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ON POSITIVE AFFINE MONOIDS

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ABSTRACT. For numerical monoids S the length of the $k[S]$-module $k[\overline{S}]/k[S]$ is always finite. This is of course because the set of holes $H(S)$ is finite, a property that does not hold in general for positive affine monoids of higher rank. We examine here in a combinatorial fashion positive affine monoids S with $H(S)$ finite, or equivalently, positive affine monoids for which the length of $k[\overline{S}]/k[S]$ is finite. This class of monoids turns out to behave in some respects like numerical monoids. In particular we describe the maximal elements in certain posets whose elements are positive affine monoids. This description provides natural "higher dimensional" versions of familiar classes of numerical monoids such as the class of symmetric numerical monoids.

1. Preliminaries and notations.

An affine monoid $S = \langle s_1, \dots , s_n\rangle$ is a finitely generated sub-monoid of $\mathbf{Z}^r$ for some $r \in \mathbf{N}$, $r \geq 1$. We denote by $\mathrm{gp}\,(S)$ the group inside $\mathbf{Z}^r$ generated by S. Observe that every element $x \in \mathrm{gp}\,(S)$ can be written as $x = s - s'$ for some elements s and s' in S and that $\mathrm{gp}\,(S)$ is free of rank at most r. The rank of S, $\mathrm{rank}\,(S)$, is by definition the rank of $\mathrm{gp}\,(S)$. We assume all affine monoids S are embedded in $\mathbf{Z}^d$ where $d = \mathrm{rank}\,(S)$.

Our main concern will be positive affine monoids: an affine monoid is called positive if zero is the only element whose inverse in $\mathrm{gp}\,(S)$ also lies in S. A positive affine monoid $S = \langle s_1, \dots , s_n\rangle$ of rank d is isomorphic to an affine monoid T inside $\mathbf{N}^d$. Thus in the sequel all positive affine monoids S will be considered to be inside $\mathbf{N}^d$ where $d = \mathrm{rank}\,(S)$.

Assume $S = \langle s_1, \dots , s_n\rangle$ is a positive affine monoid of rank one such that $\gcd\,(s_1, \dots , s_n) = 1$. Then S is called a numerical monoid.

Any affine (respectively positive affine) monoid $S = \langle s_1, \dots , s_n\rangle$ gives rise to a cone (respectively pointed cone)

$$\mathbf{R}_{\geq 0}S = \mathbf{R}_{\geq 0}\{s_1, \dots , s_n\} = \{\lambda_1 s_1 + \cdots + \lambda_n s_n \;\; \lambda_i \in \mathbf{R}_{\geq 0}\}.$$

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The dimension of $\mathbf{R}_{\geq 0}S$ is by definition the dimension of its linear span and equals $\operatorname{rank}(S)$. Recall that such a cone is the intersection of finitely many half-spaces in $\mathbf{R}^d$, where $d = \operatorname{rank}(S)$. Let H^+ be a half-space in $\mathbf{R}_{\geq 0}S$ with bounding hyperplane H. Assume H intersects $\mathbf{R}_{\geq 0}S$ and that $\mathbf{R}_{\geq 0}S$ lies entirely inside H^+. Then $F = H \cap \mathbf{R}_{\geq 0}S$ is called a face of $\mathbf{R}_{\geq 0}S$. The dimension of a face is by definition the dimension of its affine hull. A face of dimension $d-1$, where $d = \dim \mathbf{R}_{\geq 0}S$, is called a facet. The faces form a lattice under inclusion. See [**3**] for details.

If S is a positive affine monoid the set $S_+ = S \setminus \{0\}$ is called the maximal ideal of S.

The normalization of an affine monoid S, denoted $\overline{S}$, is the monoid

$$\overline{S} = \{x \in \operatorname{gp}(S);\ mx \in S \text{ for some } m \in \mathbf{N},\ m > 1\}.$$

We have $\overline{S} = \mathbf{R}_{\geq 0}S \cap \operatorname{gp}(S)$ and $\overline{S}$ is affine (respectively positive affine) when S is affine (respectively positive affine). The normalization of S is (by construction) a normal monoid. That is, a monoid T such that if $mx \in T$ for some $m \in \mathbf{N}$, $m > 1$, and $x \in \operatorname{gp}(T)$, then $x \in T$. An affine monoid is normal precisely when $S = \overline{S}$, see [**3**, Proposition 2.22]. Sometimes normal affine monoids are called integrally closed affine monoids. The terminology comes from commutative algebra, see Remark 1.3 below.

As for numerical monoids we define the set of gaps of an affine monoid S as $H(S) = \overline{S} \setminus S$. Also, we define a set $T(S)$ by

$$T(S) = \{x \in \operatorname{gp}(S);\ x \notin S,\ x + S_+ \subseteq S_+\}.$$

Remark 1.1. For numerical monoids the cardinality of the set $T(S)$ is called the type of S, denoted $\operatorname{type}(S)$, and agrees with the Cohen-Macaulay type of the corresponding monoid ring.

Assume S is an affine monoid. Then, by considering the elements $s \in S$ that lie in some bounding hyperplane of the cone $\mathbf{R}_{\geq 0}S$ and the affine form defining that hyperplane, we see that $T(S) \subseteq \mathbf{R}_{\geq 0}S$ and, in fact, $T(S) \subseteq \overline{S}$. Thus for affine monoids S we have $T(S) \subseteq H(S)$.

We associate to an affine monoid $S = \langle s_1, \ldots, s_n \rangle$ its monoid ring $k[S]$, k being a field. This is the k-algebra $k[\mathbf{t}^s;\ s \in S] \subseteq k[t_1, \ldots, t_d]$,

$d = \mathrm{rank}\,(S)$, with multiplication

$$\mathbf{t}^s \cdot \mathbf{t}^{s'} = \mathbf{t}^{s+s'}, \quad s \in S,\ s' \in S.$$

The dimension of $k[S]$ coincides with the rank of S. The ring $k[S]$ is naturally S-graded. That is, it is a graded ring with non-zero components only in degrees $s \in S$.

Two important classes of numerical monoids are the classes of symmetric and quasi-symmetric numerical monoids. These classes of numerical monoids are characterized by the fact that $T(S)$ consists of one, respectively two, elements.

If R is the polynomial ring $k[x_1, \dots, x_n]$ we can define a homomorphism

$$R \xrightarrow{\phi} k[S]$$

by $x_i \mapsto \mathbf{t}^{s_i}$. Since ϕ is surjective we have a kernel $\ker\phi = \mathfrak{p}$, and consequently an isomorphism $k[S] \cong R/\mathfrak{p}$. The ideal $\mathfrak{p}$ is a prime ideal generated by binomials.

Definition 1.2. A monoid ring $k[S] = k[\mathbf{t}^{s_1}, \dots, \mathbf{t}^{s_n}]$ corresponding to a positive affine monoid is called homogeneous if there is a vector $v \in \mathbf{Q}^d$ with

$$s_i \cdot v = 1$$

for all $i \in \{1, \dots, n\}$.

It is well known that $k[S]$ is homogeneous if and only if it is standard graded with respect to the grading

$$k[S]_i = \sum_{|b|=i} k\{(\mathbf{t}^{s_1})^{b_1} \cdots (\mathbf{t}^{s_n})^{b_n}\}, \tag{1}$$

where $|b| = b_1 + \cdots + b_n$ for any vector $b \in \mathbf{N}^n$. For details, see [**9**, Proposition 7.2.39].

Remark 1.3. Let S be an affine monoid. It is known that the integral closure $\overline{k[S]}$ of $k[S]$ in its field of fractions is precisely $k[\overline{S}]$. For details, see for example, [**8**, Proposition 7.25].

