets P_1, P_2 , form their *disjoint union* $P_1 \sqcup P_2$, in which all elements of P_1 are incomparable to all elements of P_2 .

Hanging: Given two posets P_1, P_2 and any element a in P_1 , form a new poset by *hanging P_2 below a in P_1* , that is, add to the disjoint union $P_1 \sqcup P_2$ all the order relations $p_2 < b$ for every p_2 in P_2 and b in P_1 with $b \geq_{P_1} a$.

Duplication of a hanger: Say that an element a in P is a *hanger* if P can be formed by hanging the nonempty subposet $P_2 := P_{<a}$ below a in the subposet $P_1 := P \setminus P_{<a}$. Equivalently, a is hanger in P if $P_{<a}$ is nonempty and every path in the Hasse diagram of P from an element of $P_{<a}$ to an element of $P \setminus P_{\leq a}$ must pass through a . Then one can form the *duplication of the hanger a in P* with duplicate element a' : add to the disjoint union $P \cup \{a'\}$ all order relations $p < a'$ (respectively $a' < p$) whenever $p <_P a$ (respectively $a <_P p$).

Note that when one disallows the duplication-of-hanger operation from the above list of constructions, one obtains the subclass of *forest posets*, that is, posets in which every element is covered by at most one other element.

For the sake of stating our first main result counting linear extensions, we define the notion of a naturally labeled poset P : it means that $i <_P j$ implies $i <_{\mathbf{Z}} j$. Let us also recall the *major index* statistic on a permutation $w = (w(1), \dots, w(n))$ defined by

$$\text{maj}(w) := \sum_{\substack{i=1,2,\dots,n-1: \\ w(i) > w(i+1)}} i$$

and these standard q -analogues of the number n and the factorial $n!$:

$$[n]_q := 1 + q + q^2 + \dots + q^{n-1} = \frac{1 - q^n}{1 - q}$$

$$[n]!_q := [1]_q [2]_q \cdots [n-1]_q [n]_q.$$

We give a proof of the following result by inclusion-exclusion in Section 4 and then a second proof via commutative algebra in Section 7.

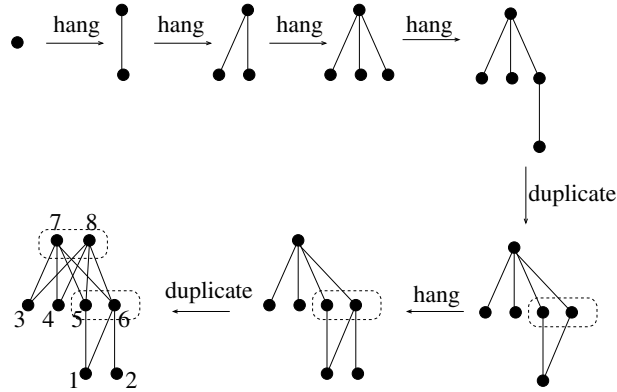


FIGURE 1. A duplicated forest built by a sequence of hanging and duplication operations.

Theorem 1.1. *Let P be a naturally labeled forest with duplications on $\{1, 2, \dots, n\}$. Then*

$$(1.2) \quad \sum_{w \in \mathcal{L}(P)} q^{\text{maj}(w)} = [n]!_q \cdot \prod_{\{J_1, J_2\} \in \Pi(P)} [|J_1| + |J_2|]_q \Big/ \prod_{J \in \mathcal{J}_{\text{conn}}(P)} [|J|]_q$$

where the product in the numerator runs over all the set $\Pi(P)$ consisting of all pairs $\{J_1, J_2\}$ of connected order ideals of P that intersect nontrivially. In particular, upon setting $q = 1$, one has

$$(1.3) \quad |\mathcal{L}(P)| = n! \cdot \prod_{\{J_1, J_2\} \in \Pi(P)} (|J_1| + |J_2|) \Big/ \prod_{J \in \mathcal{J}_{\text{conn}}(P)} |J|.$$

The products appearing in Theorem 1.1 are much more explicit than they first appear, as will be shown (see Lemma 4.1) that for a forest P with duplications, the two sets $\mathcal{J}_{\text{conn}}(P)$ and $\Pi(P)$ are easily written

down in terms of the *principal ideals* $P_{\leq p}$ and the *duplication set* $\mathcal{D}(P)$ consisting of all duplication pairs $\{a, a'\}$ that were created during the various steps that build P :

$$(1.4) \quad \begin{aligned} \mathcal{J}_{\text{conn}}(P) &= \{P_{\leq p}\}_{p \in P} \sqcup \{P_{\leq a, a'}\}_{\{a, a'\} \in \mathcal{D}(P)} \\ \Pi(P) &= \{\{P_{\leq a}, P_{\leq a'}\}\}_{\{a, a'\} \in \mathcal{D}(P)}. \end{aligned}$$

Figure 1 shows an example of a *forest with duplications* P built by a sequence of hangings and duplications; no disjoint union operations are used, yielding only one connected component. Its duplication set $\mathcal{D}(P) = \{\{5, 6\}, \{7, 8\}\}$ is shown dotted. One has the following list of cardinalities $|J|$ of connected order ideals J :

$J \in \mathcal{J}_{\text{conn}}(P)$	$P_{\leq 1}$	$P_{\leq 2}$	$P_{\leq 3}$	$P_{\leq 4}$	$P_{\leq 5}$	$P_{\leq 6}$	$P_{\leq 7}$	$P_{\leq 8}$
$ J $	1	1	1	1	2	3	7	7

$J \in \mathcal{J}_{\text{conn}}(P)$	$P_{\leq 5} \cup P_{\leq 6}$	$P_{\leq 7} \cup P_{\leq 8}$
$ J $	4	8

and this data on the pairs in $\Pi(P)$

$\{J_1, J_2\} \in \Pi(P)$	$\{P_{\leq 5}, P_{\leq 6}\}$	$\{P_{\leq 7}, P_{\leq 8}\}$
$ J_1 + J_2 $	$2 + 3 = 5$	$7 + 7 = 14$

Consequently, Theorem 1.1 implies that

$$\begin{aligned} \sum_{w \in \mathcal{L}(P)} q^{\text{maj}(w)} &= \frac{[8]_q!}{[1]_q \cdot [1]_q \cdot [1]_q \cdot [1]_q \cdot [2]_q \cdot [3]_q \cdot [7]_q \cdot [7]_q} \\ &\quad \cdot \frac{[5]_q \cdot [14]_q}{[4]_q \cdot [8]_q} \\ &= \frac{[5]_q \cdot [5]_q \cdot [6]_q \cdot [14]_q}{[7]_q} \\ &= [2]_{q^7} \cdot [5]_q \cdot [5]_q \cdot [6]_q, \end{aligned}$$

and upon setting $q = 1$, one obtains

$$|\mathcal{L}(P)| = 2 \cdot 5 \cdot 5 \cdot 6 = 300.$$

This example has been checked using the software *sage* [24], see <http://www.sagemb.org/home/pub/2701/>.

A special case of Theorem 1.1 is well known, namely, when the forest has *no duplications*, and the set $\Pi(P)$ is empty. In this case, one simply has a *forest poset*. Then equation (1.3) becomes Knuth's well-known *hook formula for linear extensions of forests* [14, subsection 5.1.4, Exercise 20], and equation (1.2) becomes Bjorner and Wachs' more general *major index q -hook formula for forests* [1, Theorem 1.2]. The derivation of these two special cases from consideration of P -partition rings was already pointed out in [2, Section 6]; see also Examples 9.7 and 9.8 below.

1.4. The ring of weak P -partitions. Although Theorem 1.1 has a simple combinatorial proof, it was not our original one. We were motivated from trying to understand the structure of the *affine semi-group ring* R_P of P -partitions, the subalgebra of the polynomial ring $k[x_1, \dots, x_n]$ spanned k -linearly by the monomials $\mathbf{x}^f := x_1^{f(1)} \dots x_n^{f(n)}$ as f runs through all weak P -partitions. In [2] this was the ring denoted R_P^{wt} . There it was noted that a minimal generating set as an algebra is given by the monomials $\mathbf{x}^J := \prod_{j \in J} x_j$ as J runs through the set $\mathcal{J}_{\text{conn}}(P)$ of nonempty connected order ideals of P . We extend this to the following result in Section 6.

Theorem 1.2. *For any poset on $\{1, 2, \dots, n\}$ and any field k , the P -partition ring R_P has minimal presentation*

$$0 \rightarrow I_P \longrightarrow S \xrightarrow{\varphi} R_P \rightarrow 0$$

in which the polynomial algebra $S = k[U_J]_{J \in \mathcal{J}_{\text{conn}}(P)}$ maps to R_P via $U_J \mapsto \mathbf{x}^J$, and the kernel ideal I_P has a minimal generating set indexed by $\{J_1, J_2\}$ in $\Pi(P)$, consisting of binomials

$$(1.5) \quad \text{syz}_{J_1, J_2} := U_{J_1} U_{J_2} - U_{J_1 \cup J_2} \cdot U_{J^{(1)}} U_{J^{(2)}} \cdots U_{J^{(t)}}$$

where the intersection $J_1 \cap J_2$ has connected component ideals $J^{(1)} \sqcup \dots \sqcup J^{(t)}$.

Example. For the poset in Figure 1, the presentation of Theorem 1.2 is $R_P = S/I_P$, where

$$(1.6) \quad S = k[U_1, U_2, U_3, U_4, U_{15}, U_{126}, U_{1234567}, U_{1234568}, U_{1256}, U_{12345678}]$$

and I_P is the ideal of S generated by

$$U_{15}U_{126} - U_{1256}U_1, \\ U_{1234567}U_{1234568} - U_{12345678}U_{1256}U_3U_4.$$

It is not hard to see (and as is explained in Corollary 5.3) how the various generating functions for (weak) P -partitions turn into Hilbert series calculations for R_P . This suggests trying to understand the structure of R_P in order to calculate its Hilbert series. One natural situation where this follows easily is when $R_P \cong S/I_P$ gives a *complete intersection presentation*, that is, the Krull dimension n of R_P plus the size $|\Pi(P)|$ of the minimal generating set for I_P sums to the Krull dimension $|\mathcal{J}_{\text{conn}}(P)|$ of S . The forward implication in the following combinatorial characterization of the complete intersection case is proven in Section 7 and used to give our second (but historically first) proof of Theorem 1.1:

Theorem 1.3. *A poset P on $\{1, 2, \dots, n\}$ is a forest with duplications if and only if $R_P = S/I_P$ is a complete intersection presentation.*

1.5. The associated graded ring. We explain in Section 5 the significance of the statistic $\nu(f)$ on a P -partition f which appeared in the unique expression (1.1) above. It turns out that $\nu(f)$ gives the \mathbf{N} -grading of the image of the monomial \mathbf{x}^f in the *associated graded ring* $\text{gr}(R_P) = \text{gr}_{\mathfrak{m}}(R_P)$ with respect to the unique \mathbf{N} -homogeneous maximal ideal $\mathfrak{m} \subset R_P$. Consequently, $\text{gr}(R_P)$ has $\mathbf{N} \times \mathbf{N}^n$ -graded Hilbert series

$$(1.7) \quad \text{Hilb}(\text{gr}(R_P), t, \mathbf{x}) = \sum_{f \in \mathcal{A}^{\text{weak}}(P)} t^{\nu(f)} \mathbf{x}^f.$$

An expression for this Hilbert series as a summation over the set $\mathcal{L}(P)$ of linear extensions of P is given in (3.1) below.¹ The following presentation and initial ideal for $\text{gr}(R_P)$ will be derived in Section 6.

¹ Assuming that P has been *naturally labeled*; see Remark 2.6.

Theorem 1.4. *For any poset on $\{1, 2, \dots, n\}$ and any field k , the associated graded ring $\text{gr}(R_P)$ has minimal presentation*

$$0 \longrightarrow I_P^{\text{gr}} \longrightarrow S \xrightarrow{\text{gr}(\varphi)} \text{gr}(R_P) \longrightarrow 0$$

in which the polynomial algebra $S = k[U_J]_{J \in \mathcal{J}_{\text{conn}}(P)}$ is mapped to $\text{gr}(R_P)$ via $U_J \mapsto \bar{\mathbf{x}}^J$, and the kernel ideal I_P^{gr} has minimal generating indexed by $\{J_1, J_2\}$ in $\Pi(P)$, consisting of the quadratic binomials and monomials

(1.8)

$$\text{syz}_{J_1, J_2}^{\text{gr}} := \begin{cases} U_{J_1}U_{J_2} - U_{J_1 \cup J_2}U_{J_1 \cap J_2} & \text{if } J_1 \cap J_2 \text{ is connected,} \\ U_{J_1}U_{J_2} & \text{if } J_1 \cap J_2 \text{ is disconnected.} \end{cases}$$

Furthermore, monomial orders exist on S for which the initial ideal of both I_P and I_P^{gr} is the squarefree quadratic monomial ideal I_P^{init} having minimal generating set indexed by $\{J_1, J_2\}$ in $\Pi(P)$, consisting of the squarefree quadratic monomials

$$(1.9) \quad \text{syz}_{J_1, J_2}^{\text{init}} := U_{J_1}U_{J_2}.$$

Example. For the poset in Figure 1, the presentation in Theorem 1.4 is $\text{gr}(R_P) = S/I_P^{\text{gr}}$, where S is as in (1.6) and I_P^{gr} is generated by the binomial

$$U_{15}U_{126} - U_{1256}U_1, \\ U_{1234567}U_{1234568}$$

while the initial ideal I_P^{init} in Theorem 1.4 is generated by the monomials

$$U_{15}U_{126}, \\ U_{1234567}U_{1234568}.$$

The existence of this quadratic initial ideal I_P^{init} has this consequence.

Corollary 1.5. *For any poset P on $\{1, 2, \dots, n\}$, the associated graded ring $\text{gr}(R_P)$ is a Koszul algebra. Thus the $\mathbf{N} \times \mathbf{N}^n$ -graded Hilbert series from (1.7) will have the property that*

$$\left[\sum_{f \in \mathcal{A}^{\text{weak}}(P)} t^{\nu(f)} \mathbf{x}^f \right]_{t \mapsto -t}^{-1}$$

is a power series in t, x_1, \dots, x_n with nonnegative coefficients; specifically, it is the $\mathbf{N} \times \mathbf{N}^n$ -graded Hilbert series of the Koszul dual algebra $\text{gr}(R_P)^\dagger$.

The remainder of this paper explains these results further. The reader interested only in the combinatorial results will find them in Sections 2–4 and can safely skip the connections to ring-theory explained in Sections 5–10. Section 11 discusses the geometry of the initial ideal I_P^{init} and its associated triangulation of the P -partition cone, relating it to graphic zonotopes and graph-associahedra. Section 12 collects some further questions.

2. Unique expressions. We discuss some old and new ways to uniquely express a P -partition, mentioned in the introduction.

