GRÖBNER STRATA IN
THE HILBERT SCHEME OF POINTS

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ABSTRACT. The present paper shall provide a framework for working with Gröbner bases over arbitrary rings k with a prescribed finite standard set Δ. We show that the functor associating to a k-algebra B the set of all reduced Gröbner bases with standard set Δ is representable and that the representing scheme is a locally closed stratum in the Hilbert scheme of points. We cover the Hilbert scheme of points by open affine subschemes which represent the functor associating to a k-algebra B the set of all border bases with standard set Δ and give reasonably small sets of equations defining these schemes. We show that the schemes parametrizing Gröbner bases are connected; give a connectedness criterion for the schemes parametrizing border bases; and prove that the decomposition of the Hilbert scheme of points into the locally closed strata parametrizing Gröbner bases is not a stratification.

1. Introduction. Let k be a ring and $S = k[x_1, \ldots, x_n]$ be the polynomial ring over k, equipped with a term order $\prec$. There are various notions of Gröbner bases, and of reduced Gröbner bases, of an ideal $I \subset S$ (see [**36**] for an overview). We use that notion of a reduced Gröbner basis which is entirely analogous to the definition in the case where k is a field. The definition will be given in Section 2. The same notion of a reduced Gröbner basis is used in [**40**], a paper which was a significant source of inspiration for the work presented here. However, not every ideal I has a reduced Gröbner basis in this sense; a reduced Gröbner basis exists if, and only if, I is a *monic* ideal. Attached to a monic ideal is its standard set, which is the set of those elements of $\mathbf{N}^n$

which do not occur as the multidegree of an element of I. For getting a feeling for monic ideals, let us look at some examples.

Example 1. Let $n = 1$ and $I = (f) \subset k[x]$ be a principal ideal. Then I is a monic ideal if, and only if, it is generated by a monic polynomial. In this case the reduced Gröbner basis of I consists of f alone. The standard set of I is $\{0, \dots, \deg(f) - 1\} \subset \mathbf{N}$.

Example 2. Let $n = 2$. We equip S with the lexicographic order such that $x_1 \succ x_2$. The ideal $I_1 \subset \mathbf{Z}[x_1, x_2]$ generated by

$$f = x_2^3 - 3x_2^2 + 2x_2,$$
$$g = x_1^2 x_2 - x_1^2 + x_1 x_2 - x_1 + x_2 - 1,$$
$$h = x_1^4 + 2x_1^3 x_2^2 - 4x_1^3 x_2 + x_1 + x_1^2 x_2^2 - x_1^2 x_2 - x_1^2$$
$$\quad - 50x_1 x_2^2 + 49x_1 x_2 + x_1 + 50x_2^2 - 49x_2 - 1$$

is monic with standard set

$$\Delta = \{(0,0), (1,0), (2,0), (3,0), (0,1), (1,1), (0,2), (1,2)\} \subset \mathbf{N}^2.$$

The reduced Gröbner basis of I_1 consists of f, g and $\widetilde{h}$, where

$$\widetilde{h} = x_1^4 + x_1^3 - x_1^2 + 8x_1 x_2^4 - 46x_1 x_2^3 + 35x_1 x_2^2 - 2x_1 x_2 + x_1$$
$$\quad + 8x_2^4 - 42x_2^3 + 20x_2^2 + 13x_2 - 1 \,.$$

Example 3. Using the notation of the previous example, the ideal

$$I_2 = (2f, g, h) \subset \mathbf{Z}[x_1, x_2]$$

is not monic, and accordingly, does not have a reduced Gröbner basis. The reason for that is the exponent $(0, 3) \in \mathbf{N}^2$, which appears as the multidegree of a monic element of I_1, but not as the multidegree of any monic element of I_2.

Before outlining our article, let us briefly summarize its main ideas:

• A good notion of Gröbner basis over an arbitrary ring is that of the reduced Gröbner basis of a monic ideal.

• The reducedness property guarantees functoriality of Gröbner bases.

• Functoriality guarantees the existence of a moduli space.

• Some familiar techniques for Gröbner bases over fields can be carried over to the setup over rings.

• S-pair criteria as such are not needed.

If B is a k-algebra and Δ is a finite standard set, we attach to B the set of all monic ideals $I \subset B[x]$ with standard set Δ. As the reduced Gröbner basis of a monic ideal is unique, we may equivalently attach to B the set of all reduced Gröbner bases in $B[x]$ with standard set Δ. It turns out that this map is functorial in B. We denote the functor by $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$. Note the dependence of this functor on both Δ and the term order $\prec$. The notation is motivated by the fact that $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$ is a subfunctor of the *Hilbert functor of points* $\mathcal{H}\mathrm{ilb}_{S/k}^{d}$. The Hilbert functor of points has been widely studied (see [**4, 16, 20, 21**] and references therein). In particular, it is well known that this functor is represented by a scheme $\mathrm{Hilb}_{S/k}^{d}$. The notions of Hilbert functor and Hilbert scheme were introduced by Grothendieck in [**15**]; see [**33**] for an introductory account of the subject. In our paper we will show that $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$ is a locally closed subfunctor of $\mathcal{H}\mathrm{ilb}_{S/k}^{d}$, hence representable by a locally closed subscheme of the Hilbert scheme. We will study an intermediate functor $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}$, which is also representable, such that in the chain of representing objects

$$(1) \qquad\qquad \mathrm{Hilb}_{S/k}^{\prec\Delta} \subset \mathrm{Hilb}_{S/k}^{\Delta} \subset \mathrm{Hilb}_{S/k}^{d},$$

the first inclusion is a closed immersion and the second inclusion is an open immersion.

The moduli spaces $\mathrm{Hilb}_{S/k}^{\Delta}$ and $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ have been studied by numerous authors, at least in the case where k is a field. In the article [**26**], the scheme $\mathrm{Hilb}_{S/k}^{\Delta}$ is called a *border basis scheme*, and in the article [**37**], the scheme $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ is called a *Gröbner basis scheme*. In the cited papers, and in [**22–25**], a theory of border bases, which generalizes the theory of Gröbner basis, is developed. In the present paper we use border bases as well, in studying the functor $\mathrm{Hilb}_{S/k}^{\Delta}$. Some of the results of the cited papers are parallel to those of our article here. Each time we state one such result, we will indicate its relation to the cited papers.

However, the two major differences between the cited papers and our treatment here are that firstly our treatment is more general, as our k is an arbitrary ring, and secondly we use the functorial language.

In fact, the starting point for the work presented here was the desire to obtain a *relative*, i.e., *functorial* notion of Gröbner bases. The desire for functoriality is what motivates the use of arbitrary rings rather than fields. However, functoriality only holds if the leading terms of a Gröbner basis are stable under arbitrary tensor products. Therefore we have to use monic ideals, and correspondingly, reduced Gröbner bases. Moreover, several of our results are novel, or stronger than previous results, even if we specialize to the case where k is a field.

The paper which bears the closest relationship to our paper here is [20]. Huibregtse studies the functors $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}$ and schemes $\mathrm{Hilb}^{\Delta}_{S/k}$, as we do here. (A standard set Δ in our notation corresponds to a *basis set* β in his, and $\mathrm{Hilb}^{\Delta}_{S/k}$ in our notation is U_{β} in his.) In this sense Huibretse also covers the functorial properties of $\mathrm{Hilb}^{\Delta}_{S/k}$. However, his viewpoint is different from ours: In [20, Lemma 7], he shows how to glue the schemes $\mathrm{Hilb}^{\Delta}_{S/k}$, for all standard sets $\Delta \subset \mathbf{N}^n$ of size d, to obtain the Hilbert scheme $\mathrm{Hilb}^{d}_{S/k}$. This implies in particular that each $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}$ is an open subfunctor of $\mathcal{H}\mathrm{ilb}^{d}_{S/k}$, and that the various functors $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}$ form an open cover of $\mathcal{H}\mathrm{ilb}^{d}_{S/k}$. These two statements are Lemma 1 and Proposition 1, respectively, in the present paper. In contrast to Huibregtse's approach, we show these statements directly at the level of functors, without using the representing schemes. Moreover, Huibretse's construction of $\mathrm{Hilb}^{\Delta}_{S/k}$ is entirely different from ours. His is based on *pseudosyzygies* and *syzygies*, whereas ours avoids the use of the S-pair criteria altogether, as was mentioned above. Furthermore, the notion of reduced Gröbner bases and monic ideals over a ring k do not appear in [20]. Accordingly, the schemes $\mathrm{Hilb}^{\prec\Delta}_{S/k}$, which we are mostly interested in here, do not appear in the cited paper. In this sense our results on the functorial properties of $\mathrm{Hilb}^{\prec\Delta}_{S/k}$ are original.

Another line of work is to study strata analogous to ours in Grothendieck's classical Hilbert scheme $\mathrm{Hilb}^{p(z)}_{\mathbf{P}^n_k}$, where $p(z)$ is a polynomial. The paper [34] is devoted to this project; the equations defining the strata are derived from Buchberger's S-pair criterion. However, the cited paper contains a few inconsistencies, as is indicated in [37] and [30]. In particular, in the latter paper, the embedding of the strata

in $\mathrm{Hilb}_{\mathbf{P}_k^n}^{p(z)}$ is elaborated upon with care. Also, it appears to be the first paper in which the term *Gröbner stratum* is used. Other papers in which related ideas appear are [**5**, **38**].

As was mentioned above, the research presented here was largely inspired by the paper [**40**]. In that paper, k is a Noetherian ring. Wibmer considers an arbitrary ideal $I \subset S$. The canonical map $k \to S/I$ corresponds to a morphism of affine schemes $\phi : \mathrm{Spec}\, S/I \to \mathrm{Spec}\, k$. The main theorem of [**40**] (Theorem 11) states the existence of a unique decomposition of $\mathrm{Spec}\, k$ into a finite number of locally closed strata such that on each stratum the reduction of I to each point of the stratum has a reduced Gröbner basis of a prescribed shape. It is striking to note the analogy of that theorem to Theorem 2 of our paper here. However, in Wibmer's setting k has to be Noetherian, whereas our setting requires no restriction on k.

Our article is organized as follows. In Section 2, we introduce the basic notions of monic ideals, reduced Gröbner bases and standard sets. In Section 3, we define the Hilbert functor $\mathcal{H}\mathrm{ilb}_{S/k}^d$ and the open subfunctors $\mathcal{H}\mathrm{ilb}_{S/k}^\Delta$, where Δ runs through all standard sets of size d. In Section 4, we thoroughly prove that these subfunctors cover the whole functor $\mathcal{H}\mathrm{ilb}_{S/k}^d$. That gives us the key to defining the subfunctor $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$ of $\mathcal{H}\mathrm{ilb}_{S/k}^\Delta$ in Section 5. In Section 6, we show that, from representability of $\mathcal{H}\mathrm{ilb}_{S/k}^\Delta$, representability of $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$ follows. In Section 7, we show that the Hilbert scheme $\mathrm{Hilb}_{S/k}^d$ is the disjoint union of the representing schemes $\mathrm{Hilb}_{S/k}^{\prec\Delta}$. Thus far, the techniques we use are non-explicit in the sense that we use abstract representability criteria for functors rather than explicit descriptions of representing schemes. Once functoriality is proved, we turn to more concrete questions. In Section 8, we write down a set of equations defining the affine schemes $\mathrm{Hilb}_{S/k}^\Delta$ and $\mathrm{Hilb}_{S/k}^{\prec\Delta}$. In Section 9, we study a few examples and improve the result of the previous section in shrinking the set of equations defining the affine schemes. In Section 10, we write down the universal objects of the functors $\mathcal{H}\mathrm{ilb}_{S/k}^\Delta$ and $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$, which are affine schemes over $\mathrm{Hilb}_{S/k}^\Delta$ and $\mathrm{Hilb}_{S/k}^{\prec\Delta}$, respectively. In Section 11, we show that $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$ is connected, we present a homogeneous variant of what we have done so far, and we give a connectedness criterion for $\mathcal{H}\mathrm{ilb}_{S/k}^\Delta$. In Section 12, we explore the transition maps between

$\mathrm{Hilb}_{S/k}^{\Delta}$ and $\mathrm{Hilb}_{S/k}^{\Pi}$, we track $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ in $\mathrm{Hilb}_{S/k}^{\Pi}$, and we show that the decomposition of Theorem 2 in general is not a stratification.

2. Notation. We start by collecting the relevant definitions and facts concerning elements and ideals in S. Throughout, a monomial order $\prec$ on S will be fixed. This is, in particular, a total order on the set of monomials $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$. Let $f \in S$, then the monomial order gives the well-known definitions of

- coefficient $\mathrm{coef}(f, x^{\alpha})$ of f at x^{α};

- support $\mathrm{supp}(f)$, which is the set of all x^{α} such that $\mathrm{coef}(f, x^{\alpha}) \neq 0$;

- leading monomial $\mathrm{LM}(f)$;

- leading coefficient $\mathrm{LC}(f)$;

- leading term $\mathrm{LT}(f)$, which equals $\mathrm{LC}(f)\mathrm{LM}(f)$;

- leading exponent (or multidegree) $\mathrm{LE}(f)$, for which $\mathrm{LM}(f) = x^{\mathrm{LE}(f)}$;

- the non-leading exponents, which are those α such that x^{α} lies in $\mathrm{supp}(f)$ but does not equal $\mathrm{LM}(f)$.

If $I \subset S$ is an ideal, we let $\mathrm{LM}(I)$ be the set of all $\mathrm{LM}(f)$, where f runs through $I - \{0\}$. This set is closed with respect to multiplication by arbitrary monomials. Analogously, we let $\mathrm{LT}(I)$ be the set of all $\mathrm{LT}(f)$, where f runs through $I - \{0\}$. This set is also closed with respect to multiplication by arbitrary monomials. Clearly if k is a field, then $\mathrm{LT}(I)$ carries the same information as $\mathrm{LM}(I)$ does, but if k is a ring, then in general $\mathrm{LT}(I)$ carries more information than $\mathrm{LM}(I)$ does.

If I is an ideal in S and x^{α} is a monomial, the set

$$\mathrm{LC}(I, x^{\alpha}) = \{\mathrm{LC}(f); f \in I - \{0\}, \mathrm{LM}(f) = x^{\alpha}\} \cup \{0\}$$

is an ideal in k.

Definition 1. An ideal I is called *monic* (see [**35**, Definition 3.3] or [**40**, Definition 4]) if the following equivalent conditions are satisfied:

- For all monomials x^{α}, the ideal $\mathrm{LC}(I, x^{\alpha})$ is either the zero ideal or the unit ideal;

- each element of $\mathrm{LM}(I)$ arises as the leading monomial of a monic $f \in I$;

- $\mathrm{LT}(I)$ is a monomial ideal;
- the sets $\mathrm{LM}(I)$ and $\mathrm{LT}(I)$ carry the same information.

Note that if k is a field, then each ideal in S is generated by finitely many monic polynomials, hence in particular monic. Also note that if k is an arbitrary ring and I is a monic ideal, then I is finitely generated.

We will mostly be working with leading exponents, more precisely, with the set $\mathrm{LE}(I)$, which is the set of all $\mathrm{LE}(f)$, where f runs through I. Clearly $\mathrm{LE}(I)$ carries the same information as $\mathrm{LM}(I)$. Therefore it carries the same information as $\mathrm{LT}(I)$ if, and only if, I is monic. In fact, we will not be working with $\mathrm{LE}(I)$ itself but rather with its complement in $\mathbf{N}^n$:

Definition 2.

- A *standard set* (or *staircase*, or *Gröbner escalier*) in $\mathbf{N}^n$ is a subset $\Delta \subset \mathbf{N}^n$ such that its complement in $\mathbf{N}^n$ is closed with respect to addition with elements of $\mathbf{N}^n$. (Equivalently, standard sets are precisely the complements of the sets $\mathrm{LE}(I)$, where I runs through all ideals in S.)

- If for a given I we have $\mathrm{LE}(I) = \mathbf{N}^n - \Delta$, we say that Δ is the *standard set attached to I*.

- If Δ is a standard set, the set $\mathscr{C}(\Delta)$ of *corners* of Δ is the set of all $\alpha \in \mathbf{N}^n - \Delta$ such that for all i, $\alpha - e_i \notin \mathbf{N}^n - \Delta$, where e_i is the i-standard basis vector.

- If Δ is a standard set, the *border* of Δ is the set $\mathscr{B}(\Delta) = \cup_{i=1}^{n}(\Delta + e_i) - \Delta$.

- (Note that if Δ is a standard set, then $\Delta \cup \mathscr{B}(\Delta)$ is a standard set as well.)

- An *edge point* of a standard set Δ is an $\varepsilon \in \Delta$ such that there exist λ and λ' in $\{e_1, \dots, e_n\}$ having the property that $\varepsilon + \lambda$ and $\varepsilon + \lambda'$ both lie in $\mathscr{B}(\Delta)$ and $\varepsilon + \lambda + \lambda'$ lies in $\mathscr{C}(\Delta \cup \mathscr{B}(\Delta))$.

- If $\varepsilon \in \Delta$ is an edge point, the vectors λ and λ' are, however, not uniquely determined. Therefore, in the situation of the last bulleted item, we call $(\varepsilon, \lambda, \lambda')$ an *edge triple*.

Figure 1 shows an example of a standard set in $\mathbf{N}^2$, along with its corners, its border and its edge points. The standard set is drawn in
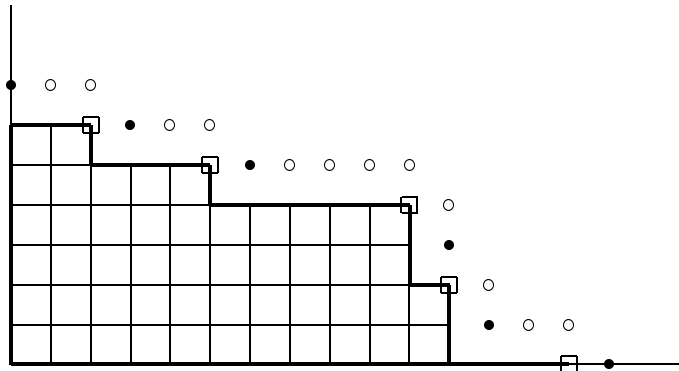
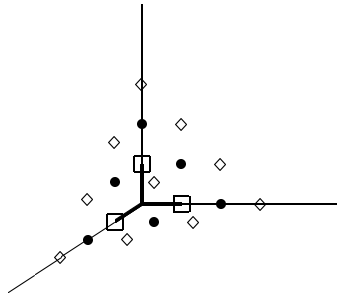FIGURE 1. A standard set, ∘ its border, • its corners, and □ its edge points.

FIGURE 2. A standard set Δ together with • elements of $\mathscr{B}(\Delta)$ and ⋄ elements of $\mathscr{C}(\Delta \cup \mathscr{B}(\Delta))$.

thick lines, the corners are marked by bullets, all other points in the border are marked by circles, and the edge points are marked by boxes.

The standard set of Figure 1 has the property that, for each edge point ε, there exist unique λ and λ' such that $(\varepsilon, \lambda, \lambda')$ is an edge triple. Figure 2 shows a standard set Δ, again drawn in thick lines, such that each edge point ε admits multiple λ and λ' making an edge triple $(\varepsilon, \lambda, \lambda')$. The border of Δ, which is at the same time the set of corners of Δ, is marked by bullets. The border of $\Delta \cup \mathscr{B}(\Delta)$, which

is at the same time the set of corners of $\Delta \cup \mathscr{B}(\Delta)$, is marked by diamonds. The edge points are marked by boxes. The edge triples of Δ are presented in Table 1.

