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ON THE SECOND POWERS OF STANLEY-REISNER IDEALS

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ABSTRACT. In this paper, we study several properties of the second power I_Δ^2 of a Stanley-Reisner ideal I_Δ of any dimension. As the main result, we prove that S/I_Δ is Gorenstein whenever S/I_Δ^2 is Cohen-Macaulay over any field K. Moreover, we give a criterion for the second symbolic power of I_Δ to satisfy (S_2) and to coincide with the ordinary power, respectively. Finally, we provide new examples of Stanley-Reisner ideals whose second powers are Cohen-Macaulay.

0. Introduction. It is proved in [**24**] that a simplicial complex Δ is a complete intersection if the third power I_Δ^3 of its Stanley-Reisner ideal is Cohen-Macaulay, using a result in [**17, 27**]. On the other hand, there is a simplicial complex Δ which is not a complete intersection such that I_Δ^2 is Cohen-Macaulay. The simplicial complex associated with a pentagon is such an example. Among one-dimensional simplicial complexes, the above example is a unique one, as shown in [**16**]. As for the two-dimensional case, such simplicial complexes are classified in [**26**]. In [**17**] a characterization of Cohen-Macaulayness of the second symbolic power $I_\Delta^{(2)}$ is given.

A main motivation of this paper is to study the Cohen-Macaulayness of the second ordinary powers of Stanley-Reisner ideals of any dimension. We consider the following two questions:

(1) What constraints does Cohen-Macaulayness of I_Δ^2 impose upon a simplicial complex Δ?

(2) Do there exist *many* simplicial complexes Δ such that I_Δ^2 are Cohen-Macaulay?

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As for the second question we give two families of examples. One is a simplicial join of pentagons; the other is a stellar subdivision of a complete intersection complex.

For the first question we treat more general properties and give necessary conditions for Cohen-Macaulayness of the square, as a result. In each section we pick up a different condition: in Sections 2, 3, and 4 we consider quasi-Buchsbaum property, Serre's condition (S_2), and unmixedness of a (symbolic) square, respectively. Summarizing results in these sections, we have the following theorem:

Theorem 0.1. *Let Δ be a simplicial complex on $[n] = \{1, 2, \dots, n\}$. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring. Suppose that S/I_Δ^2 is Cohen-Macaulay over any field K. Then the following conditions are satisfied*:

(1) *Δ is Gorenstein.*

(2) $\operatorname{diam}((\operatorname{link}_\Delta F)^{(1)}) \leq 2$ *for any face $F \in \Delta$ with* $\dim \mathit{link}_\Delta F \geq 1$.

(3) *For $F_1, F_2, F_3 \in 2^{[n]} \setminus \Delta$ there exist $G_1, G_2 \in 2^{[n]} \setminus \Delta$ such that $G_1 \cup G_2 \subset F_1 \cup F_2 \cup F_3$ and $G_1 \cap G_2 \subset F_1 \cap F_2 \cap F_3$.*

As shown in Corollary 3.3, condition (2) is equivalent to Serre's condition (S_2) of $S/I_\Delta^{(2)}$. And, as shown in Theorem 4.3, condition (3) is equivalent to the condition $I_\Delta^2 = I_\Delta^{(2)}$.

We may ask the converse:

Question 0.2. Do conditions (1), (2) and (3) imply that S/I_Δ^2 is Cohen-Macaulay?

It is known that Cohen-Macaulayness of I_Δ^2 is equivalent to Cohen-Macaulayness of $I_\Delta^{(2)}$ and $I_\Delta^2 = I_\Delta^{(2)}$. Hence the above question will be affirmative if so is the following one, which is interesting in its own right:

Question 0.3. Do conditions (1) and (2) imply that $S/I_\Delta^{(2)}$ is Cohen-Macaulay?

Stronger versions of the first question are as follows:

Question 0.4. Do conditions (1) and (3) imply that S/I_Δ^2 is Cohen-Macaulay?

Question 0.5. Do conditions (2) and (3) imply that S/I_Δ^2 is Cohen-Macaulay?

By [**16**], the above questions are true if simplicial complexes are one-dimensional.

For the case that edge ideals $I(G)$ of graphs G without isolated vertices are unmixed with the condition $2\,\mathrm{height}\, I(G) = n$, the above questions are also true. If $I(G)$ is Gorenstein, then it is a complete intersection by [**6**]. Hence $I(G)^2$ is Cohen-Macaulay and Questions 0.3 and 0.4 are affirmative. On the other hand, it is proved in [**7**] that there is some face F in the simplicial complex Δ_2 corresponding to the polarization of the second symbolic power $I(G)^{(2)}$ such that $\mathrm{link}_{\Delta_2} F$ is not strongly connected, if $I(G)$ is not a complete intersection. This implies that the polarization of $I(G)^{(2)}$ does not satisfy Serre's condition (S_2). By [**18**], $I(G)^{(2)}$ does not satisfy Serre's condition (S_2), either. It means that $I(G)$ is a complete intersection if $I(G)^{(2)}$ satisfies Serre's condition (S_2). Hence Question 0.5 is also affirmative.

Now let us summarize the organization of the paper. In Section 1, we fix the terminology which we need later.

In Section 2 we consider quasi-Buchsbaum property, which is weaker than Cohen-Macaulay property. And we prove the following theorem as a main result in this section:

Theorem 2.1. *Let Δ be a simplicial complex on $[n]$ of dimension $d-1 \geq 2$. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring. Suppose that S/I_Δ^2 is quasi-Buchsbaum over any field K. Then S/I_Δ is Gorenstein.*

Since the Cohen-Macaulay property implies Serre's condition (S_2), in Section 3 we give a criterion for $I_\Delta^{(2)}$ to satisfy (S_2), which is a generalization of [**17**, Theorem 2.3]; see Theorem 3.2 and Corollary 3.3. As an application, we show that, for Reisner's complex (a triangulation of the real projective plane) Δ, $S/I_{\Delta}^{(2)}$ satisfies (S_2) but is *not* Cohen-Macaulay.

In Section 4 we consider the problem when $I^{(2)} = I^2$ holds for a Stanley-Reisner ideal I, which is also a necessary condition for Cohen-Macaulayness of I^2. It is also discussed in [**26**]. We give a criterion for the second symbolic power to be equal to the ordinary power for Stanley-Reisner ideals in terms of the hypergraph of the generators, see Theorem 4.3. This generalizes a similar criterion for edge ideals. As an application, we show that the second powers of the edge ideals of finitely many disjoint union of pentagons are Cohen-Macaulay as in the second symbolic power case in [**17**].

In Section 5, we give examples of the complexes whose second powers of the Stanley-Reisner ideals are Cohen-Macaulay. More precisely, we prove the following theorem, which is a generalization of a two-dimensional complex in [**26**, Theorem 3.7 (iii)].

Theorem 5.4. *Let Δ be a stellar subdivision of a non-acyclic complete intersection complex Γ. Then S/I_{Δ}^2 is Cohen-Macaulay.*

1. Preliminaries.

In this section we recall several definitions and properties that we will use later. See also [**3, 14, 21, 22**].

1.1. Stanley-Reisner ideals. Let $V = [n]$. A nonempty subset Δ of the power set 2^V is called a *simplicial complex* on V if (i) $F \in \Delta$, $F' \subseteq F \Rightarrow F' \in \Delta$ and (ii) $\{v\} \in \Delta$ for all $v \in V$. An element $F \in \Delta$ is called a *face* of Δ. The dimension of F is defined by $\dim F = \sharp(F) - 1$, where $\sharp(F)$ denotes the cardinality of a set F. The dimension of Δ, denoted by $\dim \Delta$, is the maximum of the dimensions of all faces. A maximal face of Δ is called a *facet* of Δ, and let $\mathcal{F}(\Delta)$ denote the set of all facets of Δ.

