JOURNAL OF COMMUTATIVE ALGEBRA
Volume 3, Number 3, Fall 2011

INFINITARY EQUIVALENCE OF $\mathbf{Z}_p$-MODULES WITH NICE DECOMPOSITION BASES

RÜDIGER GÖBEL, KATRIN LEISTNER,
PETER LOTH AND LUTZ STRÜNGMANN

ABSTRACT. Warfield modules are direct summands of simply presented $\mathbf{Z}_p$-modules or, alternatively, are $\mathbf{Z}_p$-modules possessing a nice decomposition basis with simply presented cokernel. They have been classified up to isomorphism by their Ulm-Kaplansky and Warfield invariants. Taking a model theoretic point of view and using infinitary languages we give here a complete model theoretic characterization of a large class of $\mathbf{Z}_p$-modules having a nice decomposition basis. As a corollary, we obtain the classical classification of countable Warfield modules. This generalizes results by Barwise and Eklof.

1. Introduction. The classical theorem by Ulm [**13**] states that two countable abelian p-groups are isomorphic if and only if their numerical invariants, the Ulm-Kaplansky invariants, coincide. For uncountable (abelian) p-groups this theorem is false, however, it was Hill [**4**] and Walker [**14**] who proved that it still holds for the class of totally projective groups. In fact, the class of totally projective abelian p-groups is the largest natural class of abelian p-groups such that every member is completely determined by its Ulm-Kaplansky invariants. Passing to general abelian groups it was then Warfield [**15**] who extended Ulm's theorem to the class of Warfield modules introducing new numerical invariants, the so-called Warfield invariants. Recall that a Warfield module is a direct summand of a simply presented $\mathbf{Z}_p$-module, where $\mathbf{Z}_p$ is the ring of integers localized at the prime p. Taking a completely different point of view, it was Szmielew [**12**] who first considered abelian groups model-theoretically. The usual axioms for abelian groups can be stated in the lower predicate

2010 AMS *Mathematics subject classification.* Primary 03C52, 03C60, 13C05, Secondary 03E10, 20K21, 20K25, 20K35.

The collaboration was supported by the project No. I-963-98.6/2007 of the German-Israeli Foundation for Scientific Research & Development and by the project STR 627/6 of the German Research Foundation DFG.

Received by the editors on August 30, 2009, and in revised form on March 25, 2010.

DOI:10.1216/JCA-2011-3-3-321

calculus (i.e., in $L_{\omega\omega}$); however, in order to characterize torsion groups or simple groups languages with infinite expressions are needed. For instance, the compactness theorem shows that abelian torsion groups are not axiomatizable in $L_{\omega\omega}$. Barwise and Eklof [**1**] took up Szmielew's approach and characterized the equivalence classes of torsion abelian groups with respect to the relation of satisfying the same sentences of some infinitary language L, e.g., $L_{\omega_\alpha\omega}$. Their result uses a certain sequence of cardinals closely related to the classical Ulm-Kaplansky invariants and implies the original result by Ulm as a corollary. Later Mekler and Oikkonen [**10**] showed that these results cannot be extended to more general cases. In fact, they proved that under CH there is an abelian p-group A of size $\aleph_1$ such that no sentence of a certain infinitary language, $M_{\omega_2\omega_2}$, (whose formulas are syntactical, non-well-founded trees) can serve as a complete invariant for A.

In this paper we consider the natural question of extending the Barwise-Eklof theorem (see [**1**, Theorem 3.1]) to Warfield modules. Unfortunately, there is no reasonable class of modules including the Warfield modules that is closed under $L_{\omega_\alpha\omega}$ equivalence. However, in analogy to [**1**] we are able to completely classify the L-equivalence classes of certain $\mathbf{Z}_p$-modules admitting a nice decomposition basis for large classes of infinitary languages L. As in the group theoretic version we need additional numerical invariants deduced from the Warfield invariants and obtain the classical description of countable Warfield modules as a corollary. Naturally, our results also hold for more general local rings than $\mathbf{Z}_p$.

After completing this paper the authors learned about an unpublished Ph.D. thesis by Carol Jacoby [**6**] where results similar to our main Theorem 5.11 were obtained. Our techniques are different from those in [**6**] and more algebraic. Instead of using partial decomposition bases as in [**6**, Theorem 17] we here use global decomposition bases and therefore obtain an algebraically stronger result that is closer to Warfield's original approach from [**15**]. For instance, Lemma 3.6 yields a stronger Karp-system of I_α's that allows the lifting of partial mappings to global ones (see Theorem 4.1 and Corollary 4.3). Moreover, Jacoby assumes in [**6**, Theorem 17] that the underlying ring is complete while we are working over the non-complete ring $\mathbf{Z}_p$ (see Theorem 5.11).

2. Notations and terminology. Let g be a cardinal-valued function. Following [**1**], we define $\widehat{g}$ as follows:

$$\widehat{g}(x) = \begin{cases} g(x) & \text{if } g(x) < \aleph_0, \\ \infty & \text{if } g(x) \geq \aleph_0. \end{cases}$$

Similarly, we define

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{if } g(x) \leq \aleph_0, \\ \infty & \text{if } g(x) > \aleph_0. \end{cases}$$

All modules considered in this paper are $\mathbf{Z}_p$-modules for a fixed prime p where $\mathbf{Z}_p = \{\frac{m}{n} \in \mathbf{Q} : (n,p) = 1\}$. Let M be a module. The torsion part of M is denoted by tM. If S is a subset of M, let $\langle S \rangle$ denote the submodule of M which is generated by S. For each ordinal α, a submodule $p^\alpha M$ is defined as follows: $pM = \{pg : g \in M\}$, $p^{\alpha+1}M = p(p^\alpha M)$, and $p^\alpha M = \bigcap_{\beta<\alpha} p^\beta M$ if α is a limit ordinal. The *length* of M is the smallest ordinal τ such that $p^\tau M = p^{\tau+1}M$. With every $x \in M$, we associate its *p-height* (also called *height*) $|x|$, that is, $|x| = \alpha$ if $x \in p^\alpha M \backslash p^{\alpha+1}M$ and $|x| = \infty$ if $x \in p^\infty M = \bigcap_\alpha p^\alpha M$. Sometimes we write $|x|_M$ instead of $|x|$ to emphasize the module in which the height of x is computed. If $p^\infty M = 0$, then M is called *reduced*. A submodule N of M is called *nice* (in M) if

$$p^\alpha(M/N) = (p^\alpha M + N)/N$$

for all ordinals α. An element $x \in M$ is said to be *proper with respect to* N if x has maximal height among all elements in the coset $x+N$. In this case, we have $|x+h| = \min\{|x|,|h|\}$ for all $h \in N$. Notice that N is nice in M if and only if every coset of N in M has an element which is proper with respect to N (cf. [**9**, Proposition 1.4]). If S and T are submodules of modules M and M', respectively, then an isomorphism $f : S \to T$ is called *height-preserving* if $|f(x)|_{M'} = |x|_M$ for all $x \in S$. Let $M[p] = \{x \in M : px = 0\}$. Then we write $p^\alpha M[p]$ instead of $(p^\alpha M)[p]$. Recall that the *Ulm-Kaplansky invariants* of M are defined by

$$u_M(\alpha) = \dim p^\alpha M[p]/p^{\alpha+1}M[p]$$

for α an ordinal, and

$$u_M(\infty) = \dim p^\infty M[p].$$

If N is a submodule of M and α is an ordinal, let $N(\alpha) = p^\alpha M[p] \cap (N + p^{\alpha+1}M)$. Then the *$\alpha$-th Ulm-Kaplansky invariant of M relative to N* is defined to be

$$u_{M,N}(\alpha) = \dim p^\alpha M[p]/N(\alpha).$$

Letting $u_N^M(\alpha) = \dim N(\alpha)/p^{\alpha+1}M[p]$ we observe the equation

$$u_M(\alpha) = u_N^M(\alpha) + u_{M,N}(\alpha)$$

which clearly induces

$$\widehat{u}_M(\alpha) = \widehat{u}_N^M(\alpha) + \widehat{u}_{M,N}(\alpha).$$

An *Ulm sequence* is a sequence $\overline{\beta} = (\beta_i : i < \omega)$ where each β_i is an ordinal or the symbol ∞ such that $\beta_i < \beta_{i+1}$ for all i and we use the convention $\alpha < \infty$ whenever α is an ordinal or the symbol ∞ (see [**9**, page 59]). For $k < \omega$ we define $p^k\overline{\beta} = (\beta_{i+k} : i < \omega)$. Let $\overline{\beta} = (\beta_i : i < \omega)$ and $\overline{\gamma} = (\gamma_i : i < \omega)$ be Ulm sequences. Then we write $\overline{\beta} \leq \overline{\gamma}$ if $\beta_i \leq \gamma_i$ for all $i < \omega$. We call $\overline{\beta}$ and $\overline{\gamma}$ *equivalent* and write $\overline{\beta} \sim \overline{\gamma}$ if there exist $k, l < \omega$ such that $p^k\overline{\beta} = p^l\overline{\gamma}$. The *Ulm sequence of $x \in M$* is the sequence $u(x) = (|p^i x| : i < \omega)$, and sometimes we write $u_M(x)$ instead of $u(x)$. This sequence is said to have a *gap at an ordinal* α if there is an $i < \omega$ such that $|p^i x| = \alpha$ and $|p^{i+1}x| > \alpha + 1$.

Let $M(\overline{\beta})$ be the submodule of M defined by $M(\overline{\beta}) = \{x \in M : |p^i x| \geq \beta_i \text{ for all } i < \omega\}$. If $\beta_i \neq \infty$ for all $i < \omega$, define $M(\overline{\beta}^*) = \langle x \in M(\overline{\beta}) : |p^i x| > \beta_i \text{ for infinitely many values of } i\rangle$, and otherwise let $M(\overline{\beta}^*)$ be the torsion part of $M(\overline{\beta})$. Then the *$\overline{\beta}$-th Warfield invariant of M* is given by

$$w_M(\overline{\beta}) = \dim M(\overline{\beta})/M(\overline{\beta}^*).$$

A subset $X = \{x_i\}_{i\in I}$ of M is called a *decomposition basis for M* if all elements of X are independent and have infinite order such that $M/\langle X\rangle$ is torsion and $\langle X\rangle = \bigoplus_{i\in I}\langle x_i\rangle$ is a valuated coproduct in M (that is, if for each $x = \sum k_i x_i \in \langle X\rangle$ ($k_i \in \mathbf{Z}_p$) we have $|x|_M = \min\{|k_i x_i|_M\}$). If X and X' are decomposition bases for M, then X' is called a *subordinate of X* if every element of X' is a nonzero multiple of an element of X. Note that if M has a decomposition basis X, then $w_M(\overline{\beta})$ is the cardinality of the set

$$X^{(\overline{\beta})} = \{x \in X : u(x) \sim \overline{\beta}\}.$$

If $\langle X \rangle$ is a nice submodule of M, we call X a *nice decomposition basis for* M. Notice that a subordinate of a nice decomposition basis is nice (cf. [**9**, Lemma 3.9]). Recall that a module is called *simply presented* if it can be defined in terms of generators and relations so that the only relations are of the forms $px = 0$ and $px = y$. A *Warfield module* is a direct summand of a simply presented module or, equivalently, is a module M possessing a nice decomposition basis X with simply presented quotient $M/\langle X \rangle$.