2. Iterated Frobenius numbers. Recall from the introduction that we assume all affine monoids S are embedded in $\mathbf{Z}^d$ where $d = \operatorname{rank}(S)$ is the rank of $\operatorname{gp}(S)$. Thus in this chapter whenever $\mathbf{N}^d$ or $\mathbf{Z}^d$ occur without further explanation, d is the rank of S. Also recall that d equals the dimension of the cone $\mathbf{R}_{\geq 0}S$ generated by S. Positive affine monoids S are assumed to be embedded in $\mathbf{N}^d$, $d = \operatorname{rank} v$.

In the sequel we will use the following partial ordering:

• If a and b are two elements in a monoid S we say that $a \leq_S b$ if and only if $a + s = b$ for some element $s \in S$.

Since $\mathbf{N}^d$ is a monoid we obtain as a special case the familiar ordering $\leq_{\mathbf{N}^d}$, where for any two elements $a = (a_1, \dots, a_d)$ and $b = (b_1, \dots, b_d)$ we have $a \leq_{\mathbf{N}^d} b$ if and only if $a_i \leq b_i$ holds for every $i \in \{1, \dots, d\}$. Note that if a and b are elements of a positive affine monoid S and $a \leq_S b$, then $a \leq_{\mathbf{N}^d} b$.

Given an affine monoid S we defined in the introduction a set $T(S)$ by

$$T(S) = \{x \in \operatorname{gp}(S);\ x \notin S,\ x + S_+ \subseteq S_+\}.$$

We may deduce that $T(S)$ is finite.

Lemma 2.1. *Let $S = \langle s_1, \dots, s_n \rangle$ be an affine monoid. Then $|T(S)| < \infty$.*

Proof. Let s be any non-zero element of S and consider the S-graded ideal $(\mathbf{t}^{s+u};\ u \in T(S))$ of $\subseteq k[S]$. That $T(S)$ is finite follows since $k[S]$ is Noetherian and the minimal generators of the ideal $(\mathbf{t}^{s+u};\ u \in T(S))$ are in one-to-one correspondence with the elements in $T(S)$. The last fact follows easily since if $u_1 \in T(S)$ and $u_2 \in T(S)$, then $u_1 - u_2 \notin S$. □

Remark 2.2. In the case of numerical monoids the above lemma yields the fact (see Remark 1.1) that $|T(S)|$ equals the Cohen-Macaulay type of $k[S]$. See [**4**] for details.

Example 1. We note that S being finitely generated need not imply $H(S)$ finite. Let S be the sub-monoid of $\mathbf{N}^2$ generated by the elements

$$\{(0,2), (1,0), (1,1)\}.$$

Here $T(S)$ is the empty set but $H(S)$ consists of all points $(0, 2k+1)$, $k \in \mathbf{N}$.

Remark 2.3. For any subsets A and B of $\mathbf{Z}^d$, we denote by $A - B$ the set of differences $\{a - b \ \ a \in A,\ b \in B\}$. If the set A consists of only one element, a say, we write $a - B$ instead of $A - B$.

Let S be an affine monoid, and denote by $T_0(S) = \{h_{0,1}, \dots, h_{0,r_0}\}$ the set of maximal elements in $T(S)$ with respect to the partial order $\leq_{\mathbf{N}^d}$. We define sets $T_i(S)$ recursively as follows: assuming we have already defined $T_j(S)$, $j \in \{0, 1, \dots, i-1\}$, we define $T_i(S)$ to consist of the elements $x \in \mathbf{R}_{\geq 0} S \cap \mathrm{gp}\,(S)$ that are maximal relative $\leq_{\mathbf{N}^d}$ with the properties

- $x \notin S$.
- $x \notin T_j(S) - S$, $j \in \{0, \dots, i-1\}$.

Remark 2.4. If $T_i(S) = \varnothing$ for some number i, then $T_j(S) = \varnothing$ for all $j > i$ as well. This follows readily from the definition of the sets $T_i(S)$.

Remark 2.5. Let S be an affine monoid of rank d. A finite subset $T_0 \subseteq \mathbf{N}^d$ can satisfy $T_0 = T_0(S)$ only if T_0 is an anti-chain in the poset $(\mathbf{Z}^d, \leq_{\mathbf{N}^d})$.

Example 2. Let S be the sub-monoid of $\mathbf{N}^2$ generated by the elements

$$\{(1, k);\ k \in \mathbf{N}\}.$$

Then $T(S)$ consists of all integer points on the y-axis so $|T(S)| = \infty$ but $T_0(S) = \varnothing$. According to Remark 2.5, $T_i(S) = \varnothing$ for all $i \geq 0$.

Definition 2.6. The elements in the set $\cup_{i \geq 0} T_i(S)$ are called the iterated Frobenius numbers of S.

We now display an important property of the iterated Frobenius numbers.

Proposition 2.7. *Let S be a positive affine monoid with $H(S)$ finite. Then*
$$T(S) = \bigcup_{j\geq 0} T_j(S).$$

Proof. The elements of $T_0(S)$ belong to $T(S)$ by definition. Assume all elements in $\cup_{j=0}^{i-1} T_j(S)$ belong to $T(S)$ and consider an arbitrary iterated Frobenius number $h_i \in T_i(S)$. If $h_i \notin T(S)$ there is an element $s \in S_+$ such that $h_i + s \notin S_+$. By considering the maximal property defining h_i we conclude that $h_i + s \in T_\alpha(S) - S$ for some $\alpha \in \{0, \ldots, i-1\}$. This however yields a contradiction since if $h_\alpha \in T_\alpha(S)$ and $s' \in S$ we have
$$h_i + s = h_\alpha - s'$$
$$\Longleftrightarrow$$
$$h_i = h_\alpha - (s' + s) \in h_\alpha - S.$$
Thus $\cup_{i\geq 0} T_i(S) \subseteq T(S)$.

By the first part of the proof $T_i(S)$ can be non-empty only for a finite number of integers i. Assume $T_j(S) = \varnothing$ for $j > i$. If $T(S) \neq \cup_{j=0}^{i} T_j(S)$ the finite set
$$\left\{x \in \mathbf{R}_{\geq 0} S \cap \operatorname{gp}(S);\ x \notin S,\ x \notin \bigcup_{j\geq 0} T_j(S) - S\right\}$$
is non-empty. This however would imply $T_{i+1}(S) \neq \varnothing$, which is a contradiction. Thus $T(S) \subseteq \cup_{j\geq 0} T_j(S)$ and we are done. □

Corollary 2.8. *Let S be a positive affine monoid, and assume $h_i \in T_i(S)$ and $h_k \in T_k(S)$. Then either $h_i + h_k \in S$ or $h_i + h_k \in T_r(S)$ where $0 \leq r < \min\{i, k\}$.*

Proof. Assume $h_i + h_k \notin S$, and let $s \in S_+$. Then
$$h_i + h_k + s = h_i + (h_k + s) = h_i + s' = s''$$
where s' and s'' belong to S. Hence $h_i + h_k \in T(S)$ so $h_i + h_k \in T_r(S)$ for some $r \geq 0$. It follows from the definition of the iterated Frobenius numbers that $0 \leq r < \min\{i, k\}$. □

Remark 2.9. The property in Corollary 2.8 of the iterated Frobenius numbers will be used many times in the sequel.

Remark 2.10. For numerical monoids S the elements of the set $T(S)$ are known as Pseudo-Frobenius numbers, see [**6**]. However, due to the above proposition, the name *iterated Frobenius numbers* is motivated.

The following lemma describes for positive affine monoids $T(S)$ as a subset of $H(S)$. For numerical monoids the lemma is part of Proposition 1.19 in [**6**].