In the following, let Δ be a simplicial complex with $\dim \Delta = d - 1$, and let K be a field. Then Δ is called *pure* if all the facets of Δ have the same cardinality d. Put $f_i(\Delta) = \sharp\{F \in \Delta : \dim F = i\}$ for each $i = 0, 1, \ldots, d-1$. For each i, $\widetilde{H}_i(\Delta; K)$ (respectively, $\widetilde{H}^i(\Delta; K)$) denotes the ith reduced simplicial homology (respectively cohomology) of Δ with values in K. We omit the symbol K unless otherwise specified. The *reduced Euler characteristic* of Δ is defined by

$$\widetilde{\chi}(\Delta) = -1 + \sum_{i=0}^{d-1} f_i(\Delta) = \sum_{i=-1}^{d-1} (-1)^i \dim_K \widetilde{H}_i(\Delta).$$

For each face $F \in \Delta$, the *star* and the *link* of F are defined by

$$\begin{aligned} \operatorname{star}_\Delta F &= \{H \in \Delta \ : \ H \cup F \in \Delta\}, \\ \operatorname{link}_\Delta F &= \{H \in \operatorname{star}_\Delta F \ : \ H \cap F = \varnothing\}. \end{aligned}$$

Note that these are also simplicial complexes. For any integer k with $0 \le k \le d-1$, the k-th *skeleton* of Δ is defined by $\Delta^{(k)} = \{F \in \Delta\,;\, \dim F \le k\}$. Then $\Delta^{(k)}$ is a subcomplex of Δ with $\dim \Delta^{(k)} = k$.

The *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the squarefree monomial ideal of $S = K[x_1, \ldots, x_n]$ generated by

$$\{x_{i_1} x_{i_2} \cdots x_{i_p} \ : \ 1 \le i_1 < \cdots < i_p \le n, \ \{x_{i_1}, \ldots, x_{i_p}\} \notin \Delta\},$$

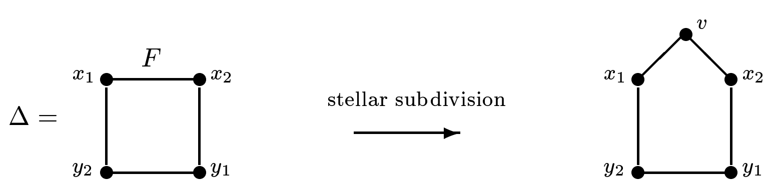
and $K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$ is called the *Stanley-Reisner ring* of Δ. Note that the Krull dimension of $K[\Delta]$ is equal to d. For any subset σ of V, x_σ denotes the squarefree monomial in $K[x_1, \ldots, x_n]$ with support σ.

For a simplicial complex Δ on V, we put $\operatorname{core} V = \{x \in V : \operatorname{star}_\Delta\{x\} \neq \Delta\}$. Moreover, we define the *core* of Δ by $\operatorname{core} \Delta = \{F \in \Delta : F \subseteq \operatorname{core} V\}$.

For a given face F of Δ with $\dim F \ge 1$ and a new vertex v, the *stellar subdivision* of Δ on F is the simplicial complex Δ_F on the vertex set $V \cup \{v\}$ defined by

$$\begin{aligned} \Delta_F = {} & \big(\Delta \setminus \{H \,|\, F \subseteq H \in \Delta\}\big) \\ & \cup \{H \cup \{v\} \,|\, H \in \Delta, \ F \not\subseteq H, \ F \cup H \in \Delta\}. \end{aligned}$$

Notice that Δ_F is homeomorphic to Δ.



Let G be a graph, which means a finite graph without loops and multiple edges. Let $V(G)$ (respectively $E(G)$) denote the set of vertices (respectively edges) of G. Put $V(G) = [n]$. Then the *edge ideal* of G, denoted by $I(G)$, is a squarefree monomial ideal of $S = K[x_1, \dots, x_n]$ defined by

$$I(G) = (x_i x_j \, : \, \{i, j\} \in E(G)).$$

For an arbitrary graph G, the simplicial complex $\Delta(G)$ with $I(G) = I_{\Delta(G)}$ is called the *complementary simplicial complex* of G.

Let G be a connected graph, and let p,q be two vertices of G. The *distance* between p and q, denoted by $\operatorname{dist}(p, q)$, is the minimal length of paths from p to q. The *diameter*, denoted by $\operatorname{diam} G$, is the maximal distance between two vertices of G. We set $\operatorname{diam} G = \infty$ if G is a disconnected graph.

Let Δ be a simplicial complex on V of dimension 1. Then Δ can be regarded as a graph on V whose edge set is defined by $E(\Delta) = \{F \in \Delta \, : \, \dim F = 1\}$.

1.2. Symbolic powers. Let I be a radical ideal of S. Let $\operatorname{Min}_S(S/I) = \{P_1, \dots, P_r\}$ be the set of the minimal prime ideals of I, and put $W = S \setminus \bigcup_{i=1}^r P_i$. Given an integer $\ell \geq 1$, the *ℓth symbolic power* of I is defined to be the ideal

$$I^{(\ell)} = I^\ell S_W \cap S = \bigcap_{i=1}^{r} P_i^\ell S_{P_i} \cap S.$$

In particular, if $I = I_\Delta$ is the Stanley-Reisner ideal of Δ, putting $P_F = (x \in [n] \setminus F)$ for each facet F, then we have

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F$$

and hence

$$I_\Delta^{(\ell)} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^\ell.$$

In general, $I^\ell \subseteq I^{(\ell)}$ holds, but the other inclusion does not necessarily hold. For instance, if $I = (x_1x_2, x_2x_3, x_3x_1)$, then

$$I^{(2)} = (x_1, x_2)^2 \cap (x_2, x_3)^2 \cap (x_1, x_3)^2 = I^2 + (x_1x_2x_3) \neq I^2.$$

Moreover, if I is a unmixed squarefree monomial ideal, then $I^{(\ell)}$ is unmixed. Thus if S/I^{ℓ} is Cohen-Macaulay (or Buchsbaum), then so is $S/I^{(\ell)}$.

1.3. Serre's condition. Let $S = K[x_1, \dots, x_n]$ and $\mathfrak{m} = (x_1, \dots, x_n)S$. Let I be a homogeneous ideal of S. For a positive integer k, S/I satisfies *Serre's condition* (S_k) if $\operatorname{depth}(S/I)_P \geq \min\{\dim(S/I)_P,\, k\}$ for every $P \in \operatorname{Spec} S/I$.

A simplicial complex Δ is called *Cohen-Macaulay* (respectively Gorenstein, (FLC) etc.) if so is $K[\Delta]$ over any field K. Moreover, if Δ is (FLC), then Δ is pure and $\operatorname{link}_{\Delta}(F)$ is Cohen-Macaulay for every nonempty face $F \in \Delta$.

A homogeneous K-algebra S/I is called *quasi-Buchsbaum* if $\mathfrak{m}H^i_{\mathfrak{m}}(S/I) = 0$ for each $i = 0, 1, \dots, \dim S/I - 1$. It is known that any quasi-Buchsbaum ring has (FLC) and the converse is also true for Stanley-Reisner rings.

1.4. Associated simplicial complex of monomial ideals. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring with natural $\mathbf{Z}^n$-graded structure. Let $\mathfrak{m} = (x_1, \dots, x_n)S$ be the unique homogeneous maximal ideal of S. Let I be a monomial ideal of S, and let $G(I)$ denote the minimal monomial generators of I. For each i, we put $\rho_i = \max\{b_i : x^{\mathbf{b}} \in G(I)\}$, where $\mathbf{b} = (b_1, \dots, b_n) \in \mathbf{N}^n$ and $x^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n}$. Then S/I can be considered as a $\mathbf{Z}^n$-graded ring.

Let $\mathbf{a} \in \mathbf{Z}^n$ be a vector. For any $\mathbf{Z}^n$-graded S-module M, $M_{\mathbf{a}}$ denotes the graded $\mathbf{a}$-component of M. We put $G_{\mathbf{a}} = \{i \in [n] : a_i < 0\}$. As $\sqrt{I}$ is a squarefree monomial ideal, there exists a simplicial complex Δ such that $I_{\Delta} = \sqrt{I}$. Then we define $\Delta(I) = \Delta$. Under this notation, a subcomplex $\Delta_{\mathbf{a}}(I)$ is defined by

$$\Delta_{\mathbf{a}}(I) = \left\{ F \in \Delta(I) : \begin{array}{l} \bullet\ F \cap G_{\mathbf{a}} = \varnothing \\ \bullet\ \text{For every } x^b \in G(I), \text{ there exists an} \\ \quad i \in [n] \setminus (F \cup G_a) \text{ s.t. } b_i > a_i. \end{array} \right\}.$$

This complex plays a key role in Takayama's formula for local cohomology modules of monomial ideals, which is known as Hochster's formula in the case of squarefree monomial ideals.

Let $I = I_\Delta$ be a squarefree monomial ideal of S. Then $I^{(\ell)}$ is a monomial ideal whose radical is equal to I. The following lemma enables us to compute $\Delta_{\mathbf{a}}(I^{(\ell)})$ easily.

