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A QUASI-LOCAL HALF-FACTORIAL DOMAIN WITH AN ATOMIC NON-HALF-FACTORIAL INTEGRAL CLOSURE

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ABSTRACT. We construct a *half-factorial* quasi-local domain R, so that its integral closure $\overline{R} = R[t]$, where $t^2, t^3 \in R$, is *atomic but not half-factorial*; $\overline{R}$ equals the seminormalization of R. Moreover, $\overline{R}$ is a quasi-local domain of bounded factorization, and every element in $\overline{R}$ of zero R-boundary is a unit in $\overline{R}$.

0. Preliminaries. For background on half-factoriality see [**2**]. A *half-factorial monoid* M is an atomic monoid in which every two decompositions into atoms of a non-unit in M have the same length. A *half-factorial domain* is a domain R so that the monoid $(R^\bullet, \cdot)$ is half-factorial.

It is well known that, unlike factorial domains, a half-factorial domain is not necessarily integrally closed (see [**3**]). Thus Valentina Barucci asked the following question [**2**]:

Is the integral closure $\overline{R}$ of a half-factorial domain R necessarily half-factorial?

The answer is negative as shown by Coykendall [**4**], who constructed a half-factorial domain so that its integral closure is not atomic. Thus Coykendall raised the question whether $\overline{R}$ is half-factorial if the domain R is half-factorial and $\overline{R}$ is atomic.

In this note we answer this question in the negative. For an integral domain A that is contained in a DVR, we extend A to a half-factorial domain $\mathrm{Hf}(A)$ (Section 1). We show that if A satisfies certain properties, then $\overline{\mathrm{Hf}(A)}$ is not half-factorial. In Section 2 we construct an integral domain A satisfying the desired properties.

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1. The functor Hf. Let A be a subring of a DVR with a $\mathbb{Z}$-valued valuation v and with field of fractions K.

Let $Z_{a,i}$ be independent indeterminates over A, where a runs over all nonzero elements of A so that $v(a) \geq 2$, and $1 \leq i < v(a)$. We adjoin to A the indeterminates $Z_{a,i}$ and the rational functions $\frac{a}{\prod_{i=1}^{v(a)-1} Z_{a,i}}$, thus obtaining a domain $F(A)$.

Hence $a = Z_{a,1} \cdots Z_{a,k-1} \frac{a}{Z_{a,1} \cdots Z_{a,k-1}}$, where $k = v(a)$. We extend the valuation v to the unique valuation of the field

$$K\big(Z_{a,i} \,:\, a \in A \text{ with } v(a) \geq 2, 1 \leq i < v(a)\big)$$

satisfying $v(Z_{a,i}) = 1$ for all the indeterminates $Z_{a,i}$. We will still denote this extension by v, and call $v(\alpha)$ the *value* of α for an element α in $K\big(Z_{a,i} \,:\, a \in A \text{ with } v(a) \geq 2, 1 \leq i < v(a)\big)$. We use this terminology and notation also in the sequel for further extensions of v.

Let $F^0(A) = A$, and let $F^{n+1}(A) = F(F^n(A))$ for $n \geq 0$. Set $F^\infty(A) = \bigcup_{n=1}^\infty F^n(A)$. We denote the set of the adjoined indeterminates $Z_{r,i}$ by $\mathbf{Z}$.

Let $S = \{u \in F^\infty(A) \mid v(u) = 0\}$ and $\mathrm{Hf}(A) = F^\infty(A)_S$ be the localization at S. Thus $S = F^\infty(A) \cap \mathrm{U}(V)$, where $\mathrm{U}(V)$ is the set of units in the valuation domain V defined by v. Hence $\mathrm{Hf}(A)$ is a quasi-local domain. Clearly Hf can be defined as an endofunctor of the category of integral domains A with a given $\mathbb{Z}$-valued valuation v on $\mathrm{Frac}(A)$, so that $v(A) \subseteq \mathbb{Z}^+$, and with valuation preserving ring homomorphisms as morphisms (these homomorphisms are necessarily monomorphisms).

