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A CHARACTERIZATION OF COFINITE COMPLEXES OVER COMPLETE GORENSTEIN DOMAINS

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ABSTRACT. Let R be a complete Gorenstein local domain, J an ideal of R of dimension one, and $N^\bullet$ a complex of R-modules bounded below. In this paper, we prove that $N^\bullet$ is a J-cofinite complex if and only if $H^i(N^\bullet)$ is a J-cofinite module for all i. Consequently, this assertion affirmatively answers the fourth question in [**4**, page 149] for an ideal of dimension one over a complete Gorenstein local domain.

1. Introduction. We assume that all rings are commutative and Noetherian with identity throughout this paper.

In this paper, we shall prove the following theorem.

Theorem 1. *Let R be a complete Gorenstein local domain of dimension d, and let J be an ideal of R of dimension one. Let $N^\bullet$ be a complex of R-modules in $\mathcal{D}^+(R)$, where $\mathcal{D}^+(R)$ is the derived category consisting of complexes bounded below. Then $N^\bullet$ is J-cofinite if and only if $H^i(N^\bullet)$ is in $\mathcal{M}(R,J)_{\text{cof}}$ for all i, where $\mathcal{M}(R,J)_{\text{cof}}$ is a category of J-cofinite modules* (*see Definition* 3 *below*).

The following question is proposed in the paper [**4**, Section 2]:

Question 1. Let R be a regular ring of dimension d and J an ideal of R. Suppose that R is complete with respect to the J-adic topology. Then does there exist an abelian category $\mathcal{M}_{\text{cof}}$ consisting of R-modules, such that elements $N^\bullet \in \mathcal{D}(R,J)_{\text{cof}}$ are characterized by the property "$H^i(N^\bullet) \in \mathcal{M}_{\text{cof}}$" for all i? Here we denote $\mathcal{D}(R,J)_{\text{cof}}$ is the essential image of $\mathcal{D}_{ft}(R)$ by the J-dualizing functor (see Definition 1 below for the definition of the dualizing functor).

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In [**9**, Theorem 2], the following result was proved:

Theorem 2 (cf. [**9**]). *Let R be a regular local ring complete with respect to the J-adic topology, J an ideal of R. Let $N^\bullet$ be in the derived category $\mathcal{D}^+(R)$. Suppose that J is of dimension one. Then $N^\bullet$ is J-cofinite if and only if $H^i(N^\bullet)$ is in $\mathcal{M}(R,J)_{\mathrm{cof}}$ for all i.*

In this paper, we shall extend this result to that over a complete Gorenstein domain. The method adopted in the proof of Theorem 1 is different from that of Theorem 2 in [**9**]. In [**9**, Theorem 2], the lemma due to Melkersson was used (see [**13**, Lemma 1·7, page 420] for Melkersson's result). On the other hand, Theorem 1 is proved by refining the lemmas due to Huneke and Koh (cf. [**7**, Lemma 4·3] and [**7**, Lemma 4·7]). Consequently, Theorem 1 affirmatively answers Question 1 for an ideal of dimension one over a complete Gorenstein domain.

2. Preliminaries. In this section, we recall the basic definitions. Let R be a ring and $\mathcal{A}$ an abelian category.

First we introduce definitions on derived categories. In this paper we mainly follow the notations like those of [**5**] (see also [**1, 10**]):

$\mathcal{M}(R)$: the category of R-modules,

$C^*(\mathcal{A})$: the category of complexes consisting of objects in $\mathcal{A}$,

$K^*(\mathcal{A})$: the homotopic category,

$\mathcal{D}^*(\mathcal{A})$: the derived category,

where * stands for $+$, $-$, b or $\varnothing$.

Now let A' be a thick abelian subcategory of $\mathcal{A}$ (that is, any extension in $\mathcal{A}$ of two objects of A' is in A'). We define $K^*_{A'}(\mathcal{A})$ (respectively $\mathcal{D}^*_{A'}(\mathcal{A})$) to be the full subcategory of $K^*(\mathcal{A})$ (respectively $\mathcal{D}^*(\mathcal{A})$) consisting of these complexes $X^\bullet$ whose cohomology objects $H^i(X^\bullet)$ are all in A'. According to the notation in [**4**, page 149], we denote $K^*_{ft}(R)$ (respectively $\mathcal{D}^*_{ft}(R)$) for $K^*_{A'}(R)$ (respectively $\mathcal{D}^*_{A'}(R)$) in the case that A' is the category consisting of all R-modules of finite type. Further, we simply write $K^*_{A'}(R)$ (respectively $\mathcal{D}^*_{A'}(R)$) in place of $K^*_{A'}(\mathcal{M}(R))$ (respectively $\mathcal{D}^*_{A'}(\mathcal{M}(R))$).