TABLE 1.

ε	λ	λ'
(1,0,0)	(1,0,0)	(0,1,0)
(1,0,0)	(1,0,0)	(0,0,1)
(1,0,0)	(0,1,0)	(0,0,1)
(0,1,0)	(0,1,0)	(1,0,0)
(0,1,0)	(0,1,0)	(0,0,1)
(0,1,0)	(1,0,0)	(0,0,1)
(0,0,1)	(0,0,1)	(1,0,0)
(0,0,1)	(0,0,1)	(0,1,0)
(0,0,1)	(1,0,0)	(0,1,0)

Definition 3. A *Gröbner basis* of an ideal I is a finite subset G of I such that the ideals in S generated by $\mathrm{LT}(I)$ and $\mathrm{LT}(g)$, for $g \in G$, agree. Note that not every ideal I necessarily admits a Gröbner basis, since k was not assumed to be Noetherian. A Gröbner basis G is called *reduced* if $\{\mathrm{LE}(g); \ g \in G\} = \mathscr{C}(\Delta)$, where $\mathrm{LE}(I) = \mathbf{N}^n - \Delta$; each $g \in G$ is monic; and all non-leading exponents of g lie in the standard set attached to I. An ideal I admits a reduced Gröbner basis if, and only if, I is monic. (See [**2**, Theorem 2.11] and [**40**, Theorem 4].)

3. The Hilbert functor of points and its standard subfunctors. Fix a positive integer d. We consider the *Hilbert functor of points*

(2)
$$\mathcal{H}\mathrm{ilb}^d_{S/k} : (k\text{-Alg}) \longrightarrow (\text{Sets})$$
$$B \longmapsto \left\{ \begin{array}{c} \phi : B[x] \to Q \text{ such that} \\ \phi \text{ is surjective and} \\ Q \text{ is a locally free } B\text{-module of rank } d \end{array} \right\} \Big/ \sim \,,$$

where $\phi : B[x] \to Q$ and $\phi' : B[x] \to Q'$ are equivalent if there exists a B-algebra isomorphism $\psi : Q \to Q'$ such that the following diagram

commutes:

$$\begin{array}{ccc} B[x] & \xrightarrow{\ \phi\ } & Q \\ {\scriptstyle\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ B[x] & \xrightarrow{\ \phi'\ } & Q' \,. \end{array}$$

Therefore ϕ and ϕ' are equivalent if, and only if, their kernels agree. In this sense, the functor $\mathcal{H}\mathrm{ilb}^d_{S/k}$ parametrizes all ideals in the polynomial ring S such that the corresponding quotient is locally free of codimension d.

At this point a remark on local freeness is in order. In the literature, one can find at least two definitions of when a B-module is locally free (see [**8**, page 137]).

The first is to demand that, for each prime ideal $\mathfrak{p} \subset B$, the localized module $M_\mathfrak{p}$ is free over the localized ring $B_\mathfrak{p}$. The second is to demand that there exist $f_1, \ldots, f_t \in B$ generating the unit ideal such that each localization $M[f_i^{-1}]$ is a free $R[f_i^{-1}]$-module. The second definition (which is used for instance in [**17**]) is stronger. However, if the module M is locally free of a finite rank d, both definitions agree, and are equivalent to the following statement: For each prime ideal $\mathfrak{p} \subset B$, there exists an $f \in B - \mathfrak{p}$ such that the localized module M_f is free over the localized ring B_f. We will use this definition in the proof of Proposition 1 below.

Our first goal is to cover the functor $\mathcal{H}\mathrm{ilb}^d_{S/k}$ by a finite collection of open subfunctors, indexed by all standard sets of size d. We shall now define these subfunctors. Given a standard set Δ, we use the shorthand notation x^Δ for the family $(x^\beta)_{\beta \in \Delta}$ and $k x^\Delta = \oplus_{\beta \in \Delta} k x^\beta$. We consider the canonical inclusion

$$\iota_\Delta : k x^\Delta \longrightarrow S.$$

Definition 4. Let Δ be a standard set of size d. We define $\mathcal{H}\mathrm{ilb}^\Delta_{S/k}$ to be the subfunctor of $\mathcal{H}\mathrm{ilb}^d_{S/k}$ which associates to each k-algebra B the set of equivalence classes of all $\phi : B[x] \to Q$ as in (2) such that the composition

$$B x^\Delta = B \otimes_k k x^\Delta \xrightarrow{\ \mathrm{id} \otimes \iota_\Delta\ } B[x] \xrightarrow{\ \phi\ } Q$$

is surjective, and therefore an isomorphism.

(The same functors are studied in [20], as was mentioned in the introduction.) In particular, all Q appearing in $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(B)$ are free B-modules of rank d. Evidently there are the following alternative descriptions of the subfunctor,

(3)
$$
\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(B) = \left\{ \begin{array}{c} \phi : B[x] \to Q \text{ such that} \\ \phi \text{ is surjective and} \\ (x^{\beta} + \ker \phi)_{\beta \in \Delta} \text{ is a } B\text{-basis of } B[x]/\ker \phi \end{array} \right\} \Big/ \sim,
$$

and also

(4) $$
\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(B) = \left\{ \begin{array}{c} \phi : B[x] \to Q \text{ such that} \\ \phi \text{ is surjective and} \\ (\phi(x^{\beta}))_{\beta \in \Delta} \text{ is a } B\text{-basis of } Q \end{array} \right\} \Big/ \sim .
$$

Upon fixing an isomorphism $Q = Bx^{\Delta}$ and requiring that $\phi \circ (\mathrm{id} \otimes \iota_{\Delta}) = \mathrm{id}$, we can rephrase the functor $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}$ as follows:

(5) $$
\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(B) = \left\{ \begin{array}{c} \phi : B[x] \to Bx^{\Delta}; \\ \phi \text{ is a } k\text{-algebra homomorphism such that} \\ \phi \circ (\mathrm{id} \otimes \iota_{\Delta}) = \mathrm{id} . \end{array} \right\} .
$$

The multiplicative structure on Bx^{Δ}, making this module a B-algebra, is induced by that on $B[x]$ by the equation $B[x]/\ker \phi = Bx^{\Delta}$. Note that by fixing the isomorphism, we pick one representative of the equivalence class modulo $\sim$. In what follows, we will shift freely between the descriptions (3), (4) and (5).

One can replace the homomorphism ι_{Δ} by an arbitrary k-module homomorphism $\phi : k^{d} \to S$ and define a functor $\mathcal{H}\mathrm{ilb}_{S/k}^{\phi}$ analogous to the above. Such functors have been used in [16, subsection 5.1]. The authors state that $\mathcal{H}\mathrm{ilb}_{S/k}^{\phi}$ is an open subfunctor of $\mathcal{H}\mathrm{ilb}_{S/k}^{d}$ and give a sketch of proof for this. For preparing the ground for the next sections, we carefully prove openness of the subfunctor $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}$ of $\mathcal{H}\mathrm{ilb}_{S/k}^{d}$ here. The proof for a functor $\mathcal{H}\mathrm{ilb}_{S/k}^{\phi}$ as in [16] is entirely analogous.

Lemma 1. *The canonical inclusion* $i : \mathcal{H}\mathrm{ilb}_{S/k}^{\Delta} \to \mathcal{H}\mathrm{ilb}_{S/k}^{d}$ *is an open embedding of functors.*

Proof. Given a k-scheme X, we denote by h_X the Hom functor which sends a k-scheme Y to the set $\mathrm{Mor}_k(Y, X)$. By [**9**, Definition VI-5], we have to check that for each k-algebra B and each morphism of functors $\psi : h_{\mathrm{Spec}\, B} \to \mathcal{H}\mathrm{ilb}^d_{S/k}$, the above horizontal arrow in the Cartesian diagram

(6)
$$\begin{array}{ccc} \mathscr{G} & \longrightarrow & h_{\mathrm{Spec}\, B} \\ \downarrow & & \downarrow{\scriptstyle\psi} \\ \mathcal{H}\mathrm{ilb}^{\Delta}_{S/k} & \xrightarrow{\quad i \quad} & \mathcal{H}\mathrm{ilb}^d_{S/k} \end{array}$$

is isomorphic to the inclusion of functors $h_U \to h_{\mathrm{Spec}\, B}$ induced by the inclusion of schemes $U \to \mathrm{Spec}\, B$, where U is an open subscheme of $\mathrm{Spec}\, B$. So let an arrow $\psi : h_{\mathrm{Spec}\, B} \to \mathcal{H}\mathrm{ilb}^d_{S/k}$ be given. By Yoneda's lemma (see [**9**, Lemma VI-1]), this is an element of $\mathcal{H}\mathrm{ilb}^d_{S/k}(B)$; therefore, the equivalence class of a surjective $\phi : B[x] \to Q$. After localizing in B at $f_1, \ldots, f_s \in B$ which generate the unit ideal, we may assume that Q is a free B-module of rank d. Further, let $\rho : k \to B$ be the structure morphism of the k-algebra B. The functor $\mathscr{G}$ in the Cartesian diagram (6) associates to each k-algebra A the set of all pairs (g, h) in $h_{\mathrm{Spec}\, B}(\mathrm{Spec}\, A) \times \mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}(A)$ such that $\psi(g) = i(h)$ in $\mathcal{H}\mathrm{ilb}^d_{S/k}$. However, g is nothing but a k-algebra homomorphism $\gamma : B \to A$, and h is nothing but (the equivalence class of) a k-algebra homomorphism $\eta : A[x] \to Q'$. Therefore, the condition $\psi(g) = i(h)$ says that the morphisms
$$\phi \otimes \gamma : A[x] \otimes_B A = B[x] \longrightarrow Q \otimes_B A$$
and
$$\eta : A[x] \longrightarrow Q'$$

are in the same equivalence class. After localizing also at certain elements of A, we may assume that Q' is free of rank d. We now fix isomorphisms $Q \otimes_B A = Ax^{\Delta}$ and $Q' = Ax^{\Delta}$ and accordingly demand that $\phi \otimes \gamma = \eta$. Then the condition making the diagram Cartesian is that η lies in $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}(A)$. In other words, we have reformulated the functor $\mathscr{G}$ as follows: $\mathscr{G}(A)$ is the set of all $\gamma : B \to A$ such that $\phi \otimes \gamma : A[x] \to Ax^{\Delta}$ is an A-algebra homomorphism and

(7) $$(\phi \otimes \gamma) \circ (\iota_{\Delta} \otimes (\gamma \circ \rho)) : Ax^{\Delta} \longrightarrow A[x] \longrightarrow Ax^{\Delta}$$

is an isomorphism. Consider the special case $B = B$, $\gamma = \mathrm{id} : B \to B$ and the composition

$$\phi \circ (\iota_\Delta \otimes \rho) : Bx^\Delta \longrightarrow B[x] \longrightarrow Bx^\Delta.$$

Let M be the matrix of this B-module homomorphism, and let $J \subset B$ be the ideal generated by $\det(M)$. Then clearly for any $\gamma : B \to A$, the composition (7) is an isomorphism if, and only if, $A = A\gamma(J)$. By Exercise VI-6 of [9], we are done.   □

As was mentioned in the introduction, the statement of Lemma 1 is implicit in [20], but not explicitly proved there.

4. The standard cover. In Section 5.2 of [16], the authors show with a very quick argument that their functors $\mathcal{H}\mathrm{ilb}^\phi_{S/k}$, where ϕ runs through all homomorphisms $B^d \to B[x]$, form an open cover of the functor $\mathcal{H}\mathrm{ilb}^d_{S/k}$, and also that there exists a finite set of such subfunctors which covers $\mathcal{H}\mathrm{ilb}^d_{S/k}$. We now show, in a constructive way, that our subfunctors $\mathcal{H}\mathrm{ilb}^\Delta_{S/k}$, which are also finite in number, suffice to cover $\mathcal{H}\mathrm{ilb}^d_{S/k}$. Our covering family of subfunctors is a subfamily of the family of [16], and is minimal.

Proposition 1. *The functors $\mathcal{H}\mathrm{ilb}^\Delta_{S/k}$, where Δ runs through all standard sets of size d, form an open cover of the functor $\mathcal{H}\mathrm{ilb}^d_{S/k}$. Moreover, this cover is minimal in the sense that when removing any member of it, the result is no longer a cover.*

Proof. We will show that, for all $B \in (k\text{-}\mathrm{Alg})$, for all prime ideals $\mathfrak{p} \subset B$ and for all $\phi \in \mathcal{H}\mathrm{ilb}^d_{S/k}(B)$, there exist a $g \in B - \mathfrak{p}$ and a standard set Δ of size d such that the localization

$$(\phi \otimes \mathrm{id}_{B_g}) \circ (\iota_\Delta \otimes \mathrm{id}_{B_g}) : B_g x^\Delta \longrightarrow B_g[x] \longrightarrow Q_g$$

is an isomorphism. This will prove that the various $\mathcal{H}\mathrm{ilb}^\Delta_{S/k}$ cover $\mathcal{H}\mathrm{ilb}^d_{S/k}$.

Let B be a k-algebra and $\phi : B[x] \to Q$ a B-algebra homomorphism representing an element of $\mathcal{H}\mathrm{ilb}^d_{S/k}(B)$, and let $\mathfrak{p} \subset B$ be a prime ideal.

We use the localization $B_{\mathfrak{p}}$ and its residue field $\kappa = B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$. Upon tensoring ϕ with $B_{\mathfrak{p}}$ and κ, respectively, we obtain the extensions

$$\begin{aligned} \phi_{\mathfrak{p}} &: B_{\mathfrak{p}}[x] \longrightarrow Q_{\mathfrak{p}}\,, \\ \phi_{\kappa} &: \kappa[x] \longrightarrow Q_{\kappa}. \end{aligned} \tag{8}$$

By assumption, Q is locally free of rank d, i.e., there exists an $f \in B - \mathfrak{p}$ such that $Q_f = \oplus_{i=1}^{d} B_f \varepsilon_i$. Localizing further, we get $Q_{\mathfrak{p}} = \oplus_{j=1}^{d} B_{\mathfrak{p}} \varepsilon_j$. Taking residue classes, we get $Q_{\kappa} = \oplus_{j=1}^{d} \kappa \varepsilon_j$. Local freeness of Q and surjectivity of ϕ imply that both maps in (8) are surjective. Since κ is a field, the ideal $\ker \phi_{\kappa}$ has a Gröbner basis with respect to $\prec$, with a standard set Δ attached to it. As Q_{κ} has dimension d, the standard set has size d. The family $x^{\beta} + \ker \phi_{\kappa}$, where β runs through Δ, is a κ-basis of $\kappa[x]/\ker \phi_{\kappa}$. Therefore the family $\phi_{\kappa}(x^{\beta})$, where β runs through Δ, is a κ-basis of Q_{κ}. From the commutative diagram

$$\begin{array}{ccc} B_{\mathfrak{p}}[x] & \xrightarrow{\ \phi_{\mathfrak{p}}\ } & Q_{\mathfrak{p}} \\ {\scriptstyle\mathrm{can}}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathrm{can}} \\ \kappa[x] & \xrightarrow{\ \phi_{\kappa}\ } & Q_{\kappa}, \end{array}$$

where the vertical arrows are canonical maps, we see that $\phi_{\mathfrak{p}}(x^{\beta})$ is a lift of $\phi_{\kappa}(x^{\beta})$ with respect to the canonical map. Nakayama's lemma (see [**8**, Corollary 4.8]) implies that the family $\phi_{\mathfrak{p}}(x^{\beta})$, where β runs through Δ, generates the $B_{\mathfrak{p}}$-module $Q_{\mathfrak{p}}$. As the rank of $Q_{\mathfrak{p}}$ is $d = \#\Delta$, this family is even a $B_{\mathfrak{p}}$-basis.

Therefore the composition

$$\phi_{\mathfrak{p}} \circ \iota_{\Delta} : B_{\mathfrak{p}} x^{\Delta} \longrightarrow B_{\mathfrak{p}}[x] \longrightarrow Q_{\mathfrak{p}} = \oplus_{i=1}^{d} B_{\mathfrak{p}} \varepsilon_i$$

is an isomorphism. Going from left to right, we write the image of the basis element x^{γ} under the composition as

$$(\phi_{\mathfrak{p}} \circ \iota_{\Delta})(x^{\beta}) = \sum_{i=1}^{d} \frac{c_{\beta,i}}{g_{\beta,i}} \varepsilon_i. \tag{9}$$

Going from right to left, we write the image of the basis element ϵ_i as

$$(\phi_{\mathfrak{p}} \circ \iota_{\Delta})^{-1}(\varepsilon_i) = \sum_{\beta \in \Delta} \frac{d_{i,\beta}}{h_{i,\beta}} x^{\beta}. \tag{10}$$

Here all $g_{\beta,i}$ and all $h_{i,\beta}$ lie in $B - \mathfrak{p}$. We set

$$h = \left(\prod_{\beta \in \Delta} \prod_{i=1}^{d} g_{\beta,i} \right) \cdot \left(\prod_{i=1}^{d} \prod_{\beta \in \Delta} h_{i,\beta} \right)$$

and $g = fh$. (Remember that f is the element of $B - \mathfrak{p}$ with respect to which we localized earlier.) Then $B_g = (B_f)_h$ and therefore $Q_g = \oplus_{i=1}^{d} B_g \varepsilon_i$. Formulas (9) and (10) define homomorphisms

$$B_{\mathfrak{p}} x^{\Delta} \longrightarrow \oplus_{i=1}^{d} B_{\mathfrak{p}} \varepsilon_i$$

and

$$\oplus_{i=1}^{d} B_{\mathfrak{p}} \varepsilon_i \longrightarrow B_{\mathfrak{p}} x^{\Delta},$$

respectively, which are obviously inverses of each other. The first assertion of the proposition is proved.

As for the second assertion, we fix a standard set Π of size d and consider the ideal $I = (x^{\alpha}; \alpha \in \mathbf{N}^n - \Pi) \subset S$. Then clearly S/I is an element of $\mathcal{H}\mathrm{ilb}_{S/k}^{\Pi}$. For all standard sets $\Delta \neq \Pi$ of the same size, there exists an element $\beta \in \Pi - \Delta$. Therefore $x^{\beta} + I$ is zero in S/I, and the family $(x^{\beta} + I)_{\beta \in \Delta}$ is not a k-basis of S/I. It follows that the functors $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}$, for all $\Delta \neq \Pi$, do not suffice to cover all of $\mathcal{H}\mathrm{ilb}_{S/k}^{d}$. $\qquad\square$

As was mentioned in the introduction, the statement of Proposition 1 is implicit in [20], but not explicitly proved there.