Lemma 2.11. *Let S be a positive affine monoid. Then we have the following*:

(i) *$T(S)$ consists of the elements of $H(S)$ that are maximal with respect to the partial order $\leq_S$.*

(ii) *$T_0(S)$ consists of the elements of $H(S)$ that are maximal with respect to the partial order $\leq_{\mathbf{N}^d}$.*

Proof. The elements of $H(S)$ that are maximal with respect to the partial order $\leq_S$ are precisely the elements that are characterized by the fact that $x+s \in S$ for any $s \in S_+$. This proves (i). The second assertion follows from the definition of $T_0(S)$ and the fact that being maximal with respect to $\leq_{\mathbf{N}^d}$ implies being maximal with respect to $\leq_S$. □

Proposition 2.12. *Let S be a positive affine monoid. Then the following are equivalent*:

(i) *$H(S)$ is finite.*

(ii) $H(S) = (T(S) - S) \cap \mathbf{N}^d$.

(iii) *If $x \in H(S)$ there is an element $s \in S$ such that $x + s \in T(S)$.*

Proof. The fact that (i) and (iii) are equivalent follows from Lemma 2.11 and Proposition 2.7. (iii) clearly implies (ii) and (ii) implies (i) since in this case $H(S)$ lies in a bounded region of $\mathbf{N}^d$. □

Consider a positive affine monoid S with $H(S)$ finite, and let x be an arbitrary non-zero element in $-\overline{S} = \{x \in \mathbf{Z}^d;\ -x \in \overline{S}\}$. Since $x \in \mathrm{gp}\,(S)$ we have $x = s - s'$ for some elements s and $s(S)$ in S. Then for any $h_0 \in T_0(S)$

(2) $0 \le_{\mathbf{N}^d} h_0 \le_{\mathbf{N}^d} h_0 - x = h_0 - (s - s') = h_0 + s' - s = s'' - s \in \mathrm{gp}\,(S).$

Now, since $H(S)$ is finite we have

$$h_0 \in T_0(S), h_0 <_{\mathbf{N}^d} y, y \in \overline{S} \Longrightarrow y \in S, \tag{3}$$

and so $h_0 - x \in S$ by (2). This proves

Corollary 2.13. *Let S be a positive affine monoid such that $H(S)$ is finite. Then*

$$-\overline{S} \subseteq \bigcap_{h_0 \in T_0(S)} (h_0 - S). \tag{4}$$

Remark 2.14. The corollary provides a generalization of the fact that all negative integers are in $g - S$ when S is a numerical monoid with Frobenius number g. Also, we may view (3) as generalizing the fact that every integer greater than the Frobenius number lies in S when S is a numerical monoid.

It is well known that numerical monoids have Cohen-Macaulay monoid rings, a property that does not hold in general for positive affine monoids, in particular not if $H(S)$ is finite non-empty and $\mathrm{rank}\,(S) \ge 2$. Indeed, Hoa and Trung have characterized the positive affine monoids that have Cohen-Macaulay monoid rings, see Theorem 2.15 below. We review the notions that are used in that theorem:

Let S be a positive affine monoid and denote by F_i, $i \in \{1, \dots, m\}$, the set of facets of the cone $\mathbf{R}_{\ge 0}S$. Put

$$S_i = \{x \in \mathrm{gp}\,(S);\ x + s \in S,\ \text{for some } s \in S \cap F_i\},$$

and $S' = \cap_{i=1}^m S_i$. Furthermore, for every subset $J \subseteq \{1, \dots, m\}$ we put

$$G_J = \bigcap_{i \notin J} S_i \setminus \bigcup_{j \in J} S_j.$$

Finally, we let π_J be the abstract simplicial complex consisting of the non-empty subsets $I \subseteq J$ for which

$$\bigcap_{i \in I} S \cap F_i \neq \{0\}.$$

Theorem 2.15 (Hoa and Trung [**7**]). *The monoid ring of a positive affine monoid S is Cohen-Macaulay if and only if*

- $S = S'$, *and*

- *for every non-empty subset $J \subseteq \{1, \dots, m\}$, either G_J is empty or else the chain complex of π_J has zero reduced homology, that is, π_J is acyclic.*

Remark 2.16. Let S be a numerical monoid. Then $\mathbf{R}_{\geq 0}S = \mathbf{R}_{\geq 0}$, so the only facet is $\{0\}$. Then $S' = S_1 = S$ so the first condition is satisfied. The second condition is trivially satisfied since there are no non-empty proper subsets of $\{1\}$. Thus follows the well-known fact that all numerical monoids have Cohen-Macaulay monoid rings.

Remark 2.17. Assume S is a positive affine monoid and $k[S]$ is not Cohen-Macaulay. Then $\operatorname{rank}(S) \geq 2$ and, as one easily sees, $T(S) \subseteq S'$.

The following results, Proposition 2.19, Corollary 2.20 and Corollary 2.21, are easy to come by in a purely algebraic way since the length of the $k[S]$-module $k[\overline{S}]/k[S]$ is finite if $H(S)$ is finite. However, we prove them here using our combinatorial tools and the following lemma.

Lemma 2.18. *Let S be a positive affine monoids with* $\operatorname{rank}(S) \geq 2$, *and let F be a facet of $\mathbf{R}_{\geq 0}S$. Then there is a non-zero element $s \in S \cap F$.*

Proof. Assume S is minimally generated by $\{s_1, \dots, s_n\}$, and let R_i, $i \in \{1, \dots, t\}$ be the set of one dimensional faces of $\mathbf{R}_{\geq 0}S$. Let x_i be any non-zero element in R_i. By [**3**, Proposition 1.20] the finite set of elements $\{x_i\}_{i=1}^t$ is, up to scalar multiples, the unique set of minimal generators of $\mathbf{R}_{\geq 0}S$. Now, since $\{s_1, \dots, s_n\}$ generate $\mathbf{R}_{\geq 0}S$ we conclude that the elements x_i can be chosen from S. $\square$

Example 3. The condition that S is finitely generated in the lemma is crucial. Consider the monoid

$$S = \langle (n, m);\ n \geq 1,\ m \geq 1 \rangle \subseteq \mathbf{N}^2.$$

The facets of $\mathbf{R}_{\geq 0}S$ are the coordinate axes. Hence no non-zero element of S can lie in a facet.

Lemma 2.18 lets us prove the following

Proposition 2.19. *Let S be a positive affine monoid with* $\operatorname{rank}(S) \geq 2$ *and assume $H(S)$ is finite. Then $S' = \overline{S}$.*

Proof. Let $\{F_1, \dots, F_k\}$ be the facets of $\mathbf{R}_{\geq 0}S$, and let $\{w_1, \dots, w_k\}$ be corresponding inner normal vectors. If $x \in S'$ let $s_i \in S \cap F_i$, $i \in \{1, \dots, k\}$, be such that $x + s_i \in S$. Since $w_i \cdot s_i = 0$ we see that $w_i \cdot x \geq 0$ for $i \in \{1, \dots, k\}$ so x lies in $\mathbf{R}_{\geq 0}S$. But since $x + s_i \in S$ it follows that $x \in \mathbf{R}_{\geq 0}S \cap \operatorname{gp}(S) = \overline{S}$ so $S' \subseteq \overline{S}$.