As stated in Proposition 1.1 below, for $a \in A$, the ring

$$A\left[Z_{a,1}, \dots, Z_{a,k-1}, \frac{a}{\prod_{i=1}^{v(a)-1} Z_{a,i}}\right]$$

can be obtained by using iteratively extended Rees algebras with respect to principal ideals: we use the notation $\widetilde{R}(A, a; Z) = A[Z, \frac{a}{Z}]$ for an integral domain A, an element $a \in A$, and an indeterminate Z over A. This notation differs from the usual one: Z and $1/Z$ are interchanged. For extended Rees algebras see, e.g., [**5**, Chapter 5] or [**1**].

The next proposition is obvious.

Proposition 1.1. *Let $Z_1, \dots, Z_n$ be independent indeterminates over a domain A, and let $a \in A$. Then*

$$A\left[Z_1, \frac{a}{Z_1}\right] = \widetilde{R}\,(A, a; Z_1),$$

and for $n > 1$ we have

$$A\left[Z_1, \dots, Z_n, \frac{a}{Z_1 \dots Z_n}\right] = \widetilde{R}\left(A\left[Z_1, \dots, Z_{n-1}, \frac{a}{Z_1 \dots Z_{n-1}}\right], \frac{a}{Z_1 \dots Z_{n-1}}; Z_n\right).$$

Proposition 1.2. *Let $0 \le n \le \infty$, and let L be a finite subset of $F^n(A)$. Then there exists a sequence*

$$(*) \qquad A = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_m$$

of A-subalgebras of $F^n(A)$ so that $L \subseteq B_m$ and for $0 \le k < m$: $B_{k+1} = B_k[X_k, (b_k/X_k)]$, where $X_k \in \mathbb{Z}$, $b_k \in B_k$, X_k is transcendental over Frac (B_k), *and $v(b_k) \ge 2$.*

Proof. Clearly, we may assume that n is finite. We use induction on n, starting with $n = 0$. Let $n > 0$. By construction, there exist elements $c_1, \dots, c_k \in F^{n-1}(A)$ so that $v(c_i) \ge 2$ for all i and so that

$$(**) \quad L \subseteq F^{n-1}(A)\left[Z_{c_j,i}(1 \le j \le k, 1 \le i < v(c_k)), \frac{c_1}{\prod_{i=1}^{v(c_1)-1} Z_{c_1,i}}, \dots, \frac{c_k}{\prod_{i=1}^{v(c_k)-1} Z_{c_k,i}}\right]$$

Moreover, in $(**)$, we may replace $F^{n-1}(A)$ by a finitely generated A-subalgebra C of $F^{n-1}(A)$. By the inductive assumption, C is contained in an A-subalgebra D of $F^{n-1}(A)$ that can be obtained from A as in

(∗). Since all the indeterminates $Z_{c_j,i}$ are algebraically independent over $\mathrm{Frac}\,(D)$, we obtain by induction on k, and using Proposition 1.1 that B can be obtained from D, so also from A as in (∗). □

The next proposition follows immediately from the construction of $\mathrm{Hf}\,(A)$ using that v is nonnegative on $\mathrm{Hf}\,(A)$.

Proposition 1.3. (1) *Each element r in* $\mathrm{Hf}\,(A)$ *of positive value is a product of $v(r)$ elements in* $\mathrm{Hf}\,(A)$ *of value* 1;

(2) *An element of* $\mathrm{Hf}\,(A)$ *is an atom if and only if its value is* 1;

(3) *An element of* $\mathrm{Hf}\,(A)$ *is a unit if and only if its value is* 0;

(4) *The domain* $\mathrm{Hf}\,(A)$ *is quasi-local, atomic and half-factorial.*

We recall that if R is a half-factorial domain, then the *R-boundary* is the unique homomorphism from the multiplicative group $\mathrm{Frac}\,(R)^{\bullet}$ to the additive group of rational integers so that, for a nonzero element $r \in R$, it coincides with the length of an atomic factorization of r (see [**4**]).

Proposition 1.4. *Let* $R = \mathrm{Hf}\,(A)$. *We have*:

(1) *The R-boundary map coincides with v*;

(2) *An element of $\overline{R}$ is a unit in $\overline{R}$ if and only if it is of R-boundary* 0;

(3) *$\overline{R}$ is quasi-local*;

(4) *$\overline{R}$ is atomic of bounded factorization.*

Proof. (1) is clear.