5. Gröbner bases in the standard subfunctors. Let us further investigate the functor $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}$. Let B be a k-algebra, $\mathfrak{p} \subset B$ a prime ideal and $\phi \in \mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(B)$. In the course of the proof of Proposition 1, we made use of polynomials lying in the ideal $\ker \phi_{\kappa}$. Since Δ is the standard set attached to the ideal $\ker \phi_{\kappa}$, each element of the reduced Gröbner basis of $\ker \phi_{\kappa}$ can be expressed as

$$(11) \qquad f_{\alpha} = x^{\alpha} + \sum_{\beta \in \Delta} c_{\alpha,\beta} x^{\beta}, \text{ where } c_{\alpha,\beta} = 0 \text{ if } \alpha \prec \beta.$$

The latter condition guarantees that $\mathrm{LE}(f_{\alpha}) = \alpha$. A priori a polynomial as in (11) exists only for all $\alpha \in \mathscr{C}(\Delta)$. The collection $\{f_{\alpha}; \alpha \in \mathscr{C}(\Delta)\}$ is

the reduced Gröbner basis, which is unique. Therefore the polynomial of (11) is unique for all $\alpha \in \mathscr{C}(\Delta)$. The following lemma (applied to $R = \kappa$, $I = \ker \phi_\kappa$) implies that a unique polynomial as in (11) exists for all $\alpha \in \mathbf{N}^n - \Delta$.

Lemma 2. *Let R be a ring and Δ a standard set. Assume that for all $\xi \in \mathscr{C}(\Delta)$, there exists a monic $f_\xi \in R[x]$ such that $\mathrm{LE}(f_\xi) = \xi$ and all non-leading exponents of f_ξ lie in Δ. Define I to be the ideal $(f_\xi; \xi \in \mathscr{C}(\Delta))$ in $R[x]$. Then the following statements hold.*

(i) For all $\alpha \in \mathbf{N}^n - \Delta$, there exists a unique $f_\alpha \in I$ such that $\mathrm{LE}(f_\alpha) = \alpha$ and all non-leading exponents of f_α lie in Δ.

(ii) All coefficients of all f_α are polynomial expressions with coefficients in $\mathbf{Z}$ of $\mathrm{coef}(f_\alpha, x^\beta)$, for $\alpha \in \mathscr{C}(\Delta)$, $x^\beta \in \mathrm{supp}(f_\alpha)$.

(iii) If $\mathrm{LE}(I) = \mathbf{N}^n - \Delta$, then I is monic with reduced Gröbner basis $(f_\xi)_{\xi \in \mathscr{C}(\Delta)}$. Moreover, the family $(f_\alpha)_{\alpha \in \mathbf{N}^n - \Delta}$ is an R-basis of the module I.

This lemma is apparently well known, at least in the case where R is a field. However, it is hard to find a reference for it in the literature, as was mentioned in the discussion after Lemma 15 in [**28**]. Its proof boils down to an inductive construction of the polynomials f_α.

We have seen that by Nakayama's lemma the family of all x^β, where β runs through Δ, is a $B_\mathfrak{p}$-basis of $B_\mathfrak{p}[x]/\ker \phi_\mathfrak{p}$. Therefore each polynomial $f_\alpha \in \ker \phi_\kappa$ as in (11), for $\alpha \in \mathbf{N} - \Delta$, has a unique lift to an element

$$\widehat{f}_\alpha = x^\alpha + \sum_{\beta \in \Delta} \widehat{c}_{\alpha,\beta} x^\beta$$

of $\ker \phi_\mathfrak{p}$. However, though $c_{\alpha,\beta} = 0$ for $\alpha \prec \beta$, the coefficients $\widehat{c}_{\alpha,\beta}$ need not be zero for $\alpha \prec \beta$.

Proposition 2. *The ideal $\ker \phi_\mathfrak{p}$ is monic with Gröbner basis $\widehat{f}_\alpha$, for $\alpha \in \mathscr{C}(\Delta)$, if, and only if, $\widehat{c}_{\alpha,\beta} = 0$ for all $\alpha \in \mathscr{C}(\Delta)$ and for all $\beta \in \Delta$ such that $\alpha \prec \beta$.*

Proof. This is a consequence of Lemma 2.   □

In the complementary case, the set $\{f_\alpha; \alpha \in \mathscr{B}(\Delta)\}$ is still the *border basis* of $\ker \phi_{\mathfrak{p}}$, in the terminology of [**25**, subsection 6.4]. In our context, border bases are best described as follows. Take a k-algebra B and a $\phi : B[x] \to Q$ in $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}$. Let $\alpha \in \mathbf{N}^n - \Delta$, then by (3), there exist unique $d_{\alpha,\beta} \in B$, for $\beta \in \Delta$, such that

$$x^\alpha + \sum_{\beta \in \Delta} d_{\alpha,\beta} x^\beta = 0 \in B[x]/\ker \phi,$$

or equivalently,

$$f_\alpha = x^\alpha + \sum_{\beta \in \Delta} d_{\alpha,\beta} x^\beta \in \ker \phi. tag12$$

The collection $\{f_\alpha; \alpha \in \mathscr{B}(\Delta)\}$ is the border basis of $\ker \phi$. If in addition $\ker \phi$ is monic with standard set Δ, then Lemma 2 implies that the collection $\{f_\alpha; \alpha \in \mathscr{C}(\Delta)\}$ is the reduced Gröbner basis of $\ker \phi$. In this sense the notion of border bases is a generalization of the notion of Gröbner bases. The goal of the next section is to exhibit that observation in the language of Hilbert functors.

6. The Gröbner subfunctors. In [**20**, Theorem 37], [**16**, Theorem 5.4] and [**4**, Theorem 2.8,] the authors show that the functor $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}$ is representable by an affine scheme. We make use of this fact in this section, denoting by $\mathrm{Hilb}^{\Delta}_{S/k}$ the representing scheme. (We will give explicit descriptions of the coordinate ring of this scheme in Sections 8 and 9.) Proposition 2 suggests considering the following elements of $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}(B)$:

Definition 5. For each k-algebra B, let $\mathcal{H}\mathrm{ilb}^{\prec\Delta}_{S/k}(B)$ be the set of equivalence classes of surjective B-algebra homomorphisms $\phi : B[x] \to Q$ such that $\ker \phi$ has a reduced Gröbner basis of the form

$$(13) \qquad f_\alpha = x^\alpha + \sum_{\beta \in \Delta,\, \beta \prec \alpha} d_{\alpha,\beta} x^\beta,$$

where α runs through $\mathscr{C}(\Delta)$.

As was mentioned in Section 2, an ideal admits a reduced Gröbner basis if, and only if, it is monic. Moreover, the equivalence class of a

surjective B-algebra homomorphism $\phi : B[x] \to Q$ is determined by its kernel, and as a monic ideal is determined by its reduced Gröbner basis. This gives us the following alternative characterizations of $\mathcal{H}\mathrm{ilb}^{\prec\Delta}_{S/k}(B)$ as:

- the set of equivalence classes of surjective $\phi : B[x] \to Q$ such that $\ker\phi$ is a monic ideal with standard set Δ.

- the set of all monic ideals in $B[x]$ with standard set Δ.

- the set of all reduced Gröbner bases in $B[x]$ with standard set Δ.

Lemma 3. $\mathcal{H}\mathrm{ilb}^{\prec\Delta}_{S/k}$ *is a subfunctor of* $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}$.

Proof. Let $\phi : B[x] \to Q$ be an element of $\mathcal{H}\mathrm{ilb}^{\prec\Delta}_{S/k}(B)$. The division algorithm (see [**7**, Sections 2, 3]) shows that the family $(x^\beta + \ker\phi)$, where β runs through Δ, is a B-basis of $B[x]/\ker\phi$. Therefore the family $\phi(x^\beta)$, where β runs through Δ, is a B-basis of Q. Hence $\phi : B[x] \to Q$ is also an element of $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}(B)$. In particular, we may assume that $Q = Bx^\Delta$.

We show that $\mathcal{H}\mathrm{ilb}^{\prec\Delta}_{S/k}$ is a functor. Let

$$\phi : B[x] \longrightarrow Bx^\Delta$$

be an element of $\mathcal{H}\mathrm{ilb}^{\prec\Delta}_{S/k}(B)$ and $\psi : B \to A$ be a k-algebra homomorphism. Tensoring is right exact, hence a surjective homomorphism

$$\phi \otimes \mathrm{id} : A[x] \longrightarrow Ax^\Delta.$$

We have to show that $\ker\phi \otimes \mathrm{id}$ is monic with standard set Δ. For this, we write the elements of the reduced Gröbner basis of $\ker\phi$ as in formula (13). We define

$$g_\alpha = x^\alpha + \sum_{\beta\in\Delta,\,\beta\prec\alpha} \psi(d_{\alpha,\beta})x^\beta,$$

for all $\alpha \in \mathscr{C}(\Delta)$. Then clearly all g_α lie in $\ker(\phi \otimes \mathrm{id})$. By Lemma 2 (i), we get a unique polynomial of the form

$$g_\alpha = x^\alpha + \sum_{\substack{\beta\in\Delta\\\beta\prec\alpha}} e_{\alpha,\beta}x^\beta$$

even for all $\alpha \in \mathbf{N}^n - \Delta$, and in particular, all these g_α lie in ker $(\phi \otimes \mathrm{id})$. Now let g be an arbitrary element of ker $(\phi \otimes \mathrm{id})$. Denote the leading term of g by cx^μ. We have to show that μ lies in $\mathbf{N}^n - \Delta$, as in this case, Lemma 2 (iii) guarantees that ker $(\phi \otimes \mathrm{id})$ is monic with standard set Δ. Consider the polynomial

$$g' = g - \sum_{\substack{\beta \in \mathbf{N}^n - \Delta, \\ \beta \prec \mu}} \mathrm{coef}(g, x^\beta) \, g_\beta.$$

Then g' lies in ker $(\phi \otimes \mathrm{id})$; its support is contained in $\Delta \cup \{\mu\}$; and its leading term is cx^μ. However, as $\phi \otimes \mathrm{id}$ lies in $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec \Delta}(A)$, we know that the family $x^\beta + \ker (\phi \otimes \mathrm{id})$, where β runs through Δ, is a basis of $A[x]/\ker (\phi \otimes \mathrm{id})$. This shows that if $\mu \in \Delta$, then $c = 0$, a contradiction. Hence $\mu \in \mathbf{N}^n - \Delta$. $\square$

Lemma 4. $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec \Delta}$ *is a Zariski sheaf.*

Proof. Let B be a k-algebra, $(U_i = \mathrm{Spec}\, B_{g_i})_{i \in I}$ an open cover of $\mathrm{Spec}\, B$ by distinguished open sets and $\phi_i \in \mathcal{H}\mathrm{ilb}_{S/k}^{\prec \Delta}(B_{g_i})$ such that for all i, j,

$$\phi_i \otimes \mathrm{id} : B_{g_i} \otimes_{B_{g_i}} B_{g_j} \longrightarrow Q_i \otimes_{B_{g_i}} B_{g_j}$$

and

$$\phi_j \otimes \mathrm{id} : B_{g_j} \otimes_{B_{g_j}} B_{g_i} \longrightarrow Q_j \otimes_{B_{g_j}} B_{g_i}$$

agree, i.e., define the same map

$$\phi_{ij} : B_{g_i g_j} \longrightarrow Q_{ij} = B_{g_i g_j} x^\Delta.$$

We write the elements of the reduced Gröbner basis of ker ϕ_i and ker ϕ_j, respectively, as

$$f_\alpha^{(i)} = x^\alpha + \sum_{\substack{\beta \in \Delta \\ \beta \prec \alpha}} d_{\alpha,\beta}^{(i)} x^\beta,$$

$$f_\alpha^{(j)} = x^\alpha + \sum_{\substack{\beta \in \Delta \\ \beta \prec \alpha}} d_{\alpha,\beta}^{(j)} x^\beta,$$

respectively, where α runs through $\mathscr{C}(\Delta)$. From Lemma 3 we know that ker ϕ_{ij} = ker $\phi_i \otimes \mathrm{id}$ = ker $\phi_j \otimes \mathrm{id}$ is monic with standard set Δ.

The images of $f_\alpha^{(i)}$ and $f_\alpha^{(j)}$, respectively, in $B_{g_i g_j}[x]$ have the following properties:

- They lie in $\ker \phi_{ij}$.
- Their leading exponent is α.
- Their non-leading exponents lie in Δ.

Therefore they are the reduced Gröbner basis of $\ker \phi_{ij}$. In particular, $f_\alpha^{(i)}$ and $f_\alpha^{(j)}$ agree on $\operatorname{Spec} B_{g_i g_j}$. The sheaf axiom for the quasi-coherent sheaf $B[x]^\sim$ on $\operatorname{Spec} B$ provides a polynomial $f_\alpha \in B[x]$ whose image in $B_{g_i}[x]$ is $f_\alpha^{(i)}$ for all i. It is clear that this polynomial takes the shape (13). Upon defining $I = (f_\alpha; \alpha \in \mathscr{C}(\Delta))$ and $\phi : S \to Q = S/I$ to be the canonical map, we have lifted the various homomorphisms ϕ_i to a homomorphism ϕ. The same line of arguments as at the end of the proof of Lemma 3 shows that I is monic with Gröbner basis f_α, where α runs through $\mathscr{C}(\Delta)$. Therefore ϕ lies in $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}(B)$.   □

Theorem 1. $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$ *is represented by a closed subscheme* $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ *of* $\mathrm{Hilb}_{S/k}^{\Delta}$.

Proof. We prove this by applying Proposition 2.9 of [17]. For this we adopt two items of the terminology of the cited paper.

- Let B be an object of (k-Alg), and let a condition on morphisms $\psi : B \to A$ in (k-Alg) be given. We say that the condition is *closed* if there exists an ideal $J \subset B$ such that $\psi : B \to A$ satisfies the condition if, and only if, ψ factors through the canonical map $B \to B/J$.

- Let B be an object of (k-Alg) and the B-algebra homomorphism $\phi : B[x] \to Q$ an object of $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(B)$. We say that a morphism $\psi : B \to A$ in (k-Alg) satisfies $V_{B,\phi}$ if the A-algebra homomorphism $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(\psi)(\phi)$, which is an element of $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(A)$, lies in $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}(A)$.

By [17, Proposition 2.9], the functor $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$ (which is a Zariski sheaf by Lemma 4) is represented by a closed subscheme of $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}$ if, and only if, for all B in (k-Alg) and all $\phi : B[x] \to Q$ in $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(B)$, the condition $V_{B,\phi}$ is closed.

Let B and ϕ as above be given. Then the family $(x^\beta + \ker \phi)_{\beta \in \mathscr{C}(\Delta)}$ is a B-basis of $B[x]/\ker \phi$. By Lemma 2, there is a unique polynomial

of the form

$$f_\alpha = x^\alpha + \sum_{\beta \in \Delta} d_{\alpha,\beta} x^\beta \in \ker \phi$$

for all $\alpha \in \mathbf{N}^n - \Delta$. Define $J \subset B$ to be the ideal generated by all $d_{\alpha,\beta}$, where α runs through $\mathbf{N}^n - \Delta$ and β runs through all elements of Δ such that $\alpha \prec \beta$.

Given a morphism $\psi : B \to A$ in (k-Alg), the homomorphism $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec \Delta}(\psi)(\phi)$ is nothing but the tensor product $\phi \otimes \mathrm{id} : A[x] \to Q \otimes_B A$. The polynomial

$$\psi(f_\alpha) = x^\alpha + \sum_{\beta \in \Delta} \psi(d_{\alpha,\beta}) x^\beta$$

is the unique element of $\ker(\phi \otimes \mathrm{id})$ such that its leading exponent is α and all non-leading exponents lie in Δ. Now $\psi : B \to A$ factors through $B \to B/J$ if, and only if, for all $\alpha \in \mathscr{C}(\Delta)$ and all $\beta \in \Delta$ such that $\alpha \prec \beta$, we have $\psi(d_{\alpha,\beta}) = 0$. This is equivalent to the ideal $(\psi(f_\alpha); \mathscr{C}(\Delta)) \subset A[x]$ being monic with reduced Gröbner basis $\{\psi(f_\alpha); \alpha \in \mathscr{C}(\Delta)\}$. Therefore $\psi : B \to A$ factors through $B \to B/J$ if, and only if, $\ker(\phi \otimes \mathrm{id})$ is monic with Gröbner basis $\psi(f_\alpha)$, where α runs through $\mathscr{C}(\Delta)$. We have proved that $V_{B,\phi}$ is a closed condition. $\square$

7. The Gröbner strata. We have proved that in the chain of inclusion (1) the first inclusion is a closed immersion and the second inclusion is an open immersion.

Definition 6. We call the locally closed subscheme $\mathrm{Hilb}_{S/k}^{\prec \Delta}$ of $\mathrm{Hilb}_{S/k}^d$ the Gröbner stratum attached to the standard set Δ.

Here is an example illustrating the difference between $\mathrm{Hilb}_{S/k}^\Delta$ and $\mathrm{Hilb}_{S/k}^{\prec \Delta}$.

Example 4. Let $k = \mathbf{Z}$, $\Delta = \{0, e_1, e_2\} \subset \mathbf{N}^2$ and $\prec$ the lexicographic order on $S = k[x_1, x_2]$ such that $x_1 \succ x_2$. The ideal

$$I_a = (x_1^2 + 2x_1 + 2x_2 + 3, \ x_1 x_2 + 2x_1 + 2x_2 + 3,$$
$$\mathbf{x_2^2 + 2x_1 + 2x_2 + 3}) \subset S$$

lies in $\mathrm{Hilb}_{S/k}^{\Delta}$ but not in $\mathrm{Hilb}_{S/k}^{\prec\Delta}$. The ideal

$$I_b = (x_1^2 + 2x_1 + 2x_2 + 3,\ x_1x_2 + 2x_1 + 2x_2 - 4,$$
$$\mathbf{x_2^2 + 2x_2 + 2}) \subset S$$

lies in $\mathrm{Hilb}_{S/k}^{\prec\Delta}$. The difference stems from the coefficient of the term x_1 in the generators printed in boldface type. That coefficient vanishes only for I_b and not for I_a. The given generators of I_b are the reduced Gröbner basis of I_b. The ideal I_a does not have a reduced Gröbner basis if $\prec$ is the term order we chose. However, if we replace that order by the graded lexicographic order, then the given generators are the reduced Gröbner basis of I_a. We will justify this in Example 9, see Section 9 below.

Gröbner strata and related objects have been studied by many authors, see [**1, 11, 12**]. The cited authors refer to these schemes as *Schubert schemes*, or *Schubert cells*. Their terminology is motivated by the analogy of the inclusion $\mathrm{Hilb}_{S/k}^{\prec\Delta} \subset \mathrm{Hilb}_{S/k}^d$ to the inclusion of a Schubert cell in the Grassmannian in the case where Δ is a subset of the standard basis $\{e_1,\ldots,e_n\} \subset \mathbf{N}^n$, augmented by $0 \in \mathbf{N}^n$. One interesting thing about Gröbner strata is the following statement.

Theorem 2. *As a topological space, the scheme $\mathrm{Hilb}_{S/k}^d$ decomposes into locally closed strata as follows,*

$$\mathrm{Hilb}_{S/k}^d = \coprod_{\Delta} \mathrm{Hilb}_{S/k}^{\prec\Delta},$$

where the disjoint union goes over all standard sets $\Delta \subset \mathbf{N}^n$ of size d.