Lemma 1.1 (Minh and Trung [**16**]). *Let I be a squarefree monomial ideal in S. Let $\ell \geq 1$ be an integer and $\mathbf{a} \in \mathbf{N}^n$. Then we have*

$$\Delta_{\mathbf{a}}(I^{(\ell)}) = \langle F \in \mathcal{F}(I) \,:\, \sum_{i \notin F} a_i \leq \ell - 1 \rangle.$$

1.5. Linkage. Let R be a Gorenstein ring, and let I, J be ideals of R. I and J are said to be *directly linked*, denoted by $I \sim J$, if there exists a regular sequence $\underline{z} = z_1, \ldots, z_h$ in $I \cap J$ such that $J = (\underline{z})\colon I$ and $I = (\underline{z})\colon J$.

Assume that I is a Cohen-Macaulay ideal of height h and $\underline{z} = z_1, \ldots, z_h$ is a regular sequence contained in I. If we set $J = (\underline{z})\colon I$, then $I = (\underline{z})\colon J$ and thus $I \sim J$.

Moreover, I is said to be *linked* to J (or I lies in the linkage class of J) if there exists a sequence of ideals of direct links

$$I = I_0 \sim I_1 \sim \cdots \sim I_r = J.$$

One can easily see that $\sim$ is an equivalence relation of ideals and any two complete intersections with the same height belong to the same class. In particular, I is called *licci* if I lies in the linkage class of a complete intersection ideal. See, e.g., [**28**] for more details.

2. Quasi-Buchsbaumness of the second powers and Gorensteinness. In this section we consider quasi-Buchsbaum property of the second power of the Stanley-Reisner ideal I_Δ. The main purpose of this section is to prove the following theorem:

Theorem 2.1. *Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and let Δ be a simplicial complex on $V = [n]$. Suppose that $d = \dim S/I_\Delta \geq 3$. If S/I_Δ^2 is quasi-Buchsbaum for any field K then Δ is Gorenstein.*

We first prove the following lemma, which is closely related to the conjecture by Vasconcelos (see also [**20**, Conjecture 3.12]): Let R be a regular local ring and I a Cohen-Macaulay ideal of R. If I is syzygetic and I/I^2 is Cohen-Macaulay, then I is a Gorenstein ideal. The following lemma easily follows from the classification theorems for simplicial complexes Δ such that S/I_Δ^2 are Cohen-Macaulay in one- and two-dimensional cases. See [**16, 26**].

Lemma 2.2. *Let Δ be a simplicial complex on $V = [n]$, and let $I_\Delta \subseteq S = K[x_1, \dots, x_n]$ denote the Stanley-Reisner ideal of Δ. If S/I_Δ^2 is Cohen-Macaulay for any field K, then Δ is Gorenstein.*

Proof. We may assume that $\Delta = \operatorname{core}\Delta$. Let K be a field and fix it. Let F be a face of Δ and put $\Gamma = \operatorname{link}_\Delta F$.

First note that S/I_Γ^2 and S/I_Δ are Cohen-Macaulay if so is S/I_Δ^2. Indeed, since S/I_Δ^2 is Cohen-Macaulay and $I_\Delta = \sqrt{I_\Delta^2}$, we have that S/I_Δ is Cohen-Macaulay; see, e.g., [**10**]. On the other hand, by localizing at $x_F = \prod_{i \in F} x_i$, we get

$$I_\Delta S[x_F^{-1}] = (I_\Gamma, x_{i_1}, \dots, x_{i_k}) S[x_F^{-1}]$$

for some variables $x_{i_1}, \dots, x_{i_k}$. Hence the assumption implies that $(I_\Gamma, x_{i_1}, \dots, x_{i_k})^2$ is a Cohen-Macaulay ideal. This yields that I_Γ^2 is also Cohen-Macaulay.

Suppose that $\dim \Gamma = 0$. Then one can take a complete graph G such that $I(G) = I_\Gamma$. Since $S/I(G)^2$ is Cohen-Macaulay, we have $I(G)^{(2)} = I(G)^2$. Hence G does not contain any triangle (e.g., see Corollary 4.5). Thus $\sharp(V(\Gamma)) = \sharp(V(G)) \leq 2$.

By the above argument, $\Lambda = \operatorname{link}_\Delta F$ is a locally complete intersection complex whenever $\dim \Lambda = 1$. Moreover, since S/I_Λ is Cohen-Macaulay and thus Λ is connected, Λ is an n-cycle or an n-pointed path; see [**25**, Proposition 1.11]. On the other hand, since $\operatorname{diam} \Lambda \leq 2$ by [**16**, Theorem 2.3], we get $n \leq 3$ if Λ is an n-pointed path. Hence $\Lambda = \operatorname{link}_\Delta F$ is Gorenstein.

Now suppose that $K = \mathbf{Z}/2\mathbf{Z}$. By [**20**, Chapter II, Theorem 5.1], $K[\Delta]$ is Gorenstein. Then we get $\widetilde{\chi}(\Delta) = (-1)^{d-1}$.

Let K be any field. Then $\widetilde{\chi}(\Delta) = (-1)^{d-1}$ because $\widetilde{\chi}(\Delta)$ does not depend on K. Therefore we conclude that Δ is Gorenstein over K by [**20**, Chapter II, Theorem 5.1] again. $\square$

A complex Δ is called a *locally Gorenstein* complex if $\mathrm{link}_\Delta\{x\}$ is Gorenstein for every vertex $x \in V$. Then the following corollary immediately follows from Lemma 2.2.

Corollary 2.3. *If S/I_Δ^2 has* (FLC) *for any field K, then Δ is a locally Gorenstein complex.*

Proof. The assumption implies that $S/I^2_{\mathrm{link}_\Delta\{x\}}$ is Cohen-Macaulay for every vertex $x \in V$. Then $\mathrm{link}_\Delta\{x\}$ is Gorenstein by Lemma 2.2. $\square$

Lemma 2.4. *Suppose $d \geq 2$. If S/I_Δ^2 is quasi-Buchsbaum, then S/I_Δ is Cohen-Macaulay.*

Proof. By assumption S/I_Δ^2 has (FLC). Then S/I_Δ has (FLC) by [**10**, Theorem 2.6] and thus it is Buchsbaum.

Now suppose that S/I_Δ is *not* Cohen-Macaulay. Then there exists an i with $0 \leq i \leq d-2$ such that $H_\mathfrak{m}^{i+1}(S/I_\Delta)_0 \cong \widetilde{H}_i(\Delta; K) \neq 0$. Then we get the following commutative diagram (see [**15**])

$$\begin{array}{ccc} H_\mathfrak{m}^{i+1}(S/I_\Delta^2)_{\mathbf{0}} & \xrightarrow{x_1} & H_\mathfrak{m}^{i+1}(S/I_\Delta^2)_{\mathbf{e}_1} \\ \downarrow & & \downarrow \\ \widetilde{H}^i(\Delta_{\mathbf{0}}(I_\Delta^2)) & \longrightarrow & \widetilde{H}^i(\Delta_{\mathbf{e}_1}(I_\Delta^2)), \end{array}$$

where the bottom map is the identity because $\Delta_{\mathbf{0}}(I^2) = \Delta_{\mathbf{e}_1}(I^2) = \Delta$ by [**24**] and the vertical maps are isomorphism. This yields $x_1 H_\mathfrak{m}^{i+1}(S/I_\Delta^2) \neq 0$. But this contradicts the assumption. $\square$

Remark 2.5. We have an analogous result in the symbolic power case. Namely, if $S/I_\Delta^{(2)}$ is quasi-Buchsbaum, then S/I_Δ is Cohen-Macaulay. The proof is almost the same since we have $\Delta_{\mathbf{0}}(I^{(2)}) = \Delta_{\mathbf{e}_1}(I^{(2)}) = \Delta$.

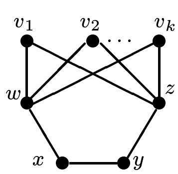
We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. By assumption and Corollary 2.3, we have that Δ is locally Gorenstein. Moreover, Δ is Cohen-Macaulay by Lemma 2.4. Take any face F of Δ with $\dim \mathrm{link}_\Delta F = 1$. As $d \geq 3$, $\mathrm{link}_\Delta F$ is given by some link of $\mathrm{link}_\Delta\{x\}$ for $x \in F$. Hence such a $\mathrm{link}_\Delta F$ is also Gorenstein. By a similar argument as in the proof of Lemma 2.2, we get the required assertion. □

The Gorensteinness of S/I_Δ does not necessarily imply the quasi-Buchsbaumness of S/I_Δ^2.