Consider an element $x \in H(S)$. By Proposition 2.12 $x = h - s$ for some elements $h \in T(S)$ and $s \in S$. Let s' be an element in $S \cap F_k$ for some facet F_k. Such an element s' exists by the lemma. Now, $h - s + s' = (h + s') - s = s'' - s \in \operatorname{gp}(S)$. If $s'' - s$ does not already lie in S we substitute s' with ns', $n \in \mathbf{N}$ being a large integer. Then, since $H(S)$ is finite, we conclude that $x = h - s \in S_k$. Since we only have a finite number of facets, it follows that $H(S) \subseteq S'$ so $S' = \overline{S}$. $\square$

Corollary 2.20. *Let S be a positive affine monoid with* $\operatorname{rank}(S) \geq 2$ *such that $k[S]$ is Cohen-Macaulay. Then $T(S) = \varnothing$. If in addition $H(S)$ is finite, then $H(S) = \varnothing$.*

Proof. Since $k[S]$ is Cohen-Macaulay $S = S'$ by Theorem 2.15. If $x \in T(S)$ we have $x + s \in S$ for all $s \in S_+$, in particular if $s \in S \cap F_i$ for some facet F_i of $\mathbf{R}_{\geq 0}S$. Hence $T(S) \subseteq S'$ so $T(S)$ must be empty. The last claim follows from Lemma 2.11. $\square$

In particular, if S is as in Corollary 2.20, rank $(S) = d$, and $H(S)$ is infinite, then there are no $\leq_{\mathbf{N}^d}$-maximal elements in $H(S)$.

Hochster proved ([**3**, Theorem 6.10]) that normal affine monoids have Cohen-Macaulay monoid rings. Using this we get

Corollary 2.21. *Let S be a positive affine monoid with* rank $(S) \geq 2$ *such that $k[S]$ is Cohen-Macaulay but not normal. Then $H(S)$ is infinite.*

Let S be a positive affine monoid. If $H(S)$ is not finite we would like to construct a positive affine monoid $\widetilde{S}$, $S \subseteq \widetilde{S} \subseteq \overline{S}$, such that $H(\widetilde{S})$ is finite and $T(\widetilde{S}) = T(S)$. It is however not clear how to proceed to obtain this. The last couple of results in this section, Proposition 2.24 and Proposition 2.25, provide a "partial answer" to this problem.

If S is a positive affine monoid with $H(S)$ infinite we can always embed S in a positive affine monoid $\widetilde{S}$ with $H(\widetilde{S})$ finite; just take $\widetilde{S} = \overline{S}$. Observe however that $H(\overline{S}) = \varnothing$ so this choice of $\widetilde{S}$ is no good if $T(S) \neq \varnothing$ since we wish for $T(S)$ to at least be a subset of $T(\widetilde{S})$. If $T(S)$ is non-empty we may however construct $\widetilde{S}$ so that $\widetilde{S} \subset \overline{S}$.

Lemma 2.22. *Let S be a positive affine monoid with $H(S)$ infinite and $T(S)$ non-empty. Then there exists a positive affine monoid $\widetilde{S}$ with $H(\widetilde{S})$ finite such that $S \subset \widetilde{S} \subset \overline{S}$.*

Proof. For any element $x = (x_1, \dots, x_d) \in \overline{S}$, let $|x| = x_1 + \cdots + x_d$. Put

$$a = 1 + \max\{|h|;\ h \in T(S)\}$$

and denote by H^+ the positive half-space

$$H^+ = \{x \in \mathbf{R}^d;\ x_1 + \cdots + x_d \geq a\}.$$

Then $P = \mathbf{R}_{\geq 0} S \cap H^+$ is a polyhedron. Also, by construction, the intersection

$$Q = \mathbf{R}_{\geq 0} S \cap H$$

of the cone $\mathbf{R}_{\geq 0} S$ and the bounding hyperplane H of H^+, is a convex polytope. By [**3**, Proposition 1.28], P is the Minkowski sum

$$P = Q + \mathbf{R}_{\geq 0} S. \tag{5}$$

We claim that $P \cap \mathrm{gp}\,(S)$ is finitely generated, by which we mean that all $x \in P \cap \mathrm{gp}\,(S)$ are positive integer combinations of a finite set of vectors. To prove this, assume $Q = \mathrm{conv}\,\{b_1, \dots, b_t\}$. By (5), an element $x \in P \cap \mathrm{gp}\,(S)$ may be written as

$$x = \alpha_1 b_1 + \cdots + \alpha_t b_t + \beta_1 s_1 + \cdots + \beta_n s_n \tag{6}$$

where $\sum_{0 \le \alpha_i \le 1} \alpha_i = 1$ and $0 \le \beta_j$ for all $i \in \{1, \dots, t\}$ and $j \in \{1, \dots, n\}$. Put $\widetilde{S} = \langle S, B \rangle$ where B is the set of elements $x \in P \cap \mathrm{gp}\,(S)$ as in (6) with $0 \le \beta_j < 1$ for all $j \in \{1, \dots, n\}$. Clearly B is a bounded set and thus B is finite. Hence $\widetilde{S}$ is positive affine. □

Lemma 2.23. *Let S be a positive affine monoid, and let $x \in H(S)$. Then $S \cup \{x\}$ is a positive affine monoid if and only if $2x \in S$ and $x \in T(S)$.*

Proof. If $2x \in S$ and $x \in T(S)$ clearly $S \cup \{x\} = \langle S, x \rangle$ so in this case $S \cup \{x\}$ is a positive affine monoid. On the other hand, if $S \cup \{x\}$ is a positive affine monoid then, since $2x \neq x$, $2x$ must belong to S and x must belong to $T(S)$. □

Proposition 2.24. *Let S be a positive affine monoid with $H(S)$ infinite. Then there is a positive affine monoid $\widetilde{S}$ such that*

(i) $S \subseteq \widetilde{S} \subseteq \overline{S}$.

(ii) $H(\widetilde{S})$ *is finite.*

(iii) $T(S) \subseteq T(\widetilde{S})$.

(iv) $T_0(S) = T_0(\widetilde{S})$.

Proof. If $T(S) = \varnothing$ we choose $\widetilde{S} = \overline{S}$ and then $\widetilde{S}$ trivially satisfies the conditions (i)–(iv). Thus assume $T(S)$ is non-empty. We define a set S_0 by

$$S_0 = S \cup \{x \in \overline{S};\ x \not\leq_{\mathbf{N}^d} h \in T(S)\}.$$

Adding any two elements from S_0 yields a new element in S_0, so S_0 is a sub-monoid of $\mathbf{N}^d$ containing S. Also, S_0 may differ from the positive affine monoid $\widetilde{S}$ constructed in Lemma 2.22 only by a finite number of elements: all elements $x \in P \cap \operatorname{gp}(S)$ from Lemma 2.22 lie in S_0 since all such x satisfy $x \not\leq_{\mathbf{N}^d} h \in T(S)$. Thus S_0 is positive affine with $H(S_0)$ finite. It is easy to see that $T(S) \subseteq T(S_0)$. If there is an element $h \in T_0(S_0) \setminus T(S)$ we put $S_1 = S_0 \cup \{h\}$. Since $2h \in S_0$ and $h \in T(S_0)$ by Lemma 2.23 this is again a positive affine monoid with $H(S_1)$ finite and by (3) we see that $T(S) \subseteq T(S_1)$. In this way we obtain in a finite number of steps a positive affine monoid S_k with $H(S_k)$ finite and $T(S) \subseteq T(S_k)$ and $T_0(S_k) \subseteq T(S)$. In fact, since the elements in $T_0(S_k)$ are $\mathbf{N}^d$-maximal in $H(S_k)$, we see that $T_0(S_k) \subseteq T_0(S)$. Then $T_0(S_k) = T_0(S)$ since $T(S) \subseteq T(S_k)$. Put $\widetilde{S} = S_k$ and we are done. □

Proposition 2.25. *Let S be a positive affine monoid with $H(S)$ finite, and let $M \subseteq T_i(S)$ for some $i \geq 0$. Then there exists a positive affine monoid $\widetilde{S}$ such that $S \subseteq \widetilde{S} \subseteq \overline{S}$ and $T_0(\widetilde{S}) = M$.*

Proof. If $M = \varnothing$ put $\widetilde{S} = \overline{S}$ and we are done. Assume $M \neq \varnothing$. Then by Lemma 2.23 and Corollary 2.8 we see that

$$S_0 = \Big(\bigcup_{j<i} T_j(S) \Big) \cup S \cup \big(T_i(S) \setminus M\big)$$

is a positive affine monoid and $S \subseteq S_0$. By Corollary 2.8 it follows that $M \subseteq T(S_0)$. Put $B_0 = T_0(S_0) \setminus M$ and $S_1 = \langle S_0, B_0 \rangle$. Again S_1 is a positive affine monoid. If $m \in M$ by (3) we see that $\{m + b;\ b \in B_0\} \subseteq S_0$ and we already know that $M \subseteq T(S_0)$. Thus, $M \subseteq T(S_1)$ and clearly $S \subset S_0 \subset S_1$. Since $H(S)$ is finite we obtain in a finite number of steps a positive affine monoid S_k such that $S \subset S_k$ and $M = T_0(S_k)$. This in fact gives $S \subset S_k \subset \bar{S}$ and, by putting $\widetilde{S} = S_k$, we are done. □

Question 1. Let S be a positive affine monoid and assume $H(S)$ is not finite. Is there always a positive affine monoid $\widetilde{S}$, $S \subseteq \widetilde{S} \subseteq \overline{S}$, such that $H(\widetilde{S})$ is finite and $T(\widetilde{S}) = T(S)$.