Next we explain the derived functor $\mathbf{R}\mathrm{Hom}^\bullet$ under these notations (cf. [**5**, page 65]). Suppose that $\mathcal{A}$ has enough injectives, and let $L \subseteq K^+(\mathcal{A})$ be the triangulated subcategory of complexes of injective objects. Then we see that for each $X^\bullet \in \mathrm{Ob}\, K(\mathcal{A})$, L satisfies the hypotheses of [**5**, Theorem 5.1, page 53] for the functor

$$\mathrm{Hom}^\bullet(X^\bullet,\ \bullet\,) : K^+(\mathcal{A}) \longrightarrow K(Ab),$$

where (Ab) is the category of abelian groups. Hence this functor has a right derived functor $\mathbf{R}_{II}\mathrm{Hom}^\bullet(X^\bullet,\ \bullet\,) : \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}(Ab)$. Since it is functorial in $X^\bullet$, we have a bi-∂-functor

$$\mathbf{R}_{II}\mathrm{Hom}^\bullet : K(\mathcal{A})^\circ \times \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}(Ab).$$

Here we find that in the first variable this functor takes acyclic complexes into acyclic complexes, and hence passes to the quotient, giving a right derived functor

$$\mathbf{R}_I\mathbf{R}_{II}\mathrm{Hom}^\bullet : \mathcal{D}(\mathcal{A})^\circ \times \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}(Ab).$$

Moreover, suppose that $\mathcal{A}$ has enough projectives. Then, by the usual process of "reversing the arrows," we see that there is also a functor

$$\mathbf{R}_{II}\mathbf{R}_I\mathrm{Hom}^\bullet : \mathcal{D}^-(\mathcal{A})^\circ \times \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(Ab).$$

Now $\mathcal{M}(R)$ has both enough injectives and enough projectives. So the two functors $\mathbf{R}_I\mathbf{R}_{II}\mathrm{Hom}^\bullet$ and $\mathbf{R}_{II}\mathbf{R}_I\mathrm{Hom}^\bullet$ are defined on $\mathcal{D}^-(R)^\circ \times \mathcal{D}^+(R)$, which are canonically isomorphic by [**5**, Lemma 6.3, page 66]. Thus we are justified in using the ambiguous notation $\mathbf{R}\mathrm{Hom}^\bullet$.

For related results and detailed proofs on derived categories and derived functors, we recommend the readers to consult [**5**].

Before defining the J-cofiniteness on complexes, we introduce the following definition (cf. [**10**, subsection 4.3, page 70]):

Definition 1. Let R be a homomorphic image of a finite dimensional Gorenstein ring, J an ideal of R, and $D^\bullet$ a dualizing complex over R. We denote by $D_J(\,-\,)$ the functor $\mathbf{R}\mathrm{Hom}^\bullet(\,-,\, \mathbf{R}\Gamma_J(D^\bullet))$ on the derived category $\mathcal{D}(R)$. In this paper, we call this functor $D_J(\,-\,)$ the *J-dualizing functor* (or the *dualizing functor* on J). Further, we often denote the ith cohomology module $H^i(D_J(\,-\,))$ by $D^i_J(\,-\,)$ for some i, according to the notation in [**4**, page 160].

The J-cofiniteness on *complexes* is defined as follows (see [**4**, Section 2, page 149] for the definition over regular rings):

Definition 2. Let R be a homomorphic image of a finite dimensional Gorenstein ring, and let J be an ideal of R. Let $N^\bullet$ be an object of the derived category $\mathcal{D}(R)$. We say $N^\bullet$ is *J-cofinite*, or for short *cofinite*, if there exists $M^\bullet \in \mathcal{D}_{ft}(R)$, such that $N^\bullet \simeq D_J(M^\bullet)$ in $\mathcal{D}(R)$. Here $D_J(\,-\,)$ is the J-dualizing functor on $\mathcal{D}(R)$ defined as above.

The J-cofiniteness on *modules* is defined as follows (cf. [**4**, pages 148, 159]):

Definition 3. Let R be a ring and J an ideal of R. We denote by $\mathcal{M}(R,J)_{\mathrm{cof}}$ the full subcategory of all R-modules N satisfying the conditions

$$(*)\qquad \begin{aligned} &\mathrm{Supp}_R(N) \subseteq V(J) \quad \text{and} \\ &\mathrm{Ext}^j_R(R/J, N) \quad \text{is of finite type, for all } j. \end{aligned}$$

An object in the category $\mathcal{M}(R,J)_{\mathrm{cof}}$ is called J-cofinite in this paper.

Here the readers should notice that the concept of J-cofiniteness for complexes does not always agree with that for modules. On the other hand, if N is an R-module and J-cofinite, then $N^\bullet$ is a J-cofinite complex, provided that R is a regular ring complete with respect to the J-adic topology (cf. [**4**, Theorem 5.1]). Here $N^\bullet$ is a complex such that $N^0 = N$ and $N^i = 0$ if $i \neq 0$. So the definition of the J-cofiniteness on *complexes* is considered to be a generalization of that on *modules* under the assumption above.

3. Refinement of lemmas by Huneke and Koh. In this section, we shall prove several lemmas.

Lemma 3. *Let R be a ring. Suppose that there is a convergent spectral sequence of R-modules:*

$$E_2^{p,q} \Longrightarrow H^{p+q}$$

in the first quadrant. If $E_2^{p,q}$ is a finitely generated R-module for all $p, q \geq 0$, then the abutment term H^n is a finitely generated R-module for all $n \geq 0$.