Our notation is standard and follows [**2**] for abelian groups and [**11**] for model theory. For terminology and background information on the theory of Warfield groups, we may refer to [**3, 5, 9, 15**].

3. Extending α-height-preserving isomorphisms between free $\mathbf{Z}_p$-modules. In this section, we establish conditions for extending isomorphisms between free $\mathbf{Z}_p$-modules which preserve heights below a given ordinal.

Definition 3.1. Let α be an ordinal or the symbol ∞, and let $\overline{\beta} = (\beta_i : i < \omega)$ be an Ulm sequence. Then the *α-initial sequence* of $\overline{\beta}$ is defined to be $\text{initial}_\alpha(\overline{\beta}) = (\gamma_i : i < \omega)$ where $\gamma_i = \beta_i$ if $\beta_i \leq \alpha$ and $\gamma_i = \infty$ if $\beta_i > \alpha$. If $\overline{\eta}$ is another Ulm sequence we call $\overline{\beta}$ and $\overline{\eta}$ *α-initially equivalent* and write

$$\overline{\beta} \sim_\alpha \overline{\eta},$$

if $\text{initial}_\alpha(\overline{\beta})$ and $\text{initial}_\alpha(\overline{\eta})$ are equivalent Ulm sequences.

Notice that $\text{initial}_\infty(\overline{\beta}) = \overline{\beta}$ and that $\overline{\beta} \sim_\infty \overline{\eta}$ exactly if $\overline{\beta} \sim \overline{\eta}$.

Definition 3.2. Let α be an ordinal or the symbol ∞. For a subset X of a module M and an Ulm sequence $\overline{\beta} = (\beta_i : i < \omega)$ we define

$$X_\alpha^{(\overline{\beta})} = \{x \in X : u_M(x) \sim_\alpha \overline{\beta}\}.$$

If $\beta_i \le \alpha$ for all $i < \omega$, then $X_\alpha^{(\overline{\beta})} = X^{(\overline{\beta})}$. Now assume that $\beta_i > \alpha$ for some $i < \omega$, and let x be an element of X. If $\infty \neq |p^i x|_M \le \alpha$ for all $i < \omega$, then $u_M(x) \not\sim_\alpha \overline{\beta}$. On the other hand, if $|p^i x|_M > \alpha$ for some $i < \omega$, then $u_M(x) \sim_\alpha \overline{\beta}$. It follows that

$$X_\alpha^{(\overline{\beta})} = X^{(\overline{\beta})} \cup \{x \in X \backslash X^{(\overline{\beta})} : |p^k x|_M > \alpha \text{ for some } k < \omega\},$$

hence $X_\alpha^{(\overline{\beta})} = \{x \in X : p^k x \in p^{\alpha+1} M \text{ for some } k < \omega\}$. Note that it may happen that $X_\alpha^{(\overline{\beta})} = X_\alpha^{(\overline{\beta}')}$ for some Ulm sequences $\overline{\beta}$ and $\overline{\beta}'$ even if $\overline{\beta}$ and $\overline{\beta}'$ are not equivalent. The idea is to identify those Ulm sequences which are equivalent below α and may be not equivalent above α.

Isomorphisms which preserve both heights below an ordinal α and Ulm sequences up to $\sim_\alpha$ will be useful. Following Barwise and Eklof [**1**] we introduce α-height-preserving isomorphisms:

Definition 3.3. Let M and N be modules, and let S and T be submodules of M and N, respectively. An isomorphism $f : S \to T$ is called *α-height-preserving* for some ordinal α if the following holds for all $x \in S$:

- If $|x|_M < \alpha$, then $|f(x)|_N = |x|_M$;
- If $|x|_M \ge \alpha$, then $|f(x)|_N \ge \alpha$.

f is called *∞-height-preserving* if f is height-preserving.

Lemma 3.4. *Let S and T be submodules of modules M and N, respectively, and let $f : S \to T$ be an α-height-preserving isomorphism where α is an ordinal or the symbol ∞. Suppose $x \in M$ and $y \in N$ such that x has order p^r modulo S, y has order p^r modulo T and $f(p^r x) = p^r y$ for some positive integer r. If either*

(i) *$r = 1$, $|x|_M = |y|_N$, x is proper with respect to S and y is proper with respect to T, or*

(ii) *$|x|_M \ge \alpha$ and $|y|_N \ge \alpha$,*

then f extends to an α-height-preserving isomorphism

$$\langle S, x\rangle \to \langle T, y\rangle$$

by sending x onto y.

Proof. All heights in this proof are computed in M and N, respectively. It is clear that, for $s \in S$ and $n \in \mathbf{Z}$, $s + nx \mapsto f(s) + ny$ defines an isomorphism $f' : \langle S, x\rangle \to \langle T, y\rangle$. First, suppose condition (i) of the lemma holds. Assume $|s + x| < \alpha$. If $|s| < \alpha$, then $\min\{|f(s)|, |x|\} = \min\{|s|, |x|\} = |s + x|$. If $|s| \geq \alpha$, then $|x| = \min\{|s + x|, |s|\} = |s + x|$ and $|f(s)| \geq \alpha$, hence $\min\{|f(s)|, |x|\} = |s + x|$. In either case we have

$$|f(s) + y| = \min\{|f(s)|, |y|\} = \min\{|f(s)|, |x|\} = |s + x|.$$

Now assume $|s + x| \geq \alpha$. Since $\min\{|s|, |x|\} = |x + s|$, this implies $|f(s) + y| = \min\{|f(s)|, |x|\} \geq \alpha$. Therefore f' is α-height-preserving. Now suppose condition (ii) holds. If $|s + nx| < \alpha$, then

$$\begin{aligned}|s + nx| &= \min\{|s + nx|, |nx|\} = |s| = |f(s)| \\ &= \min\{|f(s)|, |ny|\} = |f(s) + ny|,\end{aligned}$$

and if $|s + nx| \geq \alpha$ we have $|s| \geq \alpha$ which yields $|f(s) + ny| \geq \alpha$. This completes the proof. $\square$

Notice that in Lemma 3.4 (i) the condition "$r = 1$" is necessary: for $x = 1 \in M = \mathbf{Z}/p^2\mathbf{Z}$ and $y = (1, p) \in N = \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p^3\mathbf{Z}$ we have $|x|_M = |y|_N$ but $|px|_M \neq |py|_N$, hence $f : \{0\} \to \{0\}$ cannot be extended to a height-preserving isomorphism $\langle x\rangle \to \langle y\rangle$.

Definition 3.5. Let α be an ordinal or the symbol ∞. For a module M we define

$$w_M^\alpha(\overline{\beta}) = \sum_{\substack{\overline{\eta} \sim_\alpha \overline{\beta}, \\ \overline{\eta} \in \mathbf{U}}} w_M(\overline{\eta})$$

where $\mathbf{U}$ is a complete set of representatives of distinct equivalence classes of Ulm sequences $\overline{\eta}$ such that $w_M(\overline{\eta}) \neq 0$.

Note that if the module M has a decomposition basis X, then $w_M^\alpha(\overline{\beta})$ is the cardinality of the set $X_\alpha^{(\overline{\beta})}$. In [**6**], Warfield proved that two Warfield modules are isomorphic exactly if they have the same Ulm and Warfield invariants. The proof uses the following result: If X and Y are any decomposition bases of modules with identical Warfield invariants, then there exist subordinates X' and Y' of X and Y such that there is a bijection $X' \to Y'$ inducing a height-preserving isomorphism $\langle X' \rangle \to \langle Y' \rangle$ (see [**9**, Lemma 3.8]). In particular, it follows that if $A \subseteq X'$ and $B \subseteq Y'$ are finite sets and

$$f : \langle A \rangle \to \langle B \rangle$$

is a height-preserving isomorphism such that $f(A) = B$, then for every $x \in X'$ (respectively $y \in Y'$) there is an element $y \in Y'$ (respectively $x \in X'$) such that f extends to a height-preserving isomorphism $\langle A, x \rangle \to \langle B, y \rangle$. This extension result can be generalized:

Lemma 3.6. *Let M and N be modules with decomposition bases X and Y, respectively, such that $\widetilde{w}^{\alpha}_{M}(\overline{\beta}) = \widetilde{w}^{\alpha}_{N}(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$ where α is some fixed ordinal or the symbol ∞. Then there exist subordinates X' and Y' of X and Y, respectively, satisfying the following property: If A and B are countable subsets of X' and Y', respectively, and*

$$f : \langle A \rangle \longrightarrow \langle B \rangle$$

is an α-height-preserving isomorphism such that f maps A onto B, then for every $x \in X'$ (respectively $y \in Y'$) there is a $y \in Y'$ (respectively $x \in X'$) such that f extends to an α-height-preserving isomorphism

$$f' : \langle A, x \rangle \longrightarrow \langle B, y \rangle$$

by sending x onto y.

Proof. Since $X = \dot{\cup}_{\overline{\beta}} X^{(\overline{\beta})}_{\alpha}$, $Y = \dot{\cup}_{\overline{\beta}} Y^{(\overline{\beta})}_{\alpha}$ and each set $X^{(\overline{\beta})}_{\alpha}$, $Y^{(\overline{\beta})}_{\alpha}$ has cardinality $w^{\alpha}_{M}(\overline{\beta})$, respectively $w^{\alpha}_{N}(\overline{\beta})$, we will define the subordinates X' and Y' as disjoint unions $X' = \dot{\cup}_{\overline{\beta}} X'^{(\overline{\beta})}_{\alpha}$ and $Y' = \dot{\cup}_{\overline{\beta}} Y'^{(\overline{\beta})}_{\alpha}$. We may therefore assume $X = X^{(\overline{\beta})}_{\alpha}$ and $Y = Y^{(\overline{\beta})}_{\alpha}$ for some fixed Ulm sequence $\overline{\beta} = (\beta_i : i < \omega)$.