Proof. We have to show that each closed point of $\mathrm{Hilb}_{S/k}^d$ lies in precisely one stratum $\mathrm{Hilb}_{S/k}^{\prec\Delta}$. Let $x : \mathrm{Spec}\, F \to \mathrm{Hilb}_{S/k}^d$ be a closed point, F a field. We interpret this as an element of $\mathrm{Hilb}_{S/k}(F)$, i.e., as a surjective F-algebra homomorphism $\phi : F[x] \to Q$. The kernel of this homomorphism has a well-defined reduced Gröbner basis, and a well-defined standard set Δ. Therefore x lies in $\mathrm{Hilb}_{S/k}^{\prec\Delta}$, and not in any $\mathrm{Hilb}_{S/k}^{\prec\Pi}$, for $\Pi \neq \Delta$. $\square$

Note that in general a non-closed point of $\mathrm{Hilb}_{S/k}^d$ does not lie in any stratum $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ of (14). Indeed, a non-closed point of $\mathrm{Hilb}_{S/k}^d$ lies in some $\mathrm{Hilb}_{S/k}^{\Delta}$, and thus corresponds to a homomorphism $\phi : B[x] \to Q$ lying in $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(B)$, where B is a ring rather than a field. In particular, for all $\alpha \in \mathscr{B}(\alpha)$, there exists an f_α as in (12). However, it may happen that $d_{\alpha,\beta} \neq 0$ for some pair $\alpha \prec \beta$. If so, then the point does not lie in $\mathrm{Hilb}_{S/k}^{\prec\Delta}$.

Example 5. Consider the case $n = 2$, $d = 3$. There are three standard sets

$$\Delta_1 = \{0, e_1, 2e_1\}, \ \Delta_2 = \{0, e_1, e_2\}, \ \Delta_3 = \{0, e_2, 2e_2\}$$

of size 3 in $\mathbf{N}^2$. In Examples 9 and 10 below we will show that the three corresponding open patches $\mathrm{Hilb}_{S/k}^{\Delta_i}$ are all isomorphic to $\mathbf{A}^6$.

$$R^{\Delta_1} = k[T_{(3,0),\beta}, T_{(0,1),\beta}; \ \beta \in \Delta_1],$$
$$R^{\Delta_2} = k[T_{(2,0),\beta}, T_{(1,1),\beta}, T_{(0,2),\beta}; \ \beta \in \Delta_2 - \{0\}],$$
$$R^{\Delta_3} = k[T_{(1,0),\beta}, T_{(3,0),\beta}; \ \beta \in \Delta_1].$$

Let $K_i = \mathrm{Frac}\, R^{\Delta_i}$ and $\eta_i = \mathrm{Spec}\, K_i$ be the generic point of $\mathrm{Hilb}_{S/k}^{\Delta_i}$. The point η_i of $\mathrm{Hilb}_{S/k}^{\Delta_i}$ corresponds to the K_i-algebra $Q_i = K_i[x_1, x_2]/J_i$, where

$$\begin{aligned}
J_1 = (&x_1^3 - T_{(3,0),(2,0)}x_1^2 - T_{(3,0),(1,0)}x_1 - T_{(3,0),(0,0)}, \\
&x_2 - T_{(0,1),(2,0)}x_1^2 - T_{(0,1),(1,0)}x_1 - T_{(0,1),(0,0)}), \\
J_2 = (&x_1^2 - T_{(2,0),(1,0)}x_1 - T_{(2,0),(0,1)}x_2 - T_{(2,0),(0,0)} \\
&x_1 x_2 - T_{(1,1),(1,0)}x_1 - T_{(1,1),(0,1)}x_2 - T_{(1,1),(0,0)} \\
&x_2^2 - T_{(0,2),(1,0)}x_1 - T_{(0,2),(0,1)}x_2 - T_{(0,2),(0,0)}), \\
J_3 = (&x_1 - T_{(1,0),(0,2)}x_2^2 - T_{(1,0),(0,1)}x_2 - T_{(1,0),(0,0)}, \\
&x_2^3 - T_{(0,3),(0,2)}x_2^2 - T_{(0,3),(0,1)}x_2 - T_{(0,3),(0,0)}).
\end{aligned}$$

(The meaning of those $T_{\alpha,\beta}$ appearing in J_i which are not generators of R^{Δ_i} will be explained in Examples 9 and 10 below.) We may assume that the term order satisfies $x_1 \succ x_2$; thus, in particular $x_1^3 \succ x_2$. Since

$T_{(3,0),(0,1)} \neq 0$ in K_1, we see that the given generators of J_1 are not a reduced Gröbner basis. In fact, J_1 does not admit a reduced Gröbner basis. Consequently η_1 does not lie in any stratum of (14). Similarly, if $\prec$ is the lexicographic order, then η_2 does not lie in any stratum of (14) either, and if $\prec$ is the graded lexicographic order, then η_3 does lie in any stratum of (14).

8. Representing the functors. We start this section by briefly reviewing the construction of the affine scheme $\mathrm{Hilb}^{\Delta}_{S/k}$ given in [**4**, Theorem 2.8] and [**16**]. $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}(B)$ is the set of equivalence classes of B-algebra homomorphisms $\phi : B[x] \to Q$ such that the composition $Bx^{\Delta} \to B[x] \to Q$ is an isomorphism. Each equivalence class of $\phi : B[x] \to Q$ corresponds to precisely one B-algebra structure on the B-module Bx^{Δ}. Therefore $\mathcal{H}\mathrm{ilb}^{\Delta}_{S/k}(B)$ is reinterpreted as the set of all B-algebra homomorphisms $\phi : B[x] \to Bx^{\Delta}$ such that $\phi \circ (\iota_{\Delta} \otimes \mathrm{id}) : Bx^{\Delta} \to Bx^{\Delta}$ is the identity map. Now that we have free modules with bases, we identify ϕ with its matrix $(a_{\alpha\beta})_{\alpha \in \mathbf{N}^n, \beta \in \Delta}$, which is given by

$$(15) \qquad \phi(x^{\alpha}) = \sum_{\beta \in \Delta} a_{\alpha\beta} x^{\beta}, \text{ for all } \alpha \in \mathbf{N}^n.$$

The condition $\phi \circ (\iota_{\Delta} \otimes \mathrm{id}) = \mathrm{id}$ says that

$$a_{\alpha\beta} = \delta_{\alpha\beta}, \text{ for all } \alpha \in \mathbf{N}^n \text{ and for all } \beta \in \Delta.$$

The B-module homomorphism ϕ is a B-algebra homomorphism if, and only if, it is multiplicative. This characterization will be used in the proof of Proposition 3 below. For the time being, we use another characterization: the B-module homomorphism ϕ is a B-algebra homomorphism if, and only if, its kernel is an ideal in $B[x]$. It is easy to check that the family $x^{\alpha} - (\iota_{\Delta} \otimes \mathrm{id}) \circ \phi(x^{\alpha})$, where x^{α} runs through all monomials in $B[x]$, generates the B-module $\ker \phi$. Therefore $\ker \phi$ is an ideal in $B[x]$ if, and only if,

$$\phi(x^{\lambda}(x^{\alpha} - (\iota_{\Delta} \otimes \mathrm{id}) \circ \phi(x^{\alpha}))) = 0, \text{ for all } \lambda, \alpha \in \mathbf{N}^n.$$

Upon expressing ϕ by its matrix and using the fact that ι_{Δ} is the canonical inclusion, this condition reads as follows:

$$\sum_{\beta \in \Delta} \Big(a_{\lambda+\alpha,\beta} - \sum_{\gamma \in \Delta} a_{\alpha,\gamma} a_{\lambda+\gamma,\beta} \Big) x^{\beta} = 0, \text{ for all } \lambda, \alpha \in \mathbf{N}^n.$$

Since Bx^Δ is free with basis x^Δ, this means that

$$(16) \qquad a_{\lambda+\alpha,\beta} - \sum_{\gamma \in \Delta} a_{\alpha,\gamma} a_{\lambda+\gamma,\beta} = 0,$$

for all $\lambda, \alpha \in \mathbf{N}^n$ and for all $\beta \in \Delta$.

Clearly it suffices to let x^λ run only through $x_1, \dots, x_n$. Therefore the functor $\mathcal{H}\mathrm{ilb}^\Delta_{S/k}$ is represented by the affine scheme

$$(17) \qquad \mathrm{Hilb}^\Delta_{S/k} = \mathrm{Spec}\, R/I^\Delta,$$

where I^Δ is the ideal

$$
\begin{aligned}
I^\Delta = {} & (T_{\alpha,\beta} - \delta_{\alpha,\beta}; \alpha \in \mathbf{N}^n, \beta \in \Delta) \\
& + \left(T_{\lambda+\alpha,\beta} - \sum_{\gamma \in \Delta} T_{\alpha,\gamma} T_{\lambda+\gamma,\beta}; \alpha \in \mathbf{N}^n, \lambda \in \{e_1, \dots, e_n\}, \beta \in \Delta \right)
\end{aligned}
$$

in the polynomial ring $R = k[T_{\alpha,\beta}; \alpha \in \mathbf{N}^n, \beta \in \Delta]$.

The heart of the above described method for obtaining the coordinate ring of the scheme $\mathrm{Hilb}^\Delta_{S/k}$ is the system of equations (16). These are the *structural equations* defining the multiplicative structure on the B-algebra Bx^Δ. In contrast to this approach to the coordinate ring of $\mathrm{Hilb}^\Delta_{S/k}$, the same ring is obtained by using a border basis variant of Buchberger's S-pair criterion in the articles [**20, 22, 23, 24, 26, 37**]. In those articles, finite presentations of the coordinate ring of $\mathrm{Hilb}^\Delta_{S/k}$ are given. At the moment our approach seems weaker, as the presentation of (17) uses infinitely many generators and relations. In the next section we will see that our approach is in fact stronger. In Proposition 3 we derive a finite presentation of the coordinate ring R/I. In Theorem 3 below we will derive an improvement on this proposition.

We introduce the following notation. If $N \subset \mathbf{N}^n$ is a standard set, we write $N^{(1)} = \mathscr{B}(N)$ and, for all $i \geq 1$, $N^{(i+1)} = \mathscr{B}(N \cup N^{(1)} \cup \cdots \cup N^{(i)})$.

Proposition 3. *Let N be a standard set in $\mathbf{N}^n$ containing Δ. Then the functor $\mathcal{H}\mathrm{ilb}^\Delta_{S/k}$ is represented by the affine scheme $\mathrm{Hilb}^\Delta_{S/k} = \mathrm{Spec}\, R^\Delta$, where*

$$R^\Delta = R/I^\Delta$$

and $I^\Delta = I_1^\Delta + I_2^\Delta + I_3^\Delta$ is the sum of the ideals

$$I_1^\Delta = (T_{\alpha,\beta} - \delta_{\alpha,\beta}; \alpha \in N, \beta \in \Delta), \quad I_2^\Delta = \Big(T_{\alpha+\lambda,\beta} - \sum_{\gamma \in \Delta} T_{\alpha,\gamma} T_{\gamma+\lambda,\beta};$$

$\alpha \in N \cup N^{(1)}, \lambda \in \{e_1, \dots, e_n\}$ such that $\alpha + \lambda \in N \cup N^{(1)}, \beta \in \Delta \Big)$ and

$$I_3^\Delta = \Big(\sum_{\gamma \in \Delta} T_{\alpha,\gamma} T_{\gamma+\lambda,\beta} - \sum_{\gamma \in \Delta} T_{\alpha',\gamma} T_{\gamma+\lambda',\beta}; \alpha, \alpha' \in N^{(1)},$$

$$\lambda, \lambda' \in \{e_1, \dots, e_n\} \ s.t. \ \alpha + \lambda = \alpha' + \lambda' \in N^{(2)}, \beta \in \Delta \Big)$$

in the polynomial ring $R = k[T_{\alpha,\beta}; \alpha \in N \cup N^{(1)}, \beta \in \Delta]$.

Proof. Again we start with a B-module homomorphism $\phi : B[x] \to Bx^\Delta$, represented by its matrix $(a_{\alpha,\beta})$ as in (15). Our goal is to find constraints on the coefficients $a_{\alpha,\beta}$ which guarantee that Bx^Δ has a multiplicative structure such that ϕ is a B-algebra homomorphism. As was mentioned above, ϕ is a B-algebra homomorphism if, and only if, ϕ is multiplicative. By linearity of ϕ, this is equivalent to $\phi(x^{\alpha+\beta}) = \phi(x^\alpha)\phi(x^\beta)$ for all $\alpha, \beta \in \mathbf{N}^n$, and by an easy induction argument, the latter condition is equivalent to

$$(18) \qquad\qquad \phi(x^{\alpha+\lambda}) = \phi(x^\alpha)\phi(x^\lambda)$$

for all $\alpha \in \mathbf{N}^n$ and all $\lambda \in \{e_1, \dots, e_n\}$. Therefore, our goal is to find constraints on the coefficients $a_{\alpha,\beta}$ which guarantee that Bx^Δ has a multiplicative structure and the multiplicativity condition (18) holds for all $\alpha \in \mathbf{N}^n$ and all $\lambda \in \{e_1, \dots, e_n\}$. We will see in the course of the proof that we do not need the full matrix $(a_{\alpha,\beta})_{\alpha \in \mathbf{N}^n, \beta \in \Delta}$, but rather only the rows indexed by $\alpha \in N \cup N^{(1)}$.

Step 1. As above, the condition $\phi \circ (\iota_\Delta \otimes \mathrm{id}) = \mathrm{id}$ translates into the following constraints on the coefficients $a_{\alpha,\beta}$:

$$\forall \alpha \in N, \ \forall \beta \in \Delta : a_{\alpha,\beta} = \delta_{\alpha,\beta}.$$

Step 2. We impose the multiplicativity condition (18) on all α and $\alpha + \lambda$ which lie in $N \cup N^{(1)}$. Let us translate this into equations for the

coefficients $a_{\alpha,\beta}$. The left hand side of (18) is

$$\phi(x^{\alpha+\lambda}) = \sum_{\beta \in \Delta} a_{\alpha+\lambda,\beta} x^{\beta}.$$

The right hand side of (18) is a priori not defined before we have the multiplicative structure of Bx^{Δ} at hand. However, upon assuming that (18) holds for elements of $N \cup N^{(1)}$, we can surmount that obstacle by a trick:

$$
\begin{aligned}
(19) \quad \phi(x^{\alpha})\phi(x^{\lambda}) &= \sum_{\gamma \in \Delta} a_{\alpha,\gamma} x^{\gamma} \phi(x^{\lambda}) = \sum_{\gamma \in \Delta} a_{\alpha,\gamma} \phi(x^{\gamma})\phi(x^{\lambda}) \\
&= \sum_{\gamma \in \Delta} a_{\alpha,\gamma} \phi(x^{\gamma+\lambda}) = \sum_{\beta,\gamma \in \Delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta} x^{\beta}.
\end{aligned}
$$

Here we used the fact that $x^{\gamma} = \phi(x^{\gamma})$ if $\gamma \in \Delta$, and the multiplicativity condition (18) for γ and $\gamma + \lambda$ lying in $N \cup N^{(1)}$. The two expressions have to coincide, hence the following constraints on the coefficients $a_{\alpha,\beta}$:

$$(20) \quad \forall \alpha \in N \cup N^{(1)}, \ \forall \lambda \in \{e_1, \dots, e_n\}$$

$$\text{s.t. } \alpha + \lambda \in N \cup N^{(1)}, \ \forall \beta \in \Delta :$$

$$a_{\alpha+\lambda,\beta} = \sum_{\gamma \in \Delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta}.$$

Note that these are just (some of) the structural equations (16).

At this point *multiplicativity holds within* $N \cup N^{(1)}$ in the sense that (18) holds if $\alpha, \alpha + \lambda \in N \cup N^{(1)}$.

Step 3. We define more values of ϕ by means of equation (18). More precisely, we take $\alpha \in N^{(1)}$ and $\lambda \in \{e_1, \dots, e_n\}$ such that $\alpha + \lambda \in N^{(2)}$ and define

$$\phi(x^{\alpha+\lambda}) = \sum_{\beta,\gamma \in \Delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta} x^{\beta}.$$

Then the multiplicativity condition (18) holds for these values of α and λ, as the right hand side of the last equation is

$$
\begin{aligned}
\sum_{\beta,\gamma \in \Delta} a_{\alpha,\gamma} \phi(x^{\gamma+\lambda}) &= \sum_{\beta,\gamma \in \Delta} a_{\alpha,\gamma} \phi(x^{\gamma})\phi(x^{\lambda}) \\
&= \sum_{\beta,\gamma \in \Delta} a_{\alpha,\gamma} x^{\gamma} \phi(x^{\lambda}) = \phi(x^{\alpha})\phi(x^{\lambda}).
\end{aligned}
$$

At this point we have to make sure that the definition just given is unambiguous. This means that if $\alpha' \in N^{(1)}$ and $\lambda' \in \{e_1, \ldots, e_n\}$ are such that $\alpha + \lambda = \alpha' + \lambda'$, the definitions of $\phi(x^{\alpha+\lambda})$ and of $\phi(x^{\alpha'+\lambda'})$ coincide. This translates into the following constraints on the coefficients $a_{\alpha,\beta}$:

$$(21) \quad \forall \alpha, \alpha' \in N^{(1)}, \ \ \forall \lambda, \lambda' \in \{e_1, \ldots, e_n\}$$
$$\text{s.t. } \alpha + \lambda = \alpha' + \lambda' \in N^{(2)},$$
$$\forall \beta \in \Delta : \sum_{\gamma \in \Delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta} = \sum_{\gamma \in \Delta} a_{\alpha',\gamma} a_{\gamma+\lambda',\beta}.$$

At this point *multiplicativity also holds when passing from $N^{(1)}$ to $N^{(2)}$* in the sense that (18) holds if $\alpha \in N^{(1)}$ and $\alpha + \lambda \in N^{(2)}$.

Step 4. We claim that *multiplicativity holds within $N \cup N^{(1)} \cup N^{(2)}$* in the sense that (18) holds if $\alpha, \alpha + \lambda \in N \cup N^{(1)} \cup N^{(2)}$.

We only have to check that if both α and $\alpha + \lambda$ lie in $N^{(2)}$. In this case there exists some $\nu \in \{e_1, \ldots, e_n\}$ such that $\alpha + \lambda - \nu \in N^{(1)}$. In particular, $\nu \neq \lambda$. This implies that also $\alpha - \nu$ lies in $\mathbf{N}^n$, as that element arises from $\alpha + \lambda$ by subtracting two different standard basis elements, λ and ν. Therefore $\alpha - \nu$ in fact lies in $N^{(1)}$. We obtain

$$\phi(x^{\alpha+\lambda}) = \phi(x^{\alpha+\lambda-\nu})\phi(x^\nu) = \phi(x^{\alpha-\nu})\phi(x^\lambda)\phi(x^\nu) = \phi(x^\alpha)\phi(x^\lambda),$$

as desired. Here we used the fact that multiplicativity holds when passing from $N^{(1)}$ to $N^{(2)}$ for the outer two equalities and the fact that multiplicativity holds within $N \cup N^{(1)}$ for the inner equality.