We cannot replace the Cohen-Macaulayness of S/I_Δ^2 with that of $S/I_\Delta^{(2)}$ in Lemma 2.2 as the next example shows.

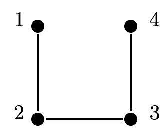
Example 2.6. Let $k \geq 2$ be a given integer. Let I be the Stanley-Reisner ideal of the following simplicial complex Δ, Then since $\operatorname{diam} \Delta \leq 2$, $S/I^{(2)}$ is Cohen-Macaulay by [**16**], but S/I^2 is not. Moreover, S/I is not Gorenstein.



In Theorem 2.1, we cannot remove the assumption that $\dim S/I_\Delta \geq 3$ as the next example shows.

Example 2.7. Put $I_\Delta = (x_1x_3, x_1x_4, x_2x_4)$, the Stanley-Reisner ideal of the 4-pointed path Δ. Then S/I_Δ^2 is Buchsbaum by [**25**, Example 2.9] and S/I_Δ is Cohen-Macaulay but not Gorenstein of dimension 2.

The following question is valid in the case that char $K = 2$, but the other cases remain open.

$\Delta =$ 

Question 2.8. If S/I_Δ^2 is Cohen-Macaulay over a fixed field K, then is Δ Gorenstein over K?

3. Cohen-Macaulayness versus (S_2) for second symbolic powers. Throughout this section, let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K. Let $\mathfrak{m} = (x_1, \dots, x_n)S$ be the unique graded maximal ideal of S with natural graded structure.

In [**24**] it is proved that, for any integer $\ell \geq 3$ and for any simplicial complex Δ on the vertex set $V = [n]$, $S/I_\Delta^{(\ell)}$ is Cohen-Macaulay if and only if it satisfies Serre's condition (S_2). So it is natural to ask the following question.

Question 3.1. Let I be the Stanley-Reisner ideal of a simplicial complex Δ on $V = [n]$. Then is $S/I^{(2)}$ Cohen-Macaulay if and only if $S/I^{(2)}$ satisfies (S_2)?

So the aim of this section is to give a criterion for $S/I_\Delta^{(2)}$ to satisfy (S_2). In order to do that, we prove the following theorem, which is a generalization of [**16**, Theorem 2.3]. Using this, we give a negative answer to the above question; see Example 3.4. Note that in the following Theorem 3.2 and Corollary 3.3 if we replace the condition that the diameter is less than or equal to 2 by the connectedness condition, then we have the corresponding condition for the original Stanley-Reisner ring instead of the second symbolic power, e.g., depth $S/I_\Delta \geq 2$ is equivalent to the connectedness of Δ if $\dim \Delta \geq 1$.

Theorem 3.2. *Let Δ be a simplicial complex with* $\dim \Delta \geq 1$. *Then the following conditions are equivalent*:

(1) depth $S/I_\Delta^{(2)} \geq 2$ (*equivalently, depth*$(S/I_\Delta^{(2)})_\mathfrak{m} \geq 2$).

(2) diam $\Delta^{(1)} \leq 2$, *where $\Delta^{(1)}$ denotes the 1-skeleton of Δ.*

Proof. Put $\Delta_{\mathbf{a}} := \langle F \in \mathcal{F}(\Delta) \,:\, \sum_{i \notin F} a_i \leq 1 \rangle$.

(1) $\Rightarrow$ (2). For given $r, s \in V = [n]$ $(r < s)$, we show that $\operatorname{dist}(r,s) \leq 2$ in $\Delta^{(1)}$. Put $\mathbf{a} = \mathbf{e}_r + \mathbf{e}_s \in \mathbf{N}^n$. Then $\Delta_{\mathbf{a}} = \langle F \in \mathcal{F}(\Delta) \,:\, r \in F \text{ or } s \in F \rangle$. Since $\operatorname{depth} S/I_\Delta^{(2)} \geq 2$, we have that $\widetilde{H}_0(\Delta_{\mathbf{a}}) = 0$ and thus $\Delta_{\mathbf{a}}$ is connected by Takayama's formula and Lemma 1.1. Hence there exists an $F \in \mathcal{F}(\Delta)$ such that $r, s \in F$ or there exist $F_r \in \mathcal{F}(\Delta)$ and $F_s \in \mathcal{F}(\Delta)$ such that $r \in F_r$, $s \in F_s$ and $F_r \cap F_s \neq \emptyset$. In any case, we get $\operatorname{dist}(r,s) \leq 2$, as required.

(2) $\Rightarrow$ (1). Assume $\operatorname{diam} \Delta^{(1)} \leq 2$. By Takayama's formula, it suffices to show that $\Delta_{\mathbf{a}}$ is connected for any $\mathbf{a} \in \{0,1\}^n$ with $\Delta_{\mathbf{a}} \neq \emptyset$; see also [**17**].

Case 1: $\sharp(\operatorname{supp} \mathbf{a}) \leq 1$. Then $\Delta_{\mathbf{a}} = \Delta$ is connected by assumption.

Case 2: $\sharp(\operatorname{supp} \mathbf{a}) = 2$. We may assume that $a_r = a_s = 1$ for some $r < s$. Then

$$\Delta_{\mathbf{a}} = \langle F \in \mathcal{F}(\Delta) \,:\, r \in F \text{ or } s \in F \rangle.$$

Since $\operatorname{diam} \Delta^{(1)} \leq 2$, we have that $\{r, s\} \in \Delta$ or there exists a $t \in V$ such that $\{r,t\}$, $\{t,s\} \in \Delta$. In the first case, if we choose a facet $F \in \mathcal{F}(\Delta)$ which contains $\{r,s\}$, then $F \in \Delta_{\mathbf{a}}$ and $r, s \in F$. In the second case, if we choose facets F_1, F_2 such that $\{r,t\} \in F_1$ and $\{s,t\} \in F_2$. Then $\Delta_{\mathbf{a}}$ is connected because F_1, $F_2 \in \Delta_{\mathbf{a}}$.

Case 3: $\sharp(\operatorname{supp} \mathbf{a}) \geq 3$. We may assume that $\sharp(\mathcal{F}(\Delta_{\mathbf{a}})) \geq 2$. Let F_1, $F_2 \in \mathcal{F}(\Delta_{\mathbf{a}})$. By assumption, $\sharp(F_i \cap \operatorname{supp}(\mathbf{a})) \geq \sharp(\operatorname{supp}(\mathbf{a})) - 1$ for each $i = 1, 2$. Then we get

$$\begin{aligned} \sharp(F_1 \cap F_2) &\geq \sharp\big(F_1 \cap \operatorname{supp}(\mathbf{a})) \cap (F_2 \cap \operatorname{supp}(\mathbf{a}))\big) \\ &\geq \sharp(\operatorname{supp}(\mathbf{a})) - 2 \geq 1. \end{aligned}$$

Hence $\Delta_{\mathbf{a}}$ is connected. □

Corollary 3.3. *Let Δ be a pure simplicial complex. Then the following conditions are equivalent*:

(1) $S/I_\Delta^{(2)}$ *satisfies* (S_2).

(2) $\operatorname{diam}((\operatorname{link}_\Delta F)^{(1)}) \leq 2$ *for any face* $F \in \Delta$ *with* $\dim \operatorname{link}_\Delta F \geq 1$.

Proof. (1) $\Rightarrow$ (2). Let F be a face of Δ with $\dim \operatorname{link}_\Delta F \geq 1$. By assumption and localization, we obtain that $S'/I^{(2)}_{\operatorname{link}_\Delta(F)}$ satisfies (S_2), where S' is a polynomial ring which corresponds to $\Gamma = \operatorname{link}_\Delta(F)$. Then depth $\operatorname{depth} S'/I^{(2)}_\Gamma \geq 2$. It follows from Theorem 3.2 that $\operatorname{diam} \Gamma^{(1)} \leq 2$, as required.

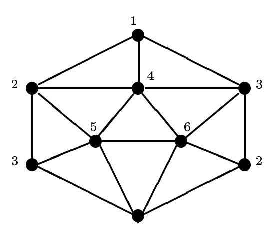
(2) $\Rightarrow$ (1). The assumption (2) preserves under localization. Hence we may assume that $S/I^{(2)}_{\operatorname{link}_\Delta\{x\}}$ satisfies (S_2). This implies that $S/I_{\operatorname{link}_\Delta\{x\}}$ also satisfies (S_2) by [**10**]. Hence $(S/I^{(2)}_\Delta)_x$ satisfies (S_2) for every variable x.

