JOURNAL OF COMMUTATIVE ALGEBRA
Volume 4, Number 1, Spring 2012

SOME REMARKS ON MULTIPLICATION AND FLAT MODULES

MAJID M. ALI

ABSTRACT. The purpose of this work is to study some properties of multiplication modules and flat modules. We give some properties of multiplication modules that characterize arithmetical rings. We investigate Ohm type properties for multiplication and flat modules, and we also characterize F-modules and FGP-modules.

1. Introduction. Throughout this paper all rings are assumed commutative with identity and all modules are unital. Let R be a ring and M an R-module. Then M is called a *multiplication* module if every submodule N of M has the form IM for some ideal I of R, [**12**]. Note that $I \subseteq [N : M]$ and hence $N = IM \subseteq [N : M]M \subseteq N$, so that $N = [N : M]M$. If K is a multiplication submodule of M, then for all submodules N of M, $N \cap K = [(N \cap K) : K]K = [N : K]K$. If M is a finitely generated faithful multiplication R-module, then M is cancelation [**28**, Corollary 1 to Theorem 9], from which it follows that $[IN : M] = I[N : M]$ for all ideals I of R and all submodules N of M. If M is a faithful multiplication module, then M is locally either zero or isomorphic to R. Thus, finitely generated faithful multiplication modules are locally isomorphic to R. Let P be a maximal ideal of R, and let $T_P(M) = \{m \in M : (1-p)m = 0 \text{ for some } m \in M\}$. Then $T_P(M)$ is a submodule of M. M is called *P-torsion* if $T_P(M) = M$. On the other hand, M is called *P-cyclic* provided there exist $m \in M$ and $q \in P$ such that $(1-q)M \subseteq Rm$. El-Bast and P.F. Smith [**14**, Theorem 1.2] showed that M is multiplication if and only if M is P-torsion or P-cyclic for each maximal ideal P of R. A multiplication module M is locally cyclic and the converse is true if M is finitely

2010 AMS *Mathematics subject classification.* Primary 13C13, 13C10, 13A15.
Keywords and phrases. Multiplication module, flat module, projective module, pure submodule, nilpotent submodule, F-module, FGP-module.
Received by the editors on September 15, 2009, and in revised form on October 18, 2009.
DOI:10.1216/JCA-2012-4-1-1

generated, [**12**, Proposition 4]. Multiplication modules have recently received considerable attention, see for example [**1–12, 14, 28**].

A submodule N of an R-module M is called *pure* if $IN = N \cap IM$ for every ideal I of R, [**26**]. If I is a pure ideal of R, then I is locally either R or zero. Several properties of pure ideals and pure submodules of multiplication modules are given in [**1, 6**]. The concepts of idempotent and nilpotent submodules were introduced by the author in [**1, 6**], respectively. A submodule N of M is *idempotent* if $N = [N : M]N$. If N is a pure submodule of a multiplication module, then N is idempotent and multiplication. The converse is true for any R-module, [**1**]. A submodule N of M is called *nilpotent* if $[N : M]^k N = 0$ for some positive integer k. An element $m \in M$ is called nilpotent if the cyclic submodule Rm is nilpotent. Several properties of idempotent and nilpotent submodules of multiplication modules are considered in [**1**].

Following [**20**, page 105], an R-module M is called a von Neumann regular module if and only if every cyclic submodule of M is a direct summand in M. It is shown [**1**, Proposition 12] that a faithful multiplication R-module M is von Neumann regular if and only if every cyclic (in fact, finitely generated) submodule N of M is idempotent in M. So the concept of faithful multiplication von Neumann regular modules generalizes von Neumann regular rings. It is proved [**1**, Corollary 11] that if R is a von Neumann regular ring and M a faithful multiplication R-module, then M is von Neumann regular. The converse is true if $M \neq PM$ for all prime ideals P of R. In particular, the converse is true if M is finitely generated, faithful and multiplication. Several properties and characterizations of faithful multiplication von Neumann regular modules are given in [**1**].

Recall that an R-module M is *flat* if, for every short exact sequence of R-modules $0 \to K \to L \to N \to 0$, the sequence $0 \to K \otimes M \to L \otimes M \to N \otimes M \to 0$ is also exact. M is flat if and only if it is locally flat. It is shown [**17**, Proposition 11.20] that M is flat if and only if $I \otimes M \cong IM$ for each finitely generated ideal I of R. Several characterizations and properties of flat submodules of multiplication modules are given by the author in [**2**]. An R-module M is *projective* if it is a direct summand of a free R-module. Projective modules are locally free and the converse is true if M is of finite presentation, [**17**]. The *trace* ideal of an R-module M is $\mathrm{Tr}M = \sum_{f \in \mathrm{Hom}\,(M,R)} f(M)$. If

M is projective, then $M = \mathrm{Tr}\,(M)M$, $\mathrm{ann}\, M = \mathrm{ann}\mathrm{Tr}\,(M)$ and $\mathrm{Tr}\,(M)$ is a pure ideal of R, [**17**, Proposition 3.30].

In this paper we investigate multiplication, flat and projective modules. Theorem 1 gives some properties of multiplication modules that characterize arithmetical rings while Theorem 3 and Proposition 6 investigate Ohm type properties for multiplication modules generalizing those for multiplication ideals, [**5**]. Theorem 5 shows that if N is a finitely generated flat submodule of a finitely generated faithful multiplication R-module M, then N is never nilpotent. Propositions 9 and 10 give necessary and sufficient conditions for the sum and intersection of a collection of flat modules to be flat.

Section 2 is concerned with F-modules and FGP -modules as a generalization of F-rings and semi-hereditary rings. An R-module M is called F-module (respectively FGP-module) if every submodule N of M is flat (respectively every finitely generated submodule of M is projective). We show that, if M is a finitely generated faithful multiplication R-module, then M is an F-module if and only if M_P is a valuation module for each prime ideal P of R and M is an FGP-module if and only if M_P is a valuation module for each prime ideal P of R and M_S is a von Neumann regular R_S-module, where S is the set of non-zero divisors of R, Proposition 16.

All rings considered in this paper are commutative with 1, and all modules are unital. For the basic concepts used, we refer the reader to [**17, 18, 20, 21, 27**].

2. Multiplication modules and flat modules.

The following theorem gives some properties of multiplication modules that characterize arithmetical rings. It generalizes [**9**, Theorem 2.1].

Theorem 1. *Let R be a ring, and let N_λ $(\lambda \in \Lambda)$ be a collection of submodules of an R-module M. Let*

$$S = \sum_{\lambda\in\Lambda} N_\lambda, \qquad N = \bigcap_{\lambda\in\Lambda} N_\lambda, \qquad A = \sum_{\lambda\in\Lambda} [N_\lambda : S]$$

and

$$B = \sum_{\lambda\in\Lambda} [N : N_\lambda].$$

(1) *If S is multiplication, then $K \cap S = \sum_{\lambda \in \Lambda} K \cap N_\lambda$ for every submodule K of M.*

(2) *If S is multiplication, then $[S : K] = \sum_{\lambda \in \Lambda} [N_\lambda : K]$ for every finitely generated submodule K of M.*

(3) *If Λ is finite and $N_\lambda + N_\mu$ is multiplication for all $\lambda \neq \mu$, then $K + N = \bigcap_{\lambda \in \Lambda} K + N_\lambda$, for every submodule K of M.*

(4) *If Λ is finite and $N_\lambda + N_\mu$ is multiplication for all $\lambda \neq \mu$, then $IN = \bigcap_{\lambda \in \Lambda} IN_\lambda$ for every ideal I of R.*

(5) *If Λ is finite and N_λ are finitely generated such that $N_\lambda + N_\mu$ is multiplication for all $\lambda \neq \mu$, then $[K : N] = \sum_{\lambda \in \Lambda} [K : N_\lambda]$ for every submodule K of M.*

Proof. (1) Obviously, $\sum_{\lambda \in \Lambda} K \cap N_\lambda \subseteq K \cap N$. Since S is multiplication, it follows by [**28**, Theorem 2] that $A + \operatorname{ann}(m) = R$ for all $m \in S$. Let $x \in K \cap S$, and let

$$H = \left\{ r \in R : rx \in \sum_{\lambda \in \Lambda} K \cap N_\lambda \right\}.$$

Assume $H \neq R$. Then there exists a maximal ideal P of R such that $H \subseteq P$. We have two cases.