(2) Let M be the maximal ideal of R of the elements of value > 0, including 0. Let u be an element of $\overline{R}$ so that $v(u) = 0$. Since u is integral over R, in an obvious notation, $\widetilde{u}$ is algebraic over $\widetilde{R} = R/M$. Let $f(X)$ be a polynomial in $R[X]$ so that $\widetilde{f}$ is the minimal polynomial of $\widetilde{u}$ in the ring $\widetilde{R}[X]$. Since $\widetilde{f}(0) \neq 0$, we obtain that $v(f(0)) = 0$; thus, $f(0)$ is a unit in R. Since u divides $f(0)$ in $\overline{R}$, it follows that u is a unit in $\overline{R}$.

On the other hand, since v is nonnegative on R, the ring R is contained in the valuation ring of v. Hence v is nonnegative also on $\overline{R}$. Thus the units in $\overline{R}$ are of value 0.

(3) Since v is nonnegative on $\overline{R}$, and since all the elements in $\overline{R}$ of value 0 are invertible, it follows that $\overline{R}$ is quasi-local.

(4) If $s = s_1 \cdots s_k$, where $s_1, \dots, s_k$ are nonunits in $\overline{R}$, then $k \leq v(s)$. Hence $\overline{R}$ is atomic of bounded factorization. □

Proposition 1.5.

$$\overline{\mathrm{Hf}\,(A)} = \overline{A}[\mathrm{Hf}\,(A)]$$

Proof. It is enough to prove that $\overline{A}[\mathrm{Hf}\,(A)]$ is integrally closed, so it suffices to prove that $\overline{A}[F^{\infty}(A)]$ is integrally closed. By Proposition 1.2, $F^{\infty}(A)$ is a directed union of A-subalgebras that are obtained from A by using finitely many iterations of extended Rees algebras with respect to principal ideals. Since an extended Rees algebra of an integrally closed domain with respect to a principal ideal is integrally closed [**5**, Propositions 1.5.2 and 5.2.1], the proposition follows. □

Proposition 1.6. *Let S be the set of elements of value* 0 *in A. Then* $\mathrm{Hf}\,(A) \cap K = A_S$.

Proof. Let c be an element in $\mathrm{Hf}\,(A) \cap K$. Let $c = r/u$, where $r, u \in F^{\infty}(A)$, and $v(u) = 0$. By Proposition 1.2, there exists a sequence $A = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_m$ of A-subalgebras of $F^{\infty}(A)$ so that $r, u \in B_m$ and for $0 \leq k < m$: $B_{k+1} = B_k[X_k, (b_k/X_k)]$, where $X_k \in \mathbf{Z}$ is transcendental over $B_k, b_k \in B_k$, and $v(b_k) \geq 2$. We prove that $c \in A_S$ by induction on m, starting with $m = 0$. Let $m > 0$. Let $X = X_m$ and $b = b_m$. Let $r = \sum_{i\geq 0} e_i X^i + \sum_{i<0} f_i(b^i/X^i)$, and $u = \sum_{i\geq 0} g_i X^i + \sum_{i<0} h_i(b^i/X^i)$, where all the sums are finite, and $e_i, f_i, g_i, h_i \in B_{m-1}$. We have $r = cu$, that is,

$$\sum_{i\geq 0} e_i X^i + \sum_{i<0} f_i \frac{b^i}{X^i} = c\biggl(\sum_{i\geq 0} g_i X^i + \sum_{i<0} h_i \frac{b^i}{X^i}\biggr).$$

Since $v(u) = 0$, we see that $v(g_0) = 0$. Also $e_0 = cg_0$, where both elements e_0 and g_0 are in B_{m-1}. By the inductive assumption we conclude that $c \in A_S$. Hence $\mathrm{Hf}\,(A) \cap K = A_S$. □

Corollary 1.7. *Let t be an element in $K \setminus A$ so that $v(t) > 0$. Then $t \notin \mathrm{Hf}\,(A)$.*

Lemma 1.8. *Let R be an integral domain contained in a* DVR *V with a $\mathbb{Z}$-valued valuation v. Assume that every element in R of value* 0 *is invertible in R. Let t be an element of* $\mathrm{Frac}\,(R) \setminus R$ *satisfying the following two conditions*:

(1) $t^2 \in R$;

(2) $v(t) > 0$.