Suppose that there exists an $i < \omega$ such that $\beta_i > \alpha$. Then $X = \{x \in X : p^k x \in p^{\alpha+1} M \text{ for some } k < \omega\}$, so there exist subordinates $X' \subseteq p^{\alpha+1} M$ of X and $Y' \subseteq p^{\alpha+1} N$ of Y satisfying the required property since $|X'| = |Y'|$ or both X' and Y' are infinite sets.

Now assume that $\infty \neq \beta_i \leq \alpha$ for all $i < \omega$. Then $X = X^{(\overline{\beta})}$ and $Y = Y^{(\overline{\beta})}$. For some Ulm sequence $\overline{\gamma}$ let $X[\overline{\gamma}] = \{x \in X : u_M(x) = \overline{\gamma}\}$.

Case 1. If $|X| = |Y| \leq \aleph_0$, then we will construct by induction subordinates X' and Y' and a countable (maybe finite) sequence of Ulm sequences

$$\overline{\beta}_1 < \overline{\beta}_2 < \cdots < \overline{\beta}_n < \cdots$$

which are all equivalent to $\overline{\beta}$ such that $X' = \dot{\cup}_n X'[\overline{\beta}_n]$ and $Y' = \dot{\cup}_n Y'[\overline{\beta}_n]$. Moreover, we will ensure that $|X'[\overline{\beta}_n]| = |Y'[\overline{\beta}_n]| = 1$. Assume that we have constructed X' and Y' as claimed. Then the unique element $x_n \in X'[\overline{\beta}_n]$ can only be mapped onto the unique element $y_n \in Y'[\overline{\beta}_n]$ by any α-height-preserving map. Thus, given an α-height-preserving isomorphism $f : \langle A \rangle \to \langle B \rangle$ such that f induces a bijection between A and B, and given $x \in X'$ (respectively $y \in Y'$) there is a unique $y \in Y'$ (respectively $x \in X'$) such that f can be extended to an α-height-preserving isomorphism by mapping x onto y (respectively y onto x).

Let $X = \{x_1, x_2, x_3, \dots\}$ and $Y = \{y_1, y_2, y_3, \dots\}$ be arbitrary enumerations of X and Y, respectively. Inductively we choose integers n_i and m_i such that

$$u_M(p^{n_i} x_i) = u_N(p^{m_i} y_i)$$

and $u_M(p^{n_i} x_i) < u_M(p^{n_{i+1}} x_{i+1})$ for all i. The desired subordinates are then given by $X' = \{p^{n_i} x_i : i = 1, 2, \dots\}$ and $Y' = \{p^{m_i} y_i : i = 1, 2, \dots\}$.

If $i = 1$, then there are n_1 and m_1 such that $u_M(p^{n_1} x_1) = u_N(p^{m_1} y_1)$ since the Ulm sequences of x_1 and y_1 are equivalent to $\overline{\beta}$. Put $\overline{\beta}_1 = u_M(p^{n_1} x_1)$.

Now assume that $n_1, m_1, n_2, m_2, \dots, n_i, m_i$ and $\overline{\beta}_1, \overline{\beta}_2, \dots, \overline{\beta}_i$ are constructed as claimed. Choose k such that $p^k x_{i+1}$ and $p^k y_{i+1}$ have Ulm sequences strictly bigger than $u_M(p^{n_i} x_i) = u_N(p^{m_i} y_i)$. Now there are l and s such that $u_M(p^{k+l} x_{i+1}) = u_N(p^{k+s} y_{i+1})$. Put $n_{i+1} = k + l$ and $m_{i+1} = k + s$ and $\overline{\beta}_{i+1} = u_M(p^{n_{i+1}} x_{i+1})$. This finishes Case 1.

Case 2. Suppose we are not in Case 1. Then, without loss of generality, we may assume $|X| \geq |Y| > \aleph_0$ since by assumption $\widetilde{w}_M(\overline{\beta}) = \widetilde{w}_N(\overline{\beta})$. The strategy is to proceed as in Case 1 with minor changes. By induction we will choose subordinates X' and Y' and a countably infinite sequence of Ulm sequences

$$\overline{\beta}_1 < \overline{\beta}_2 < \cdots < \overline{\beta}_n < \cdots$$

which are all equivalent to $\overline{\beta}$ such that $X' = \dot{\cup}_n X'[\overline{\beta}_n]$ and $Y' = \dot{\cup}_n Y'[\overline{\beta}_n]$. Moreover, this time, we will ensure that $X'[\overline{\beta}_n]$ and $Y'[\overline{\beta}_n]$ are both uncountable.

Assume for the moment that we can do this. As in Case 1 any α-height-preserving isomorphism can map elements from $X'[\overline{\beta}_n]$ only to elements from $Y'[\overline{\beta}_n]$ and vice versa (for any n). Thus, given an α-height-preserving isomorphism $f : \langle A\rangle \to \langle B\rangle$ such that f induces a bijection between A and B, the uncountability of $X'[\overline{\beta}_n]$ and $Y'[\overline{\beta}_n]$ ensures that $X'[\overline{\beta}_n]\backslash A$ and $Y'[\overline{\beta}_n]\backslash B$ are still uncountable. Therefore, for any $x \in X'\backslash A$ (respectively $y \in Y'\backslash B$) there is some $y \in Y'\backslash B$ (respectively $x \in X'\backslash A$) such that f can be extended to an α-height-preserving isomorphism by mapping x onto y (respectively y onto x).

It remains to show that we can carry on the induction. First note that the set $\{p^n\overline{\beta} : n < \omega\}$ is countable. Hence (after replacing X and Y by suitable subordinates) we have

$$X = \dot{\cup}_{i\in I_X} X[\overline{\eta}_i] \quad \textit{and} \quad Y = \dot{\cup}_{i\in I_Y} Y[\overline{\mu}_i]$$

for some $I_X, I_Y \subseteq \omega$ and Ulm sequences $\overline{\eta}_i$, $\overline{\mu}_i$ equivalent to $\overline{\beta}$. Since X and Y are uncountable there must be $\overline{\eta}_k$ and $\overline{\mu}_l$ such that $X[\overline{\eta}_k]$ and $Y[\overline{\mu}_l]$ are uncountable as well. Therefore we can write $X[\overline{\eta}_k] = \dot{\cup}_{i<\omega} X_i$ where each set X_i is uncountable. Now let $i < \omega$. For every $i \in I_X$ there exist $n_i, m_i < \omega$ such that $p^{n_i}\overline{\eta}_i = p^{m_i}\overline{\eta}_k$, and we define

$$X_i^* = \{p^{n_i}x : x \in X[\overline{\eta}_i]\} \cup \{p^{m_i}x : x \in X_i\}$$

and replace X_k^* by the set $\{p^{m_k}x : x \in X_k\}$. If $i \notin I_X$ we let $m_i = 0$ and set $X_i^* = X_i$. Then it follows that $X^* = \dot{\cup}_{i<\omega} X_i^*$ is a subordinate of X such that each X_i^* is uncountable and $X_i^* = X_i^*[p^{m_i}\overline{\eta}_k]$. Similarly, we obtain a subordinate Y^* of Y so that $Y^* = \dot{\cup}_{i<\omega} Y_i^*$ where each Y_i^* is uncountable and $Y_i^* = Y_i^*[p^{r_i}\overline{\mu}_l]$ for some r_i. It is then straightforward to see, using similar arguments as in Case 1 that we may pass to subordinates $X' = \dot{\cup}_{n<\omega} X'[\overline{\beta}_n]$ and $Y' = \dot{\cup}_{n<\omega} Y'[\overline{\beta}_n]$ satisfying

$$\overline{\beta}_1 < \overline{\beta}_2 < \cdots < \overline{\beta}_n < \cdots .$$

This finishes the proof. □

Remark 3.7. We would like to point out that, in the proof of Lemma 3.6, the construction of the sets $X_\alpha^{\prime(\overline{\beta})}$ and $Y_\alpha^{\prime(\overline{\beta})}$ shows the following:

(1) In case $|X_\alpha^{(\overline{\beta})}| = |Y_\alpha^{(\overline{\beta})}|$ for all Ulm sequences $\overline{\beta}$, the map f' can be globally extended to an α-height-preserving isomorphism $\langle X' \rangle \to \langle Y' \rangle$.

(2) Given countably infinitely many elements $a_0, a_1, \ldots \in X'$ ($b_0, b_1, \ldots \in Y'$ respectively), there exist elements $b_0, b_1, \ldots \in Y'$ ($a_0, a_1, \ldots \in X'$ respectively) such that f extends to an α-height-preserving isomorphism $\langle A, a_0, a_1, \ldots \rangle \to \langle B, b_0, b_1, \ldots \rangle$ by sending a_i onto b_i for all $i < \omega$.

(3) The lemma is still valid if "α-height-preserving" is replaced by "δ-height-preserving" for any ordinal $\delta \leq \alpha$.

Our next example shows that, in Lemma 3.6, "$\widetilde{w}_M^\alpha(\overline{\beta}) = \widetilde{w}_N^\alpha(\overline{\beta})$" cannot be replaced by "$\widehat{w}_M^\alpha(\overline{\beta}) = \widehat{w}_N^\alpha(\overline{\beta})$".

Example 3.8. Let M be a free, uncountable module. Then M has a decomposition basis X and we can write $M = \bigoplus_{x \in X} \langle x \rangle$. Now let $\{y_1, y_2, \ldots\}$ be a countably infinite subset of X, and consider the module $N = \bigoplus_{i=1}^\infty \langle y_i \rangle$ with decomposition basis $Y = \{py_1, p^2 y_2, \ldots\}$. For $\overline{\beta} = (0, 1, 2, \ldots)$ we have

$$w_M(\overline{\beta}) > \aleph_0 = w_N(\overline{\beta})$$

and $w_M(\overline{\gamma}) = 0 = w_N(\overline{\gamma})$ for all $\overline{\gamma} \not\sim \overline{\beta}$. Suppose X' and Y' are any subordinates of X and Y, respectively. Then there exists a nonnegative integer n such that X' contains infinitely many elements $x_1, x_2, \ldots$ whose Ulm sequence is equal to $p^n \overline{\beta}$. However, any subordinate of Y can only contain at most n elements of Ulm sequence $p^n \overline{\beta}$. Consequently, there exists no height-preserving isomorphism $\langle x_1, \ldots, x_{n+1} \rangle \to \langle B \rangle$ for any finite subset B of Y'.