Case 1: $A \subseteq P$. Then $\operatorname{ann}(x) \nsubseteq P$, and hence there exists a $p \in P$ such that $1 - p \in \operatorname{ann}(x)$. Hence,

$$(1 - p)x = 0 \in \sum_{\lambda \in \Lambda} K \cap N_\lambda.$$

This gives that $1 - p \in H \subseteq P$, a contradiction. Alternatively, for all $\lambda \in \Lambda$, $[N_\lambda : S] \subseteq P$, and since S is multiplication $N_\lambda = [N_\lambda : S]S \subseteq PS$, and hence $S = \sum_{\lambda \in \Lambda} N_\lambda \subseteq PS \subseteq S$, so that $S = PS$. Since $x \in S$ and S is multiplication, $Rx = IS$ for some ideal I of R. Hence,

$$Rx = IS = IPS = P(IS) = Px,$$

so there exists a $p \in P$ such that $(1 - p)x = 0$.

Case 2: $A \nsubseteq P$. There exists a $\lambda \in \Lambda$ such that $[N_\lambda : S] \nsubseteq P$. There exists a $q \in P$ such that $1 - q \in [N_\lambda : S]$, and hence $(1 - q)S \subseteq N_\lambda$. Since $x \in K \cap S$,

$$(1-q)x \in K \cap (1-q)S \subseteq K \cap N_\lambda \subseteq \sum_{\lambda \in \Lambda} K \cap N_\lambda.$$

So $1 - q \in H \subseteq P$, and this is also a contradiction. So $H = R$ and

$$x \in \sum_{\lambda \in \Lambda} K \cap N_\lambda.$$

(2) Clearly

$$\sum_{\lambda \in \Lambda} [N_\lambda : K] \subseteq [S : K].$$

Since $R = A + \operatorname{ann}(m)$ for all $m \in S$, it is easily verified that $R = A + \operatorname{ann} Y$ for all finitely generated submodules Y of S. Now, let $x \in [S : K]$ and let

$$H = \left\{ r \in R : rx \in \sum_{\lambda \in \Lambda} [N_\lambda : K] \right\}.$$

Assume $H \neq R$, so there exists a maximal ideal P of R such that $H \subseteq P$. We discuss two cases.

Case 1: $A \subseteq P$. Since $x \in [S : K]$, and hence $xK \subseteq S$, we infer that $R = A + \operatorname{ann}(xK)$. This gives that $\operatorname{ann}(xK) \nsubseteq P$, and hence there exists a $p \in P$ such that $1 - p \in \operatorname{ann}(xK)$. So $(1 - p)xK = 0$, and hence

$$(1-p)x \in [0 : K] \subseteq [N_\lambda : K] \subseteq \sum_{\lambda \in \Lambda} [N_\lambda : K].$$

Hence, $1 - p \in H \subseteq P$, a contradiction. Alternatively, if $A \subseteq P$, we get that $S = PS$. Let

$$K = \sum_{i=1}^{n} Rk_i.$$

Then for all $1 \leqslant i \leqslant n$, $xk_i \in S$, and hence $R(xk_i) = IS$ for some ideal I_i of R. Hence,

$$R(xk_i) = I_i S = I_i PS = P(I_i S) = Pxk_i,$$

and hence there exists a $p_i \in P$ with $(1-p_i)xk_i = 0$. Let

$$1-p = 1-\prod_{i=1}^{n}(1-p_i).$$

Then $(1-p)xk_i = 0$ for all $1 \leqslant i \leqslant n$, and hence $(1-p)xK = 0$. So

$$(1-p)\,x \in [0:K] \subseteq \sum [N_\lambda : K] \subseteq \sum_{\lambda\in\Lambda} [N_\lambda : K].$$

Case 2: $A \nsubseteq P$. There exists a $\lambda \in \Lambda$ with $[N_\lambda : S] \nsubseteq P$. Hence there exists a $q \in P$ such that $1-q \in [N_\lambda : S]$ and hence $(1-q)S \subseteq N_\lambda$. Since $xK \subseteq S$, we get that $(1-q)xK \subseteq (1-q)S \subseteq N_\lambda$, and hence

$$(1-q)x \in [N_\lambda : K] \subseteq \sum_{\lambda\in\Lambda} [N_\lambda : K].$$

So $1-q \in H \subseteq P$, and this is also a contradiction. Hence, $H = R$ and

$$x \in \sum_{\lambda\in\Lambda} [N_\lambda : K].$$

(3) Obviously,

$$K+N \subseteq \bigcap_{\lambda\in\Lambda} (K+N_\lambda).$$

Since Λ is finite and $N_\lambda + N_\mu$ is multiplication for all $\lambda \neq \mu$, we infer from [**9**, Theorem 2.1] that $B + \operatorname{ann}(m) = R$ for each $m \in S$. Let

$$x \in \bigcap_{\lambda\in\Lambda} (K+N_\lambda),$$

and let

$$H = \{r \in R : rx \in K+N\}.$$

Let $H \neq R$. There exists a maximal ideal P of R such that $H \subseteq P$. We discuss two cases.

Case 1: $B \subseteq P$. Since

$$x \in \bigcap_{\lambda \in \Lambda} (K + N_\lambda),$$

$x \in K + N_\lambda$ for each $\lambda \in \Lambda$. There exist $k \in K$ and $n_\lambda \in N_\lambda$ such that $x = k + n_\lambda$. Hence, $x - k = n_\lambda \in N_\lambda \subseteq S$, and hence $\operatorname{ann}(x-k) \nsubseteq P$. There exists a $p \in P$ such that $1 - p \in \operatorname{ann}(x-k)$, and hence $(1-p)(x-k) = 0$. So $(1-p)x = (1-p)k \in K \subseteq K + N$. So $1 - p \in H \subseteq P$, a contradiction.

Case 2: $B \nsubseteq P$. There exists a $\lambda \in \Lambda$ with $[N : N_\lambda] \nsubseteq P$. There exists a $q \in P$ such that $(1-q)N_\lambda \subseteq N$. It follows that $(1-q)x \in (1-q)(K+N_\lambda) \subseteq K+N$. This also gives that $1-q \in H \subseteq P$, a contradiction. So $H = R$ and $x \in K + N$.

(4) Obviously,

$$IN \subseteq \bigcap_{\lambda \in \Lambda} IN_\lambda.$$

Let

$$x \in \bigcap_{\lambda \in \Lambda} IN_\lambda,$$

and let

$$H = \{r \in R : rx \in IN\}.$$

Assume $H \neq R$. There exists a maximal ideal P of R such that $H \subseteq P$. We have $B + \operatorname{ann}(m) = R$ for each $m \in S$, [**9**, Theorem 2.1]. We discuss two cases.

Case 1: $B \subseteq P$. Then $\operatorname{ann}(x) \nsubseteq P$, and hence there exists a $p \in P$ with $(1-p)x = 0 \in IN$. So $1 - p \in H \subseteq R$, a contradiction.

Case 2: $B \nsubseteq P$. There exists a $\lambda \in \Lambda$ such that $[N : N_\lambda] \nsubseteq P$, and hence there exists a $q \in P$ such that $(1-q)N_\lambda \subseteq N$. It follows that $(1-q)x \in (1-q)IN_\lambda \subseteq IN$, and this implies that $1 - q \in H \subseteq P$, a contradiction. Thus $H = R$, and hence $x \in IN$.