Remark 2.26. Using the notation introduced in the next section, given a positive affine monoid S with $H(S)$ infinite, we ask for a positive affine monoid $\widetilde{S} \in \mathcal{S}_{T_0(S)}$ with $T(\widetilde{S}) = T(S)$.

3. Maximal objects in the poset $\mathcal{S}_{T_0}$. For symmetric (respectively quasi-symmetric, depending on the parity of g) numerical monoids one has that for all $x \in \mathbf{Z}$ either $x \in S$ or else $g - x \in S$ (respectively $x \in S$ or, $g - x \in S$, or $x = g/2$), where g is the Frobenius number of S. These two particular classes of numerical monoids are also characterized by the following fact (see [**1, 5**]): a numerical monoid S is maximal (with respect to inclusion) among the numerical monoids with fixed Frobenius number $g = g(S)$ if and only if it is symmetric (respectively quasi-symmetric). The following lemma lets us prove a similar result, Theorem 3.3, for positive affine monoids.

Lemma 3.1. *Let S be a positive affine monoid, and assume $H(S)$ is finite. For any integer $a > 1$ and any $h_i \in T_i(S)$ we have*

$$a^i h_i \in S \Longleftrightarrow a^i h_i \notin T_0(S).$$

Proof. The "only if" part is clear and the result holds for $i = 0$. Assume the result holds for $j \in \{1, \dots, i-1\}$, and consider an element $h_i \in T_i(S)$. If $a^i h_i \notin T_0(S)$ and $a^i h_i \notin S$, then $ah_i \notin S$. But $h_i <_{\mathbf{N}^d} ah_i$ so $ah_i \in h_k - S$ for some iterated Frobenius number h_k with $k < i$. But $ah_i = h_k - s$ implies that $s = 0$, so $ah_i = h_{k,l}$. Now, by the induction hypothesis, either $a^k h_k \in S$ or else $a^k h_k \in T_0(S)$. In either case, the equation

$$a^i h_i = a^{i-k-1} a^k h_k$$

yields a contradiction. □

Note in particular that it follows from the lemma that $2^{i+1} h_i \in S$ for all i and all $h_i \in T_i(S)$.

Lemma 3.2. *Let S be a positive affine monoid, and assume $h_i \in T_i(S)$, $i > 0$, is such that $2^i h_i \in T_0(S)$. Then $i = 1$.*

Proof. By Corollary 2.8 we see that all multiples $kh_i, 2 \leq k \leq 2^i - 1$ are iterated Frobenius numbers. Thus, we must have $i + 1 = 2^i$ which can only hold for $i = 0$ and $i = 1$. □

Let T_0 be a finite non-empty set of vector in $\mathbf{N}^d$ and denote by $\mathcal{S}_{T_0}$ the set of positive affine monoids S of rank d with $H(S)$ finite and $T_0(S) = T_0$. $\mathcal{S}_{T_0}$ is a partially ordered set with respect to inclusion. Let S be a non-maximal element in $\mathcal{S}_{T_0}$. According to the proof of the following theorem, a monoid $\widehat{S}$, $S \subset \widehat{S} \subseteq \overline{S}$, with $T_0(S) = T_0 = T_0(\widehat{S})$ can be constructed.

Theorem 3.3. *Let S be a positive affine monoid. Assume $H(S)$ is finite and put $T_0(S) = T_0$. Then S is maximal in $\mathcal{S}_{T_0}$ if and only if $2^i h_i \in T_0$ for all i and all $h_i \in T_i(S)$. In particular, if S is maximal in $\mathcal{S}_{T_0}$, then $T(S) = T_0(S) \cup T_1(S)$.*

Proof. Assume S is such that $2^i h_i \in T_0$ for all i and all $h_i \in T_i(S)$, and pick an element $b \notin S$, $b \in \mathrm{gp}\,(S)$. Then by Proposition 2.12 $b = h_i - s$ for some element $s \in S$ and some iterated Frobenius number h_i, and thus $h_i \in \langle S, b\rangle$. But, then $2^i h_i \in \langle S, b\rangle$ so $T_0(S) \not\subseteq T_0(\langle S, b\rangle)$ so S is maximal in $\mathcal{S}_{T_0}$.

Now assume S does not have the property that $2^i h_i \in T_0$ for all i and all $h_i \in T_i(S)$. Then, by Lemma 3.1 (with $a = 2$), there is an element $h_i \in T(S)$ with $2^i h_i \in S$. Assume $i = \min\{k \in \mathbf{N};\ \text{exists } h_k \in T_k(S), 2^k h_k \in S\}$. Note that this implies $i > 0$. Also, put $b = \min\{k \in \mathbf{N};\ kh_i \in S\}$. If $h_0 \in \langle S, h_i\rangle$ for some $h_0 \in T_0(S)$ we have $h_0 = s + nh_i$, $n \in \mathbf{N}$. Then clearly $s = 0$ so $h_0 = nh_i$. Here we must have $n > 1$ and thus $2 \le n < b \le 2^i$. Observe that this implies $3 \le b$ and $2 \le i$. Since $(b-1)h_i \notin S$, by Lemma 2.8 we have $(b-1)h_i \in T(S)$, so $(b-1)h_i = h_r \in T_r(S)$ where $0 \le r < i$. From the equation

$$bh_i <_{\mathbf{N}^d} 2(b-1)h_i = 2h_r,$$

we conclude that $2h_r \in S$. This implies $r = 0$ since $r \ge 1$ would contradict the minimality of i. Also, we may conclude that $b = 3$: considering Lemma 2.8, $b > 3$ implies that $(b-2)h_i = h_s \in T_s(S)$ for some $s \ge 1$ and that $b - 1 < 2(b-2)$. But, then

$$(b-1)h_i = h_0 <_{\mathbf{N}^d} 2(b-2)h_i = 2h_s \in S,$$

which contradicts the minimality of i.

In summary we know that $i \ge 2$, $2h_i = h_0 \in T_0(S)$, and that $3h_i \in S$. In particular this implies $h_i \notin S$ and $h_i \notin T_0(S) - S$. Since $i \ge 2$, h_i is not $\mathbf{N}^d$-maximal with the properties $h_i \notin S$ and $h_i \notin T_0(S) - S$. Thus there is a non-zero element $y \in \mathbf{N}^d$ such that $h_i + y = h_1 \in T_1(S)$. Then $2(h_i + y) = 2h_1$, but $2(h_i + y) = h_0 + 2y \in S$ so $2h_1 \in S$. This again contradicts the minimality of i.