Then an element of $R[t]$ is a unit if and only if it is of value 0*, the element t is an atom in $R[t]$, and t is associated in $R[t]$ with no element of R.*

Proof. Clearly, the elements of value > 0 in $R[t]$ are not invertible. We now show that an element $u \in R[t]$ of value 0 is a unit in $R[t]$. Since $t^2 \in R$, we have $u = a + bt$ for some elements $a, b \in R$. Since $v(u) = 0$ and $v(t) > 0$, we have $v(a) = 0$, so a is a unit in R. We see that $(a + bt)(a - bt) = a^2 - bt^2$ is a unit in R since $v(a^2 - bt^2) = 0$. Thus u is a unit in $R[t]$ as claimed.

Assume that t is associated with an element of R, thus $ut \in R$, for some element $u \in R[t]$ so that $v(u) = 0$. Let $u = u_1 + u_2 t$, where $u_1, u_2 \in R$. Thus $u_1 t = ut - u_2 t^2 \in R$, implying that $t \in R$, a contradiction.

In particular, t is not a unit in $R[t]$. Let t be a product of two nonunits in $R[t]$: $t = (a_1 + b_1 t)(a_2 + b_2 t)$, where $a_1, a_2, b_1, b_2 \in R$. Hence a_1 and a_2 are elements of value > 0. We have $t(1 - s) = a_1 a_2$ for $s = a_1 b_2 + a_2 b_1 + b_1 b_2 t$. Thus $v(s) > 0$, and $1 - s$ is a unit in $R[t]$, contradicting that t is associated with no element of R. We conclude that t is an atom in R. □

Proposition 1.9. *Let A be an integral domain that is contained in a* DVR *with a $\mathbb{Z}$-valued valuation v so that $\overline{A} = A[t]$. Assume that $t \notin A$, but $t^2 \in A$, and that $v(t) \geq 2$.*

Let $R = \mathrm{Hf}\,(A)$. *Then* $\overline{R} = R[t]$ *is not half-factorial, but is atomic of bounded factorization.*

Proof. By Corollary 1.7, we have $t \notin R$. By Proposition 1.6, we have $\overline{R} = R[t]$. Hence, by Lemma 1.8 applied to R, we see that t is an atom in $\overline{R}$, so t^2 is a product of two (equal) atoms in $\overline{R}$. On the other hand, $m := v(t^2) = (v(t))^2 \geq 4$. Since $t^2 \in A$, we see that t^2 is a product of m atoms (of value 1) in R, where $m \geq 4$. These elements are nonunits (and also atoms) in $\overline{R} = \overline{\mathrm{Hf}\,(A)}$, so $\overline{R}$ is not half-factorial. By Proposition 1.4, $\overline{R}$ is atomic of bounded factorization. ◻

2. An example. In this section we construct a domain A satisfying the assumptions of Proposition 1.9.

Let $k[t]$ be a polynomial ring over a field k. Set

$$A = k[t^2, t^3],$$

so $A = k[t^n \mid n \geq 2]$. Let v be the unique valuation on $\mathrm{Frac}\,(A)$ that is trivial on k and so that $v(t) = 2$. Clearly, $\overline{A} = A[t]$, $t \in k(t) \setminus A$, and $t^2 \in A$, so A satisfies the conditions of Proposition 1.9.

Set $K = \mathrm{Frac}\,A = k(t)$ and $R = \mathrm{Hf}\,(A)$. By Propositions 1.3–1.9, we obtain:

Example 2.1. The domain R is quasi-local and half-factorial (in particular, R is atomic). We have $\overline{R} = R[t]$, where $t \in \overline{R}$ and $t^2 \in R$. Thus $\overline{R}$ is equal to the seminormalization of R. The domain $\overline{R}$ is quasi-local and atomic of bounded factorization, but not half-factorial. Each element of zero R-boundary in $\overline{R}$ is a unit in $\overline{R}$. We have $R \cap K = A$.

This example disproves the following conjecture in [**4**]: If R is an HFD and S is an overring with the property that there are no non-units in S of boundary 0, then S is an HFD.

It seems plausible that one can start with a seminormal domain A in order to obtain a seminormal domain $\mathrm{Hf}\,(A)$ so that $\overline{\mathrm{Hf}\,(A)}$ is not half-factorial.

We leave open the following questions:

(1) If A is an HFD, and the integral closure $\overline{A}$ is Noetherian, is $\overline{A}$ an HFD?

(2) Suppose both A and $\overline{A}$ are Noetherian, and A is an HFD. Is $\overline{A}$ an HFD? In particular, if A is an affine HFD, is $\overline{A}$ an HFD?

(cf. [**2**]).

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