(5) Obviously,

$$\sum_{\lambda \in \Lambda} [K : N_\lambda] \subseteq [K : N].$$

Let $x \in [K : N]$, and let

$$H = \left\{ r \in R : rx \in \sum_{\lambda \in \Lambda} [K : N_\lambda] \right\}.$$

Since Λ is finite and $N_\lambda + N_\mu$ is multiplication, $R = B + \operatorname{ann}(m)$ for each $m \in S$. Since N_λ is finitely generated, $R = B + \operatorname{ann} N_\lambda = B$. Let P be a maximal ideal of R. Then $B \nsubseteq P$. Hence, there exist $p \in P$ and $\lambda \in \Lambda$ such that $(1-p)N_\lambda \subseteq N$. Since $x \in [K : N]$ and hence $xN \subseteq K$, we infer that $(1-p)xN_\lambda \subseteq xN \subseteq K$. So

$$(1-p)x \in [K : N_\lambda] \subseteq \sum_{\lambda \in \Lambda} [K : N_\lambda],$$

and hence $1 - p \in H \subseteq P$, a contradiction. Hence, $H = R$ and

$$x \in \sum_{\lambda \in \Lambda} [K : N_\lambda].$$

This proves the theorem. □

The next result generalizes [**9**, Corollary 2.2] to multiplication modules.

Proposition 2. *Let R be a ring and K, N submodules of a multiplication R-module M such that $K+N$ is finitely generated multiplication. Then*

$$\begin{aligned}(K+N)\,[(K \cap N) : M] &= (K \cap N)\,[(K+N) : M] \\ &= [K : M]\,N = [N : M]\,K.\end{aligned}$$

If $K + N$ is not necessarily finitely generated and M finitely generated then the result also holds.

Proof. $K + N$ is finitely generated multiplication gives that $[K : N] + [N : K] = R$, [**28**, Corollary to Theorem 2]. Since M is multiplication,

$$[K : M]\,N = [K : M]\,[N : M]\,M = [N : M]\,K.$$

It follows that

$$\begin{aligned}[K:M]\,N &= [K:M]\,[K:N]\,N + [N:M]\,[N:K]\,K \\ &\subseteq [K:M]\,(K\cap N) + [N:M]\,(K\cap N) \\ &= ([K:M]+[N:M])\,(K\cap N) \\ &\subseteq [(K+N):M]\,(K\cap N) \subseteq [K:M]\,N,\end{aligned}$$

so that $[K:M]N = [(K+N):M](K\cap N) = [(K\cap N):M](K+N)$. For the second assertion, if $K+N$ is multiplication, then $[K_P:N_P]+[N_P:K_P] = R_P$ for each maximal ideal P of R, [**28**, Corollary 2 to Theorem 1]. Since M is finitely generated multiplication, the result is true locally and hence globally. □

The next theorem gives Ohm type properties of multiplication modules. It generalizes [**5**, Theorem 2.2 and Proposition 3.1].

Theorem 3. *Let R be a ring and $N_\lambda(\lambda\in\Lambda)$ a collection of submodules of an R-module M. Let*

$$S=\sum_{\lambda\in\Lambda} N_\lambda,\quad N=\bigcap_{\lambda\in\Lambda} N_\lambda,\quad A=\sum_{\lambda\in\Lambda}[N_\lambda:S]$$

and

$$B=\sum_{\lambda\in\Lambda}[N:N_\lambda]\,.$$

(1) *If S is multiplication, then*

$$[S:M]^k\,S=\sum_{\lambda\in\Lambda}[N_\lambda:M]^k\,N_\lambda$$

for all positive integers k.

(2) *If Λ is finite and $N_\lambda+N_\mu$ is multiplication for all $\lambda\neq\mu$, then*

$$[N:M]^k\,N=\bigcap_{\lambda\in\Lambda}[N_\lambda:M]^k\,N_\lambda$$

for all positive integers k.

Proof. (1) Obviously,

$$\sum_{\lambda\in\Lambda}[N_\lambda : M]^k N_\lambda \subseteq [S : M]^k S.$$

Let $x \in [S : M]^k S$, and let

$$H = \left\{ r \in R : rx \in \sum_{\lambda\in\Lambda}[N_\lambda : M]^k N_\lambda \right\}.$$

Assume $H \neq R$. There exists a maximal ideal P of R such that $H \subseteq P$. Since S is multiplication, $A + \operatorname{ann}(m) = R$ for each $m \in S$. We discuss two cases.

Case 1: $A \subseteq P$. Since $x \in [S : M]^k S \subseteq S$, we infer that $\operatorname{ann}(x) \nsubseteq P$. There exists a $p \in P$ with $1 - p \in \operatorname{ann}(x)$. So $(1-p)x = 0 \in \sum_{\lambda\in\Lambda}[N_\lambda : M]^k N_\lambda$, and hence $1 - p \in H \subseteq P$, a contradiction.

Case 2: $A \nsubseteq P$. There exists a $\lambda \in \Lambda$ such that $[N_\lambda : S] \nsubseteq P$, and hence there exists a $q \in P$ such that $(1-q)S \subseteq N_\lambda$. It follows that $(1-q)[S : M] \subseteq [(1-q)S : M] \subseteq [N_\lambda : M]$. Hence, $(1-q)^{k+1}[S : M]^k S \subseteq [N_\lambda : M]^k N_\lambda$, and hence

$$(1-q)^{k+1} x \in [N_\lambda : M]^k N_\lambda \subseteq \sum_{\lambda\in\Lambda}[N_\lambda : M]^k N_\lambda.$$

So $(1-q)^{k+1} \in H \subseteq P$ gives $1 - q \in P$, a contradiction. This implies that $H = R$, and hence

$$x \in \sum_{\lambda\in\Lambda}[N_\lambda : M]^k N_\lambda.$$

So

$$[S : M]^k S \subseteq \sum_{\lambda\in\Lambda}[N_\lambda : M]^k N_\lambda.$$

(2) Assume Λ is finite and $N_\lambda + N_\mu$ is multiplication for all $\lambda \neq \mu$. It follows from [**28**, Theorem 2.1] that $B + \operatorname{ann}(m) = R$ for each $m \in S$. Obviously, $[N : M]^k N \subseteq \bigcap_{\lambda\in\Lambda}[N_\lambda : M]^k N_\lambda$. Let

$$x \in \bigcap_{\lambda\in\Lambda}[N_\lambda : M]^k N_\lambda,$$

and let

$$H = \left\{ r \in R : rx \in [N:M]^k N \right\}.$$

Let $H \neq R$. There exists a maximal ideal P of R such that $H \subseteq P$. We discuss two cases.

Case 1: $B \subseteq P$. So $\operatorname{ann}(x) \nsubseteq P$, and hence there exists a $p \in P$ with $(1-p) \in \operatorname{ann}(x)$. It follows that $(1-p)x = 0 \in [N:M]^k N$, and hence $1 - p \in H \subseteq P$, a contradiction.

Case 2: $B \nsubseteq P$. There exists a $\lambda \in \Lambda$ with $[N : N_\lambda] \nsubseteq P$, and hence there exists a $q \in P$ with $(1-q)N_\lambda \subseteq N$. So $(1-q)[N_\lambda : M] \subseteq [(1-q)N_\lambda : M] \subseteq [N : M]$, and hence $(1-q)^{k+1}[N_\lambda : M]^k N_\lambda \subseteq [N : M]^k N$. This gives that $(1-q)^{k+1}x \in [N : M]^k N$, and hence $(1-q)^{k+1} \in H \subseteq P$, which implies that $1 - q \in P$, a contradiction. Thus, $H = R$, and hence $x \in [N : M]^k N$. So

$$\bigcap_{\lambda \in \Lambda} [N : N_\lambda]^k N_\lambda \subseteq [N : M]^k N.$$

This concludes the proof of the theorem. □

The fact that $N_\lambda + N_\mu$ is multiplication for each $\lambda \neq \mu$ in part (2) of the above theorem is crucial. If S is multiplication, then property (2) of the theorem is not satisfied. For example, let $R = K[X^2, X^3]$, K is a field, $I = RX^2$, $J = RX^4$ and $L = RX^5$. Then $I + J + L$ is a multiplication ideal of R but $(I \cap J \cap L)^2 \neq I^2 \cap J^2 \cap L^2$.