Definition 2.1. Let P be a partial order $<_P$ on $\{1, 2, \dots, n\}$, and consider the nonnegative integers $\mathbf{N} = \{0, 1, 2, \dots\}$ as a totally ordered set with its usual order $<_{\mathbf{N}}$. Say that a map $f : P \rightarrow \mathbf{N} := \{0, 1, 2, \dots\}$ is

- a *weak P -partition* if it is weakly order-reversing: $i <_P j$ implies the inequality $f(i) \geq_{\mathbf{N}} f(j)$;
- a *P -partition* if, in addition, whenever $i <_P j$ and $i >_{\mathbf{N}} j$, one has a strict inequality $f(i) >_{\mathbf{N}} f(j)$;
- a *strict P -partition* if $i <_P j$ implies $f(i) >_{\mathbf{N}} f(j)$.

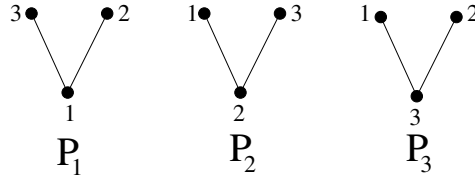
NB: This terminology is similar in spirit, but not quite the same as that used by Stanley in [21, subsections 4.5, 7.19]. We hope that the slight differences create no confusion.

Denoting by $\mathcal{A}(P), \mathcal{A}^{\text{weak}}(P), \mathcal{A}^{\text{strict}}(P)$ the sets of P -partitions, weak P -partitions, and strict P -partitions, one has the inclusions

$$(2.1) \quad \mathcal{A}^{\text{strict}}(P) \subseteq \mathcal{A}(P) \subseteq \mathcal{A}^{\text{weak}}(P).$$

One has equality in the second inclusion of (2.1) if and only if P is *naturally labeled*; similarly one has equality in the first inclusion of (2.1) if and only if P is *strictly* or *anti-naturally labeled* in the sense that $i <_P j$ implies $i >_{\mathbf{N}} j$.

Example 2.2. The three posets P_1, P_2, P_3 on $\{1, 2, 3\}$ shown below



are all isomorphic, with P_1 naturally labeled, P_3 strictly labeled and P_2 neither naturally nor strictly labeled. One has

$$\begin{aligned} \mathcal{A}(P_1) &= \{f = (f(1), f(2), f(3)) \in \mathbf{N}^3 : f(1) \geq_{\mathbf{N}} f(2), f(3)\} \\ \mathcal{A}(P_2) &= \{f = (f(1), f(2), f(3)) \in \mathbf{N}^3 : f(2) \geq_{\mathbf{N}} f(3) \text{ and} \\ &\quad f(2) >_{\mathbf{N}} f(1)\} \\ \mathcal{A}(P_3) &= \{f = (f(1), f(2), f(3)) \in \mathbf{N}^3 : f(3) >_{\mathbf{N}} f(1), f(2)\}. \end{aligned}$$

Definition 2.3. Recall that a permutation $w = (w(1), \dots, w(n))$ of $\{1, 2, \dots, n\}$ is a *linear extension* of P if the order $w(1) <_w \dots <_w w(n)$ extends P to a linear order. Denote by $\mathcal{L}(P)$ the set of all linear extensions w of P . Denote by $w_{[1,i]}$ the initial segment $\{w(1), w(2), \dots, w(i)\}$ of w thought of as a subset of $\{1, 2, \dots, n\}$. It is an order ideal of P whenever w lies in $\mathcal{L}(P)$.

For any subset $A \subset \{1, 2, \dots, n\}$, let χ_A be its characteristic function thought of as a vector in $\{0, 1\}^n$.

Proposition 2.4. *For any poset P on $\{1, 2, \dots, n\}$, and any P -partition f , there exists a unique permutation w in $\mathcal{L}(P)$ for which*

$$(2.2) \quad f(w(1)) \geq \dots \geq f(w(n))$$

and one has strict inequality $f(w(i)) > f(w(i+1))$ when $w(i) > w(i+1)$, that is, whenever i is an element of the descent set $\text{Des}(w)$. Consequently,

$$f = \sum_{i=1}^n (f(w(i)) - f(w(i+1))) \cdot \chi_{w_{[1,i]}}.$$

Proof. (See [21, Lemma 4.5.1, Theorem 7.19.4]). One takes w to be the minimum length or lexicographically earliest permutation satisfying (2.2). \square

Proposition 2.5. *For any poset P on $\{1, 2, \dots, n\}$, any weak P -partition f (and hence also any P -partition) has a unique expression as*

(i) $f = \sum_{i=k}^{\max(f)} \chi_{I_k}$ for a multiset $I_1 \supseteq \dots \supseteq I_{\max(f)}$ of nested nonempty order ideals in P , and also as

(ii) $f = \sum_{i=1}^{\nu(f)} \chi_{J_i}$ for a multiset $J_1, J_2, \dots, J_{\nu(f)}$ of nonempty connected order ideals of P which pairwise intersect trivially.

Proof. For (i), one sets $I_k := f^{-1}(\{k, k + 1, k + 2, \dots\})$ for $k = 1, 2, \dots, t := \max(f)$.

To prove (ii), one can show existence of such an expression for f by starting with the multichain $J_1 \supseteq \dots \supseteq J_t$ of order ideals from (i), and replacing each order ideal J_i with its collection of connected components. It is not hard to see that the resulting multiset of connected order ideals will pairwise intersect trivially.

To prove uniqueness of the expression in (ii), induct on $|f| := \sum_{i=1}^n f(i)$, with trivial base case $f = 0$. In the inductive step, let $f \neq 0$, and consider the set J which is the support of f as a subset of P . Because f is a P -partition, J is a nonempty order ideal. Decompose J into its connected components $J^{(1)}, J^{(2)}, \dots, J^{(c)}$, which are all connected order ideals.

If $c > 1$, then one can consider for each i the restriction $f|_{J^{(i)}}$ as a $J^{(i)}$ -partition. Since $|J^{(i)}| < |J| \leq |P|$, uniqueness follows by induction.

If $c = 1$, so that J is connected (and nonempty), then $f = \chi_J + \hat{f}$, where \hat{f} is again a P -partition, and $|\hat{f}| < |f|$. Again, uniqueness follows by induction. \square

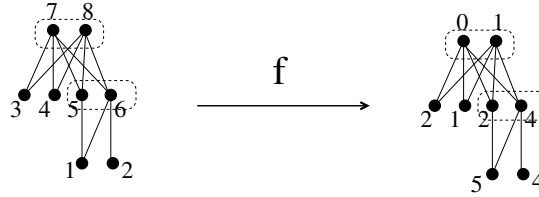
Remark 2.6. The relation between Propositions 2.4 and 2.5 is easiest when P is naturally labeled, so that a P -partition f is the same as a weak P -partition. In that case, the unique permutation w guaranteed by Proposition 2.4 has the property that the multiset of

ideals $\{I_k\}_{k=1,2,\dots,\max(f)}$ contains the order ideal $w|_{[1,i]}$ of P with multiplicity $f(w(i)) - f(w(i+1))$.

We also note that it is essentially innocuous to relabel an arbitrary poset P so that it is naturally labeled, either if the goal is to count the linear extensions $\mathcal{L}(P)$, or if the goal is to understand the ring R_P —this ring depends only upon P up to isomorphism, not on the labeling. The labeling of P only makes a difference when considering the ideal \mathcal{I}_P within R_P consider later, in Section 9.

Example 2.7. Let P be the naturally labeled poset on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ from Figure 1, and let f be the P -partition with values in the following table, as depicted below:

i	1	2	3	4	5	6	7	8
$f(i)$	5	4	2	1	2	4	0	1



Then $\max(f) = 5$ and the unique expression for f as in Proposition 2.5 (i) is $f = \sum_{j=1}^5 \chi_{I_j}$ where $\{I_1, I_2, I_3, I_4, I_5\}$ are the nested ideals shown here

$$\begin{array}{cccccc}
 I_1 & \supseteq & I_2 & \supseteq & I_3 & = & I_4 & \supseteq & I_5 \\
 \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
 \{1, 2, 3, 4, 5, 6, 8\} & & \{1, 2, 3, 5, 6\} & & \{1, 2, 6\} & & \{1, 2, 6\} & & \{1\} \\
 \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
 J_1 & & \{1, 2, 5, 6\} \sqcup \{3\} & & J_3 & & J_4 & & J_5 \\
 & & \parallel & & & & & & \\
 & & J_2 \sqcup J_6 & & & & & &
 \end{array}$$

and these decompose into the multiset of $\nu(f) = 6$ connected component ideals $\{J_1, J_2, J_3, J_4, J_5, J_6\}$ labeled above, giving the expression $f = \sum_{i=1}^6 \chi_{J_i}$ as in Proposition 2.5 (ii). The unique expression as in Proposition 2.4 has

$$\begin{aligned} w &= (w(1), w(2), w(3), w(4), w(5), w(6), w(7), w(8)) \\ &= (1, 2, 6, 3, 5, 4, 8, 7) \end{aligned}$$

and $f = 1 \cdot \chi_{w|_{[1,1]}} + 2 \cdot \chi_{w|_{[1,3]}} + 1 \cdot \chi_{w|_{[1,5]}} + 1 \cdot \chi_{w|_{[1,7]}}$.

3. Generating functions. We explain here how Proposition 2.5 suggests generating functions counting P -partitions and linear extensions according to certain statistics, which one can then specialize in various ways. We will see in Corollary 5.3 that they all have natural interpretations as Hilbert series for the P -partition ring R_P or its associated graded ring $\text{gr}(R_P)$ using different specializations of their multigradings.

Definition 3.1. Given a P -partition f , recall that $\nu(f)$ denotes the size (counting multiplicity) of the multiset $\{J_1, \dots, J_{\nu(f)}\}$ in the unique expression (1.1) whose existence is guaranteed by Proposition 2.5 (ii).

Given an order ideal J of P , let $c_P(J)$ denote the number of connected components in the Hasse diagram of the restriction $P|_J$. We also define a new descent statistic for w that depends upon the poset structure of P :

$$\text{des}_P(w) := \sum_{i \in \text{Des}(w)} c_P(w|_{[1,i]}).$$

Recall also that we have been using the notations $\mathbf{x}^f := x_1^{f(1)} \dots x_n^{f(n)}$ for $f \in \mathbb{N}^n$, and $\mathbf{x}^A := \prod_{i \in A} x_i$ for subsets $A \subseteq \{1, 2, \dots, n\}$.

Corollary 3.2. For any poset P on $\{1, 2, \dots, n\}$, one has

$$(3.1) \quad \sum_{f \in \mathcal{A}(P)} t^{\nu(f)} \mathbf{x}^f = \sum_{w \in \mathcal{L}(P)} \frac{t^{\text{des}_P(w)} \prod_{i \in \text{Des}(w)} \mathbf{x}^{w|_{[1,i]}}}{\prod_{i=1}^n (1 - t^{c_P(w|_{[1,i]})} \mathbf{x}^{w|_{[1,i]}})}.$$

Setting $t = 1$ in (3.1) gives

$$(3.2) \quad \sum_{f \in \mathcal{A}(P)} \mathbf{x}^f = \sum_{w \in \mathcal{L}(P)} \frac{\prod_{i \in \text{Des}(w)} \mathbf{x}^{w|_{[1,i]}}}{\prod_{i=1}^n (1 - \mathbf{x}^{w|_{[1,i]}})},$$

whereas setting $x_i = q$ for all i in (3.1) gives

$$(3.3) \quad \sum_{f \in \mathcal{A}(P)} t^{\nu(f)} q^{|f|} = \sum_{w \in \mathcal{L}(P)} \frac{t^{\text{des}_P(w)} q^{\text{maj}(w)}}{\prod_{i=1}^n (1 - t^{c_P(w|_{[1,i]})} q^i)}.$$

Further specializing $q = 1$ in (3.3) gives

$$(3.4) \quad \sum_{f \in \mathcal{A}(P)} t^{\nu(f)} = \sum_{w \in \mathcal{L}(P)} \frac{t^{\text{des}_P(w)}}{\prod_{i=1}^n (1 - t^{c_P(w|_{[1,i]})})}.$$

Setting both $t = 1$ and $x_i = q$ for all i in (3.1) gives

$$(3.5) \quad (1 - q)(1 - q^2) \cdots (1 - q^n) \sum_{f \in \mathcal{A}(P)} q^{|f|} = \sum_{w \in \mathcal{L}(P)} q^{\text{maj}(w)},$$

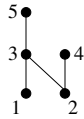
and hence, lastly,

$$(3.6) \quad \lim_{q \rightarrow 1} (1 - q)(1 - q^2) \cdots (1 - q^n) \sum_{f \in \mathcal{A}(P)} q^{|f|} = |\mathcal{L}(P)|.$$

Proof. To prove (3.1), use Proposition 2.5 (i) to write the sum on the left as a sum over w in $\mathcal{L}(P)$, and for each P -partition f , think about how many connected order ideals (counted with multiplicity) will be in the corresponding multiset from Proposition 2.5 (ii). \square

We remark that the specializations to $t = 1$ that appear in Corollary 3.2, namely (3.2) and its specializations (3.5), (3.6), are all part of Stanley's traditional P -partition theory; see [21, subsection 4.5].

Example 3.3. For this naturally labeled poset P on $\{1, 2, 3, 4, 5\}$



the expression in (3.1) can be computed using the following data

nonempty ideal $J \in \mathcal{J}(P)$	$c_P(J)$
{1}	1
{2}	1
{1, 2}	2
{2, 4}	1
{1, 2, 3}	1
{1, 2, 4}	2
{1, 2, 3, 4}	1
{1, 2, 3, 5}	1
{1, 2, 3, 4, 5}	1

$w \in \mathcal{L}(P)$	$\text{des}_P(w)$
12345	0
1235 · 4	1
124 · 35	2
2 · 1345	1
2 · 135 · 4	1 + 1 = 2
2 · 14 · 35	1 + 2 = 3
24 · 135	1

as the sum

$$\begin{aligned}
 & \sum_{f \in \mathcal{A}(P)} t^{\nu(f)} \mathbf{x}^f \\
 &= \sum_{w \in \mathcal{L}(P)} \frac{t^{\text{des}_P(w)} \prod_{i \in \text{Des}(w)} \mathbf{x}^{w|_{[1,i]}}}{\prod_{i=1}^n (1 - t^{c_P(w|_{[1,i]})} \mathbf{x}^{w|_{[1,i]}})} \\
 &= \frac{1}{(1-tx_1)(1-t^2x_1x_2)(1-tx_1x_2x_3)(1-tx_1x_2x_3x_4)(1-tx_1x_2x_3x_4x_5)} \\
 & \quad + \frac{tx_1x_2x_3x_5}{(1-tx_1)(1-t^2x_1x_2)(1-tx_1x_2x_3)(1-tx_1x_2x_3x_5)(1-tx_1x_2x_3x_4x_5)} \\
 & \quad + \frac{t^2x_1x_2x_4}{(1-tx_1)(1-t^2x_1x_2)(1-t^2x_1x_2x_4)(1-tx_1x_2x_3x_4)(1-tx_1x_2x_3x_4x_5)} \\
 & \quad + \frac{tx_2}{(1-tx_2)(1-t^2x_1x_2)(1-tx_1x_2x_3)(1-tx_1x_2x_3x_4)(1-tx_1x_2x_3x_4x_5)} \\
 & \quad + \frac{tx_2 \cdot tx_1x_2x_3x_5}{(1-tx_2)(1-t^2x_1x_2)(1-tx_1x_2x_3)(1-tx_1x_2x_3x_5)(1-tx_1x_2x_3x_4x_5)} \\
 & \quad + \frac{tx_2 \cdot t^2x_1x_2x_4}{(1-tx_2)(1-t^2x_1x_2)(1-t^2x_1x_2x_4)(1-tx_1x_2x_3x_4)(1-tx_1x_2x_3x_4x_5)} \\
 & \quad + \frac{tx_2x_4}{(1-tx_2)(1-tx_2x_4)(1-t^2x_1x_2x_4)(1-tx_1x_2x_3x_4)(1-tx_1x_2x_3x_4x_5)}
 \end{aligned}$$

which simplifies over a common denominator, after cancelations, to give

$$\frac{1 - t^2(\mathbf{x}^{(1,2,1,1,0)} + \mathbf{x}^{(1,2,1,1,1)} + \mathbf{x}^{(2,2,2,1,1)}) + t^3(\mathbf{x}^{(2,3,2,1,1)} + \mathbf{x}^{(2,3,2,2,1)})}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t\mathbf{x}^J)}.$$

The form of this last expression should be compared with Corollary 5.3 (ii).