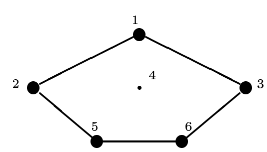
Let $P \in \operatorname{Spec}(S/I^{(2)}_\Delta)$ with $\dim (S/I^{(2)}_\Delta)_P \geq 2$. If $P \neq \mathfrak{m}$, then there exists a variable x such that $x \notin P$. Then $\operatorname{depth}(S/I^{(2)}_\Delta)_P \geq 2$ by the above argument. Otherwise, $P = \mathfrak{m}$. Since $\operatorname{diam} \Delta^{(1)} \leq 2$ by assumption, we have that $\operatorname{depth}(S/I^{(2)}_\Delta)_\mathfrak{m} \geq 2$ by Theorem 3.2. Therefore $S/I^{(2)}_\Delta$ satisfies (S_2). □

The next example shows that the (S_2)-ness of $I^{(2)}_\Delta$ does not necessarily imply its Cohen-Macaulayness.

Example 3.4 (**The triangulation of the real projective plane**). Let $I = I_\Delta$ be the Stanley-Reisner ideal of the triangulation of the real projective plane $\mathbf{P}^2$ as below. Then I_Δ is generated by the following monomials of degree 3:

$$x_1x_2x_3,\ x_1x_2x_5,\ x_1x_3x_6,\ x_1x_4x_5,\ x_1x_4x_6,\ x_2x_3x_4,\ x_2x_4x_6,\ x_2x_5x_6,\ x_3x_4x_5,\ x_3x_5x_6.$$

$\Delta =$ 

$\operatorname{link}_\Delta\{4\} =$ 

Since $\widetilde{\chi}(\Delta) = -1 + f_0 - f_1 + f_2 = -1 + 6 - 15 + 10 = 0 \neq (-1)^2$, $K[\Delta]$ is *not* Gorenstein for any field K. Moreover, Reisner proved that $K[\Delta]$ is Cohen-Macaulay if and only if $\operatorname{char} K \neq 2$.

The link of every vertex is a pentagon, and $\Delta^{(1)}$ is the complete 6-graph. Hence it follows from Corollary 3.3 that $S/I_\Delta^{(2)}$ has (S_2). But it is *not* Cohen-Macaulay, see [**17**, Example 2.8].

One can easily see that $x_1x_2x_3x_4x_5x_6 \in I_\Delta^{(2)} \setminus I_\Delta^2$. Hence S/I_Δ^2 does not satisfy (S_2).

Question 3.5. Let $I(G)$ be the edge ideal of a graph G. If $S/I(G)^{(2)}$ satisfies (S_2), then is it Cohen-Macaulay?

4. When does $I^{(2)} = I^2$ hold. In this section, we discuss when $I^{(2)} = I^2$ holds for any squarefree monomial ideal I. First we introduce the notion of special triangles.

Definition 4.1. Let I be a squarefree monomial ideal of $S = K[x_1, \dots, x_n]$. Let $G(I) = \{x^{H_1}, \dots, x^{H_\mu}\}$ be the minimal set of monomial generators, where $x^H = x_{i_1} \cdots x_{i_r}$ for $H = \{i_1, \dots, i_r\}$. Then $\mathcal{H}(I)$ is called the *associated hypergraph* of I if the vertex set of $\mathcal{H}(I)$ is V and the edge set is $\{H_1, \dots, H_\mu\}$.

Then $\{i, j, k\}$ is called a *special triangle* of $\mathcal{H}(I)$ if there exist $H_i, H_j, H_k \in \mathcal{H}(I)$ such that

$$H_i \cap \{i, j, k\} = \{j, k\},$$
$$H_j \cap \{i, j, k\} = \{i, k\},$$
$$H_k \cap \{i, j, k\} = \{i, j\}.$$

Then we say that "H_i, H_j, H_k *make a special triangle* $\{i, j, k\}$."

For instance, if $G(I)$ contains $x_1x_2L_1$, $x_2x_3L_2$, $x_3x_1L_3$ (L_1, L_2, L_3 are monomials any of which is not divided by x_1,x_2 nor x_3), then $\{1, 2, 3\}$ is a special triangle.

Remark 4.2. A special cycle is considered in [**9**], and they prove that $I^{(\ell)} = I^\ell$ holds for any $\ell \geq 1$ if there exists no special odd cycle in $\mathcal{H}(I)$.

The following is the main theorem in this section.

Theorem 4.3. *Let I be a squarefree monomial ideal. Then the following conditions are equivalent*:

(1) $I^{(2)} = I^2$ *holds.*

(2) *If there exist* $\{H_1, H_2, H_3\} \subseteq \mathcal{H}(I)$ *such that* H_1, H_2, H_3 *make a special triangle, then* $x^{H_1 \cap H_2 \cap H_3} x^{H_1 \cup H_2 \cup H_3} \in I^2$.

Remark 4.4. If no special triangles exist, then we have $I^{(2)} = I^2$. The converse is not true.

The following criterion is well known, see [**19**].

Corollary 4.5. *Let $I(G)$ denote the edge ideal of a graph G. Then $I(G)^{(2)} = I(G)^2$ holds if and only if G has no triangles* (*the cycles of length* 3).

In what follows, we prove the above theorem. First we prove the following lemma.

Lemma 4.6. *Suppose that condition* (2) *in Theorem* 4.3 *holds. Then $xI \cap (I^2 : x) \subseteq I^2$ holds for every $x \in V$.*

Proof. Suppose that there exist a variable x_1 and a monomial M such that $M \in x_1 I \cap (I^2 : x) \setminus I^2$. As $x_1 M \in I^2$, we can take N_2, $N_3 \in G(I)$ and a monomial L such that

$$x_1 M = N_2 N_3 L. \tag{4.1}$$

On the other hand, as $M \in x_1 I$, we can choose $N_1 \in G(I)$ and a monomial L' such that

$$M = N_1 L' \quad \text{and} \quad x_1 \mid L'. \tag{4.2}$$

Claim 1: $x_1 \mid N_2$, $x_1 \mid N_3$ but $x_1 \nmid N_1$. As $M \notin I^2$, x_1 does not divide L. By equations (4.1), (4.2) and $N_2 N_3 L$ is divided by x_1^2. Hence x_1 divides both N_2 and N_3 because N_i is a squarefree monomial for $i = 2, 3$. By a similar reason, we have that N_1 is not divided by x_1.

Claim 2: $N_2 \neq N_3$ and $\gcd(N_2, N_3) \mid L'$. If $N_2 = N_3$, then $x_1N_1L' = N_3^2L$ is divided by x_1N_1 and thus N_3L is divided by x_1N_1. Then $M = N_1N_2(N_3L/x_1N_1) \in I^2$. This is a contradiction. Hence $N_2 \neq N_3$.

Since $x_1N_1L' = N_2N_3L$ is divided by $\gcd(N_2, N_3)^2$, L' is divided by $\gcd(N_2, N_3)$ because x_1N_1 is squarefree.

Claim 3: There exist variables x_2, x_3 such that

$$x_2 \,\Big|\, \frac{N_3}{\gcd(N_2, N_3)}, \quad x_3 \,\Big|\, \frac{N_2}{\gcd(N_2, N_3)}, \quad x_2, x_3 \mid N_1.$$

Note that any variable which divides N_i for $i = 2, 3$ is a factor of N_1 or L'. Since $L' \notin I$, $L'/\gcd(N_2, N_3)$ is not divided by $N_3/\gcd(N_2, N_3)$. Thus there exists a variable x_2 such that $x_2 \mid N_3/\gcd(N_2, N_3)$ and $x_2 \mid N_1$. The other statement follows from a similar argument.

Take $H_i \in \mathcal{H}(I)$ such that $x^{H_i} = N_i$ for each $i = 1, 2, 3$.

Claim 4: H_1, H_2, H_3 make a special triangle $\{1, 2, 3\}$. The assertion immediately follows from Claim 1 and Claim 3. By Claim 4, we get a contradiction.