As a consequence of Theorem 3 we give the following corollary.

Corollary 4. *Let R be a ring and N a submodule of an R-module M. If N is a multiplication submodule that has no non-zero nilpotent element, then $\operatorname{ann} N = \operatorname{ann} [N : M]^k N$ for each positive integer k.*

Proof. Obviously $\operatorname{ann} N \subseteq \operatorname{ann} [N : M]^k N$. Let $N = \sum_{\alpha \in \Lambda} Rn_\alpha$. Since N is multiplication, it follows by Theorem 3 that

$$[N : M]^k N = \sum_{\alpha \in \Lambda} [Rn_\alpha : M]^k Rn_\alpha.$$

Let $y \in \operatorname{ann}[N : M]^k N$. Then

$$y \in \bigcap_{\alpha \in \Lambda} \operatorname{ann}\, [Rn_\alpha : M]^k\, Rn_\alpha,$$

and hence $y[Rn_\alpha : M]^k n_\alpha = 0$ for all $\alpha \in \Lambda$. It follows that $y^{k+1}[Rn_\alpha : M]^k n_\alpha = 0$, and hence $[Ryn_\alpha : M]^k (yn_\alpha) = 0$. Since N does not have a nilpotent, we infer that $yn_\alpha = 0$. This is true for each α, so $yN = 0$, and hence $y \in \operatorname{ann} N$. This gives that $\operatorname{ann}[N : M]^k N \subseteq \operatorname{ann} N$, and the result is proved. □

We next give the following property of flat modules.

Theorem 5. *Let R be a ring and N a submodule of a finitely generated faithful multiplication R-module M. If N is a finitely generated flat module, then N is never nilpotent.*

Proof. We first prove that, for any finitely generated flat ideal of R, $\operatorname{ann} I = \operatorname{ann} I^k$ for each positive integer k. Assume $\{J_\lambda\}_{\lambda \in \Lambda}$ is a non-empty collection of ideals of R. We show that

$$\bigcap_{\lambda \in \Lambda} J_\lambda I = \Big(\bigcap_{\lambda \in \Lambda} J_\lambda \Big) I.$$

Since I is finitely generated flat, hence finitely generated multiplication, it follows by [**14**, Corollary 1.7] that

$$\Big(\bigcap_{\lambda \in \Lambda} (J_\lambda + \operatorname{ann} I) \Big) I = \bigcap_{\lambda \in \Lambda} J_\lambda I.$$

Since $\operatorname{ann} I$ is pure, it is locally either zero or R. As I is finitely generated, it is enough to verify that

$$\bigcap_{\lambda \in \Lambda} J_\lambda I = \Big(\bigcap_{\lambda \in \Lambda} J_\lambda \Big) I$$

is true locally. Thus, we may assume that R is a local ring. If $\operatorname{ann} I = R$, then $I = 0$, and both sides of the equality collapse to zero. If $\operatorname{ann} I = 0$, then the result is obviously true. Next, let

$$\operatorname{ann} I = \sum_{\alpha \in \Lambda} Rr_\alpha.$$

It follows by [**25**, Corollary 1.4] that, for all $\alpha \in \Lambda$, there exists an ideal L_α of R such that $I = IL_\alpha$ and $r_\alpha L_\alpha = 0$. Hence, $r_\alpha \in \operatorname{ann}(L_\alpha)$, and hence

$$\operatorname{ann} I = \sum_{\alpha\in\Lambda} Rr_\alpha \subseteq \sum_{\alpha\in\Lambda} \operatorname{ann}\Big(L_\alpha\Big) \subseteq \operatorname{ann}\Big(\bigcap_{\alpha\in\Lambda} L_\alpha\Big).$$

Assume $L = \cap_{\alpha\in\Lambda} L_\alpha$. Then $\operatorname{ann} I \subseteq \operatorname{ann} L$. Since $I = IL_\alpha$, we infer that

$$I = \bigcap_{\alpha\in\Lambda} IL_\alpha = I\Big(\bigcap_{\alpha\in\Lambda} L_\alpha\Big) = IL.$$

Hence, $I \subseteq L$, and hence $\operatorname{ann} L \subseteq \operatorname{ann} I$, so that $\operatorname{ann} I = \operatorname{ann} L$. Let n be any positive integer. To show $\operatorname{ann} I = \operatorname{ann} I^n$, let $x \in \operatorname{ann} I^n$. Then $xI^n = 0$, and hence $xI^{n-1} \subseteq \operatorname{ann} I = \operatorname{ann} L$. So $0 = xI^{n-1}L = xI^{n-2}IL = xI^{n-1}$. By repeating the argument we get that $xI = 0$, and hence $x \in \operatorname{ann} I$. This gives that $\operatorname{ann} I^n \subseteq \operatorname{ann} I$. As $\operatorname{ann} I \subseteq \operatorname{ann} I^n$ is always true, $\operatorname{ann} I = \operatorname{ann} I^n$. Finally, let N be a finitely generated flat submodule of M and k a positive integer. Then $[N : M]$ is a finitely generated flat ideal of R, [**9**, Proposition 3.7]. Hence, $\operatorname{ann}[N : M] = \operatorname{ann}[N : M]^k$. Since M is faithful multiplication, we get that

$$\begin{aligned}\operatorname{ann} N = \operatorname{ann}\,[N : M] &= \operatorname{ann}\,[N : M]^{k+1}\\ &= \operatorname{ann}\,[N : M]^k\,[N : M]\\ &= \operatorname{ann}\,[N : M]^k\,[N : M]\,M\\ &= \operatorname{ann}\,[N : M]^k\,N.\end{aligned}$$

So N is not a nilpotent submodule of M. □

The next two results give other Ohm type properties for finitely generated faithful multiplication modules. The first one may be compared with [**5**, Proposition 4.3] while the second generalizes [**5**, Proposition 4.4] to multiplication modules.

Proposition 6. *Let R be a ring and K and N submodules of a finitely generated faithful multiplication R-module M. Let $K + N$ be finitely generated multiplication that has no non-zero nilpotent elements. If $[K : M]^k K = [N : M]^k N$ for some positive integer k, then*

(i) $K + \text{ann}\,(K+N)\,M = N + \text{ann}\,(K+N)\,M$.

(ii) $\text{ann}\,K = \text{ann}\,N$.

Proof. Since $K+N$ is multiplication, it follows by Theorem 3 that

$$[(K+N):M]^k\,(K+N) = [K:M]^k\,K + [N:M]^k\,N,$$

and hence

$$[(K+N):M]^k\,(K+N) = [K:M]^k\,K.$$

Hence,

$$\begin{aligned}[N:M]\,[(K+N):M]^{k-1}\,(K+N) &\subseteq [(K+N):M]^k\,(K+N)\\ &= [K:M]^k\,K.\end{aligned}$$

On the other hand,

$$\begin{aligned}[K:M]^k\,K &= [K:M]\,[K:M]^{k-1}\,K\\ &\subseteq [K:M]\left([(K+N):M]^{k-1}\,(K+N)\right).\end{aligned}$$

It follows that

$$\begin{aligned}&[N:M]\,[(K+N):M]^{k-1}\,(K+N)\\ &\qquad\subseteq [K:M]\left([(K+N):M]^{k-1}\,(K+N)\right).\end{aligned}$$