Step 5. We define more values of ϕ in analogy to Step 3: We take $\alpha \in N^{(1)}$ and $\lambda, \mu \in \{e_1, \ldots, e_n\}$ such that $\alpha + \lambda \in N^{(2)}$ and $\alpha + \lambda + \mu \in N^{(3)}$ and define

$$\phi(x^{\alpha+\lambda+\mu}) = \sum_{\beta,\gamma,\gamma' \in \Delta} a_{\alpha,\gamma'} a_{\gamma'+\lambda,\gamma} a_{\gamma+\mu,\beta} x^\beta.$$

This definition makes the identity

$$(22) \qquad\qquad \phi(x^{\alpha+\lambda+\mu}) = \phi(x^{\alpha+\lambda})\phi(x^\mu)$$

hold. We claim that the definition just given is unambiguous, i.e., that if $\alpha' \in N^{(1)}$ and $\lambda', \mu' \in \{e_1, \ldots, e_n\}$ are such that $\alpha + \lambda + \mu =$

$\alpha' + \lambda' + \mu'$, the corresponding definitions of $\phi(x^{\alpha+\lambda})$ coincide. For verifying this, we first note that we may assume that $\mu \neq \mu'$, since otherwise unambiguity is trivial. By the same argument as in Step 4, we see that $\alpha + \lambda - \mu' = \alpha' + \lambda' - \mu$ lies in $\mathbf{N}^n$. Therefore $\alpha + \lambda - \mu'$ in fact lies in $N \cup N^{(1)} \cup N^{(2)}$. Now we distinguish three cases.

Case a. $\lambda \neq \mu'$. Then by the same argument once more, $\alpha - \mu'$ lies in $\mathbf{N}^n$. It follows that $\alpha - \mu'$ lies in $N \cup N^{(1)} \cup N^{(2)}$. Therefore

$$
\begin{aligned}
\phi(x^{\alpha+\lambda+\mu}) &= \phi(x^{\alpha+\lambda})\phi(x^\mu) = \phi(x^\alpha)\phi(x^\lambda)\phi(x^\mu) \\
&= \phi(x^{\alpha-\mu'})\phi(x^{\mu'})\phi(x^\lambda)\phi(x^\mu) = \phi(x^{\alpha-\mu'+\lambda})\phi(x^{\mu'})\phi(x^\mu) \\
&= \phi(x^{\alpha'-\mu+\lambda'})\phi(x^{\mu'})\phi(x^\mu) = \phi(x^{\alpha'+\lambda'})\phi(x^{\mu'}) \\
&= \phi(x^{\alpha'+\lambda'+\mu'}).
\end{aligned}
$$

Here we used (22) for the outer equalities and the fact that multiplicativity holds within $N \cup N^{(1)} \cup N^{(2)}$ for all other equalities.

Case b. $\lambda' \neq \mu$. This is the same as the previous with the roles of primed and non-primed elements interchanged.

Case c. $\lambda = \mu'$ and $\lambda' = \mu$. Then $\alpha = \alpha'$. As $\alpha \in N^{(1)}$, there exists a $\nu \in \{e_1, \ldots, e_n\}$ such that $\alpha - \nu \in N$. Again we get a chain of equalities:

$$
\begin{aligned}
\phi(x^{\alpha+\lambda+\mu}) &= \phi(x^{\alpha+\lambda})\phi(x^{\lambda'}) = \phi(x^\alpha)\phi(x^\lambda)\phi(x^{\lambda'}) \\
&= \phi(x^{\alpha-\nu})\phi(x^\nu)\phi(x^\lambda)\phi(x^{\lambda'}) = \phi(x^{\alpha+\lambda'-\nu})\phi(x^\nu)\phi(x^\lambda) \\
&= \phi(x^{\alpha+\lambda'})\phi(x^\lambda) = \phi(x^{\alpha+\lambda'+\mu'}).
\end{aligned}
$$

Again we used (22) for the outer equalities and the fact that multiplicativity holds within $N \cup N^{(1)} \cup N^{(2)}$ for all other equalities.

At this point *multiplicativity also holds when passing from $N^{(2)}$ to $N^{(3)}$* in the obvious sense.

Induction step. Note that the proof of multiplicativity given in Step 4 was completely formal, only using the fact that multiplicativity holds within $N \cup N^{(1)}$ and when passing from $N^{(1)}$ to $N^{(2)}$. Therefore, now that we know that multiplicativity holds within $N \cup N^{(1)} \cup N^{(2)}$ and when passing from $N^{(2)}$ to $N^{(3)}$, we can imitate Step 4 and prove that multiplicativity holds within $N \cup N^{(1)} \cup N^{(2)} \cup N^{(3)}$. Analogously, Step 5 was completely formal and can be imitated for proving that

multiplicativity holds when passing from $N^{(3)}$ to $N^{(4)}$. Then we imitate Step 4 again and prove that multiplicativity holds within $N \cup N^{(1)} \cup N^{(2)} \cup N^{(3)} \cup N^{(4)}$, imitate Step 5 again for proving that multiplicativity holds when passing from $N^{(4)}$ to $N^{(5)}$, and so on. This proves that the multiplicativity condition (18) holds for all $\alpha \in \mathbf{N}^n$ and all $\lambda \in \{e_1, \dots, e_n\}$.

End of proof. We see that the B-module homomorphism $\phi : B[x] \to Bx^\Delta$ defines a B-algebra homomorphism if, and only if, the coefficients $a_{\alpha,\beta}$, where $\alpha \in N \cup N^{(1)}$ and $\beta \in \Delta$, satisfy the three conditions (19), (20) and (21) of Steps 1, 2 and 3, respectively. Therefore, an element ϕ of $\mathcal{H}\mathrm{ilb}^\Delta_{S/k}(B)$ is uniquely determined by the choice of elements $a_{\alpha,\beta} \in B$, for all $\alpha \in N \cup N^{(1)}$ and all $\beta \in \Delta$, such that (19), (20) and (21) hold. That choice corresponds to the choice of a k-algebra homomorphism $R^\Delta \to B$. $\square$

The set N of Proposition 3 can be chosen finite. Therefore the scheme $\mathrm{Hilb}^\Delta_{S/k}$ is of finite type over k, and embedded as the closed subscheme corresponding to I^Δ into affine space with coordinates $T_{\alpha,\beta}$, for $\alpha \in N$, $\beta \in \Delta$. In view of the summand I^Δ_1 of I^Δ, we see that we only need the coordinates $T_{\alpha,\beta}$, for $\alpha \in N - \Delta$, $\beta \in \Delta$, for the ambient space. The smallest possible N is Δ, hence a closed immersion of $\mathrm{Hilb}^\Delta_{S/k}$ into affine space of dimension $\#\mathscr{B}(\Delta)\#\Delta$. The same immersion is studied in [**20, 22, 23, 24, 26, 37**]. However, these articles do not use the matrices of $\phi(x^\alpha)$ but rather the polynomials

$$f_\alpha = x^\alpha + \sum_{\beta \in \Delta} d_{\alpha,\beta} x^\beta \in \ker \phi$$

(cf. (13)). These polynomials carry the same information as our matrix, as

$$(23) \qquad a_{\alpha,\beta} = \begin{cases} \delta_{\alpha,\beta} & \text{if } \alpha \in \Delta, \\ -d_{\alpha,\beta} & \text{if } \alpha \in \mathbf{N}^n - \Delta. \end{cases}$$

The work with the polynomials f_α makes syzygy criteria necessary in the cited articles. The two summands I^Δ_2 and I^Δ_3 in our ideal I^Δ correspond to the concepts of *next-door-neighbors* and *across-the-street neighbors*, respectively, in [**22**, Definition 17] and [**26**, Section 4]. We

will say more about that in the next section, in which we dispose of many of the generators of I_3^Δ. Now we focus on the Gröbner functors.

Corollary 1. *Let N, R and I^Δ be as in Proposition 3. Then the functor $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$ is represented by the affine scheme $\mathrm{Hilb}_{S/k}^{\prec\Delta} = \mathrm{Spec}\, R^{\prec\Delta}$, where*

$$R^{\prec\Delta} = R/I^{\prec\Delta}$$

and

$$I^{\prec\Delta} = I^\Delta + (T_{\alpha,\beta}; \alpha \in N \cup N^{(1)}, \beta \in \Delta, \alpha \prec \beta).$$

Proof. The additional conditions defining $I^{\prec\Delta}$ express the constraints on the subfunctor $\mathcal{H}\mathrm{ilb}_{S/k}^{\prec\Delta}$ of $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}$ which we discussed in the proof of Theorem 1, in terms of the variables $T_{\alpha,\beta}$. $\square$

Equivalently, the scheme $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ is the closed subscheme of $\mathrm{Hilb}_{S/k}^{\Delta}$ defined by the ideal in R^Δ generated by the images of all $T_{\alpha,\beta}$, for $\alpha \in N \cup N^{(1)}$ and $\beta \in \Delta$ such that $\alpha \prec \beta$. Note that for the ideal defining $\mathrm{Hilb}_{S/k}^{\prec\Delta}$, the identity

$$I^{\prec\Delta} = I^\Delta + (T_{\alpha,\beta}; \alpha \in \mathscr{C}(\Delta), \beta \in \Delta, \alpha \prec \beta)$$

holds. In other words, many of the additional conditions defining $I^{\prec\Delta}$ follow from a few basic ones. (This is a consequence of Lemma 2, applied to the ring R^Δ of Proposition 3.)

By Corollary 1, $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ is a closed subscheme of an affine space of dimension $\#\mathscr{B}(\Delta)\#\Delta$. We now cut down further the dimension of the ambient space. For this we consider the polynomial ring $R_m = k[T_{\alpha,\beta}; \alpha \in \mathscr{C}(\Delta), \beta \in \Delta, \alpha \succ \beta]$. For all $\alpha \in \mathscr{B}(\Delta) - \mathscr{C}(\Delta)$ and all $\beta \in \Delta$ such that $\alpha \succ \beta$, we define elements $T_{\alpha,\beta}$ of R_m by recursion over α as follows:

- We start with $\Gamma = \mathscr{B}(\Delta) - \mathscr{C}(\Delta)$.

- While $\Gamma \neq \varnothing$, we take the minimal element α of Γ, we find some $\nu \in \{e_1, \ldots, e_n\}$ such that $\alpha - \nu \in \mathscr{B}(\Delta)$, we define

$$\tag{24} T_{\alpha,\beta} = \sum_{\substack{\gamma \in \Delta \\ \alpha \succ \gamma + \nu \succ \beta}} T_{\alpha-\nu,\gamma} T_{\gamma+\nu,\beta},$$

where $T_{\gamma+\nu,\beta} = \delta_{\gamma+\nu,\beta}$ if $\gamma + \nu \in \Delta$, and we replace Γ by $\Gamma - \{\alpha\}$. Moreover, we define an ideal $I_m^{\prec\Delta} \subset R_m$ by the same formulas as the ideal $I^\Delta \subset R$ in Proposition 3, applied in the case where $N = \Delta$, with the following modification: In all summands of all generators of I^Δ, we replace all $T_{\alpha,\beta}$ such that $\alpha \in \Delta$ by $\delta_{\alpha,\beta}$, and delete all $T_{\alpha,\beta}$ such that $\alpha \succ \beta$.

Corollary 2. *With the above notation, we have* $\mathrm{Hilb}_{S/k}^{\prec\Delta} = \mathrm{Spec}\, R_m^{\prec\Delta}$, *where*

$$R_m^{\prec\Delta} = R_m/I_m^{\prec\Delta}.$$

Proof. We first check that our recursion is well-defined. Indeed, for all $\alpha \in \mathscr{B}(\Delta) - \mathscr{C}(\Delta)$, the existence of a standard basis element ν such that $\alpha - \nu \in \mathscr{B}(\Delta)$ follows directly from the definitions. If α is the object of consideration in one particular step of the recursion, the element $\alpha - \nu$ either lies in $\mathscr{C}(\Delta)$ or has been the object of consideration in an earlier stage of the recursion, as $\alpha \succ \alpha - \nu$. In both cases $T_{\alpha-\nu,\gamma}$ is a well-defined element of R_m. Moreover, in the sum (24) we only consider those $\gamma \in \Delta$ for which $\alpha - \nu \succ \gamma$, or equivalently, $\alpha \succ \gamma + \nu$. Therefore the element $\gamma + \nu$ either lies in Δ or has been the object of consideration in an earlier stage of the recursion. In both cases $T_{\gamma+\nu,\beta}$ is a well-defined element of R_m.

Now we return to the notation of Corollary 1 in the case where $N = \Delta$ and consider the rings R and $R^{\prec\Delta} = R/I^{\prec\Delta}$ defined there. For simplicity we write $T_{\alpha,\beta}$ for the images of the variables $T_{\alpha,\beta} \in R$ in the quotient $R^{\prec\Delta}$. In particular, $T_{\alpha,\beta} = 0$ for all $\alpha \in \mathscr{B}(\Delta)$ and all $\beta \in \Delta$ such that $\alpha \prec \beta$. Furthermore, the presence of the summand I_2^Δ in the ideal I^Δ implies that whenever $\alpha - \nu$ and α lie in $\mathscr{B}(\Delta)$, the identity

$$T_{\alpha,\beta} = \sum_{\gamma \in \Delta} T_{\alpha-\nu,\gamma} T_{\gamma+\nu,\beta}$$

holds in $R^{\prec\Delta}$. However, only those $\gamma \in \Delta$ for which $\alpha - \nu \succ \gamma$ and $\gamma + \nu \succ \beta$ make a contribution to that sum. This explains the definition of $T_{\alpha,\beta}$ given in (24). As for the definition of $I_m^{\prec\Delta}$, the replacement $T_{\alpha,\beta} = \delta_{\alpha,\beta}$ for all $\alpha \in \Delta$ is clear from the presence of the summand I_1^Δ in the ideal I^Δ; and deleting all $T_{\alpha,\beta}$ such that $\alpha \succ \beta$ stems from the equality $T_{\alpha,\beta} = 0$ in $R^{\prec\Delta}$. The assertion follows from Corollary 1. □

In the corollary we embedded $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ into an affine space of dimension

$$p = \#\{(\alpha, \beta) \in \mathscr{C}(\Delta) \times \Delta; \alpha \succ \beta\}.$$

Note that p depends on both the shape of Δ and the term order $\prec$. Example 10 below shows that, for special shapes of Δ, there exists a term order $\prec$ such that this upper bound is sharp. (For the standard set of Example 10, take $\prec$ to be the lexicographic order such that $x_1 \prec x_i$, for all $i > 1$.) This observation motivates the subscript in the ideal $I_m^{\prec\Delta}$, which stands for *minimal*. However, it is not clear if minimality holds in a strict sense:

Question 1. Given a standard set Δ, does there exist a term order $\prec$ such that p is the minimal dimension of an affine space into which $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ can be embedded?

Question 2. Given a term order $\prec$, does there exist a standard set Δ such that p is the minimal dimension of an affine space into which $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ can be embedded?

In Proposition 3 we embedded $\mathrm{Hilb}_{S/k}^{\Delta}$ into an affine space of dimension

$$q = d \# \mathscr{B}(\Delta).$$

(This dimension is obtained when letting $N = \Delta$.) For standard sets of size $d = 1$, we trivially have $\mathrm{Hilb}_{S/k}^{\Delta} = \mathbf{A}^n$; thus, the dimension of $\mathrm{Hilb}_{S/k}^{\Delta}$ equals q. It is not clear what happens for larger d:

Question 3. For which $d \in \mathbf{N}$ does there exist a standard set Δ such that q is the minimal dimension of an affine space into which all $\mathrm{Hilb}_{S/k}^{\Delta}$ can be embedded?

For special shapes of Δ, the number q is certainly not the minimal dimension which one can reach. A class of counterexamples is given by Example 10 again (for the term order $\prec$ we considered above, we have $\mathrm{Hilb}_{S/k}^{\Delta} = \mathrm{Hilb}_{S/k}^{\prec\Delta}$). Another class of counterexamples is given by Corollary 7.3.2 of [**19**], which states that if $n = 2$ and $\Delta \subset \mathbf{N}^2$ has a "sawtooth" form depicted in Figure 3 (for any parameters a, b and c), then $\mathrm{Hilb}_{S/k}^{\Delta}$ is an affine space of dimension $2d$.

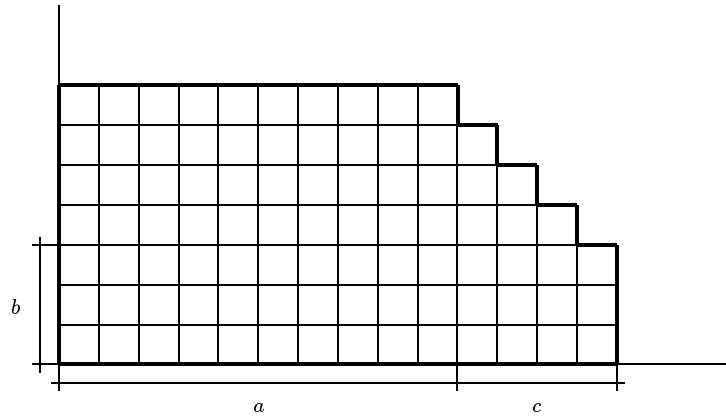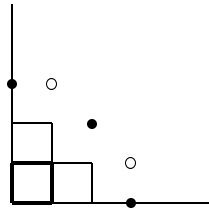FIGURE 3. A standard set of sawtooth form.

FIGURE 4. A standard set Δ together with $\Delta^{(1)}$ and $\Delta^{(2)}$.

If N is strictly larger than Δ, the dimension of the ambient space of $\mathrm{Hilb}_{S/k}^{\Delta}$ given in Proposition 3 is far from minimal, thus seeming unnecessarily large. Alas also that presentation is useful, as it leads to a compact formula for the coordinate change between two charts $\mathrm{Hilb}_{S/k}^{\Delta}$ and $\mathrm{Hilb}_{S/k}^{\Pi}$ of $\mathrm{Hilb}_{S/k}^{d}$. We will carry this out in Section 12 below.

9. A smaller set of generators. For illustrating the presentation of $\mathrm{Hilb}^{\Delta}_{S/k}$ given in the last section, we go through a few examples. These will also serve as a motivation for Theorem 3 below, which is a substantial improvement of Proposition 3. In all examples we only study $\mathrm{Hilb}^{\Delta}_{S/k}$, and not $\mathrm{Hilb}^{\prec\Delta}_{S/k}$, as the latter arises from the former by simply setting $T_{\alpha,\beta} = 0$ for all $\alpha \prec \beta$.

Example 6. Consider the following standard set Δ and its borders:

$$\Delta = \{(0,0), (1,0), (0,1), (1,1)\},$$
$$\Delta^{(1)} = \{(2,0), (0,2), (2,1), (1,2)\},$$
$$\Delta^{(2)} = \{(3,0), (3,1), (2,2), (1,3), (0,4)\}.$$

Figure 4 shows Δ, drawn in thick lines; $\Delta^{(1)}$, drawn in thin lines; and $\Delta^{(2)}$, marked by circles.