4. First proof of Theorem 1.1: Inclusion–exclusion. We begin the proof with the following lemma, partly asserted already in the introduction as (1.4). Recall that, for a forest with duplications P , we denote by $\mathcal{D}(P)$ the collection of all pairs $\{a, a'\}$ that arise by the duplication steps in the construction of P . The set $\mathcal{D}(P)$ is well defined (it does not depend upon the construction of P), as shown by the following lemma.

Lemma 4.1. *Let P be a forest with duplications on $\{1, 2, \dots, n\}$.*

(i) *The duplication pairs in $\mathcal{D}(P)$ are pairwise disjoint: for any $\{a, a'\}, \{b, b'\}$ in $\mathcal{D}(P)$, either $\{a, a'\} = \{b, b'\}$ or $\{a, a'\} \cap \{b, b'\} = \emptyset$.*

(ii) *The set $\mathcal{J}_{\text{conn}}(P)$ of nonempty connected order-ideals of P are the principal ideals $P_{\leq p}$ (for $p \in P$), and the unions $P_{\leq a} \cup P_{\leq a'}$ for $\{a, a'\}$ in $\mathcal{D}(P)$.*

(iii) *The set $\Pi(P)$ of pairs $\{J_1, J_2\}$ of connected order-ideals of P intersecting non-trivially are the pairs $\{P_{\leq a}, P_{\leq a'}\}$ for $\{a, a'\}$ in $\mathcal{D}(P)$.*

Proof. Assertion (i) is equivalent to saying that, in building up a forest with duplications, once a duplication pair $\{a, a'\}$ is created from duplicating a hanger a in a poset P , then neither a nor a' will ever be a hanger at some later stage of the construction. To see this, note that any element p in the nonempty poset $P_{<a}$ which is covered by a will also be covered by a' after the duplication. Thus, in the new poset P' after duplication, p has a single-edge path to the element a' of $P \setminus (P')_{\leq a}$ avoiding a , and similarly p has a single-edge path to the element a of $P \setminus (P')_{\leq a'}$ avoiding a' . These single-edge paths cannot be destroyed by any of the further constructions, so neither a nor a' will ever be a hanger that is later duplicated.

We prove assertions (ii) and (iii) by induction on the cardinality of P , that is, on the number of operations used in constructing P . It suffices to show that they remain true when performing any of the three construction operations. This is trivial for the disjoint union construction, and straightforward for the hanging construction.

For the duplication of a hanger operation, we argue more carefully. Assume that P' is obtained from the forest with duplications P by duplicating the hanger a , to form a new pair $\{a, a'\}$ with $\mathcal{D}(P') =$

$\mathcal{D}(P) \sqcup \{\{a, a'\}\}$. We will make use of the order-preserving surjection $\pi : P' \twoheadrightarrow P$ that collapses both a and a' to a .

For assertion (ii), note that π sends any connected order ideal J' in $\mathcal{J}_{\text{conn}}(P')$ to a connected order ideal $J := \pi(J')$ in $\mathcal{J}_{\text{conn}}(P)$. By induction, one knows that J is either of the form $J = P_{\leq p}$, or of the form $P_{\leq b} \cup P_{\leq b'}$ where $\{b, b'\}$ lies in $\mathcal{D}(P)$. It is now straightforward to check that

- if $J = P_{\leq p}$ for some $p \neq a$, then $J' = (P')_{\leq p}$,
- if $J = P_{\leq a}$, then J' is either $(P')_{\leq a}$ or $(P')_{\leq a'}$ or $(P')_{\leq a} \cup (P')_{\leq a'}$, and
- if $J = P_{\leq b} \cup P_{\leq b'}$ where $\{b, b'\}$ lies in $\mathcal{D}(P)$, then $J' = (P')_{\leq b} \cup (P')_{\leq b'}$.

Thus $\mathcal{J}_{\text{conn}}(P')$ is exactly as described.

For assertion (iii), first note that $\{(P')_{\leq a}, (P')_{\leq a'}\}$ is a pair of connected order ideals intersecting nontrivially and hence lies in $\Pi(P)$. Now assume J' is in $\mathcal{J}_{\text{conn}}(P')$, but $J' \neq (P')_{\leq a}, (P')_{\leq a'}$. We have seen above that $J' = \pi^{-1}(J)$ for some J in $\mathcal{J}_{\text{conn}}(P)$. If J contains a , then J' contains both $(P')_{\leq a}, (P')_{\leq a'}$, and hence has trivial intersection with either of them. If J does not contain a , then since a is a hanger in P , connectivity of J forces it to lie entirely in $P_{< a}$ or $P \setminus P_{\leq a}$, and will still have trivial intersection with either of $P_{\leq a}$ or $P_{\leq a'}$. This analysis shows that the pairs $\{J'_1, J'_2\}$ in $\Pi(P')$ other than $\{(P')_{\leq a}, (P')_{\leq a'}\}$ are of the form $\{\pi^{-1}(J_1), \pi^{-1}(J_2)\}$ for some pair $\{J_1, J_2\}$ in $\Pi(P)$. By induction, $\{J_1, J_2\} = \{P_{\leq b}, P_{\leq b'}\}$ for some $\{b, b'\}$ in $\mathcal{D}(P)$, and then one can check that $\{J'_1, J'_2\} = \{(P')_{\leq b}, (P')_{\leq b'}\}$. \square

The next result is the crux of Theorem 1.1 and will follow easily via inclusion-exclusion from Lemma 4.1.

Theorem 4.2. *For a forest with duplications P on n elements, one has*

$$\sum_{f \in \mathcal{A}^{\text{weak}}(P)} t^{\nu(f)} \mathbf{x}^f = \frac{\prod_{\{J_1, J_2\} \in \Pi(P)} (1 - t^2 \mathbf{x}^{J_1} \mathbf{x}^{J_2})}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t \mathbf{x}^J)}.$$

Setting $t = 1$ and $x_i = q$ for all i , this gives

$$(4.1) \quad \sum_{f \in \mathcal{A}^{\text{weak}}(P)} q^{|f|} = \frac{\prod_{\{J_1, J_2\} \in \Pi(P)} (1 - q^{|J_1| + |J_2|})}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - q^{|J|})}.$$

Proof. Given a forest with duplications P , we wish to evaluate $\sum_{f \in \mathcal{A}^{\text{weak}}(P)} t^{\nu(f)} \mathbf{x}^f$, where the sum runs over all weak P -partitions f . By Proposition 2.5, this is the same as the sum $\sum_{\{J_i\}} \prod_i t \mathbf{x}^{J_i}$ over all multisubsets $\{J_i\}$ of $\mathcal{J}_{\text{conn}}(P)$ for which the $\{J_i\}$ pairwise intersect trivially. By Lemma 4.1 this is equivalent to saying that the multiset $\{J_i\}$ contains no pair $\{P_{\leq a}, P_{\leq a'}\}$ with $\{a, a'\}$ in $\mathcal{D}(P)$. Using inclusion-exclusion, this sum then equals

$$\sum_{\mathcal{E} \subseteq \mathcal{D}(P)} (-1)^{|\mathcal{E}|} \sum_{\{J_i\}} \prod_i t \mathbf{x}^{J_i}$$

where the inside summation is over all multisubsets $\{J_i\}$ of $\mathcal{J}_{\text{conn}}(P)$ that contain at least the pair $\{P_{\leq a}, P_{\leq a'}\}$ for every $\{a, a'\}$ in \mathcal{E} . Finally, this can be rewritten

$$\begin{aligned} \sum_{\mathcal{E} \subseteq \mathcal{D}(P)} (-1)^{|\mathcal{E}|} \frac{\prod_{\{a, a'\} \in \mathcal{E}} t \mathbf{x}^{P_{\leq a}} \cdot t \mathbf{x}^{P_{\leq a'}}}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t \mathbf{x}^J)} \\ = \frac{\prod_{\{a, a'\} \in \mathcal{D}(P)} (1 - t^2 \mathbf{x}^{P_{\leq a}} \mathbf{x}^{P_{\leq a'}})}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t \mathbf{x}^J)}. \quad \square \end{aligned}$$

Proof of Theorem 1.1. Recall that, for naturally labeled posets, weak P -partitions coincide with P -partitions. Then (1.2) follows from (3.5) and (4.1). \square

5. The rings and their Hilbert series. We now change focus in the next few sections to discussing the weak P -partition ring R_P , an example of a *normal affine semigroup ring*. Good discussions of general theory on affine semigroup rings may be found in Bruns and Herzog [4, Chapter 6], Miller and Sturmfels [15, Chapter 7], Stanley [23, Chapter 1] and Sturmfels [25].

Definition 5.1. For P a poset on $\{1, 2, \dots, n\}$, let R_P be the subalgebra of the polynomial ring $k[x_1, \dots, x_n]$ which is spanned k -linearly by the monomials

$$\mathbf{x}^f := x_1^{f(1)} \dots x_n^{f(n)}$$

as f runs through all weak P -partitions. In [2] this was the ring denoted R_P^{wt} .

Let \mathfrak{m} denote the maximal ideal of R_P spanned k -linearly by all monomials \mathbf{x}^f with $f \neq 0$, so that $R_P/\mathfrak{m} \cong k$. As usual, one has the \mathfrak{m} -adic filtration

$$(5.1) \quad R_P \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \dots$$

and the associated graded ring

$$\text{gr}(R_P) := R_P/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \dots$$

In this ring $\text{gr}(R_P)$ multiplication is defined k -linearly by saying that the product of two elements \bar{f} in $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ and \bar{g} in $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is \overline{fg} in $\mathfrak{m}^{i+j}/\mathfrak{m}^{i+j+1}$.

Note that R_P has a natural \mathbf{N}^n -multigrading, in which the degree of \mathbf{x}^f is $(f(1), \dots, f(n)) \in \mathbf{N}^n$. Then its \mathbf{N}^n -graded *Hilbert series* will be

$$\text{Hilb}(R_P, \mathbf{x}) = \sum_{f \in \mathcal{A}^{\text{weak}}(P)} \mathbf{x}^f,$$

that is the same generating function² that appears in (3.2).

Note also that $\text{gr}(R_P)$ enjoys this same \mathbf{N}^n -multigrading, and even the same \mathbf{N}^n -graded *Hilbert series* as R_P , since the \mathfrak{m} -adic filtration (5.1) is a filtration by \mathbf{N}^n -homogeneous ideals.

We will always use the \mathbf{x} -variable set for the power series that are Hilbert series with respect to this \mathbf{N}^n -multigrading. In addition, one can collapse the \mathbf{N}^n -multigrading to an \mathbf{N} -grading by letting $x_i = q$ for all i . We will use the variable q for power series which are Hilbert series for this grading.

² Again assuming that P has been *naturally labeled*; see Remark 2.6.

Furthermore, $\mathrm{gr}(R_P)$ has its standard \mathbf{N} -grading in which its homogeneous component of degree i is $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. We call the t -grading and use the variable t in the corresponding Hilbert series.

In fact, one can form an even finer Hilbert series $\mathrm{Hilb}(\mathrm{gr}(R_P), t, \mathbf{x})$ that keeps track of both the t -grading and the \mathbf{N}^n -multigrading. We will see shortly that this series is exactly the right side of (3.1).

Proposition 2.5 (iii) has the following consequence. Fixing a field k , introduce a polynomial algebra $S = k[U_J]$ having generators U_J indexed by connected order ideals J of P . For the sake of considering multigraded maps, consider S as \mathbf{N}^n -multigraded, with the variable U_J having the same degree as the monomial \mathbf{x}^J , namely the characteristic vector χ_J in \mathbf{N}^n . In particular, when we collapse the grading into an \mathbf{N} -grading, the variable U_J has degree $|J|$. In addition, S admits another interesting \mathbf{N} -grading, where all U_J have degree 1, corresponding to the t -grading discussed earlier.

Corollary 5.2 (cf. [2, Proposition 7.1]). *The ring R_P is minimally generated as a k -algebra by the monomials \mathbf{x}^J as J runs through $\mathcal{J}_{\mathrm{conn}}(P)$. In particular, these maps*

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & R_P & \text{and} & S & \xrightarrow{\mathrm{gr}(\varphi)} & \mathrm{gr}(R_P) \\ U_J & \longmapsto & \mathbf{x}^J & & U_J & \longmapsto & \bar{\mathbf{x}}^J \end{array}$$

are multigraded k -algebra surjections with respect to the \mathbf{N}^n -gradings. Moreover, the second map is also \mathbf{N} -graded with respect to the t -gradings.

Proof. The fact that $\{\mathbf{x}^J\}_{J \in \mathcal{J}_{\mathrm{conn}}(P)}$ minimally generate R_P was proven in [2, Proposition 7.1], but we repeat the proof here for completeness.

The fact that they generate R_P follows from Proposition 2.5 (iii). Their *minimality* follows from the claim that the characteristic vectors χ_J for J in $\mathcal{J}_{\mathrm{conn}}(P)$ are exactly the set of primitive vectors spanning the extreme rays of the real cone nonnegatively spanned by the P -partitions.³

³ In [21, Proposition 4.6.10] such vectors are called the *completely fundamental elements* of the semigroup.

To see this claim, given J in $\mathcal{J}_{\text{conn}}(P)$, consider the Hasse diagram for J as a connected graph, and pick a spanning tree T among its edges. Then the line $\mathbf{R}\chi_J$ is exactly the intersection of the hyperplanes $x_i = 0$ for $i \notin J$, and $x_i = x_j$ for $\{i, j\}$ an edge of T . All of these hyperplanes arise as cases of equality in various half-space inequalities that define the weak P -partition cone. Hence, each such χ_J spans an extreme ray of the cone.