We conclude that $S \subset \langle S, h_i \rangle$ and $T_0(S) \subseteq T_0(\langle S, h_i \rangle)$. If $T_0(S) \neq T_0(\langle S, h_i \rangle)$ we may use the same procedure as in Proposition 2.5 and produce a positive affine monoid $\widetilde{S}$ with $S \subset \widetilde{S}$ and $T_0(S) = T_0(\widetilde{S})$. Thus S is not maximal in $\mathcal{S}_{T_0}$. The fact that $T(S) = T_0(S) \cup T_1(S)$ if S is maximal in $\mathcal{S}_{T_0}$ now follows from Lemma 3.2. □

Remark 3.4. The theorem should be compared to the situation for numerical monoids: let S be a numerical monoid. If $g = g(S)$ is odd, then S is maximal in $\mathcal{S}_g$ if and only if $T(S) = \{g\}$. If g is even, then S is maximal in $\mathcal{S}_g$ if and only if $T(S) = \{(g/2), g\}$.

Corollary 3.5. *Assume S is a positive affine monoid such that $H(S)$ is finite and $T(S) = T_0(S)$. Then S is maximal in $\mathcal{S}_{T_0}$ and provides a generalization of a symmetric numerical monoid.*

Corollary 3.6. *Let S be a positive affine monoid. Assume $H(S)$ is finite and that, for all $h_i \in T_i(S)$, there exist positive integers a_i such that $a_i h_i \in T_0(S)$. Then S is maximal in $\mathcal{S}_{T_0}$.*

Proof. Pick an element $b \in H(S)$. Then $b \in h_i - S$ for some iterated Frobenius number h_i. Thus $b = h_i - s$ for some $s \in S$, and it follows that $h_i \in \langle S, b \rangle$. Hence, since $a_i h_i = h_0 \in T_0(S)$, $T_0(S) \not\subseteq T_0(\langle S, b \rangle)$ so S is maximal in $\mathcal{S}_{T_0}$. □

Corollary 3.7. *Let S be a positive affine monoid, and assume $H(S)$ is finite. If for all $h_0 \in T_0(S)$ the coordinates of the vector h_0 have no common divisor that is even, then either S is not maximal in $\mathcal{S}_{T_0(S)}$ or else $T(S) = T_0(S)$.*

Example 4. Let S_1 and S_2 be two numerical monoids. Put S_1 on the positive x-axis and S_2 on the positive y-axis in $\mathbf{N}^2$ and fill in all integer points in the interior of $\mathbf{N}^2$. This gives a positive affine monoid

S with

$$T(S) = \{(h,0),(0,h');\ h \in T(S_1),\ h' \in T(S_2)\}.$$

Proposition 3.8. *Let S be a positive affine monoid. Then the following conditions on S are equivalent.*

(i) $S \cup \big((T_0(S) - S)) \cap \mathbf{N}^d\big) = \overline{S} \setminus \cup_{q \geq 1} T_q(S)$.

(ii) *For all $x \in \overline{S} \cup -\overline{S}$ it holds that $x \in S$ or $x \in h_0 - S$ for some $h_0 \in T_0(S)$ or $x \in T(S) \setminus T_0(S)$.*

Proof. Taking into account Corollary 2.13 the two statements are merely reformulations of each other. □

Definition 3.9. A positive affine monoid as in Proposition 3.8 is called almost symmetric.

Note in particular that $H(S)$ is finite if S is almost symmetric.

Example 5. Let S be a numerical monoid and $k \in \mathbf{N}$ a positive integer. Let $S(k)$ be the positive affine monoid that consists of all integer points in $\mathbf{N}^2$ except points of the forms (h,i) and $(0,j)$, where $h \in H(S)$, $0 \leq i \leq k$ and $1 \leq j \leq k$. It is easy to see that

$$T(S(k)) = \{(h,i);\ h \in T(S)\, i \in \{0,1,\dots,k\}\}.$$

$S(k)$ is constructed by putting S on the positive x-axis of $\mathbf{N}^2$ and then placing k "copies" of S directly above S. By construction $S(k)$ is almost symmetric if S is.

Remark 3.10. One can of course do the construction of $S(k)$ by instead putting S in the positive y-axis and placing k "copies" of S directly to the right of the positive y-axis.

Example 6. Let S be a numerical monoid. We construct a positive affine monoid S_{diag} by letting S_{diag} consist of all points $(x,y) \in \mathbf{N}^2$ such that $x + y = s \in S$. One easily verifies that $T(S_{\text{diag}}) = \{(x,y) \in \mathbf{N}^2;\ x + y = h \in T(S)\}$. Again by construction S_{diag} is almost symmetric if S is.

Example 7. The following is an example of an almost symmetric monoid S that is not derived from a numerical monoid. Let S consist of all integer points inside $\mathbf{N}^2$ except the 13 points

$$(1,0),(0,1),(1,1),(2,0),(0,2),(2,1),(1,2),(3,0),(0,3),\\ (4,0),(0,4),(4,2),(2,4).$$

Since $\overline{S}=\mathbf{N}^2$ the above 13 points are precisely the elements of $H(S)$. Clearly the set of $\mathbf{N}^2$-maximal points inside $H(S)$ is $\{(4,2),(2,4)\}$, and hence $T_0(S)=\{(4,2),(2,4)\}$. A quick computation gives

$$T_0(S)-S=\{(1,0),(0,1),(1,1),(2,0),(0,2)\}.$$

One then easily verifies that $T_1(S)=\{(4,0),(0,4),(2,1),(1,2)\}$ and $T_2(S)=\{(3,0),(0,3)\}$. Thus $S\cup((T_0(S)-S))\cap\mathbf{N}^2)=\overline{S}\setminus\cup_{q\geq 1}T_q(S)$ so S is almost symmetric.

Proposition 3.11. *Let S be a positive affine monoid and assume $H(S)$ is finite. Put $T_1'(S)=\{x\in T_1(S);\ 2x\in T_0(S)\}$. Then S is maximal in $\mathcal{S}_{T_0(S)}$ if and only if for all $x\in\overline{S}\cup-\overline{S}$ it holds that either $x\in S$, or $x\in T_0(S)-S$, or $x\in T_1'(S)$. In particular, if S is maximal in $\mathcal{S}_{T_0(S)}$ then S is almost symmetric.*

Proof. Assume S is maximal in $\mathcal{S}_{T_0(S)}$. In order to prove that for all $x\in\overline{S}\cup-\overline{S}$ it holds that either $x\in S$, or $x\in T_0(S)-S$, or $x\in T_1'(S)$, by Proposition 2.12, Corollary 2.13 and Theorem 3.3 it is sufficient to show that every element h_1-s, $h_1\in T_1(S), s\in S_+$, can be written as h_0-s', where $h_0\in T_0(S)$ and $s'\in S$. Let h_1 be an arbitrary element in $T_1(S)$. Since S is maximal in $\mathcal{S}_{T_0(S)}$ we know that $2h_1=h_0$ for some $h_0\in T_0(S)$, so $h_1=h_0-h_1$. Pick any element $s\in S_+$. Then $h_1-s=h_0-(h_1+s)=h_0-s'$, $s'\in S$.

Now assume S is a positive affine monoid such that $H(S)$ is finite and assume that for all $x\in\overline{S}\cup-\overline{S}$ it holds that either $x\in S$, or $x\in T_0(S)-S$, or $x\in T_1'(S)$. Then

$$S\cup\bigl(T_0(S)-S))\cap\mathbf{N}^d\bigr)=\overline{S}\setminus T_1'(S)$$

and clearly S is maximal in $\mathcal{S}_{T_0(S)}$. $\square$

Remark 3.12. By Proposition 3.8 and Proposition 3.11 the class of almost symmetric monoids as defined above, naturally generalizes the class of almost symmetric numerical monoids, see [**2**] for details about almost symmetric numerical monoids.