In view of the presence of I_1^{Δ} in the ideal I^{Δ} of Proposition 3, we replace the polynomial ring $k[T_{\alpha,\beta}; \alpha \in \Delta \cup \Delta^{(1)}, \beta \in \Delta]$ of that theorem by $R = k[T_{\alpha,\beta}; \alpha \in \Delta^{(1)}, \beta \in \Delta]$. Then

$$\mathrm{Hilb}^{\Delta}_{S/k} = \mathrm{Spec}\, R/I^{\Delta},$$

where I^{Δ} is the sum of the three ideals

$$
\begin{aligned}
I_{2,1}^{\Delta} = (&T_{(1,2),(0,0)} - T_{(0,2),(1,0)} T_{(2,0),(0,0)} - T_{(0,2),(1,1)} T_{(2,1),(0,0)}, \\
&T_{(1,2),(1,0)} - T_{(0,2),(1,0)} T_{(2,0),(1,0)} - T_{(0,2),(1,1)} T_{(2,1),(1,0)} \\
&- T_{(0,2),(0,0)},\ T_{(1,2),(0,1)} - T_{(0,2),(1,0)} T_{(2,0),(0,1)} \\
&- T_{(0,2),(1,1)} T_{(2,1),(0,1)},\ T_{(1,2),(1,1)} - T_{(0,2),(1,0)} T_{(2,0),(1,1)} \\
&- T_{(0,2),(1,1)} T_{(2,1),(1,1)} - T_{(0,2),(0,1)}),
\end{aligned}
$$

$$
\begin{aligned}
I_{2,2}^{\Delta} = (&T_{(2,1),(0,0)} - T_{(2,0),(0,1)} T_{(0,2),(0,0)} - T_{(2,0),(1,1)} T_{(1,2),(0,0)}, \\
&T_{(2,1),(1,0)} - T_{(2,0),(0,1)} T_{(0,2),(1,0)} - T_{(2,0),(1,1)} T_{(1,2),(1,0)}, \\
&T_{(2,1),(0,1)} - T_{(2,0),(0,1)} T_{(0,2),(0,1)} - T_{(2,0),(1,1)} T_{(1,2),(0,1)} \\
&- T_{(2,0),(0,0)},\ T_{(2,1),(1,1)} - T_{(2,0),(0,1)} T_{(0,2),(1,1)} \\
&- T_{(2,0),(1,1)} T_{(1,2),(1,1)} - T_{(2,0),(1,0)})
\end{aligned}
$$

and

$$
\begin{aligned}
I_3^\Delta = (\,&T_{(1,2),(1,0)}T_{(2,0),(0,0)} + T_{(1,2),(1,1)}T_{(2,1),(0,0)} \\
&- T_{(2,1),(0,1)}T_{(0,2),(0,0)} - T_{(2,1),(1,1)}T_{(1,2),(0,0)}, \\
&T_{(1,2),(1,0)}T_{(2,0),(1,0)} + T_{(1,2),(1,1)}T_{(2,1),(1,0)} \\
&+ T_{(1,2),(0,0)} - T_{(2,1),(0,1)}T_{(0,2),(1,0)} \\
&- T_{(2,1),(1,1)}T_{(1,2),(1,0)}, T_{(1,2),(1,0)}T_{(2,0),(0,1)} \\
&+ T_{(1,2),(1,1)}T_{(2,1),(0,1)} - T_{(2,1),(0,1)}T_{(0,2),(0,1)} \\
&- T_{(2,1),(1,1)}T_{(1,2),(0,1)} - T_{(2,1),(0,0)}, T_{(0,2),(1,0)}T_{(2,0),(1,1)} \\
&+ T_{(0,2),(1,1)}T_{(2,1),(1,1)} + T_{(0,2),(0,1)} - T_{(2,1),(0,1)}T_{(0,2),(1,1)} \\
&- T_{(2,1),(1,1)}T_{(1,2),(1,1)} - T_{(2,1),(1,0)}).
\end{aligned}
$$

The ideals $I_{2,1}^\Delta$ and $I_{2,2}^\Delta$ correspond to the equalities derived in Step 2 of the proof of Proposition 3, i.e., from the multiplicativity condition $\phi(x^{\alpha+\lambda}) = \phi(x^\alpha)\phi(x^\lambda)$ for $\alpha, \alpha+\lambda \in \Delta^{(1)}$. As for $I_{2,1}^\Delta$, we choose $\alpha = (0,2), \lambda = (1,0)$. As for $I_{2,2}^\Delta$, we choose $\alpha = (2,0), \lambda = (0,1)$.

The ideal I_3^Δ corresponds to the equalities derived in Step 3 of the proof of Proposition 3, i.e., from the unambiguity condition $\phi(x^{\alpha+\lambda}) = \phi(x^{\alpha'+\lambda'})$ for $\alpha, \alpha' \in \Delta^{(1)}$ such that $\alpha + \lambda = \alpha' + \lambda' \in \Delta^{(2)}$. The only choice for that is $\alpha = (1,2)$, $\lambda = (1,0)$, $\alpha' = (2,1)$, $\lambda' = (0,1)$.

In other words, unambiguity only has to be guaranteed at $(2,2) \in \Delta^{(2)}$. That point a corner of $\Delta \cup \Delta^{(1)}$, as we see from Figure 4. Moreover, it arises from the only edge point of Δ by addition of $(1,1)$.

Example 7. Consider the standard set Δ, along with its borders $\Delta^{(1)}$ and $\Delta^{(2)}$, as depicted in Figure 5. We define the polynomial ring R by the same formula as in the previous example. Then

$$
\mathrm{Hilb}_{S/k}^\Delta = \mathrm{Spec}\, R/I^\Delta,
$$

where

$$
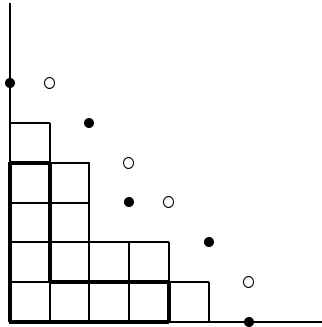I^\Delta = I_{2,1}^\Delta + \cdots + I_{2,6} + I_{3,1}^\Delta + I_{3,2}.
$$

FIGURE 5. A standard set Δ together with $\Delta^{(1)}$ and $\Delta^{(2)}$.

We do not write down the summands of I explicitly, but rather describe them as follows: The ideals $I_{2,i}^{\Delta}$ and $I_{3,j}^{\Delta}$ correspond to the equalities derived in Step 2 and Step 3, respectively, of the proof of Proposition 3, according to the following values of α, λ, and α', λ', respectively,

$I_{2,1}^{\Delta}$	$\alpha = (0,5)$	$\lambda = (1,0)$
$I_{2,2}^{\Delta}$	$\alpha = (2,3)$	$\lambda = (0,1)$
$I_{2,3}^{\Delta}$	$\alpha = (2,2)$	$\lambda = (0,1)$
$I_{2,4}^{\Delta}$	$\alpha = (2,2)$	$\lambda = (1,0)$
$I_{2,5}^{\Delta}$	$\alpha = (3,2)$	$\lambda = (1,0)$
$I_{2,6}^{\Delta}$	$\alpha = (5,0)$	$\lambda = (0,1)$

$I_{3,1}^{\Delta}$	$\alpha = (1,5)$	$\lambda = (1,0)$	$\alpha' = (2,4)$	$\lambda' = (0,1)$
$I_{3,2}^{\Delta}$	$\alpha = (4,2)$	$\lambda = (0,1)$	$\alpha' = (5,1)$	$\lambda' = (0,1)$

The interesting observation here is that the summands $I_{3,1}$ and $I_{3,2}$ correspond to the two edge points $(1,4)$ and $(4,1)$ of Δ.
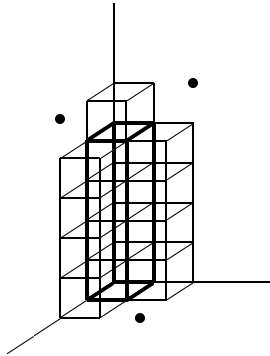
FIGURE 6. A standard set Δ together with $\Delta^{(1)}$ and three elements of $\Delta^{(2)}$.

Example 8. Consider the standard set Δ, along with its borders $\Delta^{(1)}$ and $\Delta^{(2)}$, as depicted in Figure 6. We define the polynomial ring R by the same formula as in the previous two examples. Then

$$\mathrm{Hilb}_{S/k}^{\Delta} = \mathrm{Spec}\, R/I^{\Delta},$$

where

$$I^{\Delta} = I_{2,1}^{\Delta} + \cdots + I_{2,30}^{\Delta} + I_{3,1}^{\Delta} + I_{3,2}^{\Delta} + I_{3,3}^{\Delta}.$$

The summands of I^{Δ} have analogous descriptions as in the previous example. The ideals $I_{2,i}^{\Delta}$ correspond to the 30 possible values of α, λ such that α and $\alpha + \lambda$ both lie in $\Delta^{(1)}$. The ideals $I_{3,i}^{\Delta}$ correspond to the following values of α, λ and α', λ', respectively,

$I_{3,1}^{\Delta}$	$\alpha = (1,2,0)$	$\lambda = (1,0,0)$	$\alpha' = (2,1,0)$	$\lambda' = (0,1,0)$
$I_{3,2}^{\Delta}$	$\alpha = (1,0,5)$	$\lambda = (1,0,0)$	$\alpha' = (2,0,4)$	$\lambda' = (0,0,1)$
$I_{3,3}^{\Delta}$	$\alpha = (0,4,2)$	$\lambda = (0,1,0)$	$\alpha' = (0,5,1)$	$\lambda' = (0,0,1)$

Again the summands $I_{3,1}^{\Delta}$, $I_{3,2}^{\Delta}$ and $I_{3,3}^{\Delta}$ correspond to the three edge points $(1,1,0)$, $(1,0,4)$ and $(0,4,1)$ of Δ. The following theorem explains this finding.

Theorem 3. *Let $R^\Delta = R/I^\Delta$ be the coordinate ring of $\mathrm{Hilb}^\Delta_{S/k}$ as presented in Proposition 3, where $N = \Delta$. Then the two summands I^Δ_2 and I^Δ_3, respectively, of I^Δ can be replaced by the ideals*

$$I^\Delta_{2,e} = (T_{\alpha+\lambda,\beta} - \sum_{\gamma \in \Delta} T_{\alpha,\gamma} T_{\gamma+\lambda,\beta};$$

$\alpha \in \Delta^{(1)}, \lambda \in \{e_1, \dots, e_n\}$ s.t. $\alpha + \lambda \in \Delta^{(1)}, \beta \in \Delta)$ and

$$I^\Delta_{3,e} = (\sum_{\gamma \in \Delta} T_{\varepsilon+\lambda',\gamma} T_{\gamma+\lambda,\beta} - \sum_{\gamma \in \Delta} T_{\varepsilon+\lambda,\gamma} T_{\gamma+\lambda',\beta};$$

$(\varepsilon, \lambda, \lambda')$ is an edge triple of $\Delta, \beta \in \Delta)$, respectively, in the polynomial ring R.

Proof. The statement about I^Δ_2 is easy to prove: If $\alpha \in \Delta$, then $T_{\alpha,\gamma} = \delta_{\alpha,\gamma}$; hence, $T_{\alpha+\lambda,\beta} - \sum_{\gamma \in \Delta} T_{\alpha,\gamma} T_{\gamma+\lambda,\beta} = 0$. This polynomial can therefore be eliminated from the set of generators of I^Δ_2. In the rest of the proof, we show that we may replace I^Δ_3 by $I^\Delta_{3,e}$.

We define Γ_0 to be the set of all sums $\alpha + \lambda = \alpha' + \lambda' \in \Delta^{(2)}$ where $\alpha \neq \alpha' \in \Delta^{(1)}$ and $\lambda \neq \lambda' \in \{e_1, \dots, e_n\}$. Remember that the generators of the ideal I^Δ_3 correspond to equations

(25) $$\phi(x^\alpha)\phi(x^\lambda) = \phi(x^{\alpha'})\phi(x^{\lambda'}),$$

where $\alpha + \lambda = \alpha' + \lambda' \in \Gamma_0$, in the following way: Equation (25) translates into the system of equations

$$\forall \beta \in \Delta : \sum_{\gamma \in \Delta} a_{\alpha,\gamma} a_{\gamma+\lambda,\beta} = a_{\alpha',\gamma} a_{\gamma+\lambda',\beta},$$

which system is then replaced by the generators

$$\sum_{\gamma \in \Delta} T_{\alpha,\gamma} T_{\gamma+\lambda,\beta} - \sum_{\gamma \in \Delta} T_{\alpha',\gamma} T_{\gamma+\lambda',\beta}, \beta \in \Delta$$

of I^Δ_3. Analogously, the generators of the ideal $I^\Delta_{2,e}$ correspond to equations

(26) $$\phi(x^{\alpha+\lambda}) = \phi(x^\alpha)\phi(x^\lambda),$$

where $\alpha, \alpha + \lambda \in \Delta^{(1)}$. Therefore we may assume that (26) holds for all $\alpha, \alpha + \lambda \in \Delta^{(1)}$. (As we have seen in the first paragraph of the present proof, the case where α lies in Δ gives a trivial generator of I_2^Δ. Accordingly, equation (26) is trivial if $\alpha \in \Delta$.)

Our first claim is that if (25) holds for all $\alpha + \lambda = \alpha' + \lambda' \in \Gamma_0 \cap \mathscr{C}(\Delta \cup \Delta^{(1)})$, then (25) automatically holds for all $\alpha + \lambda = \alpha' + \lambda' \in \Gamma_0$. For this we define $\Gamma = \Gamma_0 - \mathscr{C}(\Delta \cup \Delta^{(1)})$ and prove the claim by induction over Γ. Take an arbitrary $\alpha + \lambda = \alpha' + \lambda' \in \Gamma$. First we observe that there exists some $\nu \in \{e_1, \dots, e_n\}$ such that $\alpha + \lambda - \nu$ lies in $\Delta^{(2)}$, since if all $\alpha + \lambda - \nu$ were to lie either in $\Delta^{(1)}$ or outside $\mathbf{N}^n$, then $\alpha + \lambda$ would lie in $\mathscr{C}(\Delta \cup \Delta^{(1)})$, a contradiction. As $\alpha + \lambda - \nu = \alpha' + \lambda' - \nu$ lies in $\Delta^{(2)}$ and not in $\Delta^{(1)}$, the element ν equals neither λ nor λ'. Therefore both $\alpha - \nu$ and $\alpha' - \nu$ lie in $\mathbf{N}^n$, hence in $\Delta^{(1)}$. It follows that $(\alpha - \nu) + \lambda = (\alpha' - \nu) + \lambda'$ lies in Γ_0. If $(\alpha - \nu) + \lambda$ lies in $\Gamma_0 \cap \mathscr{C}(\Delta \cup \Delta^{(1)})$, then (25) holds for $(\alpha - \nu) + \lambda = (\alpha' - \nu) + \lambda'$ by assumption. If $(\alpha - \nu) + \lambda$ lies in the complement, then (25) holds for $(\alpha - \nu) + \lambda = (\alpha' - \nu) + \lambda'$ by our induction hypothesis, as $(\alpha - \nu) + \lambda \prec \alpha + \lambda$. In both cases we obtain

$$\begin{aligned}
\phi(x^\alpha)\phi(x^\lambda) &= \phi(x^{\alpha - \nu})\phi(x^\nu)\phi(x^\lambda) \\
&= \phi(x^{\alpha' - \nu})\phi(x^\nu)\phi(x^{\lambda'}) \\
&= \phi(x^{\alpha'})\phi(x^{\lambda'}).
\end{aligned}$$

Here we used (26) for the first and the last equality. Therefore (25) holds for $\alpha + \lambda = \alpha' + \lambda'$, and the first claim is proved.

Our second claim is that if (25) holds for all $\alpha + \lambda = \alpha' + \lambda' \in \Gamma_0 \cap \mathscr{C}(\Delta \cup \Delta^{(1)})$ such that $\alpha - \lambda' = \alpha' - \lambda$ lies in Δ, then (25) holds for all $\alpha + \lambda = \alpha' + \lambda' \in \Gamma_0 \cap \mathscr{C}(\Delta \cup \Delta^{(1)})$ with no additional restriction. Indeed, from $\lambda \neq \lambda'$ it follows that $\alpha - \lambda' = \alpha' - \lambda$ lies in $\mathbf{N}^n$, and therefore either in $\Delta^{(1)}$ or in Δ. In the former case we compute

$$\begin{aligned}
\phi(x^\alpha)\phi(x^\lambda) &= \phi(x^{\alpha - \lambda'})\phi(x^{\lambda'})\phi(x^\lambda) \\
&= \phi(x^{\alpha' - \lambda})\phi(x^{\lambda'})\phi(x^\lambda) \\
&= \phi(x^{\alpha'})\phi(x^{\lambda'}),
\end{aligned}$$

again using (26) for the first and the last equality. In the latter case, there is nothing to prove, as (25) holds by assumption.

Each of the remaining $\alpha + \lambda = \alpha' + \lambda'$ lies in $\Gamma_0 \cap \mathscr{C}(\Delta \cup \Delta^{(1)})$ and has the property that the element $\varepsilon = \alpha - \lambda' = \alpha' - \lambda$ lies in Δ. This is equivalent to $(\varepsilon, \lambda, \lambda')$ forming an edge triple. The theorem follows. □

Note that though $I_1^\Delta + I_2^\Delta + I_3^\Delta = I_1^\Delta + I_{2,e}^\Delta + I_{3,e}^\Delta$, the ideal $I_{2,e}^\Delta$ is strictly smaller than I_2^Δ, and the ideal $I_{3,e}^\Delta$ is strictly smaller than I_3^Δ. Of course the presentations of the coordinate ring of $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ given in Corollaries 1 and 2 can be reformulated in the spirit of Theorem 3.

Example 7 illustrates the third claim in the proof, and Example 8 illustrates the first claim. The theorem we just proved is a substantial improvement of Proposition 3, as it makes the number of generators needed for I^Δ much smaller. The concept of *across-the-corner neighbors* of [26] uses the same idea which we used in the proof of the second claim here. Yet our set of generators for I^Δ of Theorem 3 is smaller than the set of generators of the cited article.

Example 9. Let $k = \mathbf{Z}$ and $\Delta = \{0, e_1, e_2\}$ be as in Example 4. Then $\delta^{(1)} = \{2e_1, e_1 + e_2, 2e_2\}$. We use the variables $T_{\alpha,\beta}$, $\alpha \in \Delta^{(1)}$, $\beta \in \Delta$ for presenting R^Δ as in the theorem. The ideal $I_{2,e}^\Delta$ vanishes, as there are no $\alpha \in \Delta$, $\lambda \in \{e_1, e_2\}$ such that $\alpha + \lambda \in \Delta^{(1)}$. There are two edge points, e_1 and e_2, hence six generators of $I_{3,e}^\Delta$. They boil down to the following five conditions on $T_{\alpha,\beta}$:

$$
\begin{aligned}
T_{(2,0),(0,0)} &= T_{(1,1),(1,0)} T_{(2,0),(0,1)} + T_{(1,1),(0,1)}^2 \\
&\quad - T_{(2,0),(1,0)} T_{(1,1),(0,1)} - T_{(2,0),(0,1)} T_{(0,2),(0,1)}, \\
T_{(1,1),(0,0)} &= T_{(2,0),(0,1)} T_{(0,2),(1,0)} - T_{(1,1),(0,1)} T_{(1,1),(1,0)}, \\
T_{(0,2),(0,0)} &= T_{(1,1),(1,0)}^2 + T_{(1,1),(0,1)} T_{(0,2),(1,0)} \\
&\quad - T_{(0,2),(1,0)} T_{(2,0),(1,0)} - T_{(0,2),(0,1)} T_{(1,1),(1,0)}
\end{aligned}
$$

and

$$
\begin{aligned}
&T_{(2,0),(1,0)} T_{(1,1),(0,0)} + T_{(2,0),(0,1)} T_{(0,2),(0,0)} \\
&\quad - T_{(1,1),(1,0)} T_{(2,0),(0,0)} - T_{(1,1),(0,1)} T_{(1,1),(0,0)} = 0, \\
&T_{(1,1),(1,0)} T_{(1,1),(0,0)} + T_{(1,1),(0,1)} T_{(0,2),(0,0)} \\
&\quad - T_{(0,2),(1,0)} T_{(2,0),(0,0)} - T_{(0,2),(0,1)} T_{(1,1),(0,0)} = 0.
\end{aligned}
$$

Upon substituting $T_{(2,0),(0,0)}$, $T_{(1,1),(0,0)}$ and $T_{(0,2),(0,0)}$ into the last two equations, we see that the last two equations are trivial. Therefore

$\mathrm{Hilb}_{S/k}^{\Delta}$ is the six-dimensional affine space with parameters $T_{(2,0),(1,0)}$, $T_{(2,0),(0,1)}$, $T_{(1,1),(1,0)}$, $T_{(1,1),(0,1)}$, $T_{(0,2),(1,0)}$, $T_{(0,2),(0,1)}$. (Note that this also follows from Huibregtse's result on "sawtooth" standard sets, which we discussed at the end of Section 8 above.) In Example 4, we chose $T_{(2,0),(1,0)} = T_{(2,0),(0,1)} = 1$, $T_{(1,1),(1,0)} = T_{(1,1),(0,1)} = 2$, $T_{(0,2),(1,0)} = T_{(0,2),(0,1)} = 1$ for obtaining I_a. This makes I_a monic for the graded lexicographic order, but not monic for the lexicographic order where $x_1 \succ x_2$. For obtaining I_b, we replaced $T_{(0,2),(1,0)} = 1$ by $T_{(0,2),(1,0)} = 0$. This makes this ideal I_b monic for both the graded lexicographic and the lexicographic order.