Proof. Suppose that $E_2^{p,q}$ is a finitely generated R-module for all $p, q \geq 0$. Now there exists a spectral sequence of R-modules:

$$E_2^{p,q} \Longrightarrow H^{p+q},$$

which is in the first quadrant. Further it is convergent, so there is a finite filtration as follows:

$$H^l = H_0^l \supset H_1^l \supset H_2^l \supset \cdots \supset H_l^l \supset H_{l+1}^l = 0,$$

for each $l \geq 0$ satisfied with the following conditions:

(a) For all integers s with $0 \leq s \leq l$, $H_s^l / H_{s+1}^l \simeq E_\infty^{s,l-s}$.

(b) There is an integer $r \geq 2$ such that $E_r^{p,q} \simeq E_\infty^{p,q}$ for all integers $p, q \geq 0$.

Now $E_2^{p,q}$ is a finitely generated R-module for all $p, q \geq 0$, so is $E_r^{p,q} \simeq E_\infty^{p,q}$. Hence we have that $H_s^l / H_{s+1}^l \simeq E_\infty^{s,l-s}$ is a finitely generated R-module for all $0 \leq s \leq l$. By descending induction on s, one can find that H^l is a finitely generated R-module for each $l \geq 0$, as required. □

Lemma 4. *Let R be a complete Cohen-Macaulay local domain of dimension d and J an ideal of R. If the ideal J is of dimension one, then the local cohomology module $H_J^{d-1}(R)$ is J-cofinite.*

Proof. First we notice that $H_J^d(R) = 0$ by the local Lichenbaum-Hartshorne vanishing theorem, since R is a complete local domain. Further the ideal J has height $d-1$. So we see that $H_J^j(R) = 0$ if $j < d-1$ by the assumption that R is a Cohen-Macaulay local ring. Hence we have that $H_J^j(R) = 0$ if $j \neq d-1$, namely, R is a cohomological complete intersection. Then the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_R^p(R/J, H_J^q(R)) \Longrightarrow H^{p+q} = \mathrm{Ext}_R^{p+q}(R/J, R)$$

degenerates as follows: $\mathrm{Ext}_R^p(R/J, H_J^{d-1}(R)) \simeq \mathrm{Ext}_R^{p+d-1}(R/J, R)$, which is of finite type for all $p \geq 0$, as required. □

Now we refine the lemmas due to Huneke and Koh (cf. [**7**, Lemma 4·3] and [**7**, Lemma 4·7]). The finitely generated module C is arbitrary in part (iii) of the lemma below, although the module was assumed to be the first syzygy in part (iii) of [**7**, Lemma 4·3].

Lemma 5. *Let* $(R, \mathfrak{m})$ *be a complete Gorenstein local domain of dimension* d *and* J *an ideal of* R *of dimension one. Let* $\mathbf{Q}$ *be a prime ideal of* R. *Then we have the following assertions*:

(i) *if* $J + \mathbf{Q}$ *is not* $\mathfrak{m}$*-primary, then* $\mathrm{Ext}^1_R(R/\mathbf{Q}, H^{d-1}_J(R)) = 0$, *and* $\mathrm{Ext}^l_R(R/J, \mathrm{Hom}_R(R/\mathbf{Q}, H^{d-1}_J(R)))$ *is a finitely generated* R*-module for all* $l \geq 0$;

(ii) *if* $J + \mathbf{Q}$ *is* $\mathfrak{m}$*-primary, then* $\mathrm{Hom}_R(R/\mathbf{Q}, H^{d-1}_J(R)) = 0$, *and* $\mathrm{Ext}^l_R(R/J, \mathrm{Ext}^1_R(R/\mathbf{Q}, H^{d-1}_J(R)))$ *is a finitely generated* R*-module for all* $l \geq 0$;

(iii) *if* C *is an arbitrary finitely generated* R*-module, then* $\mathrm{Ext}^l_R(R/J, \mathrm{Hom}_R(C, H^{d-1}_J(R)))$ *and* $\mathrm{Ext}^l_R(R/J, \mathrm{Ext}^1_R(C, H^{d-1}_J(R)))$ *are finitely generated* R*-modules for all* $l \geq 0$. *Consequently,* $\mathrm{Ext}^l_R(R/J, \mathrm{Ext}^j_R(C, H^{d-1}_J(R)))$ *is a finitely generated* R*-module for all* $l \geq 0$ *and all* $j \geq 0$.

Proof. We may assume that J is a radical ideal. Let $J = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$ be a primary decomposition.

(i) Assume that $J + \mathbf{Q}$ is not $\mathfrak{m}$-primary. Then some prime $\mathfrak{p}_i$ contains $\mathbf{Q}$. So we may assume that $\mathfrak{p}_1 \supseteq \mathbf{Q}$.