By assumption, we get

$$\gcd(N_1, N_2, N_3)\sqrt{N_1N_2N_3} = x^{H_1 \cap H_2 \cap H_3} \cdot x^{H_1 \cup H_2 \cup H_3} \in I^2,$$

where $\sqrt{N} = x_{i_1} \cdots x_{i_r}$ for a monomial $N = x_{i_1}^{a_{i_1}} \cdots x_{i_r}^{a_{i_r}}$ $(a_{i_j} > 0)$. Since N_1 divides N_2N_3L and $x_1 \mid N_2,\ N_3$, we have

$$\sqrt{N_1N_2N_3} \,\Big|\, \frac{N_2N_3L}{x_1} = M. \tag{4.3}$$

On the other hand, since $x_1 \nmid \gcd(N_1, N_2, N_3)$, we have

$$\gcd(N_1, N_2, N_3)^2 \,\Big|\, \frac{N_2N_3}{x_1} \,\Big|\, M. \tag{4.4}$$

Hence equations (4.3) and (4.4) imply

$$\gcd(N_1, N_2, N_3)\sqrt{N_1N_2N_3} \mid M.$$

Therefore $M \in I^2$, which contradicts the choice of M. □

Now suppose that $I_x^{(2)} = I_x^2$ holds for every vertex $x \in V$. Then $I^{(2)} = I^2$ if and only if $\mathfrak{m} \notin \operatorname{Ass}(S/I^2)$. Hence the following lemma is useful when we use an induction.

Lemma 4.7 (see the proof of [**20**, Theorem 5.9]). *Let I be a squarefree monomial ideal of S with $\dim S/I \geq 1$. Now suppose that $xI \cap (I^2 : x) \subseteq I^2$ for every variable x. Then $\mathfrak{m} \notin \operatorname{Ass}_S(S/I^2)$.*

Proof. Since I^2 and $\mathfrak{m}$ are monomial ideals, it suffices to show $I^2 : M \neq \mathfrak{m}$ for every variable x and any monomial M.

Now suppose that $I^2 : M = \mathfrak{m}$ for some monomial $M \notin I^2$. Since $\mathfrak{m}M \subseteq I^2 \subseteq I$ and $\operatorname{depth} S/I > 0$, we have $M \in I$. So we may assume that $M = x_1 \cdots x_k L$, where $N = x_1 \cdots x_k \in G(I)$ and L is a monomial. By assumption, $x_k M = x_1(x_2 \cdots x_{k-1} x_k^2 L) \in I^2$. Since I is generated by squarefree monomials, we then have $x_2 \cdots x_{k-1} x_k^2 L \in I$ and hence $x_2 \cdots x_{k-1} x_k L \in I$. Hence $M \in x_1 I \cap (I^2 : x_1) \subseteq I^2$. This is a contradiction. $\square$

Proof of Theorem 4.3. First we show (2) $\Rightarrow$ (1). Suppose (2). Since this condition preserves under localization, we may assume that $(I^{(2)})_x = (I^2)_x$ for any variable x by an induction on $\dim S/I$. By the above two lemmata, we have $\mathfrak{m} \notin \operatorname{Ass}_S(S/I^2)$. Hence $I^{(2)} = I^2$, as required.

Next we show (1) $\Rightarrow$ (2). Suppose that there exists a subset $\{H_1, H_2, H_3\} \subseteq \mathcal{H}(I)$ such that H_1, H_2, H_3 make a special triangle and $x^{H_1 \cap H_2 \cap H_3} x^{H_1 \cup H_2 \cup H_3} \notin I^2$. Then it suffices to show $I^2 \subsetneq I^{(2)}$.

Put $H = H_1 \cup H_2 \cup H_3$. Let I_H be the squarefree monomial ideal of $K[x : x \in V \setminus H]$ such that $I_H S + (x \in V \setminus H) = I + (x \in V \setminus H)$. Let P be any minimal prime ideal of I_H. If $\operatorname{height} P = 1$, then there exists a vertex $j \in H_1 \cap H_2 \cap H_3$ such that $P = (x_j)$. Then $M := x^{H_1 \cap H_2 \cap H_3} x^H \in (x_j^2) = P^2$. If $\operatorname{height} P \geq 2$, then P contains two variables x_i, x_j with $i, j \in H$. Then $x^H \in P^2$ and hence $M \in P^2$. Therefore $M \in I_H^{(2)}$ but $M \notin I_H^2$ by the assumption that $M \notin I^2$. $\square$

Suppose $U \cap V = \varnothing$. Let Γ (respectively Λ) be a simplicial complex on U (respectively V). Then the *simplicial join* of Γ and Λ, denoted by $\Gamma * \Lambda$, is defined by $\Gamma * \Lambda = \{F \cup G : F \in \Delta,\ G \in \Lambda\}$. It is a simplicial complex on $U \cup V$.

The following corollary is probably well known (and hence so is Corollary 4.9), but we give a proof as an application of Theorem 4.3.

Corollary 4.8. *Let Γ be a simplicial complex on U and Λ a simplicial complex on V. Let $\Delta = \Gamma * \Lambda$ denote the simplicial join of Γ and Λ. Then Δ is a simplicial complex on $W = U \coprod V$. Put $R = K[U]$, $S = K[V]$ and $T = R \otimes_K S \cong K[W]$. Then*:

(1) *$I_\Delta^{(2)} = I_\Delta^2$ if and only if $I_\Gamma^{(2)} = I_\Gamma^2$ and $I_\Lambda^{(2)} = I_\Lambda^2$.*

(2) *T/I_Δ^2 is Cohen-Macaulay if and only if so do R/I_Γ^2 and S/I_Λ^2.*

Proof. (1) Note that $I_\Delta = I_\Gamma T + I_\Lambda T$ and $G(I_\Delta)$ is a disjoint union of $G(I_\Gamma)$ and $G(I_\Lambda)$. Thus it immediately follows from Theorem 4.3.

(2) It immediately follows from (1) and [**17**, Theorem 2.7]. □

A disjoint union of two graphs G_1 and G_2, denoted by $G_1 \coprod G_2$, is the graph G which satisfies $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Let $G = G_1 \coprod \cdots \coprod G_r$ be a disjoint union of graphs $G_1, \ldots, G_r$, and let Δ_i (respectively Δ) be the complementary simplicial complex of G_i for each $i = 1, \ldots, r$ (respectively G). Then Δ is equal to the simplicial join $\Delta_1 * \cdots * \Delta_r$.

Corollary 4.9. *Let $G = G_1 \coprod \cdots \coprod G_r$ be a disjoint union of graphs G_i for which $I(G_i)^2$ is a Cohen-Macaulay ideal. Then $I(G)^2$ is a Cohen-Macaulay ideal.*

Example 4.10. Let $G = G_1 \coprod \cdots \coprod G_r$ be a disjoint union of the pentagons G_i for $i = 1, \ldots, r$. Then $I(G)^2$ is a Cohen-Macaulay ideal.

Proof. It follows that the second symbolic power of the edge ideal of the pentagon is a Cohen-Macaulay ideal. □

5. Examples of Stanley-Reisner ideals whose square is Cohen-Macaulay. By Corollary 4.8 we know that there exists a simplicial complex Δ with arbitrary high dimension such that I_Δ^2 is non-trivially Cohen-Macaulay. We now consider the following question.

Question 5.1. For a given integer $d \geq 2$, is there a simplicial complex Δ with $\dim \Delta = d-1$ such that S/I_Δ^2 is Cohen-Macaulay and such that Δ cannot be expressed as the simplicial join of two non-empty complexes?

We give two families of examples as affirmative answers, using liaison theory. The following key proposition is due to Buchweitz [**5**]; see also Kustin and Miller [**13**]. Note that it gives a partial converse of Theorem 2.1.

Proposition 5.2 (cf. [**5**, 6.2.11], [**13**, Proposition 7.1]). *Let I be a Gorenstein homogeneous ideal in a polynomial ring S. Assume that there exist a homogeneous polynomial ring $T = S[z_1, \ldots, z_r]$ $(\deg z_i = 1)$ and a homogeneous radical ideal L such that*

(a) $S/I \cong T/(z_1, \ldots, z_r, L)$.

(b) $z_1, \ldots, z_r$ *is a regular sequence on T/L.*

(c) L *is in the linkage class of a complete intersection in T.*

Then S/I^2 is Cohen-Macaulay.

Proof. Since S/I^2 is isomorphic to the ring $T/(z_1, \ldots, z_r, L^2)$, it is enough to show that T/L^2 is Cohen-Macaulay.

Let $\mathfrak{M}$ be the unique homogeneous maximal ideal of T, and set $R = \widehat{T_{\mathfrak{M}}}$, the $\mathfrak{M}$-adic completion of $T_{\mathfrak{M}}$. As R/LR is a radical Gorenstein ideal, we can conclude that $LR/(LR)^2$ is Cohen-Macaulay, and thus $R/(LR)^2$ is Cohen-Macaulay by [**13**, Proposition 7.1]. It follows from Matijevic-Roberts theorem that T/L^2 is Cohen-Macaulay, as required. $\square$

It is well known that any Gorenstein ideal of codimension 3 lies in the linkage class of a complete intersection, see [**4, 31**] or [**28**, Theorem 4.15]. Thus we can obtain the following corollary.