Corollary 3.13. *Let S be an almost symmetric monoid. Then S is maximal in $\mathcal{S}_{T_0(S)}$ in the sense that for any monoid $S' \in \mathcal{S}_{T_0(S)}$ strictly containing S, we have $|T(S')| < |T(S)|$.*

Corollary 3.14. *Let S be an almost symmetric monoid with $T_0(S) = \{h_0\}$. Then an iterated Frobenius number h_i lies in $T(S) \setminus T_0(S)$ if and only if $h_0 - h_i \in T(S) \setminus T_0(S)$.*

Remark 3.15. If S is as in the corollary, then the elements of $\cup_{i\geq 1} T_i(S)$ occur in pairs. This fact is, in case of numerical monoids, observed already in [**5**].

Proof. We know that

$$S \cup \big((h_0 - S) \cap \mathbf{N}^d\big) = \overline{S} \setminus \bigcup_{q\geq 1} T_q(S).$$

Pick an element $x \in \cup_{q\geq 1} T_q(S)$. We see that $h_0 - x$ cannot belong to neither S nor to $h_0 - S$. Thus $h_0 - x \in \cup_{q\geq 1} T_q(S)$. □

Let $S \in \mathcal{S}_{T_0}$ be an almost symmetric monoid. We thus have

$$S \cup \big((T_0(S) - S) \cap \mathbf{N}^d\big) = \overline{S} \setminus \bigcup_{q\geq 1} T_q(S).$$

Assume S is not maximal in $\mathcal{S}_{T_0}$. Then there exists an element $h_i \in T_i(S)$ such that $\langle S, h_i\rangle \in \mathcal{S}_{T_0}$. It is natural to ask if $\langle S, h_i\rangle \in \mathcal{S}_{T_0}$ is almost symmetric. Since $\overline{S} = \overline{\langle S, h_i\rangle}$ we clearly have

$$\langle S, h_i\rangle \cup \big(T_0(S) - \langle S, h_i\rangle) \cap \mathbf{N}^d\big) = \overline{\langle S, h_i\rangle} \setminus \Big(\bigcup_{q\geq 1} T_q(S) \setminus T_{h_i}\Big), \tag{7}$$

where $T_{h_i} = \langle S, h_i\rangle \cup (T_0(S) - \langle S, h_i\rangle) \cap \mathbf{N}^d) \cap (\cup_{q\geq 1} T_q)$. It is not hard to see that

$$T_{h_i} = \{h_l - mh_i, mh_i\}_{h_l,m} \cap \Big(\bigcup_{q\geq 1} T_q(S)\Big),$$

where $h_l \in T_0(S)$ and $1 \leq m \leq m_i = \max\{k \in \mathbf{N};\ kh_i \notin S\}$.

Proposition 3.16. *Let $S \in \mathcal{S}_{T_0}$ be an almost symmetric monoid that is not maximal in $\mathcal{S}_{T_0}$, and assume $h_i \in T_i(S)$ is such that $\langle S, h_i\rangle \in \mathcal{S}_{T_0}$. Then $\langle S, h_i\rangle$ is almost symmetric precisely when*

$$\Big(\bigcup_{\alpha<i} T_\alpha(S) - mh_i\Big) \cap \Big(\bigcup_{q\geq 1} T_q(S) \setminus T_{h_i}\Big) = \varnothing,$$

where $1 \leq m \leq m_i = \max\{k \in \mathbf{N};\ kh_i \notin S\}$.

Proof. It follows from equation (7) that

$$T(\langle S, h_i\rangle) \setminus T_0(\langle S, h_i\rangle) \subseteq \bigcup_{q\geq 1} T_q(S) \setminus T_{h_i} \subseteq H(\langle S, h_i\rangle), \tag{8}$$

and $\langle S, h_i\rangle$ is almost symmetric precisely when the left inclusion is an equality. If the left inclusion is strict there is an element $h_k \in T_k(S)$ with

$$h_k \in \Big(\bigcup_{q\geq 1} T_q(S) \setminus T_{h_i}\Big) \setminus T(\langle S, h_i\rangle).$$

Since $h_k \in H(\langle S, h_i\rangle)\setminus T(\langle S, h_i\rangle)$ by Proposition 2.12 there is a non-zero element $s + mh_i \in \langle S, h_i\rangle$ such that $h_k + (s + mh_i) = h_\alpha \in T(\langle S, h_i\rangle)$. Clearly we must have $s = 0$ and $1 \leq m \leq m_i$, and $h_k + mh_i$ cannot lie in S. Thus, by Corollary 2.8, $h_\alpha \in T_\alpha(S)$ where $\alpha < \min\{i, k\}$. Hence, $h_k = h_\alpha - mh_i$ and

$$\Big(\bigcup_{\alpha<i} T_\alpha(S) - mh_i\Big) \cap \Big(\bigcup_{q\geq 1} T_q(S) \setminus T_{h_i}\Big) \neq \varnothing.$$

To prove the converse assume

$$h_k \in \Big(\bigcup_{\substack{\alpha<i\\ 1\leq m\leq m_i}} T_\alpha(S) - mh_i\Big) \cap \Big(\bigcup_{q\geq 1} T_q(S) \setminus T_{h_i}\Big).$$

Hence, h_k has the form $h_k = h_\alpha - mh_i$. If $h_k \in T(\langle S, h_i\rangle)$ in particular we have

$$h_k + mh_i = h_\alpha = s + kh_i \tag{9}$$

for some $s \in S$ and $k \in \mathbf{N}$. If $s = 0$, then $h_k = (k-m)h_i$. This yields a contradiction since then

$$(k-m)h_i \in S \quad \text{or} \quad (k-m)h_i \in T_{h_i} \quad \text{or} \quad (k-m)h_i \leq_{\mathbf{N}^d} 0.$$

On the other hand, if $s \neq 0$ it follows from (9) that $h_\alpha = s + kh_i \in S$ which is impossible since $h_\alpha \in T(S)$. Thus $h_k \notin T(\langle S, h_i\rangle)$ and the left inclusion in (8) is strict. □

Corollary 3.17. *Let $S \in \mathcal{S}_{T_0}$ be an almost symmetric monoid that is not maximal in $\mathcal{S}_{T_0}$. If there is an element $h_1 \in T_1(S)$ such that $\langle S, h_1\rangle \in \mathcal{S}_{T_0}$, then $\langle S, h_i\rangle$ is almost symmetric.*

Proof. This follows since any element

$$h_k \in \left(\bigcup_{q\geq 1} T_q(S) \setminus T_{h_1}\right) \setminus T(\langle S, h_1\rangle)$$

would by the proof of Proposition 3.16 lie in T_{h_1}, which is a contradiction. □

Corollary 3.18. *Let S be an almost symmetric numerical monoid that is not maximal in $\mathcal{S}_{g(S)}$. Then $\langle S, h_1\rangle$ is almost symmetric.*

Proof. The fact that S is not maximal in $\mathcal{S}_{g(S)}$ implies $(g/2) < h_1$. Thus $2h_1 \in S$ so $\langle S, h_1\rangle \in \mathcal{S}_{g(S)}$ and is almost symmetric by Corollary 3.17. □

4. Apéry sets. As for numerical monoids one may define the Apéry set of a positive affine monoid S with respect to any non-zero element $m \in S$:

$$\mathrm{Ap}\,(S, m) = \{x \in S;\ x - m \notin S\}.$$

For numerical monoids the following is Proposition 7 in [**5**].

Proposition 4.1. *Let S be a positive affine monoid and m a non-zero element of S. Then the following conditions on an element $t \in \mathbf{Z}^d$ are equivalent.*

(i) $t - m \in T(S)$.