First we show that $\mathrm{Ext}^l_R(R/\mathbf{Q}, H^{d-1}_J(R)) = 0$ for $l \geq 1$ by induction on r, the number of prime ideals appearing in the primary decomposition of J. For the case that $r = 1$, we have $\mathrm{Ext}^l_R(R/\mathbf{Q}, H^{d-1}_{\mathfrak{p}_1}(R)) = 0$ by [**7**, Lemma 4·4, page 427], for $\mathfrak{p}_1 \supseteq \mathbf{Q}$. Now assume that $r > 1$. Pick up $y \in \cap_{1 \leq i \leq r-1} \mathfrak{p}_i \backslash \mathfrak{p}_r$, so we have the natural injection $R_y \longrightarrow R_{\mathfrak{p}_r}$ as a ring homomorphism, hence as a homomorphism between R_y-modules. Now let T be an R-module with $\mathrm{Supp}\,(T_y) = \{\mathfrak{p}_r R_y\}$ $(= V(\mathfrak{p}_r R_y))$. Here we note that $\mathfrak{p}_r R_y$ is a maximal ideal of R_y. Consider R_y-homomorphisms among R_y-modules:

$$0 \longrightarrow K \longrightarrow T_y \longrightarrow T_{\mathfrak{p}_r} \longrightarrow C \longrightarrow 0,$$

which is exact, where K and C are the kernel and cokernel of $T_y \to T_{\mathfrak{p}_r}$, respectively. Then the R_y-module T_y has only support in $V(\mathfrak{p}_r R_y)$, so do both K and C. On the other hand, we have $K_{\mathfrak{p}_r} = 0$ and $C_{\mathfrak{p}_r} = 0$, since the natural map $T_y \to T_{\mathfrak{p}_r}$ is an isomorphism after being localized by $\mathfrak{p}_r$. So we must have $K = C = 0$. Therefore the natural map $T_y \to T_{\mathfrak{p}_r}$ is an isomorphism. Apply the above argument to $H_J^{d-1}(R)$ in place of T, so we can obtain an R_y-isomorphism $H_J^{d-1}(R)_y \simeq H_J^{d-1}(R)_{\mathfrak{p}_r}$, since the support of $H_J^{d-1}(R)_y$ is just $\{\mathfrak{p}_r R_y\}$.

Now by virtue of [**3**, Proposition 1.9, page 9], there is an exact sequence:

$$0 \longrightarrow H_{J+yR}^{d-1}(R) \longrightarrow H_J^{d-1}(R) \longrightarrow H_J^{d-1}(R)_y \longrightarrow 0.$$

Here we have $H_J^{d-1}(R)_y \simeq H_J^{d-1}(R)_{\mathfrak{p}_r} \simeq E_{R_{\mathfrak{p}_r}}(R_{\mathfrak{p}_r}/\mathfrak{p}_r R_{\mathfrak{p}_r}) \simeq E_R(R/\mathfrak{p}_r)$, which is an injective R-module. So we have $\operatorname{Ext}_R^1(R/\mathbf{Q}, H_J^{d-1}(R)_y) = 0$. Further, it follows from inductive hypothesis that $\operatorname{Ext}_R^1(R/\mathbf{Q}, H_{J+(y)}^{d-1}(R)) = 0$, since $\sqrt{J+(y)} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_{r-1}$. Hence we conclude that $\operatorname{Ext}_R^1(R/\mathbf{Q}, H_J^{d-1}(R)) = 0$.

Consider the following two spectral sequences which have the same abutment term:

$$(\sharp)_1 \qquad \operatorname{Ext}_R^p(R/J, \operatorname{Ext}_R^q(R/\mathbf{Q}, H_J^{d-1}(R))) \Longrightarrow H^{p+q},$$

$$(\sharp)_2 \qquad \operatorname{Ext}_R^p(\operatorname{Tor}_q^R(R/J, R/\mathbf{Q}), H_J^{d-1}(R)) \Longrightarrow H^{p+q},$$

so the spectral sequence $(\sharp)_1$ degenerates, since $\operatorname{Ext}_R^q(R/\mathbf{Q}, H_J^{d-1}(R)) = 0$ for $q > 0$. Two spectral sequences $(\sharp)_1$ and $(\sharp)_2$ are combined as follows:

$$E_2^{p,q} = \operatorname{Ext}_R^p(\operatorname{Tor}_q^R(R/J, R/\mathbf{Q}), H_J^{d-1}(R)) \Longrightarrow \operatorname{Ext}_R^{p+q}(R/J, \operatorname{Hom}_R(R/\mathbf{Q}, H_J^{d-1}(R))).$$

For an ideal J of dimension one, the local cohomology module $H_J^{d-1}(R)$ is J-cofinite by Lemma 4. Since the support of $\operatorname{Tor}_q^R(R/J, R/\mathbf{Q})$ is contained in $V(J)$ for all $q \geq 0$, it follows from [**7**, Lemma 4·2] that $\operatorname{Ext}_R^p(\operatorname{Tor}_q^R(R/J, R/\mathbf{Q}), H_J^{d-1}(R))$ is a finitely generated R-module for all $p \geq 0$ and $q \geq 0$. So all the E_2-terms are finitely generated R-modules. Therefore from Lemma 3, it follows that $\operatorname{Ext}_R^l(R/J, \operatorname{Hom}_R(R/\mathbf{Q}, H_J^{d-1}(R)))$ is a finitely generated R-module for all $l \geq 0$, as required.