Since $K+N$ is a finitely generated multiplication submodule of a finitely generated faithful multiplication module M, we infer from [**28**, Theorem 10] that $[(K+N):M]$ is a finitely generated multiplication module, and hence $[(K+N):M]^{k-1}(K+N)$ is finitely generated multiplication. It follows by [**28**, Corollary to Theorem 9] that

$$[N:M] \subseteq [K:M] + \text{ann}\,[(K+N):M]^{k-1}\,(K+N)\,.$$

As $K+N$ has no non-zero nilpotent element, it follows by Corollary 4 that

$$\text{ann}\,(K+N) = \text{ann}\,[(K+N):M]^{k-1}\,(K+N)\,.$$

Hence,

$$[N:M] \subseteq [K:M] + \text{ann}\,(K+N)\,,$$

and hence

$$[N:M] + \operatorname{ann}\,(K+N) \subseteq [K:M] + \operatorname{ann}\,(K+N).$$

Similarly,

$$[K:M] + \operatorname{ann}\,(K+N) \subseteq [N:M] + \operatorname{ann}\,(K+N).$$

So,

$$[K:M] + \operatorname{ann}\,(K+N) = [N:M] + \operatorname{ann}\,(K+N).$$

Since M is multiplication,

$$K + \operatorname{ann}\,(K+N)\,M = N + \operatorname{ann}\,(K+N)\,M.$$

(ii) We have

$$\begin{aligned}\operatorname{ann}\,[K:M]^k\,K &= \operatorname{ann}\,\left([K:M]^k\,K + [N:M]^k\,N\right)\\ &= \operatorname{ann}\,[(K+N):M]^k\,(K+N)\\ &= \operatorname{ann}\,(K+N) \subseteq \operatorname{ann} K\\ &= \operatorname{ann}\,[K:M]^k\,K.\end{aligned}$$

So $\operatorname{ann}[K:M]^k K = \operatorname{ann} K$. Similarly $\operatorname{ann}[N:M]^k N = \operatorname{ann} N$. Hence $\operatorname{ann} K = \operatorname{ann} N$, as required. □

Proposition 7. *Let R be a ring, and let K and N be submodules of a finitely generated faithful multiplication R-module M. Let N be a finitely generated multiplication module that has no non-zero nilpotent element. If $K+N$ is multiplication, then*

$$[K:N]^k + \operatorname{ann} N = \left[[K:M]^k\,K : [N:M]^k\,N\right] + \operatorname{ann} N$$

for each positive integer k.

Proof. Since $[N:M]^k N$ is multiplication, we infer that

$$(1)\quad [K:M]^k\,K \cap [N:M]^k\,N = \left[[K:M]^k\,K : [N:M]^k\,N\right][N:M]^k\,N.$$

As $K+N$ is multiplication, it follows by Theorem 3 that

$$[(K\cap N):M]^k\,(K\cap N)=[K:M]^k\,K\cap[N:M]^k\,N. \tag{2}$$

Since N is multiplication, $K\cap N=[K:N]N$. As M is finitely generated faithful multiplication, $[(K\cap N):M]=[[K:N]N:M]=[K:N][N:M]$. So

$$[(K\cap N):M]^k\,(K\cap N)=[K:N]^k\,[N:M]^k\,(K\cap N)\,. \tag{3}$$

Combining (1), (2) and (3), one gets that

$$\Big[[K:M]^k\,K:[N:M]^k\,N\Big]\,[N:M]^k\,N\subseteq[K:N]^k\,[N:M]^k\,N.$$

Since $[N:M]^kN$ is finitely generated multiplication and contains no non-zero nilpotent element, we infer that

$$\begin{aligned}\Big[[K:M]^k\,K:[N:M]^k\,N\Big]&\subseteq[K:N]^k+\operatorname{ann}\Big([N:M]^k\,N\Big)\\&=[K:N]^k+\operatorname{ann}N.\end{aligned}$$

So

$$\Big[[K:M]^k\,K:[N:M]^k\,N\Big]+\operatorname{ann}N\subseteq[K:N]^k+\operatorname{ann}N.$$

The other inclusion is always true, and hence

$$[K:N]^k+\operatorname{ann}N=\Big[[K:M]^k\,K:[N:M]^k\,N\Big]+\operatorname{ann}N. \qquad\square$$

The following theorem gives several properties of flat modules.

Theorem 8. *Let R be a ring and N a submodule of an R-module M.*

(1) [**22**, Theorem 4.1] *and* [**8**, Corollary 2.7]. *If M is multiplication and $\operatorname{ann} M$ is a pure ideal of R, then M is flat.*

(2) *If M is a finitely generated faithful multiplication and N a pure submodule of M, then N is flat.*

(3) *Let M be both a finitely generated faithful multiplication and von Neumann regular module. If N is a (finitely generated) submodule of M, then N is (projective) flat.*

(4) *If M is cancelation and N a non faithful projective and maximal submodule of M, then N is idempotent.*

Proof. (1) Let P be a maximal ideal of R. Since $\operatorname{ann} M$ is pure, $(\operatorname{ann} M)_P = 0_P$ or $(\operatorname{ann} M)_P = R_P$. If $R_P = (\operatorname{ann} M)_P \subseteq \operatorname{ann}(M_P) \subseteq R_P$, then $\operatorname{ann}(M_P) = R_P$. So $M_P = 0_P$ and hence M is locally flat (hence M is flat). We may assume $(\operatorname{ann} M)_P = 0_P$. It follows that $M_P \neq 0_P$. Otherwise $M_P = 0_P$ gives that for all $m \in M, (Rm)_P = 0_P$. So $\operatorname{ann}((Rm)_P) = \operatorname{ann}(Rm)_P = R_P$, and hence $(\operatorname{ann} M)_P = \cap_{m\in M} \operatorname{ann}(m)_P = R_P$. Next, since M is multiplication and $M_P \neq 0_P$, it follows by Anderson's theorem [**10**, Theorem 2.1] that M_P is cyclic and $\operatorname{ann} M_P = (\operatorname{ann} M)_P$. This shows that M_P is faithful cyclic, so $M_P \cong R_P$ and M is again locally flat, hence it is flat.

(2) N is pure means that $IN = N \cap IM$ for every ideal I of R. Let K be a submodule of N. Since N is pure,

$$[K : N]\, N = N \cap [K : N]\, M \supseteq N \cap [K : M]\, M = N \cap K \supseteq [K : N]\, N,$$

so that $K \cap N = [K : N]N$ and N is multiplication. Since M is faithful multiplication, $\operatorname{ann} N = \operatorname{ann}[N : M]$. By (1) it is enough to show that $\operatorname{ann}[N : M]$ is a pure ideal of R. From [**6**, Corollary 1.2], and [**4**, Lemma 9] $[N : M]$ is a pure ideal of R. Let $[N : M] = \sum_{\alpha\in\Lambda} Ra_\alpha$. Let P be a maximal ideal of R. We discuss two cases.

Case 1: $[N : M]_P = 0_P$. Then for each $\alpha \in \Lambda$, $(Ra_\alpha)_P = 0_P$, and hence $R_P = \operatorname{ann}((Ra_\alpha)_P) = \operatorname{ann}(a_\alpha)_P$. It follows that

$$R_P = \bigcap_{\alpha\in\Lambda} \operatorname{ann}\,(a_\alpha)_P = \left(\bigcap_{\alpha\in\Lambda} \operatorname{ann}\,(a_\alpha)\right)_P = (\operatorname{ann}\,[N : M])_P\,.$$

Case 2: $[N : M]_P = R_P$. So

$$(\operatorname{ann}\,[N : M])_P \subseteq \operatorname{ann}\,[N : M]_P = 0_P,$$

so that $(\operatorname{ann}[N:M])_P = 0_P$. This gives that $\operatorname{ann}(N:M)$ is a pure ideal of R, and the proof of (2) is completed. Alternatively, since N is pure and M is a finitely generated faithful multiplication R-module, it follows by [**6**, Theorem 1.4] that $[N:M]$ is a pure ideal of R. Hence, $[N:M]$ is a flat ideal of R, and hence $N=[N:M]M \cong [N:M]\otimes M$ is a flat submodule of M.