Since $\{\mathbf{x}^J\}_{J \in \mathcal{J}_{\text{conn}}(P)}$ is a minimal generating set for R_P as an algebra, their images $\{\bar{\mathbf{x}}^J\}_{J \in \mathcal{J}_{\text{conn}}(P)}$ by $\text{gr}(\varphi)$ give a k -basis for $\mathfrak{m}/\mathfrak{m}^2$. Hence, each such element has t -degree 1 and so the map $\text{gr}(\varphi)$ respects the t -grading. \square

This result allows us to interpret combinatorially the power of t in the power series $\text{Hilb}(\text{gr}(R_P), t, \mathbf{x})$ and to obtain some information about its form.

Corollary 5.3. *Let P be any poset on $\{1, 2, \dots, n\}$.*

(i) *The $\mathbf{N} \times \mathbf{N}^n$ -graded Hilbert series for $\text{gr}(R_P)$ is given by*

$$\text{Hilb}(\text{gr}(R_P), t, \mathbf{x}) = \sum_{f \in \mathcal{A}^{\text{weak}}(P)} t^{\nu(f)} \mathbf{x}^f.$$

(ii) *The power series in (i) can always be expressed as*

$$\frac{g(t, \mathbf{x})}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t\mathbf{x}^J)}$$

for some polynomial $g(t, \mathbf{x})$ in $\mathbf{Z}[t, \mathbf{x}]$.

(iii) *Furthermore, the generating functions appearing in Corollary 3.2 are the Hilbert series for R_P or $\text{gr}(R_P)$ with respect to their $\mathbf{N} \times \mathbf{N}^n$ -grading or $\mathbf{N} \times \mathbf{N}$ -grading or \mathbf{N}^n or \mathbf{N} -grading, where appropriate.*

Proof. For assertion (i), note that $\bar{\mathbf{x}}^J$ has t -degree 1 and \mathbf{N}^n -multidegree χ_J in $\text{gr}(R_P)$. This means that, if $f = \sum_{i=1}^{\nu(f)} \chi_{J_i}$ for connected order ideals J_i , then $\bar{\mathbf{x}}^f = \prod_{i=1}^{\nu(f)} \bar{\mathbf{x}}^{J_i}$ will have t -degree $\nu(f)$ and \mathbf{N}^n -multidegree f , as desired.

For assertion⁴ (ii), note that $\mathfrak{gr}(R_P)$ becomes a finitely-generated $\mathbf{N} \times \mathbf{N}^n$ -graded S -module where $S = k[U_J]_{J \in \mathcal{J}_{\text{conn}}(P)}$. It therefore has an $\mathbf{N} \times \mathbf{N}^n$ -graded free S -resolution,

$$0 \longrightarrow F_\ell \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathfrak{gr}(R_P) \longrightarrow 0,$$

with $F_0 = S$, and whose length ℓ is guaranteed to be at most $|\mathcal{J}_{\text{conn}}(P)|$ by Hilbert's syzygy theorem. Letting $\beta_{i,(j,\alpha)}$ denote the number of S -basis elements of the free S -module F_i having $\mathbf{N} \times \mathbf{N}^n$ -multidegree (j, α) , then

$$\begin{aligned} \text{Hilb}(R_P, t, \mathbf{x}) &= \text{Hilb}(S, t, \mathbf{x}) \cdot \sum_{i=0}^{\ell} (-1)^i \sum_{(j,\alpha) \in \mathbf{N} \times \mathbf{N}^n} \beta_{i,(j,\alpha)} t^j \mathbf{x}^\alpha \\ &= \sum_{\substack{i=0,1,\dots,\ell \\ (j,\alpha) \in \mathbf{N} \times \mathbf{N}^n}} \beta_{i,(j,\alpha)} (-1)^i t^j \mathbf{x}^\alpha \bigg/ \prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t\mathbf{x}^J). \end{aligned}$$

Thus, the numerator here is the polynomial $g(t, \mathbf{x})$. \square

6. Presentations and proofs of Theorems 1.2 and 1.4. Here we further analyze the structure of rings R_P and $\mathfrak{gr}(R_P)$, by means of the surjections φ and $\mathfrak{gr}(\varphi)$ from Corollary 5.2.

Definition 6.1. Define three ideals within the polynomial ring $S = k[U_J]_{J \in \mathcal{J}_{\text{conn}}(P)}$, each with generating sets indexed by the set $\Pi(P)$ that consists of all pairs $\{J_1, J_2\}$ of connected order ideals in P which intersect nontrivially:

$$\begin{aligned} I_P &:= (\text{syz}_{J_1, J_2})_{\{J_1, J_2\} \in \Pi(P)} \\ I_P^{\mathfrak{gr}} &:= (\text{syz}_{J_1, J_2}^{\mathfrak{gr}})_{\{J_1, J_2\} \in \Pi(P)} \\ I_P^{\text{init}} &:= (\text{syz}_{J_1, J_2}^{\text{init}})_{\{J_1, J_2\} \in \Pi(P)} \end{aligned}$$

where syz_{J_1, J_2} , $\text{syz}_{J_1, J_2}^{\mathfrak{gr}}$, $\text{syz}_{J_1, J_2}^{\text{init}}$ were defined in (1.5), (1.8) and (1.9) in the introduction.

⁴ An alternate argument for assertion (ii) is to apply [21, Prop. 4.6.11].

We will see further (Proposition 6.3) that I_P and I_P^{gr} are the kernels of the morphisms φ and $\text{gr}(\varphi)$, so that $R_P \simeq S/I_P$ and $\text{gr}(R_P) \simeq S/I_P^{\text{gr}}$. We first establish a link between these rings and S/I_P^{init} .

Proposition 6.2. *For any P on $\{1, 2, \dots, n\}$, the three rings*

$$\begin{aligned} &R_P \\ &\text{gr}(R_P) \\ &S/I_P^{\text{init}} \end{aligned}$$

share the same \mathbf{N}^n -graded Hilbert series, namely, $\sum_{f \in \mathcal{A}^{\text{weak}}(P)} \mathbf{x}^f$.

Proof. By definition, R_P has this sum $\sum_f \mathbf{x}^f$ as its \mathbf{N}^n -graded Hilbert series. Setting $t = 1$ in Corollary 5.3 (i) shows the same for $\text{gr}(R_P)$. Finally, Proposition 2.5 (ii) implies that S/I_P^{init} also has this same generating function as its \mathbf{N}^n -graded Hilbert series, since the monomials surviving in the quotient S/I_P^{init} correspond to multisets of nonempty connected order ideals that pairwise intersect trivially. \square

The relation between the monomial quotient S/I_P^{init} and the rings R_P and $\text{gr}(R_P)$ is in fact deeper than an equality of Hilbert series. Indeed, it fits into the theory of Gröbner bases (see, e.g., Sturmfels [25, Chapter 1]). Recall that a *monomial ordering* on S is a total ordering \preceq on the set of all monomials U^A in S with these properties:

- (a) \preceq has no infinite descending chains,
- (b) the monomial $1 = U^0$ is the smallest element for \preceq , and
- (c) for any monomials U^A, U^B, U^C ,

$$U^A \preceq U^B \text{ implies } U^A U^C \preceq U^B U^C.$$

Having fixed a monomial ordering \preceq , given a polynomial f in S , its *initial term* $\text{init}_{\preceq}(f)$ is its monomial with nonzero coefficient which is highest in the \preceq order. Given an ideal $I \subset S$, its *initial ideal* is the monomial ideal $\text{init}_{\preceq}(I) := (\text{init}_{\preceq}(f))_{f \in I}$.

Given a poset P , we define a total ordering \preceq on the monomials in S as follows. First choose a total order \preceq on order ideals of P such

that $|J| < |K|$ implies $J \preceq K$. Then when comparing two distinct monomials

$$(6.1) \quad \begin{aligned} \mathbf{U}_J &= U_{J_1} U_{J_2} \cdots U_{J_r} & \text{with } J_1 \preceq J_2 \preceq \cdots \preceq J_r, \\ \mathbf{U}_K &= U_{K_1} U_{K_2} \cdots U_{K_s} & \text{with } K_1 \preceq K_2 \preceq \cdots \preceq K_s, \end{aligned}$$

assume without loss of generality that $r \leq s$. Find the smallest i in $\{1, 2, \dots, r\}$ for which $J_i \neq K_i$; if no such i exists, so \mathbf{U}_J strictly divides \mathbf{U}_K , say that $\mathbf{U}_J \preceq \mathbf{U}_K$. Otherwise, if $J_i \preceq K_i$, say that $\mathbf{U}_J \preceq \mathbf{U}_K$ and, if $K_i \preceq J_i$, say that $\mathbf{U}_K \preceq \mathbf{U}_J$. It is not hard to see that such a linear order \preceq will satisfy the above properties (a), (b), (c) that define a monomial ordering.

Theorems 1.2 and 1.4 amount to the following result.

Theorem 6.3. *For a poset P on $\{1, 2, \dots, n\}$, one has these ideal equalities:*

$$\begin{aligned} I_P &= \ker(\varphi : S \longrightarrow R_P) \\ I_P^{\mathfrak{gr}} &= \ker(\mathfrak{gr}(\varphi) : S \longrightarrow \mathfrak{gr}(R_P)) \\ I_P^{\text{init}} &= \text{init}_{\preceq}(I_P) = \text{init}_{\preceq}(I_P^{\mathfrak{gr}}), \end{aligned}$$

where \preceq is the monomial ordering described above.

The first equality asserts that I_P is the toric ideal for the ring R_P with respect to its minimal generating set, in the terminology of Sturmfels [25].

Proof. Temporarily denote by $K, K^{\mathfrak{gr}}$ the kernels appearing on the right sides in the proposition:

$$\begin{aligned} K &:= \ker(\varphi : S \longrightarrow R_P); \\ K^{\mathfrak{gr}} &:= \ker(\mathfrak{gr}(\varphi) : S \longrightarrow \mathfrak{gr}(R_P)). \end{aligned}$$

One can check from the generators of I_P and $I_P^{\mathfrak{gr}}$ given in Definition 6.1 that $I_P \subseteq K$ and $I_P^{\mathfrak{gr}} \subseteq K^{\mathfrak{gr}}$. Hence, one has the inclusions

$$\begin{aligned} \text{init}_{\preceq}(I_P) &\subseteq \text{init}_{\preceq}(K) \\ \text{init}_{\preceq}(I_P^{\mathfrak{gr}}) &\subseteq \text{init}_{\preceq}(K^{\mathfrak{gr}}). \end{aligned}$$

On the other hand, since

$$\text{syz}_{J_1, J_2}^{\text{init}} = U_{J_1} U_{J_2} = \text{init}_{\preceq}(\text{syz}_{J_1, J_2}) = \text{init}_{\preceq}(\text{syz}_{J_1, J_2}^{\text{gr}}),$$

one concludes that

$$I_P^{\text{init}} \subseteq \text{init}_{\preceq}(I_P), \text{init}_{\preceq}(I_P^{\text{gr}}).$$

These various ideal inclusions lead to towers of surjections

$$(6.2) \quad \begin{array}{ccccc} S/I_P^{\text{init}} & \twoheadrightarrow & S/\text{init}_{\preceq}(I_P) & \twoheadrightarrow & S/\text{init}_{\preceq}(K) \\ S/I_P^{\text{init}} & \twoheadrightarrow & S/\text{init}_{\preceq}(I_P^{\text{gr}}) & \twoheadrightarrow & S/\text{init}_{\preceq}(K^{\text{gr}}). \end{array}$$

Recall, that for any homogeneous ideal I of S and any monomial ordering \preceq , the initial ideal $\text{init}_{\preceq}(I)$ has the property that S/I and $S/\text{init}_{\preceq}(I)$ share the same Hilbert series. Together with Proposition 6.2 this shows all these quotient rings

$$\begin{array}{ll} S/K & (\cong R_P) \\ S/K^{\text{gr}} & (\cong \text{gr}(R_P)) \\ S/I_P^{\text{init}} & \\ S/\text{init}_{\preceq}(K) & \\ S/\text{init}_{\preceq}(K^{\text{gr}}) & \end{array}$$

share the same \mathbf{N}^n -multigraded Hilbert series. One concludes that all of the surjections in the towers (6.2) are isomorphisms. Thus,

$$\begin{array}{l} I_P^{\text{init}} = \text{init}_{\preceq}(I_P) = \text{init}_{\preceq}(K) \\ I_P^{\text{init}} = \text{init}_{\preceq}(I_P^{\text{gr}}) = \text{init}_{\preceq}(K^{\text{gr}}), \end{array}$$

and the generators for I_P, I_P^{gr} given in their definitions form Gröbner bases with respect to \preceq for the ideals K, K^{gr} . This implies $I_P = K$ and $I_P^{\text{gr}} = K^{\text{gr}}$. \square

Proposition 6.4. *Each of the three ideals I_P, I_P^{gr} and I_P^{init} is generated minimally by the generating sets appearing in Definition 6.1 indexed by $\Pi(P)$.*

Proof. We give the argument by contradiction for why the generator

$$\text{syz}_{J_1, J_2} = U_{J_1} U_{J_2} - U_{J_1 \cup J_2} \prod_{i=1}^t U_{J^{(i)}}$$

of I_P cannot be redundant; the arguments for I_P^{gr} and I_P^{init} are similar and even easier. If syz_{J_1, J_2} were redundant, then it could be expressed as a sum

$$\text{syz}_{J_1, J_2} = \sum_{\substack{\{K_1, K_2\} \in \Pi_P \\ \{K_1, K_2\} \neq \{J_1, J_2\}}} f_{K_1, K_2} \cdot \text{syz}_{K_1, K_2}$$

where the f_{K_1, K_2} are some polynomials in the variables U_J of S . Since the monomial $U_{J_1} U_{J_2}$ appears on the left, it must appear on the right, say in the term $f_{K_1, K_2} \cdot \text{syz}_{K_1, K_2}$, forcing one of the two monomials $U_{K_1} U_{K_2}$ or $U_{K_1 \cup K_2} \prod_{i=1}^m U_{K^{(i)}}$ in syz_{K_1, K_2} to divide $U_{J_1} U_{J_2}$. Since $U_{J_1} U_{J_2}$ is quadratic, this forces either the equality of sets

- $\{J_1, J_2\} = \{K_1, K_2\}$, a contradiction, or
- $m = 1$ (that is, $K_1 \cap K_2 = K^{(1)}$ is connected) and $\{K_1 \cup K_2, K_1 \cap K_2\} = \{J_1, J_2\}$. This is again a contradiction because J_1 and J_2 have non-trivial intersection, that is, neither one is included in the other. \square

We close this section by discussing the situation when $\text{gr}(R_P) \cong R_P$.

Corollary 6.5. *The following are equivalent for a poset P on $\{1, 2, \dots, n\}$:*

- (i) *One has $I_P^{\text{gr}} = I_P$ and $\text{gr}(R_P) \cong R_P$.*
- (ii) *The toric ideal $I_P = \ker(S \xrightarrow{\varphi} R_P)$ is homogeneous for the standard \mathbf{N} -grading on S in which each U_J has degree one.*
- (iii) *Every pair $\{J_1, J_2\}$ of connected order ideals that intersects nontrivially has $J_1 \cap J_2$ connected.*

Proof. The equivalence of (i) and (ii) is easy and well known. For the equivalence of (ii) and (iii), note that the minimal generator syz_{J_1, J_2} is homogeneous if and only if $t = 1$, that is, if and only if $J_1 \cap J_2$ is connected. Now apply Proposition 6.4. \square

An important special case of this situation where $\text{gr}(R_P) \cong R_P$ was studied by Hibi [13], namely, when *every* nonempty order ideal is connected. We leave the straightforward proof of the following proposition to the reader.