(ii) We shall show that if $J + \mathbf{Q}$ is $\mathfrak{m}$-primary, then $\operatorname{Ext}_R^l(R/J, \operatorname{Ext}_R^1(R/\mathbf{Q}, H_J^{d-1}(R)))$ is a finitely generated R-module for all $l \geq 0$.

First we notice that $\mathbf{Q} \not\subseteq \cup_{1 \leq i \leq r} \mathfrak{p}_i$, since $J + \mathbf{Q}$ is $\mathfrak{m}$-primary. Pick up $x \in \mathbf{Q} \setminus \cup_{1 \leq i \leq r} \mathfrak{p}_i$, so $J + xR$ is $\mathfrak{m}$-primary. Now since the ring $(R, \mathfrak{m})$ is a domain by assumption, we can consider the exact sequence:

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0.$$

Then we have the short exact sequence

$$0 \longrightarrow H_J^{d-1}(R) \xrightarrow{x} H_J^{d-1}(R) \longrightarrow H_J^{d-1}(R/xR) \longrightarrow 0.$$

We note that $H_J^{d-1}(R/xR) = H_{J+xR}^{d-1}(R/xR) = H_{\mathfrak{m}/xR}^{d-1}(R/xR) = E_{R/xR}(R/\mathfrak{m})$, for R/xR is a Gorenstein local ring. Since x is contained in $\mathbf{Q}$, we have $\operatorname{Hom}_R(R/\mathbf{Q}, H_J^{d-1}(R)) = 0$ and isomorphisms:

$$\begin{aligned} \operatorname{Ext}_R^1(R/\mathbf{Q}, H_J^{d-1}(R)) &\simeq \operatorname{Hom}_R(R/\mathbf{Q}, H_J^{d-1}(R/xR)) \\ &\simeq \operatorname{Hom}_R(R/\mathbf{Q}, E_{R/xR}(R/\mathfrak{m})). \end{aligned}$$

Eventually we conclude that $\operatorname{Ext}_R^q(R/\mathbf{Q}, H_J^{d-1}(R)) = 0$ for $q \neq 1$. Hence two spectral sequences $(\sharp)_1$ and $(\sharp)_2$ are combined as follows:

$$\begin{aligned} E_2^{p,q} = \operatorname{Ext}_R^p(\operatorname{Tor}_q^R(R/J, R/\mathbf{Q}), H_J^{d-1}(R)) \\ \Longrightarrow \operatorname{Ext}_R^{p+q}(R/J, \operatorname{Ext}_R^1(R/\mathbf{Q}, H_J^{d-1}(R))). \end{aligned}$$

Then we have that $\operatorname{Ext}_R^p(R/J, \operatorname{Ext}_R^1(R/\mathbf{Q}, H_J^{d-1}(R)))$ is finitely generated for all $p \geq 0$ by repeating the same argument as in the proof of (i), as required.

(iii) Before proving part (iii) of the lemma, we notice that the injective dimension of $H_J^{d-1}(R)$ is not greater than one. So it is enough to show that the extension R-modules $\operatorname{Ext}_R^p(R/J, \operatorname{Ext}_R^1(C, H_J^{d-1}(R)))$ and $\operatorname{Ext}_R^p(R/J, \operatorname{Hom}_R(C, H_J^{d-1}(R)))$ are finitely generated for all $p \geq 0$, in order to prove the assertion.

Let C be an arbitrary finitely generated R-module, and take a prime filtration of C:

$$0 = N_{s+1} \subset N_s \subset N_{s-1} \subset \cdots \subset N_1 \subset N_0 = C,$$

with short exact sequences $0 \to N_{i+1} \to N_i \to R/\mathbf{Q}_i \to 0$ for each i $(0 \leq i \leq s)$. To prove part (iii) of the lemma, we proceed by descending induction on i. If $i = s$, the assertion follows from parts (i) and (ii) of this lemma. Now suppose that $\mathrm{Ext}_R^p(R/J, \mathrm{Ext}_R^l(N_{i+1}, H_J^{d-1}(R)))$ is finitely generated for all $p \geq 0$ and $l \geq 0$. From the above short exact sequence, we have the long exact sequence:

$$(\natural) \quad \begin{aligned} 0 &\longrightarrow \mathrm{Hom}\,(R/\mathbf{Q}_i, H_J^{d-1}(R)) \longrightarrow \mathrm{Hom}\,(N_i, H_J^{d-1}(R)) \\ &\longrightarrow \mathrm{Hom}\,(N_{i+1}, H_J^{d-1}(R)) \longrightarrow \mathrm{Ext}_R^1(R/\mathbf{Q}_i, H_J^{d-1}(R)) \\ &\longrightarrow \mathrm{Ext}_R^1(N_i, H_J^{d-1}(R)) \longrightarrow \mathrm{Ext}_R^1(N_{i+1}, H_J^{d-1}(R)) \\ &\longrightarrow 0, \end{aligned}$$

for each $i \geq 0$. We must show that $\mathrm{Ext}_R^p(R/J, \mathrm{Ext}_R^l(N_i, H_J^{d-1}(R)))$ is finitely generated for all $p \geq 0$ and $l \geq 0$. To do so, we divide the proof into two cases.