4. Model-theoretic preliminaries. Throughout this paper, L will denote an ordinary first order language with identity, finitary relation and function symbols and constant symbols. In addition, we assume that L has a variable v_α for every ordinal α. Examples for atomic formulas in L are terms like "$x = a$" or "$R(x, a, z)$" where x, a, z are constant symbols or variables of L and R a relation symbol of L. In order to define the language L_∞ (often denoted by $L_{\infty\omega}$) which we will refer to mostly, we define for each ordinal α a collection L_α of formulas as follows: L_α is the smallest collection F of formulas which contains the atomic formulas and is closed under the following logical operations:

(L1) If $\varphi \in F$, then $\neg\varphi \in F$.

(L2) If $\Phi \subseteq F$, then $\bigwedge \Phi, \bigvee \Phi \in F$.

(L3) If $\varphi \in L_\beta$ for $\beta < \alpha$ and v is a variable, then $\exists v\, \varphi$, $\forall v\, \varphi \in F$.

In (L2) $\bigwedge \Phi$ ($\bigvee \Phi$, respectively) denotes the conjunction (disjunction, respectively) of all elements of the set Φ. Notice that Φ can be of any cardinality. Following [**1**], we let $L_\infty = \bigcup_\alpha L_\alpha$. The *quantifier rank* $qr(\varphi)$ of a formula $\varphi \in L_\infty$ is defined to be the least ordinal α such that $\varphi \in L_\alpha$. Note that the quantifier rank of a formula can be seen as a measure of its complexity because it 'counts' the number of nested quantifiers which occur in the formula. The notion of a *sentence* is defined, as usual, as a formula containing no *free* variables, i.e., containing no variables not *bound* by a quantifier.

A model for a language is understood to be a set A, whose elements, in combination with the language-specific identity, functions, relations, etc., satisfy the axioms of the language and in which every possible sentence has a distinct truth value. Models are denoted by $\mathfrak{A} = \langle A, \dots \rangle$. If $\varphi \in L_\infty$ is a formula with at most n variables, $a_1, \dots, a_n \in A$ and $\varphi(a_1, \dots, a_n)$ is true, we write $\mathfrak{A} \models \varphi[a_1, \dots, a_n]$, and accordingly for a sentence φ which is true, $\mathfrak{A} \models \varphi$.

Let α be an ordinal or the symbol ∞. Then two models $\mathfrak{A} = \langle A, \dots \rangle$ and $\mathfrak{B} = \langle B, \dots \rangle$ for L_∞ are called L_α-*equivalent*, and we write $\mathfrak{A} \equiv_\alpha \mathfrak{B}$, if for all sentences $\varphi \in L_\alpha$ we have

$$\mathfrak{A} \models \varphi \text{ if and only if } \mathfrak{B} \models \varphi.$$

L_α-equivalent models can be characterized using partial isomorphisms between them:

Theorem 4.1 [**8**]. *Let $\mathfrak{A} = \langle A, \dots \rangle$ and $\mathfrak{B} = \langle B, \dots \rangle$ be models for L_∞ and δ an ordinal or the symbol ∞. Then the following are equivalent*:

(i) $\mathfrak{A} \equiv_\delta \mathfrak{B}$;

(ii) *For each ordinal $\alpha \le \delta$ there is a non-empty set I_α of isomorphisms on substructures of $\mathfrak{A}$ into $\mathfrak{B}$ such that*

(a) *if* $\alpha \leq \beta$*, then* $I_\beta \subseteq I_\alpha$*;*

(b) *if* $\alpha + 1 \leq \delta$*,* $f \in I_{\alpha+1}$ *and* $a \in A$ *(*$b \in B$*, respectively), then* f *extends to a map* $f' \in I_\alpha$ *such that* $a \in \text{domain}\,(f')$ *(*$b \in \text{range}\,(f')$*, respectively).*

The next result can be found in [**1**]:

Corollary 4.2. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be countable models for* L_∞ *such that* $\mathfrak{A} \equiv_{\omega_1} \mathfrak{B}$*. Then* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are isomorphic.*

Suppose $\mathfrak{A}'$ and $\mathfrak{B}'$ are substructures of $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}' \equiv_\delta \mathfrak{B}'$. If there are corresponding sets I_α $(\alpha \leq \delta)$ as in Theorem 4.1 (ii) consisting entirely of δ-height-preserving isomorphisms, then we write $\mathfrak{A}' \equiv_\delta^h \mathfrak{B}'$. We obtain the following corollary to Lemma 3.6:

Corollary 4.3. *Let* M *and* N *be modules with decomposition bases* X *and* Y*, respectively, such that* $\widetilde{w}_M^\delta(\overline{\beta}) = \widetilde{w}_N^\delta(\overline{\beta})$ *for all Ulm sequences* $\overline{\beta}$ *where* δ *is some fixed ordinal or the symbol* ∞*. Then there exist subordinates* X' *and* Y' *of* X *and* Y *such that* $\langle X' \rangle \equiv_\delta^h \langle Y' \rangle$*.*

Proof. Let X' and Y' be the subordinates of X and Y obtained in Lemma 3.6. For every ordinal α, let I_α be the set I of all δ-height-preserving isomorphisms

$$f : \langle A \rangle \longrightarrow \langle B \rangle$$

where A and B are finite subsets of X' and Y' such that $f(A) = B$. Then Lemma 3.6 shows that for L_δ and the modules $\langle X' \rangle$ and $\langle Y' \rangle$, the sets I_α satisfy condition (ii) from Theorem 4.1; hence, $\langle X' \rangle \equiv_\delta^h \langle Y' \rangle$. To see this, let $f : \langle A \rangle \to \langle B \rangle$ be a map in I such that A and B are finite subsets of X' and Y' and $f(A) = B$, and let $x \in \langle X' \rangle$. Then $x = \sum_{i=1}^{m} n_i x_i$ for some $x_i \in X'$ and $n_i \in \mathbf{Z}_p$. Thus, by Lemma 3.6, there is an extension $f' \in I$ of f which maps $\langle A, x_1, \dots, x_m \rangle$ onto $\langle B, y_1, \dots, y_m \rangle$ for some $y_i \in Y'$. Clearly, $x \in \text{domain}\,(f')$. If $y \in \langle Y' \rangle$, then by symmetry f extends to a map $f^* \in I$ with $y \in \text{range}\,(f^*)$. $\square$

Since a wide range of modules can be described as direct sums of special modules (cf. Example 3.8) in order to study these it is helpful to know that L_α-equivalence is invariant under this construction.

If $\bigoplus_{i\in I}\mathfrak{A}_i$ denotes the direct sum of the models $\mathfrak{A}_i$, then $\mathfrak{A}_i \equiv_\alpha \mathfrak{B}_i$ for each $i \in I$ implies $\bigoplus_{i\in I}\mathfrak{A}_i \equiv_\alpha \bigoplus_{i\in I}\mathfrak{B}_i$. Also, if $\mathfrak{A}$ is a model for L_∞ and I and J are infinite index sets, the equality $\mathfrak{B}_i = \mathfrak{A} = \mathfrak{C}_j$ for each $i \in I$ and $j \in J$ implies $\bigoplus_{i\in I}\mathfrak{B}_i \equiv_\infty \bigoplus_{j\in J}\mathfrak{C}_j$ [**1**, Corollary 1.7, Lemma 1.8].

Recall that every ordinal α can be written as $\alpha = \omega\delta + n$ where δ is a unique ordinal and $n < \omega$. Barwise and Eklof [**1**] showed that for a p-group G, "$x \in p^\alpha G$" can be expressed by a formula of quantifier rank δ ($\delta + 1$, respectively) if $n = 0$ ($n > 0$, respectively) and that for $m < \omega$, the statements "rank$\,(p^\alpha G) \geq m$" and "$u_G(\alpha) \geq m$" can be expressed by the sentences $\varphi_{\alpha,m}$ and $\psi_{\alpha,m}$ of quantifier rank $\delta+m$ and $\delta + m + 1$, respectively, as the statements are equivalent to

$$\exists x_1 \cdots \exists x_m \left(\bigwedge_{i=1}^{m} x_i \in p^{\omega\delta}G \;\wedge\; p^n x_1, \dots , p^n x_m \text{ are independent}\right)$$

and

$$\exists x_1 \cdots \exists x_m \left(\bigwedge_{i=1}^{m} x_i \in p^{\omega\delta+n}G \;\wedge\; px_i = 0 \;\wedge\; x_1, \dots , x_m \right.$$
$$\left. \text{are independent modulo } p^{\omega\delta+n+1}G \right).$$

Note that the expression "$x_1, \dots , x_m$ are independent" is of quantifier rank zero since it is equivalent to an infinite chain of subjunctions, which itself is equivalent to an infinite chain of disjunctions and conjunctions of atomic formulas. Then "$x_1, \dots , x_m$ are independent modulo $p^{\omega\delta+n+1}G$" is of quantifier rank $\delta + 1$. It is clear that these results carry over to $\mathbf{Z}_p$-modules. Similarly, we can express facts about Warfield invariants: let $\overline{\beta} = (\beta_i : i < \omega)$ be an Ulm sequence. First, assume that $\beta_i \neq \infty$ for all $i < \omega$ and write $\beta_i = \omega\delta_i + n_i$ where δ_i is an ordinal and $n_i < \omega$. Define

$$\delta_i' = \begin{cases} \delta_i & \text{if } n_i = 0 \\ \delta_i + 1 & \text{if } n_i > 0. \end{cases}$$

Then "$x \in M(\overline{\beta})$" can be expressed by a formula $\theta_{\overline{\beta}}(x)$ of quantifier rank $\xi = \sup\{\delta'_i : i < \omega\}$:

$$\theta_{\overline{\beta}}(x) = \bigwedge_{i<\omega} (p^i x \in p^{\beta_i} M).$$

The statement "$x \in M(\overline{\beta}^*)$" means that x can be expressed as a linear combination of elements $x_i \in M(\overline{\beta})$ which satisfy $|p^j x_i| > \beta_j$ for infinitely many $j \in \omega$ and therefore is equivalent to $\exists k < \omega\ \exists x_1 \cdots \exists x_k$ $(x = \sum_{i=1}^k \lambda_i x_i$ for some $\lambda_1, \dots, \lambda_k \in \mathbf{Z} \wedge \bigwedge_{i=1}^k (x_i \in M(\overline{\beta}) \wedge\ p^j x_i \in p^{\beta_j+1} M$ for infinitely many $j < \omega))$; hence, it can be expressed by a formula $\theta_{\overline{\beta}^*}(x)$ of quantifier rank $\xi + \omega$ (the correct quantifier rank of this formula was pointed out by Carol Jacoby [**7**]).