Proposition 6.6. *A finite poset P has every nonempty order ideal connected if and only if P has a minimum element $\hat{0}$. Furthermore, in this case,*

- *the two decompositions in Proposition 2.5 (ii) and (iii) coincide,*
- *the statistic $\nu(f)$ on P -partitions f equals the maximum value $\max(f)$,*
- *the statistic $\text{des}_P(w)$ on linear extensions w in $\mathcal{L}(P)$ is independent of the poset structure P and equals the descent number $\text{des}(w) := |\text{Des}(w)|$,*
- *the ring $R_P \cong \text{gr}(R_P)$ is the same as the Hibi ring introduced in [13] but associated with the poset $P \setminus \hat{0}$. In other words,*

$$R_P \cong \text{gr}(R_P) \cong k[y_J]_{J \in \mathcal{J}(P \setminus \hat{0})} / (y_{J_1} \cdot y_{J_2} - y_{J_1 \cup J_2} \cdot y_{J_1 \cap J_2})_{J_1, J_2 \in \mathcal{J}(P)}.$$

7. Second proof of Theorem 1.1: Complete intersections. We give here a second proof, via our ring presentations, of the precursor Theorem 4.2, rather than Theorem 1.1 itself.

This proof uses some basic notions of commutative algebra that we shall recall here: we refer to Stanley [23, Section I.5] for more details on this subject.

The Krull dimension $\dim(A)$ of a finitely generated commutative k -algebra A is the maximum cardinality d of a subset $\{\theta_1, \dots, \theta_d\}$ in A which are algebraically independent over k . If A is \mathbf{N} -graded, then the Krull dimension coincides with the multiplicity of the pole $z = 1$ in the Hilbert series $\text{Hilb}(A, z)$. In particular, when several algebras share the same Hilbert series, they also share the same Krull dimension.

Let $\theta_1, \dots, \theta_\ell$ be homogeneous elements in a graded k -algebra A . Then one has the inequality

$$(7.1) \quad \dim(A/(\theta_1 A + \dots + \theta_\ell A)) \geq \dim A - \ell.$$

If A is *Cohen-Macaulay* (which is for example the case of a polynomial algebra over a field), then equality in (7.1) occurs if and only if for each $i = 1, 2, \dots, \ell$, one has that θ_i is a non-zero-divisor in the quotient $A/(\theta_1 A + \dots + \theta_{i-1} A)$. Such a sequence $(\theta_1, \dots, \theta_\ell)$ is called an *A-regular sequence*.

We now have all the necessary tools to give our second proof of Theorem 4.2.

Proof of Theorem 4.2. For any poset P on $\{1, 2, \dots, n\}$, the affine semigroup ring R_P of P -partitions has Krull dimension n , since the cone of P -partitions is n -dimensional. But then $\mathfrak{gr}(R_P)$ and S/I_P^{init} also have Krull dimension n , since Proposition 6.2 asserts that they have the same \mathbf{N}^n -graded Hilbert series.

Now the presentation for any of the three rings $R_P, \mathfrak{gr}(R_P), S/I_P^{\text{init}}$ in Theorems 1.2 and 1.4 exhibits them as quotients of $S = k[U_J]_{J \in \mathcal{J}_{\text{conn}}(P)}$, which has Krull dimension $|\mathcal{J}_{\text{conn}}(P)|$, by an ideal $(I_P, I_P^{\mathfrak{gr}}$ or $I_P^{\text{init}})$ having $|\Pi(P)|$ minimal generators. Hence, one always has the inequality

$$(7.2) \quad |\mathcal{J}_{\text{conn}}(P)| - |\Pi(P)| \geq n$$

and equality occurs if and only if this is a *complete intersection presentation*, meaning that the ideal generators in each case form an S -regular sequence.

When these are complete intersection presentations, one obtains the following Hilbert series calculation for $\mathfrak{gr}(R_P)$

$$\begin{aligned} \left(\sum_{f \in \mathcal{A}^{\text{weak}}(P)} t^{\nu(f)} \mathbf{x}^f \right) &= \text{Hilb}(\mathfrak{gr}(R_P), t, \mathbf{x}) \\ &= \frac{\prod_{\{J_1, J_2\} \in \Pi(P)} (1 - t^2 \mathbf{x}^{J_1} \mathbf{x}^{J_2})}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t \mathbf{x}^J)} \end{aligned}$$

by iterating the relation

$$\text{Hilb}(R/(\theta), t) = (1 - t^{\deg(\theta)}) \cdot \text{Hilb}(R, t)$$

which holds for a nonzero divisor θ in a (multi-)graded ring R ; see [23, Section I.5, page 25]. It only remains to note that when P is a forest

with duplications, Proposition 1.4 shows $|\mathcal{J}_{\text{conn}}(P)| = n + |\mathcal{D}(P)|$ and $|\Pi(P)| = |\mathcal{D}(P)|$. Equality in (7.2) follows. \square

8. Koszulity. We discuss here an immediate consequence of I_P^{gr} having a quadratic initial ideal I_P^{init} , coming from the theory of *Koszul algebras*. The reader is referred to Fröberg [9] and the book by Polishchuk and Positselski [17] for background on Koszul algebras.

Corollary 8.1. *For any poset P on $\{1, 2, \dots, n\}$, the graded ring $A = \text{gr}(R_P)$ is a Koszul algebra. In other words, (R_P, \mathfrak{m}) is nongraded Koszul in the sense considered by Fröberg [8].*

In particular, the $\mathbf{N} \times \mathbf{N}^n$ -multigraded Hilbert series $\text{Hilb}(A, t, \mathbf{x})$ described in Corollary 5.3 has the property that $\text{Hilb}(A, -t, \mathbf{x})^{-1}$ lies in $\mathbf{N}[t, \mathbf{x}]$, as it is the Hilbert series for the Koszul dual algebra A^\dagger .

Proof. It is well known (see, e.g., [7, Proposition 3]) that having an initial ideal generated by quadratic monomials, as is the case with $I_P^{\text{init}} = \text{init}_{\leq}(I_P^{\text{gr}})$, suffices to imply Koszulity. The relation between the Hilbert series of a Koszul ring A and its Koszul dual A^\dagger is also standard. \square

Example 8.2. Since Theorem 4.2 implies that a forest with duplications P has

$$\text{Hilb}(R_P, t, \mathbf{x}) = \frac{\prod_{\{J_1, J_2\} \in \Pi(P)} (1 - t^2 \mathbf{x}^{J_1} \mathbf{x}^{J_2})}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t \mathbf{x}^J)},$$

one sees that

$$\text{Hilb}(R_P, -t, \mathbf{x})^{-1} = \frac{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 + t \mathbf{x}^J)}{\prod_{\{J_1, J_2\} \in \Pi(P)} (1 - t^2 \mathbf{x}^{J_1} \mathbf{x}^{J_2})}$$

which manifestly lies in $\mathbf{N}[t, \mathbf{x}]$.

Example 8.3. The naturally labeled poset P from Example 3.3 had $\text{Hilb}(R_P, t, \mathbf{x})$ equal to

$$\frac{1 - t^2 (\mathbf{x}^{(1,2,1,1,0)} + \mathbf{x}^{(1,2,1,1,1)} + \mathbf{x}^{(2,2,2,1,1)}) + t^3 (\mathbf{x}^{(2,3,2,1,1)} + \mathbf{x}^{(2,3,2,2,1)})}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t \mathbf{x}^J)},$$

and hence $\text{Hilb}(R_P, -t, \mathbf{x})^{-1}$ equal to

$$\frac{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 + t\mathbf{x}^J)}{1 - (t^2(\mathbf{x}^{(1,2,1,1,0)} + \mathbf{x}^{(1,2,1,1,1)} + \mathbf{x}^{(2,2,2,1,1)}) + t^3(\mathbf{x}^{(2,3,2,1,1)} + \mathbf{x}^{(2,3,2,2,1)})}$$

which again manifestly lies in $\mathbf{N}[t, \mathbf{x}]$.

9. The ideal of P -partitions, and the MAJ formula for forests. When the poset P is not naturally labeled, the P -partitions $\mathcal{A}(P)$ form a proper subset of the affine semigroup $\mathcal{A}^{\text{weak}}(P)$ of weak P -partitions. In fact, this subset $\mathcal{A}(P)$ is a *semigroup ideal*, in the sense that

$$\mathcal{A}^{\text{weak}}(P) + \mathcal{A}(P) = \mathcal{A}(P).$$

Definition 9.1. For a poset P on $\{1, 2, \dots, n\}$, let $\mathcal{I}_P \subset R_P$ denote the ideal of the affine semigroup ring R_P spanned k -linearly by the monomials \mathbf{x}^f where f runs through $\mathcal{A}(P)$.

From the R_P -module filtration $\mathcal{I}_P \supset \mathfrak{m}\mathcal{I}_P \supset \mathfrak{m}^2 \supset \dots$, one can form the *associated $\text{gr}(R_P)$ -graded module*

$$\text{gr}(\mathcal{I}_P) = \mathcal{I}_P/\mathfrak{m}\mathcal{I}_P \oplus \mathfrak{m}\mathcal{I}_P/\mathfrak{m}^2\mathcal{I}_P \oplus \mathfrak{m}^2\mathcal{I}_P/\mathfrak{m}^3\mathcal{I}_P \oplus \dots$$

Recall that Corollary 5.2 showed that R_P , $\text{gr}(R_P)$, respectively, were generated as k -algebras by the collection of monomials $\{\mathbf{x}^J\}_{J \in \mathcal{J}_{\text{conn}}(P)}$ and their images within $\mathfrak{m}/\mathfrak{m}^2$, respectively. Similarly, Proposition 2.4 shows that the ideal \mathcal{I}_P within R_P is finitely generated, by the monomials

$$(9.1) \quad \left\{ \prod_{i \in \text{Des}(w)} \mathbf{x}^{w|_{[1,i]}} : w \in \mathcal{L}(P) \right\},$$

and hence their images within $\mathcal{I}_P/\mathfrak{m}\mathcal{I}_P$ will generate $\text{gr}(\mathcal{I}_P)$ as a $\text{gr}(R_P)$ -module. As it is finitely generated, we can deduce the following result exactly as in Corollary 5.3.

Corollary 9.2. *The $\mathbf{N} \times \mathbf{N}^n$ -graded Hilbert series for $\text{gr}(\mathcal{I}_P)$ is*

$$\text{Hilb}(\text{gr}(\mathcal{I}_P), t, \mathbf{x}) = \sum_{f \in \mathcal{A}(P)} t^{\nu(f)} \mathbf{x}^f$$

and can always be expressed in the form

$$\frac{h(t, \mathbf{x})}{\prod_{J \in \mathcal{J}_{\text{conn}}(P)} (1 - t\mathbf{x}^J)}$$

for some polynomial $h(t, \mathbf{x})$ in $\mathbf{Z}[t, \mathbf{x}]$.

Remark 9.3. Note that the monomials in (9.1) will not necessarily generate \mathcal{I}_P *minimally* in general. For example, let $P = P_3$ be the poset with order relations $3 <_P 1, 2$ among those in Example 2.2. Then

$$\mathcal{L}(P) = \{3 \cdot 12, \quad 3 \cdot 2 \cdot 1\}$$

where here dots have been added indicating descents. The generating set for \mathcal{I}_P described in (9.1) is in this case $\{x_3, \quad x_3 \cdot x_2 x_3\}$. However, it is easy to check (or see Proposition 9.5 below) that in this case \mathcal{I}_P is the principal ideal within $R_P = k[x_3, x_1 x_3, x_2 x_3, x_1 x_2 x_3]$ generated by the single monomial $\{x_3\}$.

Although we do not know a *minimal* generating set in general for the ideal \mathcal{I}_P within R_P , it turns out to be easy to characterize when \mathcal{I}_P is *principal*, that is, generated by a single element. This is equivalent to the existence of a minimum P -partition f_{\min} in $\mathcal{A}(P)$ with the property that

$$f_{\min} + \mathcal{A}^{\text{weak}}(P) = \mathcal{A}(P).$$

Such a characterization was provided by Stanley (see [21, Lemma 4.5.12]) in the special case where P is *strictly labeled*; we explain here the obvious modification of his characterization for the general case.

Definition 9.4. We define a candidate for f_{\min} , the function $\delta : P \rightarrow \mathbf{N}$ whose value $\delta(i)$ is the maximum over all saturated chains in $P_{>i}$ of the number of *strict* covering relations in the chain, that is, covering relations $i <_P j$ for which $i >_{\mathbf{N}} j$. It is easily checked both that

- (a) δ lies in $\mathcal{A}(P)$, and
- (b) every f in $\mathcal{A}(P)$ has $f(i) \geq \delta(i)$ for all i .

Say that the poset P on $\{1, 2, \dots, n\}$ satisfies the *labeled- δ -chain condition*⁵ if, for every i , all saturated chains in $P_{\geq i}$ have the same number of strict covering relations, namely, $\delta(i)$.

Proposition 9.5. *The P -partition ideal \mathcal{I}_P within the (weak) P -partition ring R_P is a principal ideal if and only if P satisfies the labeled- δ -chain condition. Furthermore, in this case $f_{\min} = \delta$.*

Proof. The second assertion follows from properties (a) and (b) above: if f_{\min} exists, then (a) implies $\delta \geq f_{\min}$, while (b) implies $f_{\min} \geq \delta$.

For the first two assertions, note that the values of δ satisfy

$$(9.2) \quad \delta(i) \begin{cases} = 0 & \text{if } i \text{ is maximal in } P \\ \geq \delta(j) & \text{if } i \prec_P j \text{ and } i \succ_{\mathbf{N}} j \text{ for some } j \\ \geq \delta(j) + 1 & \text{if } i \prec_P j \text{ and } i \succ_{\mathbf{N}} j \text{ for some } j. \end{cases}$$

It is then easily seen that the labeled- δ -chain condition is equivalent to the assertion that changing the inequalities in (9.2) to equalities gives a well-defined recursive formula for δ .

Thus, when the labeled- δ -chain condition holds, any f in $\mathcal{A}(P)$ has $f - \delta$ in $\mathcal{A}^{\text{weak}}(P)$: the recursive formula for δ shows that $f - \delta$ is weakly decreasing along each covering relation $i \prec_P j$.

Conversely, if the labeled- δ -condition fails, then there exists some covering relation $i \prec_P j$ for which the inequality in (9.2) is strict. In this case, one can check that the function defined by

$$f(k) = \begin{cases} \delta(k) & \text{if } k \in P_{\geq i} \text{ but } k \neq j, \\ \delta(k) + 1 & \text{if } k \in P \setminus P_{\geq i} \text{ or } k = j, \end{cases}$$

gives an element f of $\mathcal{A}(P)$ with the property that $f - \delta$ does not lie in $\mathcal{A}^{\text{weak}}(P)$:

$$(f - \delta)(i) = 0 > 1 = (f - \delta)(j)$$

so $f - \delta$ fails to be weakly order-reversing along the cover relation $i \prec_P j$. \square

⁵ The reason for this terminology is that, in the special case where P is strictly labeled, it was called the *δ -chain condition* by Stanley in [21, subsection 4.5].