Example 10. For $\Delta = \{0, \dots, (d-1)e_1\} \subset \mathbf{N}^n$, the scheme $\mathrm{Hilb}_{S/k}^{\Delta}$ is the dn-dimensional affine space with coordinates $T_{de_1,\beta}, T_{e_2,\beta}, \dots,$ $T_{e_n,\beta}$, for $\beta \in \Delta$.

We just give a hint for the proof of this: First consider the case $n = 2$. For $a = 0, \dots, d-2$, the equations $\phi(x^{(a+1,1)}) = \phi(x^{(a,1)})\phi(x^{(1,0)})$ give explicit formulas for all $T_{(a+1,1),\beta}$ as polynomials in $T_{(0,1),\gamma}$ and $T_{(d,0),\gamma}$, $\gamma \in \Delta$. In the system of polynomial equations corresponding to the equation $\phi(x^{(d-1,1)})\phi(x^{(1,0)}) = \phi(x^{(d,0)})\phi(x^{(0,1)})$, replace each $T_{(a+1,1),\beta}$, for $a = 0, \dots, d-2$, by the polynomial expression from above. Then it turns out that the above system of equations is trivial. The assertion follows in the case $n = 2$. For larger n, fix an $i \in \{2, \dots, n\}$ and derive analogous formulas for $T_{(a+1)e_i,1)\beta}$ as polynomials in $T_{e_i,\gamma}$ and $T_{(d,0),\gamma}$, $\gamma \in \Delta$. The system of polynomial equations corresponding to the equation $\phi(x^{(d-1)e_1+e_i)})\phi(x^{e_1}) = \phi(x^{de_1})\phi(x^{e_i})$ is trivial again. By Theorem 3, these equations, for $i = 2, \dots, n$, are all we have to study. The assertion follows for all n.

10. The universal objects. Equation (23) describes the transition between the matrix $(a_{\alpha\gamma})$ of a homomorphism $\phi \in \mathcal{H}\mathrm{ilb}_{S/k}^{\prec \Delta}(B)$ and the elements f_α of the kernel of ϕ. Together with Corollary 1 and Proposition 3, this enables us to directly write down the universal objects.

Proposition 4. (i) *Let R^Δ be the coordinate ring of $\mathrm{Hilb}_{S/k}^{\Delta}$ as presented in Proposition 3 or Theorem 3. Then the universal object of*

the representable functor $\mathcal{H}\text{ilb}^{\Delta}_{S/k}$ *is the affine scheme*

$$U^{\Delta} = \operatorname{Spec} R^{\Delta}[x] \Big/ \Big(x^{\alpha} - \sum_{\beta \in \Delta} T_{\alpha,\beta} x^{\beta}; \alpha \in N \cup N^{(1)} \Big)$$

over $\text{Hilb}^{\Delta}_{S/k} = \operatorname{Spec} R^{\Delta}$.

(ii) *Let* $R^{\prec \Delta}$ *be the coordinate ring of* $\text{Hilb}^{\prec \Delta}_{S/k}$ *as presented in Corollaries 1 or 2. Then the universal object of the representable functor* $\mathcal{H}\text{ilb}^{\prec \Delta}_{S/k}$ *is the affine scheme*

$$U^{\prec \Delta} = \operatorname{Spec} R^{\prec \Delta}[x] \Big/ \Big(x^{\alpha} - \sum_{\beta \in \Delta,\, \beta \prec \alpha} T_{\alpha,\beta} x^{\beta}; \alpha \in \mathscr{C}(\Delta) \Big)$$

over $\text{Hilb}^{\prec \Delta}_{S/k} = \operatorname{Spec} R^{\prec \Delta}$.

Proof. (i) is clear. As for (ii), the only thing we have to prove is that for generating the ideal

$$\Big(x^{\alpha} - \sum_{\beta \in \Delta, \beta \prec \alpha} T_{\alpha,\beta} x^{\beta}; \alpha \in N \cup N^{(1)} \Big),$$

it suffices to take all $\alpha \in \mathscr{C}(\Delta)$. However, in $R^{\prec \Delta}$ we have $T_{\alpha,\beta} = 0$ whenever $\alpha \prec \beta$. Therefore, the assertion follows from Lemma 2 (i). □

Note that the theorem gives us

$$U^{\Delta} \hookrightarrow \mathbf{A}^{n}_{\text{Hilb}^{\Delta}_{S/k}}, \qquad U^{\prec \Delta} \hookrightarrow \mathbf{A}^{n}_{\text{Hilb}^{\prec \Delta}_{S/k}},$$

respectively, as closed subschemes. Moreover, the coordinate rings of U^{Δ} and $U^{\prec \Delta}$, respectively, are free over R^{Δ} and $R^{\prec \Delta}$, respectively, by definition of the functor $\mathcal{H}\text{ilb}^{\Delta}_{S/k}$. In particular, the morphisms $U^{\Delta} \to \text{Hilb}^{\Delta}_{S/k}$ and $U^{\prec \Delta} \to \text{Hilb}^{\prec \Delta}_{S/k}$ are automatically flat.

Proposition 4 makes the statement precise that *the scheme* $\text{Hilb}^{\prec \Delta}_{S/k}$ *is the parametrizing space of all reduced Gröbner bases in S with standard sets* Δ: A point $\operatorname{Spec} B \to \text{Hilb}^{\prec \Delta}_{S/k}$ is a homomorphism $R^{\prec \Delta} \to B$. In

other words, we assign to the variables $T_{\alpha,\beta}$ values $a_{\alpha,\beta} \in B$ which satisfy the structural equations (16). Then we define

$$f_\alpha = x^\alpha - \sum_{\substack{\beta \in \Delta \\ \beta \prec \alpha}} a_{\alpha,\beta} x^\beta.$$

Geometrically this means that we consider the Cartesian diagram

$$
\begin{array}{ccc}
\operatorname{Spec} B[x]/(f_\alpha; \alpha \in \mathscr{C}(\Delta)) & \longrightarrow & U^{\prec \Delta} \\
\downarrow & & \downarrow \\
\operatorname{Spec} B & \longrightarrow & \operatorname{Hilb}_{S/k}^{\prec \Delta}.
\end{array}
$$

The structural equations guarantee that the polynomials f_α are a reduced Gröbner basis. Equivalently, by Buchberger's S-pair criterion (see [**7**, Sections 2, 6]), the S-pairs of the various f_α reduce to zero modulo all f_α.

11. Connectedness.

Proposition 5. *There exists a morphism*

$$g : \mathbf{A}_k^1 \times_{\operatorname{Spec} k} \operatorname{Hilb}_{S/k}^{\prec \Delta} \to \operatorname{Hilb}_{S/k}^{\prec \Delta}$$

such that

- *the restriction of g to $(t-1) \times_{\operatorname{Spec} k} \operatorname{Hilb}_{S/k}^{\prec \Delta}$ is the identity; and*

- *for each point p of $\operatorname{Hilb}_{S/k}^{\prec \Delta}$, the restriction of g to $\mathbf{A}_k^1 \times_{\operatorname{Spec} k} p$ is a curve on $\operatorname{Hilb}_{S/k}^{\prec \Delta}$ which connects p with the point defined by the monomial ideal $(x^\alpha; \alpha \in \mathbf{N}^n - \Delta) \subset S$.*

In particular, if $\operatorname{Spec} k$ is connected, then $\operatorname{Hilb}_{S/k}^{\prec \Delta}$ is connected as well.

We omit the proof of this, just noting that it follows the lines of the well-known construction of curves in Hilbert schemes. What is needed for the proof is found in the first few pages of [**3**], in [**8**, Section 15.8] and in Lemma 2 above. A crucial part is the existence of a linear map

$\ell : \mathbf{Z}^n \to \mathbf{Z}$ such that $\ell(\alpha) > \ell(\beta)$ for all $\alpha \in \Delta^{(1)}$ and $\beta \in \Delta$ such that $T_{\alpha,\beta} \neq 0$ in the ring $R_m^{\prec \Delta}$. This is guaranteed by [**3**, Chapter 1, Section 1], or [**8**, Exercise 15.12]. Upon writing $\mathbf{A}_k^1$ as $\operatorname{Spec} k[t]$, the ring homomorphism $R^{\prec \Delta} \to k[t] \otimes_k R^{\prec \Delta}$ corresponding to g sends each $T_{\alpha,\beta}$ to $t^{\ell(\alpha)-\ell(\beta)} T_{\alpha,\beta}$.

Can we carry this construction over from $\operatorname{Hilb}_{S/k}^{\prec \Delta}$ to $\operatorname{Hilb}_{S/k}^{\Delta}$? Here is one obvious case in which we can. We say that Δ is a *corner cut* if there exists a linear map $\ell : \mathbf{Z}^n \to \mathbf{Z}$ such that $\ell(\alpha) > \ell(\beta)$ for all $\alpha \in \mathbf{N}^n$ and all $\beta \in \Delta$. If $\prec$ is any term order, we define a new term order $\prec_\ell$ by first grading the elements of $\mathbf{N}^n$ with respect to the weight ℓ and then using $\prec$ as a tie-breaker. If Δ is a corner cut and ℓ is the corresponding linear map, then obviously $\operatorname{Hilb}_{S/k}^{\Delta} = \operatorname{Hilb}_{S/k}^{\prec_\ell \Delta}$. In particular, $\operatorname{Hilb}_{S/k}^{\Delta}$ is connected.

However, the condition on Δ being a corner cut is very restrictive. Let us replace it by a slightly weaker condition. We say that Δ a *weak corner cut* if there exists a linear map $\ell : \mathbf{Z}^n \to \mathbf{Z}$ such that $\ell(\alpha) \geq \ell(\beta)$ for all $\alpha \in \mathbf{N}^n$ and all $\beta \in \Delta$.

Example 11. Remove from the set $\{\alpha \in \mathbf{N}^n; |\alpha| \leq r\}$ any subset S of $\{\alpha \in \mathbf{N}^n; |\alpha| = r\}$; then the remaining set Δ is a standard set. Δ is a weak corner cut but not a corner cut.

In order to show connectedness if Δ is a weak corner cut, we need some more notation. Consider the functor

$$\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta,\ell} : (k\text{-Alg}) \longrightarrow (\text{Sets})$$
$$B \longmapsto \left\{ \begin{array}{c} \phi : B[x] \to Q \text{ in } \mathcal{H}ilb_{S/k}^{\Delta}(B) \text{ such that} \\ \ker \phi \text{ is homogeneous w.r.t. } \ell \end{array} \right\}.$$

Upon using the notation of Proposition 3 and Theorem 3, we see that $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta,\ell}$ is representable by the affine subscheme $\operatorname{Hilb}_{S/k}^{\Delta,\ell}$ of $\operatorname{Hilb}_{S/k}^{\Delta}$ defined by the ideal $(T_{\alpha,\beta}; \alpha \in \Delta \cup \Delta^{(1)}, \beta \in \Delta, \ell(\alpha) \neq \ell(\beta))$ in the coordinate ring R^Δ.

Proposition 6. (i) $\operatorname{Hilb}_{S/k}^{\Delta,\ell}$ *is an affine space.*

(ii) *If Δ is a weak corner cut and $\operatorname{Spec} k$ is connected, then $\operatorname{Hilb}_{S/k}^{\Delta}$ is connected.*

Proof. As for (i), we refer to [**26**, Theorem 5.3], where the same statement is proved. As for (ii), we start with the same construction as in Proposition 5: We define a ring homomorphism $R^\Delta \to k[t] \otimes_k R^\Delta$ by sending each $T_{\alpha,\beta}$ to $t^{\ell(\alpha)-\ell(\beta)} T_{\alpha,\beta}$ and obtain a morphism g : $\mathbf{A}^1_k \times_{\mathrm{Spec}\, k} \mathrm{Hilb}^\Delta_{S/k} \to \mathrm{Hilb}^\Delta_{S/k}$. If p is an arbitrary point in $\mathrm{Hilb}^{\prec\Delta}_{S/k}$, then $g((t),p)$ is a point in the closed subscheme $\mathrm{Hilb}^{\prec\Delta,\ell}_{S/k}$ of $\mathrm{Hilb}^{\prec\Delta}_{S/k}$. Then the assertion follows from (i). $\square$

The weak corner cut property is not necessary for $\mathrm{Hilb}^\Delta_{S/k}$ to be connected. If $n = 2$, or if $d \leq 7$, then $\mathrm{Hilb}^d_{S/k}$ is known to be irreducible. In these cases all open subschemes $\mathrm{Hilb}^\Delta_{S/k}$, regardless of the shape of Δ, are irreducible, thus connected. References for the case $n = 2$ include [**13**], [**32**, Theorem 18.7] and [**18**, Theorem 8.11]. The original reference for the case $d \leq 7$ is [**31**]. The authors of [**6**] prove that the upper bound $d \leq 7$ for irreducibility of $\mathrm{Hilb}^d_{S/k}$ is sharp. More precisely, Theorem 1.1 of the cited paper states that, for $d \leq 8$, the scheme $\mathrm{Hilb}^d_{S/k}$ is reducible if, and only if, $d = 8$ and $n \geq 4$. In that case there exists a component of dimension $8n - 7$ in $\mathrm{Hilb}^d_{S/k}$ which consists of local algebras isomorphic to homogeneous algebras with Hilbert function $(1, 4, 3)$. Therefore the candidates for Δ such that $\mathrm{Hilb}^\Delta_{S/k}$ might be connected are those in $\mathbf{N}^4$ with Hilbert function $(1, 4, 3)$. These are found in the following list:

$$\Delta_1 = \{0, e_1, e_2, e_3, e_4, e_1 + e_2, e_1 + e_3, e_2 + e_3\},$$
$$\Delta_2 = \{0, e_1, e_2, e_3, e_4, e_1 + e_2, e_1 + e_3, 2e_1\},$$
$$\Delta_3 = \{0, e_1, e_2, e_3, e_4, e_1 + e_2, e_1 + e_3, 2e_2\},$$
$$\Delta_4 = \{0, e_1, e_2, e_3, e_4, e_1 + e_2, 2e_1, 2e_2\},$$
$$\Delta_5 = \{0, e_1, e_2, e_3, e_4, e_1 + e_2, 2e_1, 2e_3\},$$
$$\Delta_6 = \{0, e_1, e_2, e_3, e_4, e_1 + e_2, 2e_3, 2e_4\},$$
$$\Delta_7 = \{0, e_1, e_2, e_3, e_4, 2e_1, 2e_2, 2e_3\}.$$

However, it is easy to see that all these sets are weak corner cuts. Moreover, it is easy to see that all standard sets of dimension $n \geq 5$ and size $d = 8$ are weak corner cuts or corner cuts. Therefore the question whether or not $\mathrm{Hilb}^\Delta_{S/k}$ is connected concerns those Δ of dimension $n \geq 3$ and size $d \geq 9$ which are not (weak) corner cuts, for instance

$$\Delta = \{0, e_1, 2e_1, 3e_1, e_2, 2e_2, 3e_2, e_3, 2e_3\} \subset \mathbf{N}^3.$$

For all Δ in question, the presentation of the coordinate ring of $\mathrm{Hilb}^{\Delta}_{S/k}$ given by Theorem 3 is much too large for testing connectedness of $\mathrm{Hilb}^{\Delta}_{S/k}$ by computational means. It is not known to the author if any of the $\mathrm{Hilb}^{\Delta}_{S/k}$ in question is connected or not.

12. Changing the charts. Let Δ and Π be two standard sets of size d. We embed them into standard sets $\Delta \subset N \subset \mathbf{N}^n$ and $\Pi \subset M \subset \mathbf{N}^n$. The union $N \cup M$ is a standard set containing both Δ and Π. We write the coordinate ring of $\mathrm{Hilb}^{\Delta}_{S/k}$ as in Proposition 3, with N replaced by $N \cup M$:

$$R^{\Delta} = k[T_{\alpha,\beta}; \alpha \in N \cup M \cup (N \cup M)^{(1)}, \beta \in \Delta]/I^{\Delta}.$$

We write the coordinate ring of $\mathrm{Hilb}^{\Pi}_{S/k}$ in an analogous way, replacing the matrix $T = (T_{\alpha,\beta})$ of indeterminates by a matrix of new variables $U = (U_{\alpha,\xi})$:

$$R^{\Pi} = k[U_{\alpha,\xi}; \alpha \in N \cup M \cup (N \cup M)^{(1)}, \xi \in \Pi]/I^{\Pi}.$$

We decompose the indexing set of the rows as follows:

$$N \cup M \cup (N \cup M)^{(1)} = (\Delta \cap \Pi) \coprod (\Delta - \Pi) \coprod (\Pi - \Delta) \coprod \rho.$$

Accordingly, we decompose the matrix T into the blocks

$$T = \begin{array}{c} \\ \Delta \cap \Pi \\ \Delta - \Pi \\ \Pi - \Delta \\ \rho \end{array} \begin{array}{cc} \Delta \cap \Pi & \Delta - \Pi \\ \left(\begin{array}{cc} E & 0 \\ 0 & E \\ T_{31} & T_{32} \\ T_{41} & T_{42} \end{array} \right), \end{array}$$

and the matrix U into the blocks

$$U = \begin{array}{c} \\ \Delta \cap \Pi \\ \Delta - \Pi \\ \Pi - \Delta \\ \rho \end{array} \begin{array}{cc} \Delta \cap \Pi & \Pi - \Delta \\ \left(\begin{array}{cc} E & 0 \\ U_{21} & U_{22} \\ 0 & E \\ U_{41} & U_{42} \end{array} \right) \end{array}$$

where E is the identity matrix. The symbols to the left of the rows and above the columns of T and U indicate the sets by which the respective submatrices are indexed.