Since the statement

$$w_M(\overline{\beta}) \geq m$$

is true if and only if $\exists x_1 \cdots \exists x_m\ (x_1, \dots, x_m \in M(\overline{\beta})\ \wedge\ x_1, \dots, x_m$ are independent modulo $M(\overline{\beta}^*))$, it can be expressed by a sentence $\theta_{\overline{\beta},m}$ of quantifier rank $\xi + \omega + m$.

Now suppose the module M has a decomposition basis X. If $\infty \neq \beta_i \leq \alpha$ for all $i < \omega$, then $w_M^\alpha(\overline{\beta}) = w_M(\overline{\beta})$ and otherwise

$$\begin{aligned} w_M^\alpha(\overline{\beta}) &= |\{x \in X : p^k x \in p^{\alpha+1} M \text{ for some } k < \omega\}| \\ &= \operatorname{rank}\left(\bigoplus_{x \in X} (\langle x \rangle \cap p^{\alpha+1} M)\right) \end{aligned}$$

which coincides with the rank of $\langle X \rangle \cap p^\alpha M$ and therefore with the torsion-free rank of $p^\alpha M$. Therefore, by consulting $\theta_{\overline{\beta},m}$ and $\varphi_{\alpha,m}$ "$w_M^\alpha(\overline{\beta}) \geq m$" (where $m < \omega$) can be expressed by a sentence $\psi_{\alpha,\overline{\beta},m}$ whose quantifier rank is

$$\begin{array}{ll} \xi + \omega + m & \text{if } \infty \neq \beta_i \leq \alpha \text{ for all } i < \omega \\ \delta + m & \text{if } \beta_i > \alpha \text{ for some } i < \omega. \end{array}$$

Our observations yield one direction of some classifications, of which (i) was already formulated in [**1**] and (ii) can be found in [**7**]:

Lemma 4.4. *Let M and N be modules. Suppose $M \equiv_\lambda N$ where λ is a limit ordinal. Then*:

(i)

(a) $\widehat{u}_M(\alpha) = \widehat{u}_N(\alpha)$ *if* $\alpha < \omega\lambda$.

(b) *If* length $(M) < \omega\lambda$ *and* length $(N) < \omega\lambda$, *then* $\widehat{u}_M(\infty) = \widehat{u}_N(\infty)$.

(ii) *If M and N have decomposition bases and if $\alpha < \omega\lambda$ where $\lambda = \omega\gamma$ and γ is a limit ordinal, then $\widehat{w}_M^\alpha(\overline{\beta}) = \widehat{w}_N^\alpha(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$.*

Proof. Bear in mind $\alpha = \omega\delta + n < \omega\lambda$ yields $\delta + k < \lambda$ for all $k < \omega$. Hence, (i) (a) follows since $\psi_{\alpha,m}$, for finite m is an element of L_λ, and with (L1) $\neg\psi_{\alpha,m}$ and also $u_M(\alpha) < m + 1 = \neg\psi_{\alpha,m+1}$, are in L_λ, too. Therefore, $M \equiv_\lambda N$ yields $M \models \psi_{\alpha,m}, \neg\psi_{\alpha,m+1} \Leftrightarrow N \models \psi_{\alpha,m}, \neg\psi_{\alpha,m+1}$, which implies $u_M(\alpha) = u_N(\alpha)$. Since $m < \omega$ we achieve this equation only for the generalized Ulm-Kaplansky invariant $\widehat{u}_M(\alpha)$.

For (b), observe that if length $(M) := \eta < \omega\lambda$ we have $x \in p^\infty M \Leftrightarrow x \in p^\eta M$ which is a formula of quantifier rank $< \lambda$ and therefore an element of L_λ. Then, "$u_M(\infty) \geq m$" is a formula similar to $\varphi_{\alpha,m}$ and thus in L_λ, too. The assertion then follows as in (a). In (ii), we write $\delta = \omega\delta' + n'$ and have

$$\xi + \omega + m \leq \delta + \omega + m < \omega\delta' + \omega + \omega = \omega(\delta' + 2) < \omega\gamma = \lambda$$

and $\delta + m < \lambda$; hence, for any $m < \omega$ the formula $\psi_{\alpha,\overline{\beta},m}$ is in L_λ and the assertion follows similar to (i). □

5. L_α-equivalence of $\mathbf{Z}_p$-modules with nice decomposition bases.

Definition 5.1. Let α be an ordinal or the symbol ∞. For modules M and N with decomposition bases X and Y, let $\mathrm{Prs}_\alpha^{X,Y}$ denote the set of all α-height-preserving isomorphisms

$$f : E \longrightarrow F$$

where E and F are finitely generated submodules of M and N, respectively, such that the following is true: there exist generators $x_1, \dots, x_n$

of E, generators $y_1, \dots, y_n$ of F, and a positive integer $k \leq n$ such that $x_1, \dots, x_k \in X \cup \{0\}$ and $y_1, \dots, y_k \in Y \cup \{0\}$ and

- $f(x_i) = y_i$ for all $i = 1, \dots, n$;
- in case $k < n$ the submodules $E_i = \langle x_1, \dots, x_i \rangle$ and $F_i = \langle y_1, \dots, y_i \rangle$ satisfy the following properties for $i = k, \dots, n-1$:

(a) $|E_{i+1}/E_i| = |F_{i+1}/F_i| = p$,

(b) if $|x_{i+1}|_M < \alpha$ or $|y_{i+1}|_N < \alpha$, then x_{i+1} is proper with respect to $\langle X, E_i \rangle$ and y_{i+1} is proper with respect to $\langle Y, F_i \rangle$.

Notice that if $|x_{i+1}|_M < \alpha$ or $|y_{i+1}|_N < \alpha$ (see property (b)), then $|x_{i+1}|_M = |y_{i+1}|_N$ because f is α-height-preserving and maps x_{i+1} onto y_{i+1}. For modules M and N with decomposition bases X and Y, we have $\mathrm{Prs}_\beta^{X,Y} \subseteq \mathrm{Prs}_\alpha^{X,Y}$ whenever α and β are ordinals with $\alpha < \beta$. It is clear that each set $\mathrm{Prs}_\alpha^{X,Y}$ is non-empty since it contains the map $0 \mapsto 0$. Now Lemma 3.6 can be extended as follows:

Lemma 5.2. *Let M and N be modules with decomposition bases X and Y, respectively, such that $\widetilde{w}_M^\alpha(\overline{\beta}) = \widetilde{w}_N^\alpha(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$ where α is some fixed ordinal or the symbol ∞. Let X' and Y' be subordinates of X and Y as in Lemma* 3.6 *and assume that $f : E \to F$ is a map in $\mathrm{Prs}_\alpha^{X',Y'}$. Then for every $x \in X'$ ($y \in Y'$, respectively) there is a $y \in Y'$ ($x \in X'$, respectively) such that f extends to a map*

$$f' : \langle E, x \rangle \longrightarrow \langle F, y \rangle$$

with $f' \in \mathrm{Prs}_\alpha^{X',Y'}$ by sending x onto y.

Proof. We prove the assertion by induction on $m = n - k$ (cf. Definition 5.1). The case $m = 0$ was shown in Lemma 3.6, so we assume that the claim is true for $m \geq 0$, and suppose

$$f : E_{k+m+1} = \langle x_1, \dots, x_{k+m+1} \rangle \longrightarrow F_{k+m+1} = \langle y_1, \dots, y_{k+m+1} \rangle$$

is a map in $\mathrm{Prs}_\alpha^{X',Y'}$ as in Definition 5.1. Let $x \in X'$ ($y \in Y'$, respectively). By induction hypothesis, $f|_{E_{k+m}} : E_{k+m} \to F_{k+m}$ extends to a map

$$f^* : \langle E_{k+m}, x \rangle \longrightarrow \langle F_{k+m}, y \rangle$$

in $\mathrm{Prs}_\alpha^{X',Y'}$ with $f^*(x) = y \in Y'$ ($f^{*-1}(y) = x \in X'$, respectively). In case $|x_{k+m+1}| = |y_{k+m+1}| < \alpha$ the element x_{k+m+1} is proper with respect to $\langle E_{k+m}, x\rangle$ and has order p modulo $\langle E_{k+m}, x\rangle$, and the same is true for y_{k+m+1} and $\langle F_{k+m}, y\rangle$. By Lemma 3.4, the map f^* extends to an α-height-preserving isomorphism

$$\begin{aligned} f' : \langle E_{k+m}, x, x_{k+m+1}\rangle = \langle E_{k+m+1}, x\rangle &\to \langle F_{k+m}, y, y_{k+m+1}\rangle \\ &= \langle F_{k+m+1}, y\rangle \end{aligned}$$

by sending x_{k+m+1} onto y_{k+m+1}; hence, f' extends f. It is clear that x_{i+1} has order p modulo $\langle E_i, x\rangle$ and y_{i+1} has order p modulo $\langle F_i, x\rangle$ for $i = k, \dots, k+m$. Therefore, $f' \in \mathrm{Prs}_\alpha^{X',Y'}$, and the induction is complete. □

The following result will be useful:

Lemma 5.3. *Let M and N be modules with nice decomposition bases X and Y, respectively, such that $\widetilde{w}_M^\alpha(\overline{\beta}) = \widetilde{w}_N^\alpha(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$ where α is some fixed ordinal or the symbol ∞. Let X' and Y' be subordinates of X and Y as in Lemma* 3.6, *and assume that $f : E \to F$ is a map in $\mathrm{Prs}_\alpha^{X',Y'}$. If $x \in M\backslash E$ and $px \in E$, then f extends to a map $f^* : E^* \to F^*$ in $\mathrm{Prs}_\alpha^{X',Y'}$ for which there is an element $x^* \in M$ that is proper with respect to $\langle X', E^*\rangle$ and has order p modulo E^* such that $\langle E^*, x\rangle = \langle E^*, x^*\rangle$ and $\langle X', E^*\rangle = \langle X', E\rangle$.*