(3) Let $N=\sum_{\alpha\in\Lambda} Rm_\alpha$. By [**1**, Lemma 8] we have that $\operatorname{ann}(m_\alpha) = \operatorname{ann}(e_\alpha)$ for some idempotent e_α of R. So

$$\operatorname{ann} N = \bigcap_{\alpha\in\Lambda} \operatorname{ann}\,(m_\alpha) = \bigcap_{\alpha\in\Lambda} \operatorname{ann}\,(e_\alpha).$$

We show that $\operatorname{ann} N$ is pure. It is enough to show that $\operatorname{ann} N$ is locally either R or zero. Thus, we may assume that R is local. Hence, either e_α or $1-e_\alpha$ is a unit. If e_α is a unit for some α, then $0=\operatorname{ann}(e_\alpha)\supseteq \operatorname{ann} N$, so that $\operatorname{ann} N = 0$. If $1-e_\alpha$ is a unit for all $\alpha\in\Lambda$, then $e_\alpha(1-e_\alpha)=0$ gives that $e_\alpha = 0$ for all $\alpha\in\Lambda$. Hence $\operatorname{ann}(e_\alpha)=R$, and hence $\operatorname{ann} N = R$. Since M is a von Neumann regular module, it follows by [**1**, Proposition 12] that $Rm_\alpha = Rf_\alpha m_\alpha = f_\alpha M$ for some idempotent f_α of R. Since M is a finitely generated faithful multiplication (hence cancelation),

$$[N:M] = \left[\sum_{\alpha\in\Lambda} f_\alpha M : M\right] = \sum_{\alpha\in\Lambda} Rf_\alpha.$$

So $[N:M]$ is an ideal of R that is generated by idempotents and hence $[N:M]$ is a multiplication, [**18**]. Since $\operatorname{ann}[N:M] = \operatorname{ann} N$ is pure, $[N:M]$ is a flat ideal of R. So $N=[N:M]M\cong[N:M]\otimes M$ is a flat submodule of M. For the case where N is finitely generated, $[N:M]=Rf$ for some idempotent f, so $N=fM$ is a projective submodule of M.

(4) Suppose N is a non faithful projective. Then $N=\operatorname{Tr}(N)N$, $\operatorname{Tr}(N)$ is a pure ideal of R and $\operatorname{ann} N = \operatorname{ann}\operatorname{Tr}(N)$, [**17**]. Next, since

$$N = \operatorname{Tr}\,(N)\,N \subseteq \operatorname{Tr}\,(N)\,M \subseteq M,$$

and N is maximal, either $N=\operatorname{Tr}(N)M$ or $\operatorname{Tr}(N)M = M$. If $\operatorname{Tr}(N)M=M$ and M is a cancelation, $\operatorname{Tr}(N)=R$, and hence $\operatorname{ann} N = \operatorname{ann}\operatorname{Tr}(N) = 0$, a contradiction. So $N=\operatorname{Tr}(N)M$, and hence $[N : M] = \mathrm{Tr}(N)$. So $N = \mathrm{Tr}(N)N = [N : M]N$ is an idempotent submodule of M. This finishes the proof of the theorem. □

We have three remarks on Theorem 8. First, let I be a projective ideal of R. Then I is a multiplication, [**28**]. Also, $I = \mathrm{Tr}(I)I$, $\mathrm{Tr}(I)$ is a pure ideal of R and $\mathrm{ann}\, I = \mathrm{ann}\,\mathrm{Tr}(I)$, [**17**]. Since $\mathrm{Tr}(I)$ is pure, it follows by the proof of part (2) of Theorem 8 that $\mathrm{ann}\, I = \mathrm{ann}\,\mathrm{Tr}(I)$ is a pure ideal of R. So I is a multiplication with pure annihilator. Hence, I is flat. This gives an alternative proof to the fact that projective ideals are flat. Second, in fact if M is a multiplication module such that $\mathrm{ann}\, M$ is pure, then for each maximal ideal P of R either $N_P = 0_P$ or $N_P = M_P$, [**6**, Theorem 1.1]. So N is locally flat and hence N is flat. This generalizes part (2) of the theorem. Third, let M be cancelation and N a maximal flat submodule of M. For all $0 \neq r \in \mathrm{ann}\, N$, there exists an ideal $L = L_r$ of R with $N = LN$ and $rL = 0$, [**25**, Corollary 1.4]. So $N = LN \subseteq LM \subseteq M$. Since N is maximal, $N = LM$ or $LM = M$. If $LM = M, L = R$ and $\mathrm{ann}\, L = 0$. But $0 \neq r \in \mathrm{ann}\, L$. Hence $N = LM$, and hence $[N : M] = L$. So $N = LN = [N : M]N$ and N is idempotent. This generalizes part (4) of the theorem.

The next two results give necessary and sufficient conditions for the sum and intersection of flat modules to be flat. Compare with [**28**, Theorem 8].

Proposition 9. *Let R be a ring, and let $N_i (1 \leq i \leq n)$ be a finite collection of finitely generated submodules of a finitely generated faithful multiplication R-module M. Let $N_i + N_j$ be a flat submodule for all $i < j$.*

(1) *$S = \sum_{k=1}^{n} N_k$ is a finitely generated flat submodule of M.*

(2) *Assuming further that N_i are flat submodules, then $N = \cap_{k=1}^{n} N_k$ is a finitely generated flat submodule of M.*

Proof. (1) Since M is a multiplication, $N_i = [N_i : M]M$, and hence

$$N_i + N_j = [N_i : M]\, M + [N_j : M]\, M.$$

As M is finitely generated faithful multiplication, hence cancelation, we get that

$$[(N_i + N_j) : M] = [N_i : M] + [N_j : M]$$

is a finitely generated flat (hence multiplication) ideal of R, [**7**, Proposition 3.7]. It follows by [**28**, Proposition 4] that

$$R = [[N_i : M] : [N_j : M]] + [[N_j : M] : [N_j : M]]\,,$$

and hence $R = [N_i : N_j] + [N_j : N_i]$. By [**9**, Lemma 1.1], we have that

$$R = \sum_{i=1}^{n} [N_i : S] \subseteq \sum_{i \neq j} [(N_i + N_j) : S] \subseteq R,$$

so that $\sum_{i \neq j} [(N_i + N_j) : S] = R$. To prove S is flat, it is enough to prove the result locally. Thus, we may assume that R is local. Hence, there exist k, $l \in \{1, \dots, n\}$ with $k \neq l$ such that $S = N_k + N_l$ is flat.

(2) As mentioned above, $[N_i : N_j] + [N_j : N_i] = R$. Hence,

$$\sum_{k=1}^{n} [N : N_k] = R,$$

[**9**, Lemma 1.1]. It follows that there exists an $x_k \in [N : N_k]$ with $\sum_{k=1}^{n} x_k = 1$. So,

$$N = \sum_{k=1}^{n} x_k N \subseteq \sum_{k=1}^{n} x_k N_k \subseteq N,$$

so that

$$N = \sum_{k=1}^{n} x_k N_k$$

is finitely generated. To show that N is flat it is enough to prove it locally. Thus, we may assume that R is local. Since $\sum_{k=1}^{n} [N : N_k] = R$, there exists a $k \in \{1, \dots, n\}$ such that $N = N_k$. Hence N is flat. □

Proposition 10. *Let R be a ring, and let $N_i (1 \leq i \leq n)$ be a finite collection of submodules of an R-module M such that $[N_i : N_j] + [N_j : N_i] = R$ for all $i < j$. If $N_i + N_j$ is flat, then*