Corollary 5.3. *Let $I_\Delta \subseteq S$ be a Gorenstein Stanley-Reisner ideal of codimension 3. Then S/I_Δ^2 is Cohen-Macaulay.*

In the rest of this section we prove the second power of the Stanley-Reisner ideal of a stellar subdivision of any non-acyclic complete intersection complex is Cohen-Macaulay. In what follows, as vertices of simplicial complexes we use indeterminates instead of natural numbers for convenience. Let Γ be a non-acyclic complete intersection simplicial complex whose Stanley-Reisner ideal is

$$I_\Gamma = (x_{11}x_{12}\cdots x_{1i_1}, x_{21}x_{22}\cdots x_{2i_2}, \dots, x_{\mu 1}x_{\mu 2}\cdots x_{\mu i_\mu}).$$

Let $\mathcal{F}(\Gamma)$ be the set of all facets of Γ. Then

$$\begin{aligned}\mathcal{F}(\Gamma) = \{\{&x_{11}, \dots, \widehat{x_{1k_1}}, \dots, x_{1i_1}, x_{21}, \dots, \widehat{x_{2k_2}}, \dots, x_{2i_2}, \dots, \\ &x_{\mu 1}, \dots, \widehat{x_{\mu k_\mu}}, \dots, x_{\mu i_\mu}\} \\ &\mid 1 \le k_1 \le i_1, 1 \le k_2 \le i_2, \dots, 1 \le k_\mu \le i_\mu\}.\end{aligned}$$

Let Δ be the *stellar subdivision* of Γ on

$$F = \{x_{11}, \dots, x_{1j_1}, x_{21}, \dots, x_{2j_2}, \dots, x_{p1}, \dots, x_{pj_p}\},$$

where $1 \le p \le \mu$ and $1 \le j_1 < i_1, \dots, 1 \le j_p < i_p$ and $j_1 + \cdots + j_p \ge 2$.

Let v be the new added vertex. Then

$$\begin{aligned}\mathcal{F}(\Delta) &= \{G \in \mathcal{F}(\Gamma) \mid G \not\supset F\} \cup \{\{v\} \cup G \setminus \{w\} \mid G \supset F, w \in F\} \\ &= \{\{x_{11}, \dots, \widehat{x_{1k_1}}, \dots, x_{1i_1}, x_{21}, \dots, \widehat{x_{2k_2}}, \dots, x_{2i_2}, \dots, \\ &\qquad x_{\mu 1}, \dots, \widehat{x_{\mu k_\mu}}, \dots, x_{\mu i_\mu}\} \\ &\quad \mid 1 \le k_1 \le i_1, 1 \le k_2 \le i_2, \dots, 1 \le k_\mu \le i_\mu \\ &\qquad \text{with } 1 \le k_1 \le j_1 \text{ or } 1 \le k_2 \le j_2 \text{ or } \cdots \text{ or } 1 \le k_p \le j_p\} \\ &\cup \{\{v, x_{11}, \dots, \widehat{x_{1k_1}}, \dots, x_{1i_1}, x_{21}, \dots, \widehat{x_{2k_2}}, \dots, x_{2i_2}, \dots, \\ &\qquad x_{\mu 1}, \dots, \widehat{x_{\mu k_\mu}}, \dots, x_{\mu i_\mu}\} \setminus \{w\} \\ &\quad \mid j_1 + 1 \le k_1 \le i_1, j_2 + 1 \le k_2 \le i_2, \dots, j_p + 1 \le k_p \le i_p \\ &\qquad 1 \le k_{p+1} \le i_{p+1}, \dots, 1 \le k_\mu \le i_\mu, w \in F\}\end{aligned}$$

and

$$I_\Delta = (I_\Gamma, x_F, vx_{1j_1+1}\cdots x_{1i_1}, vx_{2j_2+1}\cdots x_{2i_2}, \dots, vx_{pj_p+1}\cdots x_{pi_p})$$

is an ideal of a polynomial ring

$$S = k[x_{11}, \dots, x_{1i_1}, x_{21}, \dots, x_{2i_2}, \dots, x_{\mu 1}, \dots, x_{\mu i_\mu}, v].$$

Applying Proposition 5.2 to this ideal $I = I_\Delta$, we obtain the following theorem. The two-dimensional case is proved in [**26**].

Theorem 5.4. *Let $\Delta = \Gamma_F$ be the stellar subdivision of the non-acyclic complete intersection complex Γ as above. Then S/I_Δ^2 is Cohen-Macaulay.*

Proof. Consider the variables $\underline{z} = z_1, z_2, \ldots, z_N$, where $N = j_1 + \cdots + j_p - 1$, and put $Z = z_1 \cdots z_N$. Moreover, we set

$$\begin{array}{lcllcl} X_1 & = & x_{1,1} \cdots x_{1,j_1}, & Y_1 & = & x_{1,j_1+1} \cdots x_{1,i_1}, \\ X_2 & = & x_{2,1} \cdots x_{2,j_2}, & Y_2 & = & x_{2,j_2+1} \cdots x_{1,i_2}, \\ & \vdots & & & \vdots & \\ X_p & = & x_{p,1} \cdots x_{p,j_p} & Y_p & = & x_{p,j_p+1} \cdots x_{p,i_p}, \\ & & & Y_{p+1} & = & x_{p+1,1} \cdots x_{p+1,i_{p+1}}, \\ & & & & \vdots & \\ & & & Y_\mu & = & x_{\mu,1} \cdots x_{\mu,i_\mu} \end{array}$$

and

$$L = (I_\Gamma, vY_1, \ldots, vY_p, vZ - x_F) \subseteq T = S[\underline{z}].$$

Then $I_\Gamma = (X_1Y_1, \ldots, X_pY_p, Y_{p+1}, \ldots, Y_\mu)$, $I_\Delta = (I_\Gamma, x_F, vY_1, \ldots, vY_p)$ and S/I_Δ is isomorphic to $T/(\underline{z}, L)$.

In what follows, we show that L lies in the linkage class of a complete intersection (i.e., licci). Firstly, we can easily prove the following equality:

$$(I_\Gamma, Z) \colon (Y_1, \ldots, Y_\mu, Z) = (I_\Gamma, Z, x_F). \tag{5.1}$$

Secondly we show the following equality:

$$L = (I_\Gamma, vZ - x_F) \colon (I_\Gamma, Z, x_F). \tag{5.2}$$

To end this, it is enough to show the right-hand side is contained in L. Let $\alpha \in (I_\Gamma, vZ - x_F) \colon (I_\Gamma, Z, x_F)$. Then there exists $\beta \in T$ such that $\alpha Z - \beta(vZ - x_F) \in I_\Gamma$. Then $\beta \in (I_\Gamma, Z) \colon x_F = (Y_1, \ldots, Y_\mu, Z)$. In particular, we can write $\beta = \sum_{i=1}^{\mu} \gamma_i Y_i + \delta Z$ for some γ_i, $\delta \in T$. It follows that

$$Z\left[\alpha - \sum_{i=1}^{p} \gamma_i(vY_i) - \delta(vZ - x_F)\right] \in I_\Gamma.$$

As Z is a nonzero divisor on $T/I_\Gamma T$, we conclude that $\alpha \in L$.

In equations (5.1), (5.2), both (I_Γ, Z) and $(I_\Gamma, vZ - x_F)$ are complete intersection ideals of the same height $\mu + 1$ as $(Y_1, \dots, Y_\mu, Z)$ or L. Hence L is licci.

In order to prove that S/I_Δ^2 is Cohen-Macaulay by Proposition 5.2, it is enough to show that $\underline{z}$ is a regular sequence on T/L and that T/L is reduced. By the above proof, we have that L is licci and $\dim T/L = \dim T/(Y_1, \dots, Y_\mu, Z)$. In particular, L is Cohen-Macaulay and $\dim T/L = i_1 + \cdots + i_\mu - \mu + N$.