Proof. Since X' is a nice decomposition basis for M and finite extensions of nice submodules are nice, $\langle X', E\rangle$ is a nice submodule of M and contains therefore an element a such that $x^* = x + a$ is proper with respect to $\langle X', E\rangle$. There are elements $x_1^*, \dots, x_s^* \in X'$ such that $E^* = \langle E, x_1^*, \dots, x_s^*\rangle$ contains both a and px. By Lemma 5.2, f has an extension $f^* : E^* \to F^*$ in $\mathrm{Prs}_\alpha^{X',Y'}$. Then $\langle E^*, x\rangle = \langle E^*, x^*\rangle$, $\langle X', E^*\rangle = \langle X', E\rangle$, and x^* is proper with respect to E^* and has order p modulo E^*. □

Lemma 5.4. *Let A be a submodule of a module M, and suppose $x \in M$ is proper with respect to A and has height $\beta \neq \infty$. Then:*

(i) *If $|px| > \beta + 1$, there is an element $y \in p^{\beta+1}M$ such that $x - y \in p^\beta M[p]$ and $x - y \notin A + p^{\beta+1}M$.*

(ii) *If x has order p modulo A, then* $u_{M,A}(\beta) = u_{M,\langle A,x\rangle}(\beta) + 1$ *and* $u_{M,A}(\alpha) = u_{M,\langle A,x\rangle}(\alpha)$ *if* $\alpha \neq \beta$.

Proof. (i) If $|px| > \beta + 1$, then $px = py$ for some $y \in p^{\beta+1}M$; hence, $x - y \in p^{\beta}M[p]$. For any $a \in A$ we have

$$|x - y + a| = \min\{|x + a|, |y|\} = |x + a| \leq \beta;$$

therefore, $x - y \notin A + p^{\beta+1}M$.

(ii) Let $A_1 = \langle A, x\rangle$. If the coset $x + A$ contains an element x' with $|x'| = \beta$ and $|px'| > \beta + 1$, we replace x by x'.

Case I: $|px| > \beta + 1$**.** By (i), there exists an element $y \in p^{\beta+1}M$ such that $x - y \in p^{\beta}M[p]$ and $x - y \notin A + p^{\beta+1}M$. Then

$$x - y \notin A(\beta) = p^{\beta}M[p] \cap (A + p^{\beta+1}M),$$

and therefore $A_1(\beta)/A(\beta) = \langle A, x - y\rangle(\beta)/A(\beta) \cong \mathbf{Z}/p\mathbf{Z}$. But then

$$\dim p^{\beta}M[p]/A(\beta) = \dim \frac{p^{\beta}M[p]/A(\beta)}{A_1(\beta)/A(\beta)} + \dim A_1(\beta)/A(\beta)$$

shows that $u_{M,A}(\beta) = u_{M,A_1}(\beta) + 1$. If $\alpha < \beta$ we have $x \in p^{\beta}M \subseteq p^{\alpha+1}M$ which implies $A_1(\alpha) = A(\alpha)$. Now assume that for some $\alpha > \beta$ there exists an element in $A_1(\alpha)\backslash A(\alpha)$. Then we can find elements $a \in A$ and $g \in p^{\alpha+1}M$ such that $a + kx + g \in p^{\alpha}M[p]$ for some positive integer $k < p$. But then there are integers m and n such that $mk + np = 1$; hence, $ma + x - npx \in p^{\alpha}M$. Since x is proper with respect to A, this yields

$$\alpha \leq |ma + x - npx| \leq |x| = \beta,$$

a contradiction. It follows that $A_1(\alpha) = A(\alpha)$. Therefore $u_{M,A}(\alpha) = u_{M,A_1}(\alpha)$ if $\alpha \neq \beta$.

Case II: $|px| = \beta + 1$**.** Assume $A_1(\beta)\backslash A(\beta)$ is non-empty. As before, we can find a positive integer $k < p$ and elements $a \in A$ and $g \in p^{\beta+1}M$ such that $a + kx + g \in p^{\beta}M[p]$, so there are integers m and n satisfying

$ma + x - npx + mg \in p^\beta M[p]$. But then $\beta \leq |ma + x| \leq |x|$; thus, $ma + x$ has height β. Moreover,

$$|p(ma + x)| = |np^2 x - mpg| > \beta + 1,$$

so we can replace x by $ma + x$ and are in Case I. If $\alpha < \beta$ or $\alpha > \beta$, we conclude that $A_1(\alpha) = A(\alpha)$ as in the previous case; hence, we obtain $u_{M,A}(\alpha) = u_{M,A_1}(\alpha)$ if $\alpha \neq \beta$. □

Lemma 5.5. *Let M and N be modules with nice decomposition bases X and Y, respectively, such that $\widetilde{w}^\alpha_M(\overline{\beta}) = \widetilde{w}^\alpha_N(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$ where α is some fixed ordinal or the symbol ∞, and let X' and Y' be subordinates of X and Y as in Lemma* 3.6*. Assume that $f : E \to F$ is a map in $\mathrm{Prs}^{X',Y'}_\alpha$. Let $A = \langle X' \rangle$ and $B = \langle Y' \rangle$, and suppose γ is an ordinal $< \alpha$. Then there is an $m < \omega$ such that*

$$u_{M,A}(\gamma) = u_{M,A+E}(\gamma) + m \text{ and } u_{N,B}(\gamma) = u_{N,B+F}(\gamma) + m.$$

In particular, if $\widehat{u}_{M,A}(\gamma) = \widehat{u}_{N,B}(\gamma)$, then $\widehat{u}_{M,A+E}(\gamma) = \widehat{u}_{N,B+F}(\gamma)$.

Proof. If $E \subseteq A$, then $F \subseteq B$ and there is nothing to show. Now assume that $E \nsubseteq A$. Then $F \nsubseteq B$, and we write $E = \langle x_1, \dots , x_k, x_{k+1}, \dots , x_n \rangle$ and $F = \langle y_1, \dots , y_k, y_{k+1}, \dots , y_n \rangle$ as in Definition 5.1. Let $A_i = \langle A, x_1, \dots , x_i \rangle$ and $B_i = \langle B, y_1, \dots , y_i \rangle$. Then every coset $x_{i+1} + A_i$ has an element x'_{i+1} of maximal height since A_i is nice in M, so there are $a_1, \dots , a_s \in X'$ such that

$$E_i = \langle x_1, \dots , x_i, a_1, \dots , a_s \rangle$$

contains $x_{i+1} - x'_{i+1}$ for all $i = k, \dots , n - 1$. By Lemma 5.2, there are elements $b_1, \dots , b_s \in Y'$ such that f extends to a map

$$f' : \langle E, a_1, \dots , a_s \rangle \longrightarrow \langle F, b_1, \dots , b_s \rangle$$

in $\mathrm{Prs}^{X',Y'}_\alpha$ by sending each a_i onto b_i. Letting $F_i = \langle y_1, \dots, y_i, b_1, \dots, b_s \rangle$, we have

$$\langle E_i, x'_{i+1} \rangle = \langle E_i, x_{i+1} \rangle = E_{i+1}$$

and

$$\langle F_i, f'(x'_{i+1})\rangle = \langle F_i, y_{i+1}\rangle = F_{i+1}$$

and x'_{i+1} is proper with respect to E_i for all $i = k, \dots, n-1$.

Suppose $|x'_{i+1}|_M \leq \gamma$. Since $\gamma < \alpha$ and $f'(E_{i+1}) = F_{i+1}$, we have $|f'(x'_{i+1})|_N = |x'_{i+1}|_M$ and $f'(x'_{i+1})$ is proper with respect to F_i. If $|x'_{i+1}|_M > \gamma$, then $|f'(x'_{i+1})|_N > \gamma$ in which case $E_i + p^{\gamma+1}M = \langle E_i, x'_{i+1}\rangle + p^{\gamma+1}M$ and $F_i + p^{\gamma+1}N = \langle F_i, f'(x'_{i+1})\rangle + p^{\gamma+1}N$; therefore,

$$u_{M,E_i}(\gamma) = u_{M,\langle E_i, x'_{i+1}\rangle}(\gamma) \text{ and } u_{N,F_i}(\gamma) = u_{N,\langle F_i, f'(x'_{i+1})\rangle}(\gamma).$$

Now we apply Lemma 5.4 (ii) repeatedly and obtain $u_{M,A}(\gamma) = u_{M,A+E}(\gamma) + m$ and $u_{N,B}(\gamma) = u_{N,B+F}(\gamma) + m$ for some $0 \leq m \leq n-k$. □

The next result generalizes [**1**, Lemma A.3.2].

Lemma 5.6. *Let M and N be modules with decomposition bases X and Y, respectively, and let ν be an ordinal such that* $\text{length}\,(tN) \geq \omega(\nu+1)$. *Let $f : E \to F$ be a map in $\text{Prs}^{X,Y}_{\omega(\nu+1)}$. If $x \in M\backslash E$ with $p^{r+1}x \in E$ for some $r < \omega$ and $|x| \geq \omega\nu$, then f extends to a map*

$$f' : \langle E, x\rangle \longrightarrow \langle F, y\rangle$$

in $\text{Prs}^{X,Y}_{\omega\nu}$ by sending x onto y.