Case 1. If $J + \mathbf{Q}_i$ is not $\mathfrak{m}$-primary, then $\mathrm{Ext}_R^1(R/\mathbf{Q}_i, H_J^{d-1}(R)) = 0$ and an R-module $\mathrm{Hom}_R(R/\mathbf{Q}_i, H_J^{d-1}(R))$ is J-cofinite by part (i). Then we obtain a short exact sequence and an isomorphism from the above exact sequence:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}\,(R/\mathbf{Q}_i, H_J^{d-1}(R)) \longrightarrow \mathrm{Hom}\,(N_i, H_J^{d-1}(R)) \\ &\longrightarrow \mathrm{Hom}\,(N_{i+1}, H_J^{d-1}(R)) \longrightarrow 0, \\ &\mathrm{Ext}_R^1(N_i, H_J^{d-1}(R)) \simeq \mathrm{Ext}_R^1(N_{i+1}, H_J^{d-1}(R)). \end{aligned}$$

Now $\mathrm{Ext}_R^l(N_{i+1}, H_J^{d-1}(R))$ is J-cofinite for all $l \geq 0$. So $\mathrm{Ext}_R^l(N_i, H_J^{d-1}(R))$ is also J-cofinite for all $l \geq 0$. Therefore, it follows from part (i) that $\mathrm{Ext}_R^p(R/J, \mathrm{Ext}_R^l(N_i, H_J^{d-1}(R)))$ is finitely generated for all $p \geq 0$ and $l \geq 0$.

Case 2. If $J + \mathbf{Q}_i$ is $\mathfrak{m}$-primary, then it holds that $\mathrm{Hom}_R(R/\mathbf{Q}_i, H_J^{d-1}(R)) = 0$ and $\mathrm{Ext}_R^1(R/\mathbf{Q}_i, H_J^{d-1}(R))$ is J-cofinite by part (ii). In this case, we have that $\mathbf{Q}_i \not\subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$. Further we see that $\mathrm{Ext}_R^1(R/\mathbf{Q}_i, H_J^{d-1}(R))$ is artinian, since its support is in $V(\mathfrak{m})$ and J-cofinite (hence all the Bass numbers of this module are finite by [**7**, Lemma 4·1, page 426]).

Now we divide the sequence ($\natural$) into kernels and cokernels:

($\natural$)$_1$
$$0 \to \mathrm{Hom}_R(N_i, H_J^{d-1}(R)) \to \mathrm{Hom}_R(N_{i+1}, H_J^{d-1}(R)) \to X^1 \to 0,$$
($\natural$)$_2$
$$0 \to X^1 \to \mathrm{Ext}_R^1(R/\mathbf{Q}_i, H_J^{d-1}(R)) \to Y^1 \to 0,$$
($\natural$)$_3$
$$0 \to Y^1 \to \mathrm{Ext}_R^1(N_i, H_J^{d-1}(R)) \to \mathrm{Ext}_R^1(N_{i+1}, H_J^{d-1}(R)) \to 0,$$

since $\mathrm{Hom}_R(R/\mathbf{Q}_i, H_J^{d-1}(R)) = 0$ by part (ii). Here $\mathrm{Ext}_R^1(R/\mathbf{Q}_i, H_J^{d-1}(R))$ is artinian, as are both X^1 and Y^1 in the sequence ($\natural$)$_2$. Since $\mathbf{Q}_i \not\subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$, we can pick up an element $w \in \mathbf{Q}_i \backslash \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \cdots \cup \mathfrak{p}_r$. Then there is a short exact sequence:

$$0 \longrightarrow R/J \xrightarrow{w} R/J \longrightarrow R/J + (w) \longrightarrow 0,$$

from which we obtain a long exact sequence:

$$\cdots \longrightarrow \mathrm{Ext}_R^l(R/J + (w), X^1) \longrightarrow \mathrm{Ext}_R^l(R/J, X^1) \xrightarrow{w} \mathrm{Ext}_R^l(R/J, X^1) \longrightarrow \cdots .$$

We note that the element w annihilates $\mathrm{Ext}_R^1(R/\mathbf{Q}_i, H_J^{d-1}(R))$, as $w \in \mathbf{Q}_i$. So it follows from the sequence ($\natural$)$_2$ that the element w annihilates X^1, that is, $wX^1 = 0$. Hence the above long exact sequence collapses to the short exact sequence:

($\flat$) $$0 \longrightarrow \mathrm{Ext}_R^{l-1}(R/J, X^1) \longrightarrow \mathrm{Ext}_R^l(R/J + (w), X^1) \longrightarrow \mathrm{Ext}_R^l(R/J, X^1) \longrightarrow 0,$$