On the other hand,

$$\begin{aligned}\dim T/(\underline{z}, L) &= \dim S/I_\Delta = \dim S/(I_\Gamma, v) = i_1 + \cdots + i_\mu - \mu \\ &= \dim T/L - N.\end{aligned}$$

This implies that $\underline{z}$ is a regular sequence on T/L. Moreover, as $T/(\underline{z}, L)$ is reduced, so is T/L, as required. □

Remark 5.5. The above Gorenstein ideals are obtained from the so-called Herzog ideals (see [**8, 11, 12, 13**]) and T/L is called the *Kustin-Miller unprojection ring* ([**2**]). Moreover, the assertion of Theorem 5.4 says that the quotient algebras of those ideals are *strongly unobstructed*.

Example 5.6 (Cross Polytope). Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the fundamental vectors of the d-dimensional Euclidean space $\mathbf{R}^d$. Then the convex hull $\mathcal{P} = \mathrm{CONV}(\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \dots, \pm\mathbf{e}_d\})$ is called the *cross d-polytope*. Let Γ be the boundary complex of the cross d-polytope $\mathcal{P}$. Let $W = \{x_1, \dots, x_d, y_1, \dots, y_d\}$. For a sequence $\mathbf{i} = [i_1, \dots, i_m]$ with $1 \le i_1 < \cdots < i_m \le d$, we assign a subset of W

$$F_{\mathbf{i}} = \big\{x_{i_1}, \dots, x_{i_m}\big\} \cup \big\{y_j \,:\, j \in [d] \setminus \{i_1, \dots, i_m\}\big\}.$$

Then Γ can be regarded as a simplicial complex on W such that

$$\mathcal{F}(\Gamma) = \{F_{\mathbf{i}} : m = 0, 1, \dots, d,\ 1 \le i_1 < \cdots < i_m \le d\},$$

and it is a $(d-1)$-dimensional complete intersection complex with

$$I_\Gamma = (x_1y_1, x_2y_2, \ldots, x_dy_d).$$

Let v be a new vertex, and choose a facet $F_{[1,2,\ldots,d]} = \{x_1, \ldots, x_d\}$ of Γ. Let Δ be the stellar subdivision of Γ on F. Then Δ is a $(d-1)$-dimensional Gorenstein complex on $V = W \cup \{v\}$ and its geometric realization of Δ is homeomorphic to $\mathbf{S}^{d-1}$. The above theorem says that the second power of

$$I = (x_1y_1, x_2y_2, \ldots, x_dy_d,\ vy_1, \ldots, vy_d,\ x_1x_2\cdots x_d)$$

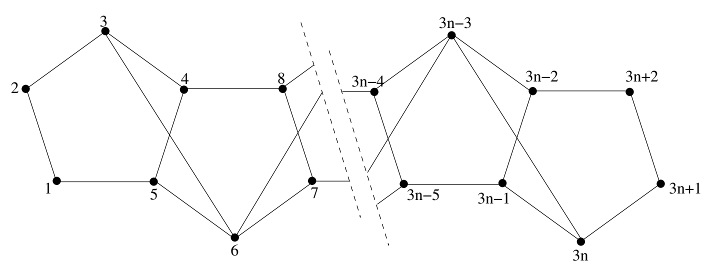
is Cohen-Macaulay, but the third power is not if $d \geq 2$ because the third power of the Stanley-Reisner ideal $(x_1y_1, x_2y_2, vy_1, vy_2, x_1x_2)$ of a pentagon is not.

In the last part of the paper, we give candidates of edge ideals $I(G)$ for which $S/I(G)^2$ is Cohen-Macaulay (but $S/I(G)^3$ is not by [**19**]). The case $n = 2$ is mentioned in [**26**, Theorem 3.7 (iv)].

Conjecture 5.7. *Let G be a graph on the vertex set $V = \{x_1, x_2, \ldots, x_{3n+2}\}$ with*

$$I(G) = \big(x_1x_2, \{x_{3k-1}x_{3k}, x_{3k}x_{3k+1}, x_{3k+1}x_{3k+2}, x_{3k+2}x_{3k-2}\}_{k=1,2,\ldots,n}, \{x_{3\ell-3}x_{3\ell}\}_{\ell=2,3,\ldots,n}\big).$$

Then $S/I(G)^2$ is Cohen-Macaulay but $S/I(G)^3$ is not.



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REFERENCES

1. R. Achilles and W. Vogel, *Über vollständige Durchschnitte in lokalen Ringen*, Math. Nachr. **89** (1979), 285–298.

2. J. Böem and S.A. Papadakis, *Stellar subdivisions and Stanley-Reisner rings of Gorenstein complexes*, preprint available from arXiv:0912.215v1 [math.AC].

3. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1993.

4. D. Buchsbaum and D. Eisenbud, *Algebraic structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), 447–485.

5. R. Buchweitz, *Contributions á la théorie des singularités*, Ph.D. thesis, University of Paris, 1981.

6. M. Crupi, G. Rinaldo and N. Terai, *Cohen-Macaulay edge ideal whose height is half of the number of vertices*, Nagoya Math. J. **201** (2011), 117–131.

7. M. Crupi, G. Rinaldo, N. Terai and K. Yoshida, *Effective Cowsik-Nori theorem for edge ideals*, Comm. Alg. **38** (2010), 3347–3357.

8. J. Herzog, *Certain complexes associated to a sequence and a matrix*, Manuscripta Math. **12** (1974), 217–248.

9. J. Herzog, T. Hibi, N.V. Trung and X. Zheng, *Standard graded vertex cover algebras, cycles and leaves*, Trans. Amer. Math. Soc. **291** (2008), 6231–6249.

10. J. Herzog, Y. Takayama and N. Terai, *On the radical of a monomial ideal*, Arch. Math. **85** (2005), 397–408.

11. C. Huneke, *Linkage and Koszul homology of ideals*, Amer. J. Math. **104** (1982), 1043–1062.

12. A.R. Kustin and M. Miller, *Multiplicative structure on resolutions of algebras defined by Herzog ideals*, J. London Math. Soc. **28** (1983), 247–260.

13. ———, *Deformation and linkage of Gorenstein algebras*, Trans. Amer. Math. Soc. **284** (1984), 501–534.

14. H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

15. N.C. Minh and Y. Nakamura, *Buchsbaum properties of symbolic powers of Stanley-Reisner ideals of dimension one*, J. Pure Appl. Algebra **215** (2011), 161–167.

16. N.C. Minh and N.V. Trung, *Cohen-Macaulayness of powers of two-dimensional squarefree monomial ideals*, J. Algebra **322** (2009), 4219–4227.

17. ———, *Cohen-Macaulayness of monomial ideals and symbolic powers of Stanley-Reisner ideals*, Adv. Math. **226** (2011), 1285–1306.

18. S. Murai and N. Terai, *H-vectors of simplicial complexes with Serre's conditions*, Math. Res. Lett. **16** (2009), 1015–1028.

19. G. Rinaldo, N. Terai, and K. Yoshida, *Cohen-Macaulayness for symbolic power ideals of edge ideals*, submitted.

20. A. Simis, W.V. Vasconcelos and R.H. Villarreal, *On the ideal theory of graphs*, J. Algebra **167** (1994), 389–416.

21. R.P. Stanley, *Combinatorics and commutative algebra*, Second edition, Birkhäuser, Boston, 1996.

22. J. Stückrad and W. Vogel, *Buchsbaum rings and applications*, Springer-Verlag, Berlin, 1986.

23. Y. Takayama, *Combinatorial characterization of generalized Cohen-Macaulay monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **48** (2005), 327–344.

24. N. Terai and N.V. Trung, *Cohen-Macaulayness of large powers of Stanley-Reisner ideals*, preprint available from arXiv:1009.0833v1 [math.AC].

25. N. Terai and K. Yoshida, *Locally complete intersection Stanley-Reisner ideals*, Illinois J. Math. **53** (2009), 413–429.

26. N.V. Trung and T.M. Tuan, *Equality of ordinary and symbolic powers of Stanley-Reisner ideals*, J. Algebra **328** (2011), 77–93.

27. M. Varbaro, *Symbolic powers and matroids*, preprint, 2010.

28. W.V. Vasconcelos, *Arithmetic of blowup algebras*, London Math. Soc. Lect. Note Series, Cambridge University Press, Cambridge, 1994.

29. R.H. Villarreal, *Monomial algebras*, Pure Appl. Math., Marcel Dekker, New York, 2001.

30. R. Waldi, *Vollständige Durchschnitte in Cohen-Macaulay-Ringen*, Arch. Math. (Basel) **31** (1978/1979), 439–442.

31. J. Watanabe, *A note on Gorenstein rings of embedding codimension three*, Nagoya Math. J. **50** (1973), 227–232.

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