When P satisfies the labeled- δ -chain condition, let $\text{maj}(P) := |\delta| = \sum_{i=1}^n \delta(i)$. The following is then simply a translation of Proposition 9.5.

Corollary 9.6. *A poset P on $\{1, 2, \dots, n\}$ satisfies*

$$\sum_{f \in \mathcal{A}(P)} \mathbf{x}^f = \mathbf{x}^{f_{\min}} \sum_{f \in \mathcal{A}^{\text{weak}}(P)} \mathbf{x}^f$$

for some vector f_{\min} in \mathbf{N}^n if and only if P satisfies the labeled- δ -chain condition. In this case, $f_{\min} = \delta$, and one has

$$\sum_{w \in \mathcal{L}(P)} q^{\text{maj}(w)} = q^{\text{maj}(P)} \cdot (1 - q)(1 - q^2) \cdots (1 - q^n) \cdot \text{Hilb}(R_P, q).$$

Example 9.7. Recall from the introduction that a forest poset P is one in which an element is covered by at most one other element. Thus, any forest poset P on $\{1, 2, \dots, n\}$ always satisfies the labeled- δ -chain condition, since for each i there is only one maximal chain in $P_{\geq i}$.

Note also that, for forest posets, since no duplications are used in their construction, $\mathcal{D}(P)$ is empty, so that $\Pi(P)$ is empty, and $\mathcal{J}_{\text{conn}}(P)$ is simply the set of all principal order ideals $P_{\leq i}$. Thus one concludes in this case, from Theorem 1.1 and Corollary 9.6 that, for arbitrarily labeled forest posets P ,

$$\sum_{w \in \mathcal{L}(P)} q^{\text{maj}(w)} = q^{\text{maj}(P)} \frac{[n]!_q}{\prod_{i=1}^n [|P_{\leq i}]!_q}.$$

This is the major index q -hook formula for forests of Björner and Wachs [1, Theorem 1.2]. See also [2, Section 6].

Example 9.8. More generally, there is an easy sufficient (but not necessary) condition on the labeling of a forest with duplications P to make it satisfy the labeled- δ -chain condition: for every duplication pair $\{a, a'\}$ and every duplicated pair of covering edges (i.e., either of the form $b \leq_P a, a'$ or of the form $a, a' \leq_P b$), assume that both covering edges in the pair have the same weak/strict nature, that is either $b <_{\mathbf{N}} a, a'$ or $b >_{\mathbf{N}} a, a'$.

Then for such labelings of a forest with duplications, one has

$$\sum_{w \in \mathcal{L}(P)} q^{\text{maj}(w)} = q^{\text{maj}(P)} [n]!_q \cdot \prod_{\{J_1, J_2\} \in \Pi(P)} [|J_1| + |J_2|]_q \Big/ \prod_{J \in \mathcal{J}_{\text{conn}}(P)} [|J|]_q.$$

Remark 9.9. Because they are normal affine semigroup rings, a result of Hochster [4, Theorem 6.3.5 (a)] implies that the weak P -partition rings R_P are *always* Cohen-Macaulay. We have seen that R_P is a complete intersection whenever P is a forest with duplications, and it will be shown in the next section that the converse also holds.

Thus, one might ask for a combinatorial characterization of when R_P has the intermediate property of being *Gorenstein*, that is, the *canonical module* $\Omega(R_P)$ is isomorphic to R_P itself. This was already answered by Stanley's work on the δ -chain condition that was mentioned earlier, as we now explain.

A result [4, Theorem 6.3.5 (b)], often attributed both to Danilov and to Stanley, implies that the canonical module $\Omega(R_P)$ is isomorphic to the ideal within R_P spanned k -linearly by the monomials \mathbf{x}^f as f runs through the set $\mathcal{A}^{\text{strict}}(P)$ of all strict P -partitions. Hence $\Omega(R_P) \cong R_P$ exactly when

$$\mathcal{A}^{\text{strict}}(P) = f_{\min} + \mathcal{A}^{\text{weak}}(P)$$

for some f_{\min} . Stanley showed that such an f_{\min} exists (and equals δ) exactly when P satisfies his original δ -chain condition, that is, for every i , all maximal chains in $P_{\geq i}$ have the same length.

10. Characterizing complete intersections: Proof of Theorem 1.3. Recall that in the second proof of Theorem 1.1 in Section 7, it was noted that any of the presentations of three rings R_P , $\mathfrak{gr}(R_P)$, S/I_P^{init} given in Theorem 1.2 had the same number of generators and relations. Thus, any of these is a complete intersection presentation if and only if it is true for all three of them; we will say that P is a *c.i. poset* when this holds. It was further shown there that forests with duplication P are c.i. posets. Our goal now is to show that

this property *characterizes* forests with duplication. In the process, we will encounter more equivalent characterizations, including one by forbidden induced subposets (Theorem 10.5).

10.1. Nearly principal ideals. Given a subset A of a elements in a poset P , let $I(A)$ denote the smallest order ideal of P containing A , that is,

$$I(A) := \{p \in P : \text{there exists } a \in A \text{ with } p \leq_P a\}.$$

Recall also that $\Pi(P)$ denotes the set of pairs $\{J_1, J_2\}$ of nonempty connected order ideals of P that intersect nontrivially.

Definition 10.1. Define the set $\mathcal{B}(P)$ of all connected, *nonprincipal* order ideals of P , and define a map

$$\begin{aligned} \Pi(P) &\xrightarrow{\pi} \mathcal{B}(P) \\ \{J_1, J_2\} &\longmapsto J_1 \cup J_2. \end{aligned}$$

It is easy to check that π is well defined. It is also surjective: any non-principal connected order ideal J with maximal elements j_1, j_2, \dots, j_m for $m \geq 2$ can be written as the union $J = J_1 \cup J_2$ where $J_1 := I(j_1)$ and $J_2 := I(j_2, \dots, j_m)$.

Say that an order ideal J in $\mathcal{B}(P)$ is *nearly principal* if its fiber $\pi^{-1}(J)$ for this surjective map π contains only one element. In other words, J is connected, nonprincipal, and there is a unique (unordered) pair $\{J_1, J_2\}$ of connected ideals that intersect nontrivially with union $J_1 \cup J_2 = J$.

It turns out that one can be much more explicit about the nature of nearly principal ideals; see Proposition 10.4 below. But our immediate goal is to show how they help characterize the posets P for which $R_P \cong S/I_P$ is a complete intersection presentation.

Proposition 10.2. *For any poset P on $\{1, 2, \dots, n\}$, the following are equivalent:*

- (i) *any or all of the presentations $R_P \cong S/I_P$ and $\text{gr}(R_P) \cong S/I_P^{\text{gr}}$ and S/I_P^{init} are complete intersection presentations.*

- (ii) $|\Pi(P)| = |\mathcal{B}(P)| = |\mathcal{J}_{\text{conn}}(P)| - |P|$.
- (iii) *The surjection $\pi : \Pi(P) \rightarrow \mathcal{B}(P)$ is a bijection.*
- (iv) *Every connected order ideal of P is either principal or nearly principal.*

Proof. The equivalence of (i) and (ii) already appeared in the second proof of Theorem 1.1. The equivalence between (ii) and (iii) is trivial, since by definition one has the equality $|\mathcal{B}(P)| = |\mathcal{J}_{\text{conn}}(P)| - |P|$. The equivalence of (iii) and (iv) is immediate from the definition of a nearly principal ideal. \square

Say that Q is an (*induced*) *subposet* of P if one has an injective map $i : Q \rightarrow P$ for which $i(q) \leq_P i(q')$ if and only if $q \leq_Q q'$. Condition (iv) of Proposition 10.2 lets one deduce the following.

Corollary 10.3. *Induced subposets of c.i.-posets are c.i.-posets.*

Proof. Given an injective map $i : Q \rightarrow P$ as above, and an order ideal J of Q which is connected (respectively principal, respectively nearly principal), one readily checks that the order ideal $I(i(J))$ of P is connected (respectively principal, nearly principal). Thus if the subposet Q is not c.i., then it contains a connected order ideal J which is neither principal nor nearly principal by Proposition 10.2 (iv), and then P contains the connected order ideal $I(i(J))$ which is neither principal nor nearly principal, so that P is also not c.i. \square

Corollary 10.3 implies that c.i.-posets are exactly the posets avoiding some family of “forbidden” posets as induced subposets. This forbidden family might, *a priori*, be infinite.⁶ Our next goal is to show that c.i.

⁶ For example, consider the family of *crown posets* $\{C_n\}_{n \geq 2}$, where C_n has $2n$ elements $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ and relations

$$a_1 < b_1 > a_2 < b_2 > a_3 < b_3 > \cdots < b_{n-2} > a_{n-1} < b_{n-1} > a_n < b_n > a_1.$$

No two crowns C_i, C_j for $i \neq j$ contains one another as an induced subposet, so the family of posets avoiding crowns as induced posets is not characterized by avoiding some finite subfamily.

posets are characterized by avoiding the three posets P_1, P_2, P_3 shown in Theorem 10.5 below. For this, it helps to start with a more explicit description of nearly principal order ideals.

Proposition 10.4. *A connected nonprincipal order ideal J of a finite poset P is nearly principal if and only if*

- (a) *it has exactly two maximal elements j_1, j_2 , and*
- (b) *for every common lower bound $\ell <_P j_1, j_2$, the open intervals $] \ell, j_1[$ and $] \ell, j_2[$ coincide.*

Proof. For the “only if” assertion, let J be a connected nonprincipal order ideal in P that fails one of the two conditions above.

- If J fails (a), having distinct maximal elements j_1, j_2, \dots, j_m with $m \geq 3$, then it can be written in at least two ways as a union of connected order ideals intersecting nontrivially:

$$\begin{aligned} J &= I(j_1) \cup I(j_2, j_3, j_4, \dots, j_m) \\ &= I(j_2) \cup I(j_1, j_3, j_4, \dots, j_m) \end{aligned}$$

Hence, J is not nearly principal.

- If J satisfies (a) so that it has two maximal elements j_1 and j_2 , but fails (b) by having a lower bound $\ell <_P j_1, j_2$ and an element k of $] \ell, j_1[$ not lying in $] \ell, j_2[$, then J can again be written in at least two ways as a union of connected order ideals intersecting nontrivially

$$\begin{aligned} J &= I(j_1) \cup I(j_2) \\ &= I(j_1) \cup I(j_2, k). \end{aligned}$$

Note that $I(j_2, k)$ is connected because it is the union of two principal ideals that both contain ℓ . This shows J is not nearly principal.

For the “if” assertion, assume that J is a connected nonprincipal ideal satisfying conditions (a) and (b) above. We wish to show that, given any expression $J = J_1 \cup J_2$ where J_1 and J_2 are connected order ideals intersecting nontrivially, one can reindex so that $J_1 = I(j_1)$ and $J_2 = I(j_2)$. By condition (a), one can reindex without loss of generality so that $j_1 \in J_1 \setminus J_2$ and $j_2 \in J_2 \setminus J_1$. Therefore, $I(j_1) \subseteq J_1$, so it only remains to show the reverse inclusion, that is, $J_1 \setminus I(j_1)$ is empty. If not,

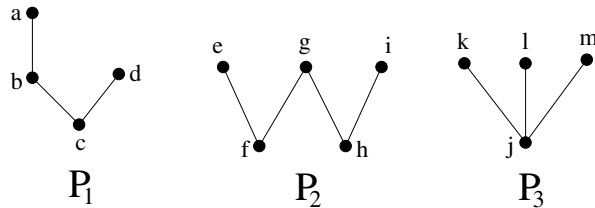
then by the connectivity of J_1 , there must exist k, ℓ with $k \in J_1 \setminus I(j_1)$ and $\ell \in I(j_1)$ such that k, ℓ are comparable in P .

If $k <_P \ell$, then together with $\ell \leq_P j_1$, this contradicts $k \notin I(j_1)$.

If $\ell <_P k$, then note that $k \in J = I(j_1, j_2)$ together with $k \notin I(j_1)$ forces $k \leq j_2$. Thus, $\ell <_P k \leq j_2$ so that ℓ is a lower bound for j_1, j_2 . However, then k lies in $] \ell, j_2[$ but not in $] \ell, j_2[$, contradicting condition (b). \square

10.2. Two further characterizations of c.i. posets.

Theorem 10.5. *The c.i. posets are those which do not contain any of the following three posets $\{P_1, P_2, P_3\}$ as induced subposets:*



Proof. Each of P_1, P_2 and P_3 is not a c.i.-poset because it is itself an order ideal $J = P_i$ which is connected but neither principal nor nearly principal. For example, one can exhibit these two different decompositions into connected order ideals intersecting nontrivially:

$$\begin{aligned}
 P_1 &= I(a) \cup I(d) = I(a) \cup I(b, d) \\
 P_2 &= I(e) \cup I(g, i) = I(e, g) \cup I(i) \\
 P_3 &= I(k) \cup I(\ell, m) = I(k, \ell) \cup I(m).
 \end{aligned}$$

By Proposition 10.2 (iv) and it only remains to show that, if a poset P contains a connected nonprincipal order ideal J failing either of the conditions (a), (b) in Proposition 10.4, then P contains one of P_1, P_2, P_3 as induced subposets.

First, assume J fails condition (a), having distinct maximal elements j_1, j_2, \dots, j_m with $m \geq 3$. Then connectivity of J forces $I(j_1) \cap I(j_2, j_3, \dots, j_m)$ to contain at least one element, which we will denote ℓ_1 , and reindex so that $\ell_1 \leq_P j_1, j_2$. Again, connectivity of J forces $I(j_1, j_2) \cap I(j_3, j_4, \dots, j_m)$ to contain at least one element, which we will denote ℓ_2 , and without loss of generality, one can again reindex so that $\ell_2 \leq_P j_2, j_3$. Now there are three cases:

- if $\ell_1 \leq_P j_3$ (which holds in particular if $\ell_1 \leq_P \ell_2$), then $\{j_1, j_2, j_3, \ell_1\}$ induces a subposet of P isomorphic to P_3 ;
- in a symmetric way, if $\ell_2 \leq_P j_1$ (which holds in particular if $\ell_2 \leq_P \ell_1$), then $\{j_1, j_2, j_3, \ell_2\}$ induces a subposet of P isomorphic to P_3 ;
- otherwise, ℓ_1, ℓ_2 are incomparable in P and $\{j_1, j_2, j_3, \ell_1, \ell_2\}$ induces a subposet of P isomorphic to P_2 .

Finally, assume that J satisfies condition (a), so that $J = I(j_1, j_2)$, but J fails condition (b), due to the existence of a lower bound $\ell <_P j_1, j_2$ and (without loss of generality by re-indexing) some element k in $] \ell, j_1[$ but not in $] \ell, j_2[$. Then $\{j_1, j_2, k, \ell\}$ induce a subposet of P isomorphic to P_1 . \square

Theorem 10.6. *The set of c.i. posets is exactly the set of forests with duplications.*

Proof. It has already been proved in Section 7 that a forest with duplication is a c.i. poset. Conversely, given a c.i. poset P , we will show by induction on $|P|$ that it is a forest with duplications.