(1) *$S = \sum_{k=1}^{n} N_k$ is flat.*

(2) *$N = \cap_{k=1}^{n} N_k$ is flat if and only if N_k are flat.*

Proof. We only prove the "only if" part of (2). We have

$$\sum_{k=1}^{n}[N : N_k] = R.$$

Let N be flat. To show that N_k are flat, it is enough to prove the result locally. Thus, we may assume that R is a local ring. There exists an $l \in \{1, \dots, n\}$ such that $[N : N_l] = R$, and hence $N_l = N$ is flat. Next, $R = [N : N_l] \subseteq [N_k : N_l] = [N_k : N_k + N_l]$, so that $R = [N_k : N_k + N_l]$ for all $k \neq l$. It follows that $N_k = N_k + N_l$ is flat. □

3. F-modules and FGP-modules. Let R be a commutative ring with unity. R is called a *P.P.*, (respectively *P.F.*) ring if every principal ideal of R is projective (respectively flat). Equivalently, R is *P.P.* (respectively *P.F.*) if and only if, for all $a \in R$, $\mathrm{ann}\,(a) = \mathrm{Re}$ for some idempotent e of R (respectively $\mathrm{ann}\,(a)$ is a pure ideal of R), [**15, 16, 23**]. It is shown [**15, 23**] that R is a *P.F.* ring if and only if, for all prime ideals P of R, R_P is an integral domain and R is a *P.P.* ring if and only if, for all prime ideals P of R, R_P is an integral domain and K, the total quotient ring of R, is a von Neumann regular ring. Thus, R is a *P.P.* ring if and only if R is *P.F.* and K is von Neumann regular. An R-module M is said to be *C.P.* (respectively *C.F.*) if every cyclic submodule of M is projective (respectively flat). Equivalently, for each $m \in M$, $\mathrm{ann}\,(m) = \mathrm{Re}$ for some idempotent e of R (respectively $\mathrm{ann}\,(m)$ is a pure ideal of R). It is shown [**1**, Lemma 8] that a faithful von Neumann regular module is a *C.P.* (hence *C.F.*) module. It is also proved that R is *P.P.* if and only if every projective R-module is *C.P.* and R is *P.F.* if and only if every flat R-module is *C.F.*, [**15**]. We start this section with the following result.

Proposition 11. *Let R be a ring.*

(1) *R is P.F. if and only if every multiplication R-module with pure annihilator is C.F.*

(2) *Let M be a finitely generated faithful multiplication R-module. Then R is P.F. if and only if M is C.F.*

Proof. (1) Follows by the first part of Theorem 8.

(2) Let M be a $C.F.$ module. Let P be a prime ideal of R. Since M is a finitely generated faithful multiplication, $M_P \cong R_P$. If M is an R-$C.F.$ module, then M_P is an R_P-$C.F.$ module. For, let $x \in M_P$. There exists a $y \in M$ with $Rx = (Ry)_P$. Hence, $\operatorname{ann}(x) = \operatorname{ann}((Ry)_P) = \operatorname{ann}(y)_P$. Since $\operatorname{ann}(y)$ is pure in R, $\operatorname{ann}(x)$ is pure in R_P. So R_P is a $P.F.$ ring. Since R_P is local, R_P is an integral domain and hence R is a $P.F.$ ring. The converse follows by (1). □

The following result generalizes the fact that R is $P.F.$ if and only if R_P is an integral domain for all prime ideals P of R to the module case.

Proposition 12. *Let R be a ring and M an R-module. If M is C.F., then M_P is a torsion free R_P-module for all prime ideals P of R. The converse is true if M is finitely generated, faithful and a multiplication.*

Proof. Let M be $C.F.$, and let P be a prime ideal of R. Then M_P is a $C.F.$-R_P-module. Let $0 \neq x \in M_P$. There exists a $y \in M$ with $\operatorname{ann}(x) = \operatorname{ann}(y)_P$, and hence $\operatorname{ann}(x)$ is pure in R_P. Since R_P is local and $\operatorname{ann}(x) \neq R_P$, $\operatorname{ann}(x) = 0_P$. Hence, M_P is torsion-free. Conversely, by Proposition 11, it is enough to show that R is a $P.F.$ ring. Since M_P is torsion-free and $M_P \cong R_P$, R_P is an integral domain; hence, R is $P.F.$ □

Proposition 13. *Let R be a ring and M an R-module.*

(1) *R is P.P. if and only if every faithful multiplication R-module is a C.P-module.*

(2) *Let M be a finitely generated faithful multiplication R-module. R is P.P. if and only if M is C.P. and M_S is a von Neumann regular module, where S is the set of non-zero divisors of R.*

Proof. (1) Let R be $P.P.$ Let M be a faithful multiplication. Let $m \in M$; then $Rm = IM$ for some finitely generated ideal I of R, [**24**, Note 3.7]. Let

$$I = \sum_{i=1}^{n} Ra_i.$$

Then

$$\operatorname{ann} I = \bigcap_{i=1}^{n} \operatorname{ann}\ (a_i) = \bigcap_{i=1}^{n} R\,(1-e_i) = \mathrm{Re},$$

where

$$e = 1 - \prod_{i=1}^{n} (1-e_i)$$

is an idempotent element of R. Since M is faithful multiplication, $\operatorname{ann}(m) = \operatorname{ann} I = \mathrm{Re}$. So M is $C.P.$ Conversely, consider R to be a faithful multiplication R-module. Then R is a $C.P.$ R-module; hence, R is a $P.P.$ ring.

(2) Assume that M is a finitely generated faithful multiplication R-module. Let R be $P.P.$ By (1), M is $C.P.$ Next, $K = R_S$ is a von Neumann regular ring. It follows by [**1**, Corollary 11] that M_S is a von Neumann regular module. Conversely, since M_S is a finitely generated faithful von Neumann regular R_S-module, it follows by [**1**, Corollary 11] that R_S is a von Neumann regular ring. If M is $C.P.$, then it is $C.F.$ and, by Proposition 11, R is $P.F.$ So, R_P is an integral domain for each prime ideal P of R, and hence R is $P.P.$ □

Proposition 14. *Let R be a ring and M an R-module. If M is $C.P.$, then M_P is torsion-free for every prime ideal P of R and M_S is a von Neumann regular module, where S is the set of non-zero divisor of R. The converse is true if M is finitely generated, faithful and a multiplication.*

Proof. If M is $C.P.$, then M is $C.F.$; hence, M_P is torsion-free. If M is $C.P.$, then M_S is a $C.P.R_S$-module. Since every non-zero divisor of R_S is a unit, we infer from [**13**] that M_S is a von Neumann regular an R_S-module. Conversely, assume M is finitely generated faithful and multiplication such that M_P is torsion-free; then $R_P \cong M_P$ is an integral domain. Moreover, M_S is a finitely generated faithful multiplication von Neumann regular R_S-module. So R_S is a von Neumann regular ring, [**1**, Corollary 11]. Hence, R is $P.P.$ and, by Proposition 13, M is $C.P.$ □

A ring R is *semi-hereditary* (respectively F-ring) if and only if every finitely generated ideal of R is projective (respectively every ideal is flat). It is well known that, if every finitely generated ideal of a ring R is flat then every ideal is flat. So, R is an F-ring if and only if every finitely generated ideal is flat. It is well known that R is semi-hereditary (respectively F-ring) if and only if R_P is a valuation domain for each prime ideal P of R and K, the total quotient ring of R, is von Neumann regular (respectively R_P is a valuation domain). So R is semi-hereditary if and only if R is an F-ring and K is a von Neumann regular ring, [**15**]. We say that M is an F-module if every finitely generated submodule of M is flat and M is FGP if every finitely generated submodule of M is projective.