for each $l \geq 1$ and the isomorphism $\mathrm{Hom}_R(R/J, X^1) \simeq \mathrm{Hom}_R(R/J + (w), X^1)$. Since X^1 is artinian and $\mathrm{Supp}\,(R/J + (w)) \subseteq V(\mathfrak{m})$, $\mathrm{Ext}_R^l(R/J + (w), X^1)$ is finitely generated for all $l \geq 0$. Hence $\mathrm{Ext}_R^l(R/J, X^1)$ is finitely generated for all $l \geq 0$ by the sequence ($\flat$), namely, X^1 is J-cofinite. Since both X^1 and $\mathrm{Ext}_R^1(R/\mathbf{Q}_i, H_J^{d-1}(R))$ are J-cofinite, Y^1 is J-cofinite by the sequence ($\natural$)$_2$. Then by the sequence ($\natural$)$_3$, we deduce that $\mathrm{Ext}_R^1(N_{i+1}, H_J^{d-1}(R))$ is J-cofinite if and only if $\mathrm{Ext}_R^1(N_i, H_J^{d-1}(R))$ is J-cofinite. Similarly, since X^1 is J-cofinite, $\mathrm{Hom}_R(N_{i+1}, H_J^{d-1}(R))$ is J-cofinite if and only if

$\operatorname{Hom}_R(N_i, H_J^{d-1}(R))$ is J-cofinite by the sequence $(\natural)_1$. Therefore, $\operatorname{Ext}_R^p(R/J, \operatorname{Ext}_R^l(N_i, H_J^{d-1}(R)))$ is finitely generated for all $p \geq 0$ and $l \geq 0$. The proof of part (iii) is completed. □

Remark 1. In the proof of part (i) of Lemma 5, we find the following. For $r > 1$, there is an exact sequence:

$$(\dagger) \quad 0 \to H^{d-1}_{\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_{r-1}}(R) \to H^{d-1}_{\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r}(R) \to E_R(R/\mathfrak{p}_r) \to 0.$$

One might expect that $H^{d-1}_{\mathfrak{p}_1}(R) \simeq E_R(R/\mathfrak{p}_1)$ from the above exact sequence, which is the case that $r = 1$, but we can never obtain that from the sequence $(\dagger)$. There is an exact sequence:

$$0 \longrightarrow H^{d-1}_{\mathfrak{p}_1}(R) \longrightarrow E_R(R/\mathfrak{p}_1) \longrightarrow E_R(R/\mathfrak{m}) \longrightarrow 0.$$

Lemma 6. *Let R and J be as in Theorem* 1. *Let M be an R-module. Then we have the following equalities*:

(i) $D_J^{d-1}(M) = \operatorname{Hom}_R(M, H_J^{d-1}(R))$;

(ii) $D_J^d(M) = \operatorname{Ext}_R^1(M, H_J^{d-1}(R))$.

Proof. Let $E^\bullet$ be an injective resolution of R. We note that $\Gamma_J(E^\bullet)$ is an injective resolution of $\ker \Gamma_J(d^{d-1}) = H_J^{d-1}(R)$. Then we have commutative diagrams:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Hom}_R(M, \ker \Gamma_J(d^{d-1})) & \longrightarrow & \operatorname{Hom}_R(M, \Gamma_J(E^{d-1})) & \longrightarrow & \operatorname{Hom}_R(M, \Gamma_J(E^d)) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \ker \operatorname{Hom}_R(1_M, \Gamma_J(d^{d-1})) & \longrightarrow & \operatorname{Hom}_R(M, \Gamma_J(E^{d-1})) & \longrightarrow & \operatorname{Hom}_R(M, \Gamma_J(E^d)),
\end{array}$$

and

$$\begin{array}{ccccccc}
\operatorname{Hom}_R(M, \Gamma_J(E^{d-1})) & \longrightarrow & \operatorname{Hom}_R(M, \Gamma_J(E^d)) & \longrightarrow & \operatorname{Ext}_R^1(M, \ker \Gamma_J(d^{d-1})) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\operatorname{Hom}_R(M, \Gamma_J(E^{d-1})) & \longrightarrow & \operatorname{Hom}_R(M, \Gamma_J(E^d)) & \longrightarrow & \operatorname{Coker} \operatorname{Hom}_R(1_M, \Gamma_J(d^{d-1})) & \longrightarrow & 0.
\end{array}$$

By chasing the diagrams, we have $D_J^{d-1}(M) = H^{d-1}(D_J(M)) = \operatorname{Hom}_R(M, H_J^{d-1}(R))$ and $D_J^d(M) = H^d(D_J(M)) = \operatorname{Ext}_R^1(M, H_J^{d-1}(R))$, since it holds that $H^{d-1}(D_J(M)) = \ker \mathrm{Hom}_R(1_M, \Gamma_J(d^{d-1}))$ and $H^d(D_J(M)) = \mathrm{Coker}\ \mathrm{Hom}_R(1_M, \Gamma_J(d^{d-1}))$ by definition. The proof is completed. □