Proposition 7. *Let* $\Delta \subset N$ *and* $\Pi \subset M$ *be standard sets.*

(i) $\mathrm{Hilb}_{S/k}^{\Delta} \cap \mathrm{Hilb}_{S/k}^{\Pi}$ *is the open subscheme*

$$\mathrm{Spec}\, R^{\Delta} - \mathbf{V}(\det(T_{32})).$$

(ii) *The gluing morphism* $\psi_{\Delta,\Pi}$ *which identifies the intersection as an open subscheme of* $\mathrm{Hilb}_{S/k}^{\Delta}$ *with an open subscheme of* $\mathrm{Hilb}_{S/k}^{\Pi}$ *is given by the homomorphism*

$$U \longmapsto T = U \cdot \begin{pmatrix} E & 0 \\ T_{31} & T_{32} \end{pmatrix}$$

between the coordinate rings.

Proof. We only prove the first assertion, as the proof of the second is similar. Take a k-algebra B and a homomorphism which lies in both $\mathcal{H}\mathrm{ilb}_{S/k}^{\Delta}(B)$ and $\mathcal{H}\mathrm{ilb}_{S/k}^{\Pi}(B)$. This homomorphism is represented by two surjections ϕ and ϕ', respectively, such that there exists an isomorphism Ψ making the following diagram commutative:

$$
\begin{array}{ccccc}
Bx^{\Pi} & \longrightarrow & B[x] & \xrightarrow{\phi'} & Bx^{\Pi} \\
\Psi \downarrow {\scriptstyle \cong} & & {\scriptstyle \mathrm{id}} \downarrow & & \Psi \downarrow {\scriptstyle \cong} \\
Bx^{\Delta} & \longrightarrow & B[x] & \xrightarrow{\phi} & Bx^{\Delta}.
\end{array}
$$

For all $\alpha \in \Pi - \Delta$, consider the elements $f_{\alpha} \in \ker \phi$ of equation (12). From the commutative diagram above it follows that

$$(27) \qquad \Psi(x^{\alpha}) = \begin{cases} x^{\alpha} & \text{if } \alpha \in \Pi \cap \Delta, \\ -\sum_{\beta \in \Delta} d_{\alpha,\beta} x^{\beta} & \text{if } \alpha \in \Pi - \Delta. \end{cases}$$

Indeed, the first line is immediate; as for the second line, if $\alpha \in \Pi - \Delta$, then $\phi(x^{\alpha} + \sum_{\beta \in \Delta} d_{\alpha,\beta} x^{\beta}) = 0$, i.e.,

$$\Psi(x^{\alpha}) = \Psi(\phi'(x^{\alpha})) = \phi(x^{\alpha}) = -\sum_{\beta \in \Delta} d_{\alpha,\beta} \phi(x^{\beta}) = -\sum_{\beta \in \Delta} d_{\alpha,\beta} x^{\beta}.$$

As for the inverse of Ψ, we define the polynomials

$$g_\alpha = x^\alpha + \sum_{\beta \in \Pi} e_{\alpha,\beta} x^\beta \in \ker \phi'$$

in analogy to (12) and obtain

$$\Psi^{-1}(x^\alpha) = \begin{cases} x^\alpha & \text{if } \alpha \in \Pi \cap \Delta, \\ -\sum_{\beta \in \Delta} e_{\alpha,\beta} x^\beta & \text{if } \alpha \in \Delta - \Pi. \end{cases}$$

The given homomorphism lies in both $\mathrm{Hilb}_{S/k}^\Delta$ and $\mathrm{Hilb}_{S/k}^\Pi$ if, and only if, the linear map (27) is invertible. We see that this condition is equivalent to the matrix

$$(d_{\alpha,\beta})_{\alpha \in \Pi - \Delta, \, \beta \in \Delta - \Pi}$$

being invertible. $\square$

Note that this implies that $\mathrm{Hilb}_{S/k}^\Delta \cap \mathrm{Hilb}_{S/k}^\Pi$ is the open locus where the matrices

$$T^\square = \begin{pmatrix} E & 0 \\ T_{31} & T_{32} \end{pmatrix}, \text{ and } U^\square = \begin{pmatrix} E & 0 \\ U_{21} & U_{22} \end{pmatrix}$$

are inverse to each other. This is also proved in [**20**, subsection 2.2].

Consider the matrix of indeterminates $T = (T_{\alpha,\beta})$ from the above discussion. The rows and columns of that matrix are indexed by elements of $\mathbf{N}^n$, which are ordered by the term order $\prec$. By Corollary 1, $\mathrm{Hilb}_{S/k}^{\prec \Delta}$ is the closed subscheme of $\mathrm{Spec}\, R^\Delta$ on which T is a *lower triangular matrix with respect to* $\prec$. We obtain:

Corollary 3. *The intersection* $\mathrm{Hilb}_{S/k}^{\prec \Delta} \cap \mathrm{Hilb}_{S/k}^\Pi$ *is the locally closed subscheme of* $\mathrm{Hilb}_{S/k}^\Pi$ *in which*

- $U^\square$ *is the inverse of* $T^\square$, *and*
- T *is a lower triangular matrix with respect to* $\prec$.

It is thus tempting to suspect that whenever $\mathrm{Hilb}_{S/k}^{\prec \Delta} \cap \mathrm{Hilb}_{S/k}^\Pi$ is not empty, the boundary in $\mathrm{Hilb}_{S/k}^\Pi$ of $\mathrm{Hilb}_{S/k}^{\prec \Delta} \cap \mathrm{Hilb}_{S/k}^\Pi$ contains $\mathrm{Hilb}_{S/k}^{\prec \Pi}$.

Indeed, if U is lower triangular, then in particular its submatrix $U^\square$ is lower triangular; "the inverse of a lower triangular matrix is a lower triangular matrix," thus on the intersection $\mathrm{Hilb}_{S/k}^\Delta \cap \mathrm{Hilb}_{S/k}^\Pi$, equation $T^\square U^\square = E$ would assert that both $T^\square$ and $U^\square$ are lower triangular. Moreover, "the product of two lower triangular matrices is lower triangular", thus on the intersection, equations $T = UT^\square$ and $U = TU^\square$ of Proposition 7 would assert that both T and U are lower triangular.

However, the first quoted assertion only holds for square matrices in which rows and columns are indexed by the same totally ordered set. This is not the case for $T^\square$ and $U^\square$, whose rows and columns are indexed by Δ and Π, respectively. (The second quoted assertion is true for all products AB of matrices A and B indexed by subsets of a totally ordered set such that the indexing set of the columns of A equals the indexing set of the rows of B.) Moreover, if T and U were both lower triangular in a point of the intersection, then that point would lie in both $\mathrm{Hilb}_{S/k}^{\prec\Delta}$ and $\mathrm{Hilb}_{S/k}^{\prec\Pi}$, in contradiction to Theorem 2. Therefore the heuristics of the last paragraph do not prevail. The most we can hope for is that for all standard sets Δ and Π of size d, we have

$$
(28) \qquad
\begin{aligned}
&\partial_{\mathrm{Hilb}_{S/k}^\Pi}(\mathrm{Hilb}_{S/k}^{\prec\Delta} \cap \mathrm{Hilb}_{S/k}^\Pi) = \varnothing \text{ or} \\
&\partial_{\mathrm{Hilb}_{S/k}^\Pi}(\mathrm{Hilb}_{S/k}^{\prec\Delta} \cap \mathrm{Hilb}_{S/k}^\Pi) \supset \mathrm{Hilb}_{S/k}^{\prec\Pi},
\end{aligned}
$$

as was indicated above, where $\partial_A(B)$ denotes the boundary in A of B.

If this was true, the decomposition (14) of Theorem 2 would be a *stratification* in the following technical sense: Let X be a topological space and $Y_i \subset X$, for $i \subset \mathcal{I}$, be locally closed subsets such that

$$
(29) \qquad X = \coprod_{i \in \mathcal{I}} Y_i.
$$

Then (29) is called a *stratification* if for all $i \in \mathcal{I}$, there exists an $\mathcal{I}(i) \subset \mathcal{I}$ such that

$$
(30) \qquad \overline{Y_i} = \coprod_{j \in \mathcal{I}(i)} Y_j,
$$

where the bar denotes the closure. The connection between condition (28) and (14) being a stratification is established by the following result:

Lemma 5. *Let X be a topological space, $X = \cup_{i \in \mathcal{I}} X_i$ an open cover and $Y_i \subset X_i$ closed such that (29) holds. Then (29) is a stratification if, and only if, for all $i, j \in \mathcal{I}$, we have either $\partial_{X_j}(Y_i \cap X_j) = \varnothing$ or $\partial_{X_j}(Y_i \cap X_j) \supset Y_j$.*

Proof. First we show the "only if" direction. By (30), we have $\partial_X Y_i = \coprod_{j \in \mathcal{I}(i) - \{i\}} Y_j$. We obtain

$$\partial_{X_k}(Y_i \cap X_k) = (\partial_X Y_i) \cap X_k = \coprod_{j \in \mathcal{I}(i) - \{i\}} Y_j \cap X_k.$$

Upon considering $j = k$, the claim follows.

As for the "if" direction, we have

$$\overline{Y_i} = Y_i \coprod \partial_X Y_i = Y_i \coprod \cup_{j \in \mathcal{I}} \partial_{X_j}(Y_i \cap X_j) \supset Y_i \coprod \left(\coprod_{j \in \mathcal{I}'(i)} Y_j \right).$$

Here the last inclusion comes from our hypothesis, $\mathcal{I}'(i)$ being the set of all j such that $\partial_{X_j}(Y_i \cap X_j) \neq \varnothing$. If this inclusion was strict, a j would exist such that $\partial_{X_j}(Y_i \cap X_j) \supsetneq Y_j$. Take a point x from the difference. Then in particular x lies in $\partial_X Y_i$. Moreover, by (29), a k exists such that $x \in Y_k$. Therefore x lies in $(\partial_X Y_i) \cap X_k = \partial_{X_k}(Y_i \cap X_k)$. In particular, that set is not empty; thus, k lies in $\mathcal{I}'(i)$. It follows that the inclusion above is not strict.   □

Let us return to the situation we studied above, i.e., the decomposition (14) of Theorem 2. Denote by $\mathcal{S}$ the set of all standard sets of size d. Then the decomposition (14) is a stratification if, and only if, for all $\Delta \in \mathcal{S}$,

$$(31) \qquad \overline{\mathrm{Hilb}_{S/k}^{\prec \Delta}} = \coprod_{\Pi \in \mathcal{S}(\Delta)} \mathrm{Hilb}_{S/k}^{\prec \Pi},$$

where union is taken over all Δ in

$$\mathcal{S}(\Delta) = \{\Pi \in \mathcal{S}; \overline{\mathrm{Hilb}_{S/k}^{\prec \Delta}} \cap \mathrm{Hilb}_{S/k}^{\prec \Pi} \neq \varnothing\}.$$

Here is a negative result on the question wether or not (14) is a stratification.

Proposition 8. *If $\prec$ is the lexicographic order on S and $\dim \operatorname{Hilb}^d_{S/k} > dn$, then the decomposition of Theorem 2 is not a stratification. In particular, if $n = 3$ and $d \geq 102$ or $n = 4$ and $d \geq 25$, the decomposition of Theorem 2 is not a stratification.*

Proof. We order the variables such that $x_1 \succ \cdots \succ x_n$ and consider the standard set $\Delta = \{0, \ldots, (r-1)e_n\} \subset \mathbf{N}^n$. Remember that by Example 10, $\operatorname{Hilb}^\Delta_{S/k}$ has dimension dn. Furthermore, $\operatorname{Hilb}^\Delta_{S/k} = \operatorname{Hilb}^{\prec \Delta}_{S/k}$, as $\alpha \succ \beta$ for all $\alpha \in \mathbf{N}^n - \Delta$ and all $\beta \in \Delta$ in the lexicographic order. We claim that

$$(32) \qquad\qquad \forall \Pi \in \mathcal{S} : \overline{\operatorname{Hilb}^{\prec \Delta}_{S/k}} \cap \operatorname{Hilb}^{\prec \Pi}_{S/k} \neq \varnothing.$$

For proving that, we use a few schemes introduced and discussed in [**29**, Section 5]. The first is

$$\operatorname{Hilb}^{d,0}_{S/k} = ((\mathbf{A}^n_k)^d - \Lambda)/S_r,$$

where for $i = 1, \ldots, r$, we denote by $(x^{(i)}_1, \ldots, x^{(i)}_n)$ the coordinates on the i-th copy of $\mathbf{A}^n$ in the product; where $\Lambda = \cup_{i \neq j} \mathbf{V}(x^{(i)}_1 - x^{(j)}_1, \ldots, x^{(i)}_n - x^{(j)}_n)$ is the *large diagonal* in the product; and where S_r is the symmetric group acting on the product in the obvious way. $\operatorname{Hilb}^{d,0}_{S/k}$ is an open subscheme of $\operatorname{Hilb}^d_{S/k}$, and the functor associated to the scheme $\operatorname{Hilb}^{d,0}_{S/k}$ is given by

$$\mathcal{H}\mathrm{ilb}^{d,0}_{S/k}(B) = \left\{ \begin{array}{c} \text{closed subschemes } Z \subset \mathbf{A}^n_B \text{ such that} \\ p : Z \to \operatorname{Spec} B \text{ is finite étale of degree } d \end{array} \right\},$$

where p is the restriction to Z of the projection $\mathbf{A}^n_B \to \operatorname{Spec} B$. (In contrast to that, the functor $\mathcal{H}\mathrm{ilb}^d_{S/k}$ sends a k-algebra B to the set of all closed subschemes $Z \subset \mathbf{A}^n_B$ such that the restriction of the projection $p = \mathbf{A}^n_B \to \operatorname{Spec} B$ is finite flat of degree d. Finite flatness of $p : Z = \operatorname{Spec} B[x]/I \to \operatorname{Spec} B$ translates to local freeness of the B-algebra $B[x]/I$. Therefore, the additional requirement in $\mathcal{H}\mathrm{ilb}^{d,0}_{S/k}$ is *unramifiedness*.) Moreover, we use the subscheme

$$\operatorname{Hilb}^{\prec \Delta, 0}_{S/k} = \operatorname{Hilb}^\Delta_{S/k} \cap \operatorname{Hilb}^{d,0}_{S/k}$$

of $\mathrm{Hilb}_{S/k}^{d,0}$ and its analogue for Π instead of Δ. The functor associated to that scheme sends a k-algebra B to the set of all étale $p : Z = \mathrm{Spec}\, B[x]/I \to \mathrm{Spec}\, B$ such that $I \subset B[x]$ is monic with standard set Δ. (As $\mathrm{Hilb}_{S/k}^{\Delta} = \mathrm{Hilb}_{S/k}^{\prec\Delta}$, the functor equivalently sends a k-algebra B to the set of all étale $p : Z = \mathrm{Spec}\, B[x]/I \to \mathrm{Spec}\, B$ such that $B[x]/I$ is free with basis x^{Δ}.)

We fix a homomorphism from k to a field k' having at least d elements and a bijection $\{0, \dots, d-1\} \to B$, where $B \subset k'$ has d elements. That induces a bijection $\{0, \dots, d-1\}^n \to B^n$, whose restriction to Π induces a bijection $\Pi \to C$, where $C \subset B^n \subset (k')^n$ has d elements. Let $I \subset k'[x]$ be the ideal defining C. Then Corollary 10 of [**27**] says that the ideal I is monic with standard set Π. Therefore $\xi = k'[x]/I$ is a k'-rational closed point of $\mathrm{Hilb}_{S/k}^{\prec\Pi}$. We shall prove that ξ lies in the closure in $\mathrm{Hilb}_{S/k}^{d,0}$ of $\mathrm{Hilb}_{S/k}^{\prec\Delta,0}$. Then ξ will also lie in the closure in $\mathrm{Hilb}_{S/k}^{d}$ of $\mathrm{Hilb}_{S/k}^{\prec\Delta}$, and (32) will be proved.

For this we denote by

$$\pi : (\mathbf{A}_k^n)^d - \Lambda \longrightarrow \mathrm{Hilb}_{S/k}^{d,0}$$

the canonical morphism. As $\mathrm{Hilb}_{S/k}^{\prec\Delta} = \mathrm{Hilb}_{S/k}^{\Delta}$, and therefore $\mathrm{Hilb}_{S/k}^{\prec\Delta,0}$ is an open subscheme of $\mathrm{Hilb}_{S/k}^{d,0}$, it follows that $\pi^{-1}(\mathrm{Hilb}_{S/k}^{\Delta,0}) = (\mathbf{A}_k^n)^d - \Lambda - A$ for some closed $A \subset (\mathbf{A}_k^n)^d$. (In fact, A is the scheme associated to the ideal $(x_n^i - x_n^j; i \neq j)$, but we will not need that here.) Remember that we have to show that for all open $U \subset \mathrm{Hilb}_{S/k}^{d,0}$ such that $\xi \in U$, we have $U \cap \mathrm{Hilb}_{S/k}^{\prec\Delta,0} \neq \varnothing$. Let η be an element of $\pi^{-1}(\xi)$. (The choice of η corresponds to the choice of a labeling of the elements of E.) It clearly suffices to show that for all open $V \subset (\mathbf{A}_k^n)^d - \Lambda$ such that $\eta \in V$, we have $V \cap ((\mathbf{A}_k^n)^d - \Lambda - A) \neq \varnothing$. Upon writing $V = (\mathbf{A}_k^n)^d - \Lambda - A'$, for some closed $A' \subset (\mathbf{A}_k^n)^d$, it follows that $V \cap ((\mathbf{A}_k^n)^d - \Lambda - A) = (\mathbf{A}_k^n)^d - \Lambda - A - A'$. That intersection is empty if, and only if, $\Lambda \cup A \cup A' = (\mathbf{A}_k^n)^d$. But this is impossible as all three summands are closed subschemes of $(\mathbf{A}_k^n)^d$ and strictly smaller than the ambient scheme.

Now that (32) is proved, we conclude as follows: If the decomposition in question was a stratification, (32) would say that $\mathcal{S}(\Delta) = \mathcal{S}$. Moreover, each $\mathrm{Hilb}_{S/k}^{\prec\Pi}$, being a subscheme of the closure of $\mathrm{Hilb}_{S/k}^{\prec\Delta}$,

would have a dimension at most dn. Therefore by Theorem 2 and (31), the dimension of

$$\mathrm{Hilb}^d_{S/k} = \coprod_{\Pi \in \mathcal{S}} \mathrm{Hilb}^{\prec \Pi}_{S/k} = \coprod_{\Pi \in \mathcal{S}(\Delta)} \mathrm{Hilb}^{\prec \Pi}_{S/k} = \overline{\mathrm{Hilb}^{\prec \Delta}_{S/k}}.$$

would be dn. This proves the first assertion of the proposition. As for the second assertion, we know from (1) of [21] that $\dim \mathrm{Hilb}^d_{S/k} > dn$ if $n = 3$ and $d \geq 102$ or $n = 4$ and $d \geq 25$. □

The last proposition raises some interesting questions. The *good component* $\mathscr{G}^d_{S/k}$ of $\mathrm{Hilb}^d_{S/k}$ is the schematic closure of the subscheme $\mathrm{Hilb}^{d,0}_{S/k}$. (As for the construction of $\mathscr{G}^d_{S/k}$, see [10, 39].) In particular, $\mathscr{G}^d_{S/k}$ is of dimension dn; thus, the argument we used in the last proof does not work if we replace the full Hilbert scheme by the good component.

Question 4. Is the decomposition

$$\mathscr{G}^d_{S/k} = \coprod_{\Delta} (\mathscr{G}^d_{S/k} \cap \mathrm{Hilb}^{\prec \Delta}_{S/k}),$$

induced by the decomposition of Theorem 2, a stratification?

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