The base case $|P| = 1$ is trivial. In the inductive step, if P contains no two comparable elements, then P is a disjoint union of posets with one element, and hence a forest with duplication. Otherwise, let a be a non minimal element of P , and we consider two cases.

Case 1. Every element a' incomparable to a in P has $I(a') \cap I(a) = \emptyset$.

In this case, consider the (nonempty) induced subposets $P_{<a}$ and $P \setminus P_{<a}$ in P . Both are c.i. posets by Proposition 10.3, and both have fewer elements than P , so they are forests with duplication by

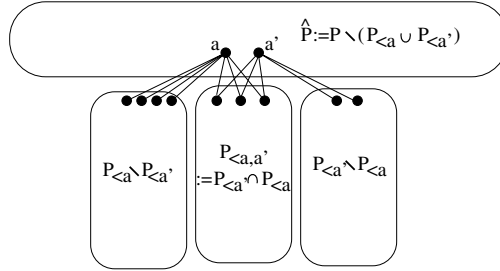
induction. And, it is straightforward to check that, in this situation, P is isomorphic to the poset obtained by hanging $P_{<a}$ below a in $P \setminus P_{<a}$. Therefore, P is also a forest with duplication.

Case 2. There exists an element a' incomparable to a in P for which $I(a') \cap I(a) \neq \emptyset$.

In this case, decompose P into four induced subposets

$$(10.1) \quad P = \widehat{P} \sqcup P_{<a,a'} \sqcup P_{<a} \setminus P_{<a,a'} \sqcup P_{<a'} \setminus P_{<a,a'}$$

where $\widehat{P} := P \setminus (P_{<a} \cup P_{<a'})$, and where $P_{<a} \setminus P_{<a'}$ and $P_{<a'} \setminus P_{<a}$ are allowed to be empty, but $P_{<a,a'}$ is not. This decomposition is depicted schematically here:



We will show that P is isomorphic to the poset Q built by this process:

- (1) Start with $\widehat{P} \setminus \{a'\}$.
- (2) Hang $P_{<a,a'}$ below a in $\widehat{P} \setminus \{a'\}$.
- (3) Duplicate the hanger a in the result, with duplicate element denoted a' .
- (4) Hang $P_{<a} \setminus P_{<a'}$ (if it is nonempty) below a , and hang $P_{<a'} \setminus P_{<a}$ (if it is nonempty) below a' in the resulting poset.

Since $\widehat{P} \setminus \{a'\}$ and $P_{<a,a'}$, and $P_{<a} \setminus P_{<a'}$ and $P_{<a'} \setminus P_{<a}$ are all induced subposets of P , they are all c.i. posets by Proposition 10.3. Since they have smaller cardinality than P , they are all forests with duplication by induction. Therefore, Q is also a forest with duplication.

It only remains to show that P is isomorphic to Q . Their underlying sets are the same. It should also be clear that, by construction, P and Q have the same restrictions to the last three pieces on the right side of (10.1). For the first piece \widehat{P} this is also true, for the following reason: since $P_{<a,a'}$ is assumed to contain at least one element ℓ , any element b of P will have $b >_P a$ if and only $b >_P a'$, else $\{b, a, a', \ell\}$ would induce a subposet of P isomorphic to P_1 .

Now given two elements x, y lying in two *different* pieces from the decomposition (10.1), one must check that x, y are related the same way in P and Q . This is checked case-by-case, according to the two pieces in which they lie.

x lies in $P_{<a} \setminus P_{<a'}$ and y lies in $P_{<a'} \setminus P_{<a}$.

Here transitivity implies that x, y are incomparable both in P and in Q .

x lies in $P_{<a} \setminus P_{<a'}$ or $P_{<a'} \setminus P_{<a}$ and y lies in $P_{<a,a'}$.

Then x, y are incomparable in Q . But the same holds in P : if $x <_P y$ then it would contradict $x \notin P_{<a,a'}$ by transitivity, and if $y <_P x$, then $\{a, a', x, y\}$ induces a subposet of P isomorphic to P_1 .

x lies in $P_{<a} \setminus P_{<a'}$ and y lies in \widehat{P} .

Then $y \leq_Q x$ and $y \leq_P x$ are both impossible by transitivity. Thus one must check that $x \leq_Q y$ if and only if $x \leq_P y$. One has $x \leq_Q y$ if and only if $a \leq_P y$, and it is true that $a \leq_P y$ implies $x \leq_P y$ by transitivity. Thus, it remains to check the converse: $a \not\leq_P y$ implies $x \not\leq_P y$. Assuming $a \not\leq_P y$, if one had $x \leq_P y$; then pick ℓ to be any element of the nonempty subset $P_{<a,a'}$. Either $\ell \not\leq y$ and $\{y, x, a, a', \ell\}$ induces a subposet of P isomorphic to P_2 , or $\ell \leq y$ and $\{\ell, y, a, a'\}$ induces a subposet isomorphic to P_3 . Contradiction.

x lies in $P_{<a'} \setminus P_{<a}$ and y lies in \widehat{P} .

Swapping the roles of a, a' puts one in the case just considered.

x lies in $P_{<a,a'}$ and y lies in \widehat{P} .

Again $y \leq_Q x$ and $y \leq_P x$ are both impossible by transitivity. Thus one must check that $x \leq_Q y$ if and only if $x \leq_P y$. One has $x \leq_Q y$ if and only if either $a \leq_P y$ or $a' \leq_P y$. Furthermore, either $a \leq_P y$ or $a' \leq_P y$ will imply $x \leq_P y$ by transitivity. Thus, it remains to check the converse: if both $a \not\leq_P y$ and $a' \not\leq_P y$ then this forces $x \not\leq_P y$. This

follows since otherwise if $x \leq_P y$ then $\{y, a, a', x\}$ induces a subposet isomorphic to P_3 in P .

This completes the proof that P is isomorphic to the forest with duplication Q . \square

11. Geometry of I_P^{init} , graph-associahedra and graphic zonotopes. Our goal in this section is to explain the geometry underlying Proposition 2.5 (ii) and the initial ideal I_P^{init} , in terms of a subdivision of the cone of P -partitions. We explain how

- the cone of P -partitions is the normal cone \mathcal{N}_ω at a particular vertex ω in the graphic zonotope \mathcal{Z}_G associated to the Hasse diagram graph G of P ,
- the normal fan of \mathcal{Z}_G is refined by the (simplicial) normal fan of Carr and Devadoss's *graph-associahedron* $\mathcal{P}_{\mathcal{B}(G)}$ associated to G , and
- the initial ideal I_P^{init} is exactly the Stanley-Reisner ideal $I_{\Delta(P)}$ for the simplicial complex Δ_P describing the triangulation of the cone \mathcal{N}_ω by the normal fan of $\mathcal{P}_{\mathcal{B}(G)}$.

Definition 11.1. Let Δ_P denote the simplicial complex having the squarefree monomial ideal I_P^{init} in the polynomial algebra $S = k[U_J]_{J \in \mathcal{J}_{\text{conn}}(P)}$ as its *Stanley-Reisner ideal* I_{Δ_P} . By definition, this means that Δ_P is the abstract simplicial complex with vertex set indexed by the collection $\mathcal{J}_{\text{conn}}(P)$ of nonempty connected order ideals J in P , and a subset $\{J_1, \dots, J_d\}$ forms a $(d-1)$ -simplex of Δ_P if and only if the $\{J_i\}$ pairwise intersect trivially (either disjointly, or nested).

Recall that a *flag (or clique) complex* is an abstract simplicial complex Δ on a vertex set V having the following property: whenever a subset $\sigma \subset V$ has every pair $\{i, j\} \subset \sigma$ spanning an edge of Δ , then the entire subset σ spans a simplex of Δ .

We refer the reader to Stanley [23, Sections III.2 and III.10] for the notions of *shellability* and *regular triangulations* used in the next result.

Proposition 11.2. *For any poset P on $\{1, 2, \dots, n\}$, the simplicial complex Δ_P is a flag simplicial complex, giving a regular triangulation of a shellable $(n-2)$ -dimensional ball.*

Proof. The fact that Δ_P is a flag comes from the fact that I_P^{init} is generated by (squarefree) *quadratic* monomials. The fact is that it gives a regular (and hence shellable) triangulation of a ball comes from a general result of Sturmfels on initial ideals and regular triangulations; see [25, Chapter 8]. \square

We wish to relate Δ_P to the *normal fans* of two polytopes associated to the (undirected) graph G on vertex set $\{1, 2, \dots, n\}$ which is the Hasse diagram of P :

- the *graphic zonotope* Z_G , and
- the *graph-associahedron* $\mathcal{P}_{\mathcal{B}(G)}$ of Carr and Devadoss [5].

For a discussion of polytopes, normal fans and zonotopes, see Ziegler’s book [26, Chapter 7]; for graphic zonotopes and graph-associahedron, see [19, Sections 5–7].

Recall that, for two subsets $A, B \subset \mathbf{R}^n$, their *Minkowski sum* is

$$A + B = \{a + b : a \in A, b \in B\}.$$

Definition 11.3. The *graphic zonotope* Z_G is the Minkowski sum of the line segments $\{[0, e_i - e_j]\}_{\{i,j\} \in E}$. In particular, taking $G = K_n$, one has that Z_{K_n} is the n -dimensional *permutohedron*.

Definition 11.4. The *graphical building set* $\mathcal{B}(G)$ is the collection of all nonempty vertex subsets $J \subseteq \{1, 2, \dots, n\}$ for which the vertex-induced subgraph $G|_J$ is connected.

The *graph-associahedron* $\mathcal{P}_{\mathcal{B}(G)}$ is the Minkowski sum of the simplices

$$\{\text{conv}(\{e_j\}_{j \in J}) : J \in \mathcal{B}(G)\}$$

where here $\text{conv}(A)$ denotes the convex hull of the vectors in A .

Recall that, for a convex polytope \mathcal{P} in $V = \mathbf{R}^n$, its *normal fan* $\mathcal{N}(\mathcal{P})$ is the collection of cones in the dual space V^* which partitions linear functionals according to the face of \mathcal{P} on which they achieve their maximum value. We will repeatedly use the following well-known fact about normal fans of Minkowski sums.

Proposition 11.5 (see, e.g., Ziegler [26, Proposition 7.12]). *The Minkowski sum $\mathcal{P}_1 + \cdots + \mathcal{P}_d$ has normal fan $\mathcal{N}(\mathcal{P}_1 + \cdots + \mathcal{P}_d)$ equal to the common refinement of the normal fans $\mathcal{N}(\mathcal{P}_1), \dots, \mathcal{N}(\mathcal{P}_d)$.*

Proposition 11.6. *Let G be a graph on vertex set $\{1, 2, \dots, n\}$.*

(i) *The normal fan $\mathcal{N}(\mathcal{Z}_G)$ is the collection of cones in \mathbf{R}^n cut out by the graphic arrangement of hyperplanes $\{x_i = x_j\}_{\{i,j\} \in E}$.*

(ii) *In particular, when G is the complete graph K_n , this graphic arrangement is the usual type A_{n-1} braid or Weyl chamber arrangement.*

(iii) *The braid arrangement $\mathcal{N}(\mathcal{Z}_{K_n})$ refines the normal fan $\mathcal{N}(\mathcal{P}_{\mathcal{B}(G)})$.*

(iv) *The normal fan $\mathcal{N}(\mathcal{P}_{\mathcal{B}(G)})$ in turn refines the normal fan $\mathcal{N}(\mathcal{Z}_G)$.*

Proof. Assertion (i) is well known and follows from the fact that the hyperplane $x_i = x_j$ is normal to the line segment $[0, e_i - e_j]$, see e.g., [19, Section 5].

Assertion (ii) is simply a definition of the type A_{n-1} braid arrangement, as the collection of all hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$.

Assertion (iii) is asserting another well-known fact: that $\mathcal{P}_{\mathcal{B}(G)}$ is a *generalized permutohedron* in the sense of Postnikov [18]; see [19, Example 6.2]. This follows from Proposition 11.5 by checking that each simplex $\text{conv}(\{e_j\}_{j \in J})$ has its normal fan refined by the braid arrangement. The latter holds because a typical edge of $\text{conv}(\{e_j\}_{j \in J})$ between vertex e_i and vertex e_j is normal to the hyperplane $x_i = x_j$.

Assertion (iv) follows from Proposition 11.6 by noting that, for each edge $\{i, j\}$ of G , the normal hyperplane $x_i = x_j$ to the Minkowski summand $[0, e_i - e_j]$ of \mathcal{Z}_G is the normal hyperplane to the Minkowski summand $\text{conv}(\{e_i, e_j\})$ of $\mathcal{P}_{\mathcal{B}(G)}$. \square

We next review basic facts about the structure of the normal fans for the permutohedron \mathcal{Z}_{K_n} , graphic zonotope \mathcal{Z}_G , and graph associahedron $\mathcal{P}_{\mathcal{B}(G)}$, all inside \mathbf{R}^n .

Permutohedron. Rays in the normal fan $\mathcal{N}(\mathcal{Z}_{K_n})$ are indexed by nonempty proper subsets J of $\{1, 2, \dots, n\}$; such a ray is the nonnegative span of the characteristic vector χ_J in \mathbf{R}^n . The maximal cones are indexed by permutations $w = (w_1, \dots, w_n)$ and defined by the inequalities $x_{w_1} \geq x_{w_2} \geq \cdots \geq x_{w_n}$. A ray indexed by a subset J lies in the cone indexed by w if and only if $J = w_{[1,i]}$ for some $i = 1, 2, \dots, n-1$.

Graphic zonotope. Maximal cones in the normal fan $\mathcal{N}(\mathcal{Z}_G)$, or vertices in the graphic zonotope, are indexed by *acyclic orientations* ω of the graph G ; such a cone corresponds to the subset of \mathbf{R}^n defined by the conjunction of the inequalities $x_i \geq x_j$ whenever ω directs an edge of G as $i \rightarrow j$. In slightly different terms, the transitive closure of an acyclic orientation ω gives a partial order P_ω on $\{1, 2, \dots, n\}$, and the maximal cone \mathcal{N}_ω of $\mathcal{N}(\mathcal{Z}_G)$ corresponding to ω is the cone of (weak) P_ω -partitions. The decomposition of Proposition 2.4 comes from expressing this cone \mathcal{N}_ω as the union of the maximal cones of $\mathcal{N}(\mathcal{Z}_{K_n})$ corresponding to permutations w in the set of linear extensions $\mathcal{L}(P_\omega)$.