Proposition 15. *Let R be a ring and M a faithful multiplication R-module.*

(1) *If R is an F-ring, then M is an F-module. The converse is true if we assume further that M is finitely generated.*

(2) *If M is finitely generated, then M is an FGP-module if and only if R is a semi-hereditary ring.*

Proof. (1) Let $\Phi : R \to M$ be defined by $\Phi(a) = am$, $a \in R$. Then Φ is a ring homomorphism and onto. Let R be an F-ring. Let N be a finitely generated submodule of M. Then $\Phi^{-1}(N)$ is a finitely generated ideal of R. Hence, $\Phi^{-1}(N)$ is flat, and hence a multiplication. It follows that $N = \Phi(\Phi^{-1}(N))$ is a multiplication. Moreover, since M is faithful multiplication, it is easy to verify that $\operatorname{ann} N = \operatorname{ann} \Phi^{-1}(N)$ is a pure ideal of R. Hence, N is flat by Theorem 8 and M is an F-module. Alternatively, if N is a finitely generated submodule of M, then $N = IM$ for some finitely generated ideal I of R. Since R is an F-ring, I is flat. Since M is faithful multiplication, M is flat. Hence, $N = IM \cong I \otimes M$ is flat, and hence M is an F-module. Conversely, let M be F-module. Let P be a prime ideal of R. Then M_P is an F-R_P-module. Since M is a finitely generated faithful multiplication, $M_P \cong R_P$. Hence, R_P is an F-ring, and hence a *P.F.* ring. Since R_P is local, R_P is an integral domain, and hence R is a *P.F.*-ring. Let

$$I = \sum_{i=1}^{n} Ra_i$$

be a finitely generated ideal of R. Then

$$\operatorname{ann} I = \bigcap_{i=1}^{n} \operatorname{ann}\,(a_i)$$

is a pure ideal of R, [**6**, Corollary 1.3]. Next, since R_P is an F-ring, $R_P \cong (R_P)_P$ is a valuation domain. Hence, I_P is principal, and hence I is a multiplication, [**10, 11**]. So I is flat and R is an F-ring. Alternatively, if I is an ideal of R, IM is a flat submodule of M and by [**7**, Proposition 3.7] $I = [IM : M]$ is a flat ideal of R. So R is an F-ring.

(2) Let M be FGP. Let I be a finitely generated ideal of R. Then IM is finitely generated submodule of M. So IM is projective and, by [**7**, Proposition 3.7], $I = [IM : M]$ is projective. So R is semi-hereditary. Conversely, let N be a finitely generated submodule of M. Then $[N : M]$ is a finitely generated ideal of R, [**28**, Theorem 10]. Hence, $[N : M]$ is projective, and hence $N = [N : M]M \cong [N : M] \otimes M$ is projective. So M is FGP. □

An R-module M is called a *valuation* if, for all $0 \neq m,\, n \in M$, either $Rm \subseteq Rn$ or $Rn \subseteq Rm$, [**3**]. Consequently, if M is finitely generated, then M is cyclic. The next result shows under certain conditions that, if every 2-generated submodule of an R-module M is cyclic, then M is a valuation module.

Proposition 16. *Let R be a local ring and M a torsion-free R-module. If every 2-generated submodule of M is cyclic, then M is a valuation module.*

Proof. Let $0 \neq m, n \in M$. Then $Rm + Rn = Rk$ for some $0 \neq k \in M$. Hence, $k = am + bn$ for some $a, b \in R$. Since $Rm \subseteq Rk$ and $Rn \subseteq Rk$, there exist $c, d \in R$ with $m = ck$ and $n = dk$. So $k = ack + bdk$, and hence $k(1 - ac - bd) = 0$. Since M is torsion-free, $ac + bd = 1$. Since R is local, either c or d is a unit. Hence, $Rm = Rk$ or $Rn = Rk$. So $Rm \subseteq Rn$ or $Rn \subseteq Rm$ and M is a valuation module. □

We close our work by the following result characterizing F- and FGP-multiplication modules.

Proposition 17. *Let R be a ring and M a finitely generated faithful multiplication R-module.*

(1) *M is an F-module if and only if M_P is a valuation torsion-free module for each prime ideal P of R.*

(2) *M is an FGP module if and only if M_P is a valuation torsion-free module for each prime ideal P of R and M_S is a von Neumann regular module, where S is the set of non-zero divisors of R.*

Proof. (1) Let M be an F-module. Then R is an F-ring. Hence, R_P is a valuation domain, and further $M_P \cong R_P$ is a valuation torsion-free module. The statement is reversible.

(2) If M is an FGP-module, then M is an F-module and, by (1), M_P is a torsion-free valuation module. Also, M is FGP implies that M is a *C.P.* module, and hence M_S is a *C.P.*-R_S-module. Since every non-zero divisor of R_S is a unit, M_S is von Neumann regular. Conversely, since $M_P \cong R_P$, R_P is a valuation domain. Moreover, M_S is a finitely generated faithful multiplication von Neumann regular module. So, by [**1**, Corollary 11], R_S is a von Neumann regular ring. Hence R is semi-hereditary, and by Proposition 15 M is FGP. □

REFERENCES

1. M.M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Comm. Algebra **36**(2008), 4620–4642.

2. ———, *Idealization and theorems of D.D. Anderson*, II, Comm. Algebra **35** (2007), 2767–2793.

3. ———, *Invertibility of multiplication modules*, New Zealand J. Math. **35** (2006), 17–29.

4. ———, *Multiplication modules and tensor product*, Beitr. Algebra Geom. **47** (2006), 305–327.

5. ———, *The Ohm type properties of multiplication ideals*, Beitr. Algebra Geom. **37** (1996), 399–414.

6. M.M. Ali and D.J. Smith, *Pure submodules of multiplication modules*, Beitr. Algebra Geom. **44** (2004), 61–74.

7. ———, *Some remarks on multiplication and projective modules*, Comm. Algebra **32** (2004), 3897–3909.

8. ———, *Projective, flat and multiplication modules*, New Zealand J. Math. **31** (2002), 115–129.

9. ———, *Finite and infinite collections of multiplication modules*, Beitr. Algebra Geom. **42**(2001), 557–573.

10. D.D. Anderson, *Some remarks on multiplication ideals* II, Comm. Algebra **28** (2000), 2577–2583.

11. D.D. Anderson, *Some remarks on multiplication ideals*, Math. Japon. **4** (1980), 463–469.

12. B. Barnard, *Multiplication modules*, J. Algebra **71** (1981), 174–178.

13. I.J. Cheatham and J.R. Smith, *Regular modules and semisimple modules*, Pacific J. Math. **65** (1976), 315–323.

14. Z.A. El-Bast and P.F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), 755–779.

15. S. Endo, *Note on P.P. rings*, Nagoya Math. J. **17** (1960), 167–170.

16. ———, *On semihereditary rings*, J. Math. Soc. Japan **13** (1961), 109–119.

17. C. Faith, *Algebra* I: *Rings, modules and categories*, Springer-Verlag, New York, 1981.

18. R. Gilmer, *Multiplicative ideal theory*, Kingston, Queen's, 1992.

19. A. Hattori, *A foundation of the torsion theory over general rings*, Nagoya Math. J. **17** (1960), 147–158.

20. F. Kash, *Modules and rings*, Academic Press, London, 1982.

21. M.D. Larsen and P.J. McCarthy, *Multiplicative theory of ideals*, Academic press, London, 1971.

22. A.G. Naoum, *Flat modules and multiplication modules*, Period. Math. Hungar. **21** (1990), 309–317.

23. ———, *A note on P.F. rings*, Dirasat, **12** (1985), 195–198.

24. A.G. Naoum and F.H. Al-Alwan, *Dedekind modules*, Comm. Algebra **24** (1996), 397–412.

25. A.G. Naoum, A. Mahmood and F.H. Alwan, *On projective and flat modules*, Arab J. Math. **7** (1986), 43–53.

26. P. Ribenboim, *Algebraic numbers*, Wiley, New York, 1972.

27. J.J. Rotman, *An introduction to homological algebra*, Academic Press, London, 1979.

28. P.F. Smith, *Some remarks on multiplication modules*, Arch. Math. **50** (1988), 223–235.

Department of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36, P.C. 123 Al-Khod, Sultanate of Oman
Email address: mali@squ.edu.om