(ii) *t is maximal in* $\mathrm{Ap}\,(S,m)$ *with respect to the partial order $\leq_S$.*

Proof. Assuming (i) we have $t = h_{i,j} + m$ for some iterated Frobenius number $h_{i,j}$ so, clearly, $t \in \mathrm{Ap}\,(S,m)$. Let $s \in S_+$, and consider the element $t+s-m = (h_{i,j}+m)+s-m$. Since this element belongs to S we see that t is maximal in $\mathrm{Ap}\,(S,m)$. If (ii) holds, then $t+s-m \in S$ for every $s \in S_+$, that is, $t - m \in T(S)$. □

Corollary 4.2. *Let S be a positive affine monoid and $m \in S_+$. Then there is a one-to-one correspondence between the elements of $T(S)$ and the elements of* $\mathrm{Ap}\,(S,m)$ *that are maximal relative $\leq_S$. In particular,* $\mathrm{Ap}\,(S,m)$ *is finite.*

Proof. This follows from Proposition 4.1. □

Just as for numerical monoids, the set $T_0(S)$ can be described using $\mathrm{Ap}\,(S,m)$ for any non-zero element $m \in S$.

Proposition 4.3. *Let S be a positive affine monoid. For any non-zero element $m \in S$ we have*

$$T_0(S) = \max_{\leq_{\mathbf{N}^d}}\{x \in \mathrm{Ap}\,(S,m)\} - m.$$

Proof. We know that $T(S) \subseteq \mathrm{Ap}\,(S,m) - m$. On the other hand, all elements in the set $\mathrm{Ap}\,(S,m) - m$ belong to $H(S)$, so the result follows from Lemma 2.11. □

5. Special case of numerical monoids. In this section we confine ourselves to numerical monoids. We present here separately versions of a few results seen in the previous sections since in case of numerical monoids one may say a bit more. Recall from the introduction that in case of numerical monoid S the cardinality of the set $T(S)$ is called the type of S and is denoted by $\operatorname{type} S$. Also, $\operatorname{type} S$ equals the Cohen-Macaulay type of $k[S]$.

For numerical monoids S all negative integers belong to $g(S)-S$. The definition of the iterated Frobenius numbers thus takes the following form:

Definition 5.1. Let S be a numerical monoid with Frobenius number $g(S)$. Put $h_0(S) = g(S)$, and define the iterated Frobenius numbers $h_i(S)$ by

$$h_i(S) = \max\{x \in \mathbf{Z};\ x \notin S,\ x \notin h_j(S) - S, j \in \{0, \dots, i-1\}\}.$$

Remark 5.2. The number $h_1(S)$ is the number $h(S)$ explored already in [**5**].

Since a numerical monoid is an affine monoid, we have

Proposition 5.3. *Let S be a numerical monoid, and assume there are $r+1$ iterated Frobenius numbers $\{h_r, \dots, h_1, g\}$. Then* $\operatorname{type} S = r+1$ *and*

$$T(S) = \{h_r, \dots, h_1, g\}.$$

Remark 5.4. In particular we see that h_i is the $(i+1)$st largest element in $T(S)$.

Observe that for numerical monoids $\operatorname{gp}(S) = \mathbf{Z}$. Using this we recall from Proposition 3.8 that a numerical monoid satisfying the equivalent conditions in Proposition 5.5 below, is called an almost symmetric numerical monoid.

Proposition 5.5. *Let $S = \langle s_0, \dots, s_t \rangle$ be a numerical monoid with $T(S) = \{h_r, \dots, h_1, g\}$. Then the following conditions on S are equivalent*:

(i) $S \cup (g - S) = \mathbf{Z} \setminus \{h_r, \dots, h_1\}$.

(ii) *For all* $x \in \mathbf{Z}$ *it holds that* $x \in S$ *or* $x \in g - S$ *or* $x \in \{h_r, \dots, h_1\}$.

(iii) *If* $x + y = g$, *then either* $x \in S$ *or* $y \in S$, *or both* x *and* y *belong to* $\{h_r, \dots, h_1\}$.

(iv) *There are equally many elements in the set*

$$\{0, 1, \dots, g\} \setminus \{h_r, \dots, h_1\}$$

from S *as there are from outside* S.

(v) $h_i - S_+ \subseteq g - S_+$ *for all* $0 \leq i \leq r$.

Proof. We first consider the case when g is odd. (i) and (ii) are equivalent by Proposition 3.8. (ii) implies (iii) since if neither x nor y lies in S, then $x + y = g$ implies that neither x nor y lies in $g - S$ and hence both x and y lie in $\{h_r, \dots, h_1\}$. (iii) implies (i) since for any integer x, we have that if neither x nor $g - x$ lies in S, they both belong to the set $\{h_r, \dots, h_1\}$. The fact that (iii) and (iv) are equivalent is easily verified considering the equation $x + y = g$ and Corollary 3.14. Finally, by considering the equation

$$S \cup (g - S) \cup \cdots \cup (h_r - S) = \mathbf{Z}$$

we see that (i) and (v) are equivalent since $h_i \notin g - S$ for any i other than 0.

The case g even follows in the exact same way considering Remark 5.7 and Lemma 5.8 below. □

Remark 5.6. Proposition 5.5 partially generalizes Lemma 1 and Lemma 3 in [**5**].

Remark 5.7. In order for the proof of Proposition 5.5 to go through also in the case g even, we need to know that $g/2$ belongs to $T(S)$ in case S satisfies the equivalent conditions (i) and (ii). Lemma 5.8 below shows that this is the case. However, observe that $g/2$ does not always belong $T(S)$ if $g(S)$ is even. The numerical monoid $\langle 3, 11, 13 \rangle$ is an example of this since $g(\langle 3, 11, 13 \rangle) = 10$ but $5 + 3 = 8 \notin S$.

Lemma 5.8. *Let S be an almost symmetric numerical monoid. Then*

(i) type S *is odd whenever* $g(S)$ *is odd.*

(ii) type S *is even whenever* $g(S)$ *is even and in particular, in this case* $g/2$ *always belongs to* $T(S)$.

Proof. Clearly we may assume S is not maximal in $\mathcal{S}_g$. By Corollary 3.18 we know that $\langle S, h_1\rangle$ is almost symmetric and Proposition 3.16 gives that $T(\langle S, h_1\rangle) = T(S) \setminus \{g - h_1, h_1\}$. The result follows by induction on r. □

Example 8. Consider the numerical monoid $S = \langle 8, 12, 14, 15, 17, 18, 21, 27\rangle$. We have $g(S) = 19$ and $T(S) = \{6, 9, 10, 13, 19\}$ so S is not symmetric. However, $\mathbf{N} \cap (g - S) = \{1, 2, 3, 4, 5, 7, 11\}$ so we see that S is almost symmetric.

Example 9. Consider the numerical monoid $S = \langle 4, 17, 18, 23\rangle$. We have $g(S) = 19$, $\mathbf{N}\cap(g-S) = \{1, 2, 3, 7, 11, 15\}$ and $T(S) = \{13, 14, 19\}$. Since $5 \notin S$, $5 \notin g - S$ and $5 \notin T(S)$ we conclude that S is not almost symmetric.

Proposition 5.9. *Let* $S = \langle s_0, \dots, s_t\rangle$ *be an almost symmetric numerical monoid with* g *odd (respectively even) and assume* type $S = 2l + 1$ *(respectively* $2(l + 1)$*). Then there exists a strict sequence of almost symmetric numerical monoids*

$$S \subset S_1 \subset \cdots \subset S_l$$

such that $g(S) = g(S_i)$ *for all* $i \in \{1, \dots, l\}$ *and with* S_l *symmetric (respectively quasi-symmetric).*

Proof. We may assume that S is not maximal in $\mathcal{S}_g$. As in Lemma 5.8 we have $T(\langle S, h_1\rangle) = T(S)\setminus\{g-h_1, h_1\}$ and $\langle S, h_1\rangle$ is almost symmetric. The result follows by induction on r. □

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