Graph associahedron. Rays in the normal fan $\mathcal{N}(\mathcal{P}_{\mathcal{B}(G)})$ are a subset of the rays in $\mathcal{N}(\mathcal{Z}_{K_n})$: one only includes the rays indexed by nonempty proper subsets J of $\{1, 2, \dots, n\}$ for which the vertex-induced subgraph $G|_J$ is *connected*. In other words, J is required to be an element of the *graphical building set* $\mathcal{B}(G)$. A collection of rays $\{J_1, \dots, J_t\}$ spans a cone in $\mathcal{N}(\mathcal{P}_{\mathcal{B}(G)})$ if and only if pairwise one has that J_i, J_k intersect trivially (either they are disjoint or nested) and if disjoint, then $J_i \cup J_k$ induces a disconnected subgraph $G|_{J_i \cup J_k}$ (that is, $J_1 \cup J_2$ is not in $\mathcal{B}(G)$). Such collections form the simplices in what is called the *nested set complex* $\Delta_{\mathcal{B}(G)}$ for the building set $\mathcal{B}(G)$.

Proposition 11.7. *Given a poset P on $\{1, 2, \dots, n\}$, with Hasse diagram G , let ω be the acyclic orientation having P as its transitive closure.*

Then the simplicial complex Δ_P having $I_{\Delta_P} = I_P^{\text{init}}$ describes the triangulation of the P -partition maximal cone \mathcal{N}_ω in the fan $\mathcal{N}(\mathcal{Z}_G)$ by cones of the normal fan $\mathcal{N}(\mathcal{P}_{\mathcal{B}(G)})$.

Proof. Temporarily let Γ_P denote the simplicial complex describing the triangulation of \mathcal{N}_ω in the fan $\mathcal{N}(\mathcal{Z}_G)$ by cones of the normal fan $\mathcal{N}(\mathcal{P}_{\mathcal{B}(G)})$. We wish to show that $\Delta_P \cong \Gamma_P$. As a preliminary reduction, assume that P is connected: when P is a disjoint union $P_1 \sqcup P_2$ of two other posets, one can check that

$$\begin{aligned} \Delta_P &\cong \Delta_{P_1} * \Delta_{P_2} \\ \Gamma_P &\cong \Gamma_{P_1} * \Gamma_{P_2} \end{aligned}$$

where here $*$ denotes the *simplicial join* operation, cf. [19, Remark 6.7].

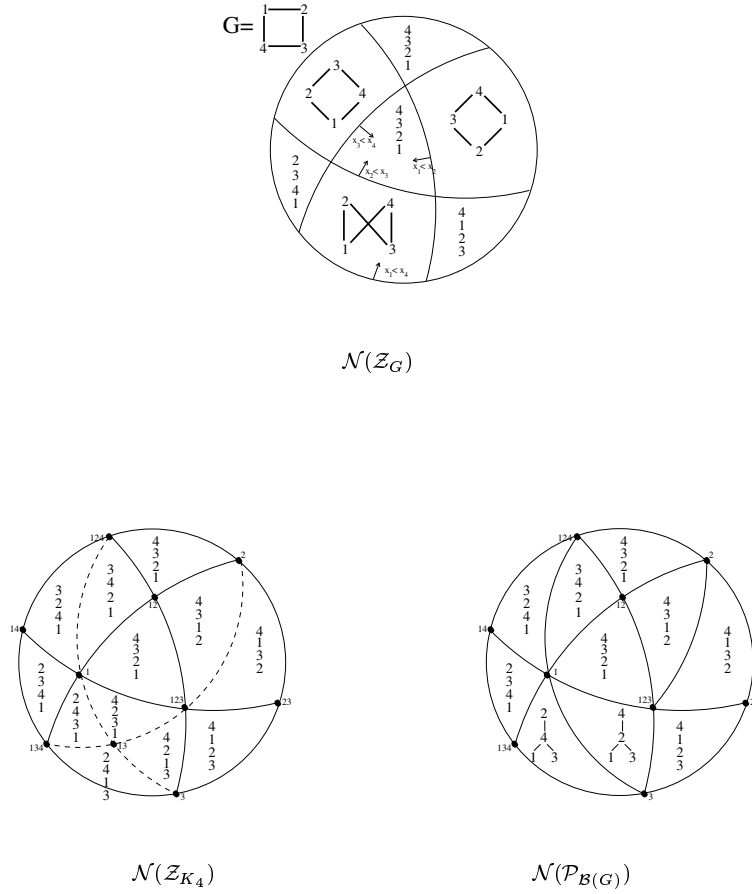


FIGURE 2. Normal fans for the graphic zonotope \mathcal{Z}_G , the permuthedron \mathcal{Z}_{K_n} , and the graph associahedron $\mathcal{P}_{\mathcal{B}(G)}$, for the graph G shown, having $n = 4$ vertices. The normal fans live in \mathbf{R}^4 but are depicted inside the hyperplane $x_1 + x_2 + x_3 + x_4 = 0$ via their intersection with the hemisphere of the unit sphere in which $x_1 \geq x_4$. Note that $\mathcal{N}(\mathcal{Z}_{K_n})$ refines $\mathcal{N}(\mathcal{P}_{\mathcal{B}(G)})$, and the latter refines $\mathcal{N}(\mathcal{Z}_G)$.

Since Δ_P is a flag complex, it suffices to check that Γ_P is also a flag complex, and that their 1-skeleta (=vertices and edges) are isomorphic.

Recall that Δ_P has vertex set given by the set $\mathcal{J}_{\text{conn}}(P)$ of connected order ideals J in P , with two vertices $\{J_1, J_2\}$ spanning an edge of Δ_P if and only if the order ideals J_1, J_2 intersect trivially (either disjoint, or nested).

On the other hand, Γ_P is the subcomplex of the nested set complex $\Delta_{\mathcal{B}(G)}$ indexing the cones of $\mathcal{N}(\mathcal{P}_{\mathcal{B}(G)})$ that lie in the cone \mathcal{N}_ω . Note that a cone lies in \mathcal{N}_ω if and only if each of its extreme rays lies in \mathcal{N}_ω . Thus, Γ_P is a vertex-induced subcomplex of the flag complex $\Delta_{\mathcal{B}(G)}$ and hence is itself flag.

Vertices of $\Delta_{\mathcal{B}(G)}$ are indexed by nonempty proper subsets J of $\{1, 2, \dots, n\}$ for which $G|_J$ is connected. The extra condition that J indexes a ray inside \mathcal{N}_ω is equivalent to χ_J being a weak P -partition, that is, J is an order ideal of P . Thus vertices of Γ_P are indexed by the connected order ideals J in $\mathcal{J}_{\text{conn}}(P)$, the same indexing set as the vertices of Δ_P .

The condition for a pair of connected order ideals $\{J_1, J_2\}$ to index an edge in the nested set complex $\Delta_{\mathcal{B}(G)}$ is that they intersect trivially (either disjointly or nested) and if disjoint then they furthermore have $G|_{J_1 \cup J_2}$ not in $\mathcal{B}(G)$, so that $J_1 \cup J_2$ is not a connected order ideal. But it is impossible for two order ideals J_1, J_2 of P to be disjoint and have $J_1 \cup J_2$ a connected ideal: this would imply that there is some Hasse diagram edge connecting them, giving an order relation between some pair of elements $\{j_1, j_2\}$ with j_i in J_i for $i = 1, 2$, and would force either j_1 or j_2 to lie in the intersection $J_1 \cap J_2$. Thus, $\{J_1, J_2\}$ index an edge of Γ_P if and only if they intersect trivially, that is, if and only if they index an edge of Δ_P . Hence, Δ_P and Γ_P are isomorphic flag complexes. \square

The maximal cones in $\mathcal{P}_{\mathcal{B}(G)}$ correspond to what were called $\mathcal{B}(G)$ -trees in [18, Section 7] and [19, subsection 8.1]. This means that the maximal simplices of the triangulation Δ_P will correspond to what we might call P -forests: forest posets F in which every principal ideal $F_{\leq i}$ is a connected order ideal of P , and whenever i, j are incomparable in the poset F , one has that the ideal $F_{\leq i} \cup F_{\leq j}$ of P is disconnected.

Example 11.8. For the poset P on $\{1, 2, 3, 4\}$ in which $1, 3 <_P 2, 4$, the Hasse diagram is the graph G shown in Figure 2. The acyclic orientation ω of G whose transitive closure gives P corresponds to a quadrangular cone \mathcal{N}_ω which is the lowest on the page among the three quadrangular cones depicted in $\mathcal{N}(\mathcal{Z}_G)$. This cone \mathcal{N}_ω is subdivided into four cones in $\mathcal{N}(\mathcal{Z}_{K_4})$, corresponding to the set of linear extensions $\mathcal{L}(P) = \{1324, 1342, 3124, 3142\}$. On the other hand, the cone \mathcal{N}_ω is subdivided into only two cones in $\mathcal{N}(\mathcal{P}_{\mathcal{B}(G)})$, labeled in the figure by the two $\mathcal{B}(G)$ -trees $1, 3 < 2 < 4$ and $1, 3 < 4 < 2$.

Note that unlike the usual triangulation of the cone \mathcal{N}_ω of P -partitions corresponding to the order complex $\Delta\mathcal{J}(P)$ that was discussed in subsection 1.1, the maximal cones in the triangulation Δ_P are not unimodular. In fact, each such maximal cone corresponding to some P -forest F will decompose into $|\mathcal{L}(F)|$ different unimodular cones from the triangulation by $\Delta\mathcal{J}(P)$, that is, from the normal fan $\mathcal{N}(\mathcal{Z}_{K_n})$ of the permutohedron.

12. Other questions. Here we collect some questions and problems left unresolved in this work.

12.1. Resolving the rings R_P over S and Ferrers posets. The following problem is motivated by our desire to count linear extensions for more posets P .

Problem 12.1. Find more posets P where one can compute $\text{Hilb}(R_P, \mathbf{x})$, possibly by writing down an explicit S -resolution of R_P , or $\text{gr}(R_P)$ or S/I_P^{init} .

One particular instance originally motivated us but has proven elusive so far. Given a number partition λ , consider the finite poset $P = P_\lambda$ on the set of squares (i, j) in the Ferrers diagram for λ , partially ordered componentwise, with the square $(1, 1)$ as the maximum element. Gansner [10] showed how the Hillman-Grassl algorithm proves an interesting hook formula that counts weak P -partitions f by an intermediate multigrading, where one specializes the variable $x_{i,j}$ associated

with square (i, j) to the variable y_{i-j} recording its *content* $i - j$:

$$(12.1) \quad \sum_{f \in \mathcal{A}^{\text{weak}}(P_\lambda)} \prod_{(i,j) \in \lambda} y_{i-j}^{f(i,j)} = \prod_{(i,j) \in \lambda} \left(1 - \prod_{(i',j') \in H(i,j)} y_{i'-j'} \right)^{-1},$$

where here $H(i, j)$ denotes the set of squares of λ lying in the hook of square (i, j) .

Question 12.2. For these posets $P = P_\lambda$, can we explain (12.1) via an analysis of the structure of the ring R_P , or $\text{gr}(R_P)$ or S/I_P^{init} that leads to its Hilbert series? Is one of these rings easy to resolve over S , for example?

12.2. Further structure for the ideal \mathcal{I}_P of P -partitions. It can be shown (e.g., using [16, Proposition 3]) that, for any poset P on $\{1, 2, \dots, n\}$, the ideal $\mathcal{I}(P)$ of P -partitions is a Cohen-Macaulay module, either over the ring R_P of weak P -partitions, or over the polynomial algebra $S = k[U_J]_{J \in \mathcal{J}_{\text{conn}}(P)}$. This raises several related questions about the modules $\mathcal{I}(P)$, beginning with the issue of their minimal generating sets, raised in Remark 9.3.

Problem 12.3. Describe the minimal monomial generators for $\mathcal{I}(P)$ over R_P .

Beyond minimal generating sets, one ultimately wants the following.

Problem 12.4. Given any poset P on $\{1, 2, \dots, n\}$, describe for $\mathcal{I}(P)$

(i) an explicit resolution of \mathcal{I}_P as an S -module or an R_P -module, or both, and

(ii) the multigraded Betti numbers in the *minimal* free resolutions, that is, the multigraded vector spaces $\text{Tor}_*^S(\mathcal{I}_P, k)$ and $\text{Tor}_*^{R_P}(\mathcal{I}_P, k)$.

Of course, there are similar questions one can ask about the associated graded ring $\text{gr}(R_P)$ and associated graded modules $\text{gr}(\mathcal{I}_P)$ over it, and over S .

Example 12.5. Consider the poset $P = P_2$ from Example 2.2, having order relations $2 <_P 1, 3$. Then $S = k[U_2, U_{12}, U_{23}, U_{123}]$, and

$$(12.2) \quad \begin{aligned} \text{Hilb}(S, \mathbf{x}) &= \frac{1}{(1-x_2)(1-x_1x_2)(1-x_2x_3)(1-x_1x_2x_3)} \\ \text{Hilb}(S, q) &= \frac{1}{(1-q)(1-q^2)^2(1-q^3)}. \end{aligned}$$

It turns out that the generating set $\{x_2, x_2x_3\}$ described in (9.1) for the ideal \mathcal{I}_P is minimal in this case, leading to the following minimal free S -resolution

$$\begin{array}{ccccccc} & S(- (0, 2, 1)) & & S(- (0, 1, 0)) & & & \\ 0 \longrightarrow & \oplus & \xrightarrow{A} & \oplus & \longrightarrow & \mathcal{I}_P & \\ & S(- (1, 2, 1)) & & S(- (0, 1, 1)) & & & \\ & & & e_2 & \longmapsto & x_2 & \\ & & & e_{23} & \longmapsto & x_2x_3 & \end{array}$$

where

$$A = \begin{bmatrix} U_{23} & -U_{123} \\ -U_2 & U_{12} \end{bmatrix}.$$

Together with the Hilbert series for S given in (12.2), this allows one to calculate

$$\begin{aligned} \text{Hilb}(\mathcal{I}_P, \mathbf{x}) &= \frac{x_2 + x_2x_3 - x_2^2x_3 - x_1x_2^2x_3}{(1-x_2)(1-x_1x_2)(1-x_2x_3)(1-x_1x_2x_3)} \\ \text{Hilb}(\mathcal{I}_P, q) &= \frac{q + q^2 - (q^3 + q^4)}{(1-q)(1-q^2)^2(1-q^3)} = \frac{q + q^2}{(1-q)(1-q^2)(1-q^3)} \\ \sum_{w \in \mathcal{L}(P)} q^{\text{maj}(w)} &= q^2 + q^3. \end{aligned}$$

Lastly, given Stanley's characterization for when R_P is Gorenstein discussed in Section 9 above, it is reasonable to ask the following.

Problem 12.6. Characterize when \mathcal{I}_P is Gorenstein, that is, when one has an isomorphism $\Omega(\mathcal{I}_P) \cong \mathcal{I}_P$, up to a shift in grading.

This should be approachable, as the canonical module $\Omega(\mathcal{I}_P)$ has a simple description (via [16, Proposition 3]): it is the ideal within R_P spanned k -linearly by the monomials \mathbf{x}^f as f runs through those weak P -partitions $f : P \rightarrow \mathbf{N}$ for which

$$\begin{aligned} f(i) &\geq_{\mathbf{N}} f(j) \text{ if } i \triangleleft_P j \\ f(i) &>_{\mathbf{N}} f(j) \text{ if } i <_N j. \end{aligned}$$

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