4. Proof of the main theorem. Now we prove the main theorem.

Proof of Theorem 1. Suppose that $N^\bullet$ $(\in \mathcal{D}^+(R))$ is a J-cofinite complex. Then there is a complex $M^\bullet \in \mathcal{D}^-_{\mathrm{ft}}(R)$ such that $N^\bullet \simeq D_J(M^\bullet)$ by definition. Now there is a spectral sequence:

$$E_2^{p,q} = H^p(D_J(H^{-q}(M^\bullet))) \Longrightarrow H^n = H^n(D_J(M^\bullet)),$$

which is obtained by the double complex $\mathrm{Hom}\,(M^\bullet, \Gamma_J(E^\bullet))$. Then there is a finite filtration:

$$(\star) \qquad H^n = H^n_0 \supset H^n_1 \supset H^n_2 \supset \cdots \supset H^n_n \supset H^n_{n+1} = 0,$$

satisfied with the following conditions for each $n \geq 0$: (a) For an integer s with $0 \leq s \leq n$, $H^n_s/H^n_{s+1} \simeq E_\infty^{s,n-s}$. (b) There is an integer $r \geq 2$ such that $E_r^{p,q} \simeq E_\infty^{p,q}$ for all integers $p, q \geq 0$. Since $E_2^{p,q} = 0$ for $p \neq d-1, d$ and all q, we have $E_r^{p,q} = 0$ for $p \neq d-1, d$ and all q and all $r \geq 2$. Hence all the differentials that come into and go out of $E_r^{d-1,n}$ and $E_r^{d,n}$ are zero for all $r \geq 2$. So we have, for all q, $E_r^{s,q} \simeq E_\infty^{s,q}$, which is zero for $s \neq d, d-1$. Further, $H^n_s/H^n_{s+1} \simeq E_\infty^{s,n-s} = 0$ for $s \neq d-1, d$, that is, $H^n_s = H^n_{s+1}$ for $s \neq d-1, d$.

Case $s = d-1$**.** Now we have isomorphisms $H^n_{d-1}/H^n_d \simeq E_\infty^{d-1,n-d+1} \simeq E_2^{d-1,n-d+1}$. Hence there is an exact sequence

$$0 \longrightarrow H^n_d \longrightarrow H^n_{d-1} \longrightarrow E_2^{d-1,n-d+1} \longrightarrow 0.$$

Further there are isomorphisms $H^n_{d-1} \simeq \cdots \simeq H^n_0 = H^n$. So we obtain the exact sequence

$$\text{(d-1)} \qquad 0 \longrightarrow H^n_d \longrightarrow H^n \longrightarrow E_2^{d-1,n-d+1} \longrightarrow 0.$$

Case $s = d$**.** Next we have that $H^n_d/H^n_{d+1} \simeq E_\infty^{d,n-d} \simeq E_2^{d,n-d}$. So there is an exact sequence

$$0 \longrightarrow H^n_{d+1} \longrightarrow H^n_d \longrightarrow E_2^{d,n-d} \longrightarrow 0.$$

Further there are isomorphisms $H^n_{d+1} \simeq \cdots \simeq H^n_n \simeq H^n_{n+1} = 0$. Hence we have the equality $H^n_d = E_2^{d,n-d}$. Therefore, we obtain the following exact sequence:

$$(\sharp) \qquad 0 \longrightarrow E_2^{d,n-d} \longrightarrow H^n \longrightarrow E_2^{d-1,n-d+1} \longrightarrow 0,$$

for each n, combining the sequences (d-1) with the above equality.

In order to prove that $H^n = H^n(D_J(M^\bullet)) \in \mathcal{M}(R,J)_{\mathrm{cof}}$, it suffices to show that $E_2^{p,q} = H^p(D_J(H^{-q}(M^\bullet))) \in \mathcal{M}(R,J)_{\mathrm{cof}}$ for all p and q by the short exact sequence ($\sharp$). We can consider a single R-module N as a complex $N^\bullet$ such that $N^0 = N$ and $N^i = 0$ if $i \neq 0$. Replacing $H^{-q}(M^\bullet)$ with M, we shall prove the theorem for the case where the complex $M^\bullet$ consists of a single module M of finite type over R, since $H^{-q}(M^\bullet)$ is an R-module of finite type by definition.

Now suppose that M is a finitely generated R-module. Then we have $D_J^{d-1}(M) = \mathrm{Hom}_R(M, H_J^{d-1}(R))$ and $D_J^d(M) = \mathrm{Ext}^1_R(M, H_J^{d-1}(R))$ by Lemma 6. It follows from part (iii) of Lemma 5 that R-modules $\mathrm{Hom}_R(M, H_J^{d-1}(R))$ and $\mathrm{Ext}^1_R(M, H_J^{d-1}(R))$ are J-cofinite. So $D_J^{d-1}(M)$ and $D_J^d(M)$ are in $\mathcal{M}(R,J)_{\mathrm{cof}}$. Therefore, $D_J^j(M)$ is in $\mathcal{M}(R,J)_{\mathrm{cof}}$ for all j.

Conversely, let $N^\bullet$ be an object of $\mathcal{D}^+(R)$, satisfying $H^i(N^\bullet) \in \mathcal{M}(R,J)_{\mathrm{cof}}$ for all i. Then $\mathrm{Ext}^j(R/J, H^q(N^\bullet))$ is of finite type for all j. From the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(R/J, H^q(N^\bullet)) \Longrightarrow H^{p+q} = \mathrm{Ext}^{p+q}(R/J, N^\bullet),$$

we deduce that the abutment terms H^n are also of finite type for all n by Lemma 3. Therefore it follows from [**4**, Theorem 5.1, page 154] that $N^\bullet$ is a cofinite complex. The proof of Theorem 1 is completed. □

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