Proof. Let r be the smallest integer ≥ 0 such that $p^{r+1}x \in E$. Then $|p^{r+1}x| \geq \omega\nu + r + 1$; hence, $|f(p^{r+1}x)| \geq \omega\nu + r + 1$, so we can write $f(p^{r+1}x) = p^{r+1}y_0$ for some $y_0 \in p^{\omega\nu}N$. If $p^r y_0 \notin F$ we let $y = y_0$. Now suppose $p^r y_0 \in F$, and let B be a basic subgroup of $p^{\omega\nu+r}(tN)$ (see [**2**, Vol. I, page 139]). It is clear that $B[p] \subseteq (p^{\omega\nu+r}N)[p]$. Assume the latter group is finite. Then $B[p]$ is finite; thus, B is finite and we can write $p^{\omega\nu+r}(tN) = B \oplus D$ for some divisible group D (see [**2**, Theorem 27.5]); hence, $\text{length}\,(tN) < \omega(\nu+1)$, a contradiction. Consequently, $(p^{\omega\nu+r}N)[p]$ is an infinite group, and therefore $p^{\omega\nu+r}N[p] \not\subseteq F[p]$. Then there is a $y_1 \in p^{\omega\nu}N$ such that $p^r y_1 \notin F$ and $p^{r+1}y_1 = 0$. Letting $y = y_0 + y_1$, we obtain

$$|y| \geq \omega\nu,\ p^r y \notin F \quad \text{and} \quad p^{r+1}y = f(p^{r+1}x).$$

By Lemma 3.4, $f \in \mathrm{Prs}_{\omega\nu}^{X,Y}$ extends to an $\omega\nu$-height-preserving isomorphism

$$f' : \langle E, x\rangle \longrightarrow \langle F, y\rangle$$

by sending x onto y. Since $f' \in \mathrm{Prs}_{\omega\nu}^{X,Y}$, the proof is complete. □

The following result will be needed (see [**15**, Lemma 5.1] in which the terminology *lower decomposition basis* is used for a subordinate X' satisfying the stated properties; note that $u^M_{\langle X'\rangle}(\alpha)$ corresponds to the dimension of $I_\alpha([X'])$ in Warfield's notation).

Lemma 5.7 [**15**]. *Let M be a module possessing a decomposition basis X. Then X has a subordinate X' such that for every ordinal α, $u^M_{\langle X'\rangle}(\alpha)$ is finite or $u_M(\alpha) = u_{M,\langle X'\rangle}(\alpha)$.*

Lemma 5.8. *Let M and N be modules with nice decomposition bases X and Y, respectively, and let α be a limit ordinal or the symbol ∞. Suppose $\widehat{u}_M(\delta) = \widehat{u}_N(\delta)$ for all ordinals $\delta < \alpha$ and $\widetilde{w}^\alpha_M(\overline{\beta}) = \widetilde{w}^\alpha_N(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$. Then there exist subordinates X' and Y' of X and Y, respectively, such that every map $f : E \to F$ in $\mathrm{Prs}_\alpha^{X',Y'}$ satisfies the following conditions*:

(i) *If $x \in M\backslash E$ with $px \in E$ and $\infty \neq \sup\{|x+a| : a \in \langle X', E\rangle\} < \alpha$, then f extends to $f' \in \mathrm{Prs}_\alpha^{X',Y'}$ such that $x \in \mathrm{domain}\,(f')$.*

(ii) *If $y \in N\backslash F$ with $py \in F$ and $\infty \neq \sup\{|y+b| : b \in \langle Y', F\rangle\} < \alpha$, then f extends to $f' \in \mathrm{Prs}_\alpha^{X',Y'}$ such that $y \in \mathrm{range}\,(f')$.*

Proof. First, we show that there exist subordinates X' and Y' of X and Y, respectively, such that $\langle X'\rangle \equiv^h_\alpha \langle Y'\rangle$ and $\widehat{u}_{M,\langle X'\rangle}(\delta) = \widehat{u}_{N,\langle Y'\rangle}(\delta)$ for all ordinals $\delta < \alpha$. Consider the equation

$$\widehat{u}_M(\delta) = \widehat{u}^M_{\langle X\rangle}(\delta) + \widehat{u}_{M,\langle X\rangle}(\delta),$$

and note that $u^M_{\langle X\rangle}(\delta)$ is the number of elements in X whose Ulm sequences have a gap at δ (cf. [**15**, page 341]). By Lemma 5.7, we can find subordinates X^* of X and Y^* of Y such that for every ordinal $\delta < \alpha$ the following is true: If both $u^M_{\langle X^*\rangle}(\delta)$ and $u^N_{\langle Y^*\rangle}(\delta)$ are infinite, then

$$\widehat{u}_{M,\langle X^*\rangle}(\delta) = \widehat{u}_{N,\langle Y^*\rangle}(\delta).$$

Notice that this statement remains true after replacing X^* and Y^* by subordinates X' and Y' as in Lemma 3.6. Then, for every ordinal $\delta < \alpha$, X' and Y' have the same number of elements whose Ulm sequences have a gap at δ whenever $u^M_{\langle X' \rangle}(\delta)$ or $u^N_{\langle Y' \rangle}(\delta)$ is finite. Setting $A = \langle X' \rangle$ and $B = \langle Y' \rangle$, we obtain $A \equiv^h_\alpha B$ and

$$\widehat{u}_{M,A}(\delta) = \widehat{u}_{N,B}(\delta)$$

for all ordinals $\delta < \alpha$. The modules $A + E$ and $B + F$ are nice in M and N, respectively, hence in (i) and (ii) of the lemma, "sup" can be replaced by "max."

Let $f : E = \langle x_1, \dots , x_n \rangle \to F = \langle y_1, \dots , y_n \rangle$ be a map in $\mathrm{Prs}_\alpha = \mathrm{Prs}_\alpha^{X',Y'}$, and suppose x is an element of $M \backslash E$ with $px \in E$ and

$$\infty \neq \beta = \max\{|x + a| : a \in A + E\} < \alpha.$$

If $x \in A + E$, then Lemma 5.2 yields an extension $f' \in \mathrm{Prs}_\alpha$ of f with $x \in \mathrm{domain}\,(f')$. Now suppose $x \notin A + E$. Then x has order p modulo $A + E$. By Lemma 5.3, f extends to a map $f^* : E^* \to F^*$ in Prs_α for which there is an element $x^* \in M$ such that x^* is proper with respect to $A + E^*$ and has order p modulo E^*, $\langle E^*, x \rangle = \langle E^*, x^* \rangle$ and $A + E^* = A + E$. If possible, we choose $f^* \in \mathrm{Prs}_\alpha$ and x^* so that $|px^*| > |x^*| + 1$.

To simplify notation we now write x for this element x^* and $f : E \to F$ for the function $f^* : E^* \to F^*$. Then $\infty \neq |x| = \beta < \alpha$ and $px \in p^{\beta+1}M$ which implies $f(px) \in p^{\beta+1}N$.

Case I: $|px| > \beta + 1$**.** By Lemma 5.4 (i), there exists an element $x' \in p^{\beta+1}M$ such that $x - x' \in p^\beta M[p]$ and $x - x' \notin A + E + p^{\beta+1}M$; thus, $\widehat{u}_{M,A+E}(\beta) \neq 0$. Since $\beta < \alpha$, we can apply Lemma 5.5 and obtain $\widehat{u}_{N,B+F}(\beta) \neq 0$. Then there exists an element $z \in p^\beta N[p]$ with $z \notin B + F + p^{\beta+1}N$; hence, $|z| = \beta$ and $|z + h| \leq \beta$ for all $h \in B + F$. Since α is a limit ordinal or $\alpha = \infty$, we have $\beta + 2 < \alpha$, so there is an element $w \in p^{\beta+1}N$ such that $f(px) = pw$. Then

$$|w + z + h| = \min\{|w|, |z + h|\} \leq \beta = |w + z|$$

for all $h \in B + F$; hence, $w + z$ is proper with respect to $B + F$. Notice that $w + z \notin B + F$; otherwise, $|w| = |z - (w + z)| \leq \beta$ by the previous observation, contradicting the fact that $|w| \geq \beta + 1$. By Lemma 3.4 we can extend f to an α-height-preserving isomorphism

$$f' : \langle E, x\rangle \longrightarrow \langle F, w + z\rangle$$

with $f'(x) = w + z$. Clearly, $f' \in \mathrm{Prs}_\alpha$.

Case II: $|px| = \beta + 1$. Then $\beta + 1 \neq \infty$; therefore, $f(px) = pw$ for some $w \in N$ with $|w| = \beta$. Suppose there exists an element $z \in B + F$ such that $|w + z| \geq \beta + 1$. Then $|z| = \beta$ and there are elements $y_{n+1}, \dots , y_m \in Y'$ such that $z \in F' = \langle F, y_{n+1}, \dots , y_m\rangle$. By Lemma 5.2, there are $x_{n+1}, \dots , x_m \in X'$ so that f extends to a map

$$\overline{f} : E' = \langle E, x_{n+1}, \dots , x_m\rangle \to F'$$

in Prs_α. Letting $c = \overline{f}^{-1}(z)$, we have $|x + c| \geq \beta = \max\{|x + a| : a \in A + E\}$. Notice that $c \in A + E' = A + E$; therefore, $x + c$ is proper with respect to $A + E'$. Since $x + c$ has order p modulo E' and $|p(w + z)| > \beta + 1$ yields $|p(x + c)| > \beta + 1$, we can replace x by $x + c$ and are in Case I. Therefore, we may assume that $|w + z| \leq \beta = |w|$ for all $z \in B + F$, i.e., w is proper with respect to $B + F$. By Lemma 3.4, f extends to an α-height-preserving isomorphism

$$f' : \langle E, x\rangle \longrightarrow \langle F, w\rangle$$

by mapping x to w. Again, $f' \in \mathrm{Prs}_\alpha$. The second assertion follows immediately; hence, the proof is complete. □

Theorem 5.9. *Let M and N be reduced modules with nice decomposition bases. If $\widehat{u}_M(\delta) = \widehat{u}_N(\delta)$ for all ordinals δ and $\widetilde{w}_M(\overline{\beta}) = \widetilde{w}_N(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$, then $M \equiv_\infty N$.*

Proof. Let $\mathrm{Prs}_\infty^{X',Y'}$ be as in Lemma 5.8, and put $I_\delta = \mathrm{Prs}_\infty^{X',Y'}$ for every ordinal δ. Since M and N are reduced, we can apply Lemma 5.8. Then Theorem 4.1 shows that $M \equiv_\infty N$. □

The following fact will be useful:

Lemma 5.10. *Let M be a module. If* length $(tM) = \alpha$, *then* length $(M) \leq \alpha + \omega$.

We are now able to prove our main result:

Theorem 5.11. *Let M and N be modules with nice decomposition bases and let δ be an ordinal. Suppose*

(i) $\widehat{u}_M(\alpha) = \widehat{u}_N(\alpha)$ *for all ordinals* $\alpha < \omega\delta$;

(ii) *if* length $(tM) < \omega\delta$, *then* $\widehat{u}_M(\infty) = \widehat{u}_N(\infty)$;

(iii) $\widetilde{w}_M^{\omega\nu}(\overline{\beta}) = \widetilde{w}_N^{\omega\nu}(\overline{\beta})$ *for all ordinals* $\nu \leq \delta$ *and all Ulm sequences* $\overline{\beta}$.

Then $M \equiv_\delta N$. The converse holds if $\delta = \omega\gamma$ where γ is a limit ordinal and if $\widetilde{w}_M^{\omega\delta}(\overline{\beta}) = \widetilde{w}_N^{\omega\delta}(\overline{\beta})$ and $(w_M^{\omega\nu}(\overline{\beta}) \leq \aleph_0 \Leftrightarrow w_N^{\omega\nu}(\overline{\beta}) \leq \aleph_0)$ for all ordinals $\nu < \delta$ and all Ulm sequences $\overline{\beta}$.

Proof. Let M and N be modules with nice decomposition bases X and Y, respectively, satisfying (i)–(iii).

Case I: length $(tM) < \omega\delta$ (this implies length $(tN) < \omega\delta$ by (i)). We show that in this case $M \equiv_\infty N$. By Lemma 5.10, there is a $\nu < \delta$ such that length $(M) \leq \omega(\nu + 1)$; hence, $w_M(\overline{\beta}) = w_M^{\omega(\nu+1)}(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$, and a corresponding statement is true for N. Now write $\overline{\infty} = (\infty, \infty, \dots)$,

$$X = X_\infty \dot{\cup} X_r \quad \text{and} \quad Y = Y_\infty \dot{\cup} Y_r$$

with $X_\infty = \{x \in X : u(x) \sim \overline{\infty}\}$ and $Y_\infty = \{y \in Y : u(y) \sim \overline{\infty}\}$. Then

$$w_M(\overline{\infty}) = |X_\infty| \quad \text{and} \quad w_N(\overline{\infty}) = |Y_\infty|,$$

so by our remarks on direct sums in Section 4 we obtain

$$\begin{aligned} p^\infty M &\cong \bigoplus_{w_M(\overline{\infty})} \mathbf{Q} \oplus \bigoplus_{u_M(\infty)} \mathbf{Z}(p^\infty) \equiv_\infty \bigoplus_{w_N(\overline{\infty})} \mathbf{Q} \oplus \bigoplus_{u_N(\infty)} \mathbf{Z}(p^\infty) \\ &\cong p^\infty N. \end{aligned}$$

By [**20**, Theorem 21.2], there are reduced modules M_r and N_r such that $M = p^\infty M \oplus M_r$, $N = p^\infty N \oplus N_r$, $\langle X_r \rangle \subseteq M_r$, and

$\langle Y_r \rangle \subseteq N_r$. Then X_r and Y_r are nice decomposition bases for M_r and N_r, respectively. We have $w_{M_r}(\overline{\beta}) = 0 = w_{N_r}(\overline{\beta})$ for any Ulm sequence $\overline{\beta} \sim \overline{\infty}$, and

$$\widehat{w}_{M_r}(\overline{\beta}) = \widehat{w}_M(\overline{\beta}) = \widehat{w}_N(\overline{\beta}) = \widehat{w}_{N_r}(\overline{\beta})$$

whenever $\overline{\beta} \not\sim \overline{\infty}$. Since $\widehat{u}_{M_r}(\alpha) = \widehat{u}_M(\alpha) = \widehat{u}_N(\alpha) = \widehat{u}_{N_r}(\alpha)$ for all ordinals α, Theorem 5.9 yields $M_r \equiv_\infty N_r$. Therefore, $M \equiv_\infty N$.

Case II: length $(tM) \geq \omega\delta$ (which implies length $(tN) \geq \omega\delta$). Let X' and Y' be subordinates of X and Y, respectively, as in Lemma 5.8. For any ordinal $\nu \leq \delta$ we define I_ν to be the set of all maps $f : E \to F$ in $\mathrm{Prs}_{\omega\nu} = \mathrm{Prs}_{\omega\nu}^{X',Y'}$. For $\nu + 1 \leq \delta$ let

$$f : E \longrightarrow F$$

be a map in $I_{\nu+1}$, and suppose that x is an element of $M \backslash E$. We will extend f to a map $f' \in I_\nu$ with $x \in$ domain (f').

Let $A = \langle X' \rangle$, and let r be the smallest integer ≥ 0 such that $p^{r+1}x \in A + E$. Then $\widetilde{w}_M^{\omega(\nu+1)}(\overline{\beta}) = \widetilde{w}_N^{\omega(\nu+1)}(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$ because $\nu + 1 \leq \delta$. Therefore, we can apply Lemma 5.2 to extend f to a map $f^* : E^* \to F^*$ in $\mathrm{Prs}_{\omega(\nu+1)}$ such that $p^{r+1}x \in E^*$. To simplify notation, we write $f : E \to F$ for $f^* : E^* \to F^*$.

Suppose $|x| \geq \omega\nu$. Then by Lemma 5.6, f extends to some map $f : \langle E, x \rangle \to \langle F, y \rangle$ in I_ν. Now assume $|x| < \omega\nu$.

Case IIa: Suppose that for all $m = 0, \ldots, r$ we have

$$\max\{|p^m x + z| : z \in \langle A + E, p^{m+1}x \rangle\} < \omega\nu.$$

Then we use Lemma 5.8 repeatedly to obtain an extension f' of f in $I_{\nu+1} \subseteq I_\nu$ with $x \in$ domain (f').

Case IIb: Now assume that there exists $0 \leq m \leq r$ and an element $z \in \langle A + E, p^{m+1}x \rangle$ such that $|p^m x + z| \geq \omega\nu$. Let m be the smallest such integer. Using Lemma 5.2 again we extend f to a map

$$\overline{f} : \overline{E} \longrightarrow \overline{F}$$

in $I_{\nu+1}$ with $z \in \langle \overline{E}, p^{m+1}x \rangle$. By Lemma 5.6, $\overline{f}$ extends to a map $\overline{f}' \in I_\nu$ whose domain is $\langle \overline{E}, p^m x + z \rangle$. Now write $z = e + \lambda p^{m+1} x$ where

$e \in \overline{E}$ and $\lambda \in \mathbf{Z}_p$. Then $(1+\lambda p)p^m x = p^m x + z - e \in \langle \overline{E}, p^m x + z\rangle$. Since $1+\lambda p$ is a unit in the ring $\mathbf{Z}_p$, it follows that $\langle \overline{E}, p^m x\rangle = \langle \overline{E}, p^m x + z\rangle$. Finally, we use Lemma 5.8 repeatedly to extend $\overline{f}'$ to some map $f' \in I_\nu$ whose domain contains the elements $p^{m-1}x, \dots, px, x$.

In view of Lemma 5.8 (ii) the conditions of Theorem 4.1 (ii) are satisfied, and we conclude that $M \equiv_\delta N$. The last part of the theorem follows from Lemma 4.4. □

Corollary 5.12 [**1**]. *Let G and H be p-groups, and let δ be an ordinal. Suppose*

(i) $\widehat{u}_G(\alpha) = \widehat{u}_H(\alpha)$ *for all ordinals* $\alpha < \omega\delta$;

(ii) *if* $\operatorname{length}(G) < \omega\delta$, *then* $\widehat{u}_G(\infty) = \widehat{u}_H(\infty)$.

Then $G \equiv_\delta H$. If δ is a limit ordinal, the converse also holds.

Since two countable groups G and H are isomorphic if $G \equiv_{\omega_1} H$ (see Corollary 4.2), Theorem 5.11 yields the following:

Corollary 5.13 [**15**]. *Two countable Warfield modules M and N are isomorphic if and only if $u_M(\alpha) = u_N(\alpha)$ for all ordinals α, $u_M(\infty) = u_N(\infty)$, and $w_M(\overline{\beta}) = w_N(\overline{\beta})$ for all Ulm sequences $\overline{\beta}$.*

Acknowledgments. The authors wish to thank the referee for all the valuable comments and suggestions which helped to improve the presentation of this paper. The authors also would like to thank Carol Jacoby for helpful comments and suggestions.

REFERENCES

1. J. Barwise and P.C. Eklof, *Infinitary properties of abelian torsion groups*, Ann. Math. Logic **2** (1970), 25–68.

2. L. Fuchs, *Infinite Abelian groups*, Vol. I, Academic Press, New York, 1970; Vol. II, Academic Press, New York, 1973.

3. ———, *Abelian p-groups and mixed groups*, University of Montreal Press, Montreal, 1980.

4. P. Hill, *On the classification of abelian groups*, 1967, unpublished.

5. R. Hunter, F. Richman and E. Walker, *Warfield modules*, Lecture Notes Math. **616**, Springer Verlag, New York, 1976.

6. C. Jacoby, *The classification in $L_{\infty\omega}$ of groups with partial decomposition bases*, Ph.D. thesis, University of California, Irvine, 1980.

7. ———, unpublished note, 2010.

8. C. Karp, *Finite quantifier equivalence*, in *The theory of models*, North-Holland, Amsterdam, 1965.

9. P. Loth, *Classifications of Abelian groups and Pontrjagin duality*, in *Algebra, logic and application series* **10**, Gordon and Breach Science Publishers, Amsterdam, 1998.

10. A. Mekler and J. Oikkonen, *Abelian p-groups with no invariants*, J. Pure Appl. Algebra **87** (1993), 51–59.

11. P. Rothmaler, *Introduction to model theory*, prepared by Frank Reitmaier, translated and revised from the 1995 German original by the author, in *Algebra, logic and application series* **15**, Gordon and Breach Science Publishers, Amsterdam, 2000.

12. W. Szmielew, *Elementary properties of Abelian groups*, Fund. Math. **41** (1954), 203–271.

13. H. Ulm, *Zur Theorie der abzählbar-unendlichen abelschen Gruppen*, Math. Ann. **107** (1933), 774–803.

14. E.A. Walker, *Ulm's theorem for totally projective groups*, Proc. Amer. Math. Soc. **37** (1973), 387–392.

15. R. Warfield, *Classification theory of abelian groups II: Local theory*, Lecture Notes Math. **874**, Springer Verlag, New York, 1981.

Fakultät für Mathematik, Universität Duisburg-Essen, D 45117 Essen, Germany
Email address: ruediger.goebel@uni-duisburg-essen.de

Fakultät für Mathematik, Universität Duisburg-Essen, D 45117 Essen, Germany
Email address: katrin.leistner@uni-duisburg-essen.de

Department of Mathematics, Sacred Heart University, Fairfield, CT 06825
Email address: lothp@sacredheart.edu

Fakultät für Mathematik, Universität Duisburg-Essen, D 45117 Essen, Germany
Email address: lutz.struengmann@uni-duisburg-essen.de