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ON CODIMENSION-ONE $\mathbf{A}^1$-FIBRATION WITH RETRACTION

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ABSTRACT. In this paper we prove some results on the sufficiency of codimension-one fiber conditions for a flat algebra with a retraction to be locally $\mathbf{A}^1$ or at least an $\mathbf{A}^1$-fibration.

1. Introduction. Throughout this paper, R will denote a commutative ring with unity and $R^{[n]}$ a polynomial ring in n variables over R. Let A be an R-algebra. We shall use the notation $A = R^{[n]}$ to mean that A is isomorphic, as an R-algebra, to a polynomial ring in n variables over R.

For a prime ideal P of R, $k(P)$ will denote the residue field R_P/PR_P and A_P will denote the ring $S^{-1}A$, where $S = R \backslash P$. Thus $A \otimes_R k(P) = A_P/PA_P$.

A finitely generated flat R-algebra A is said to be an $\mathbf{A}^1$*-fibration* over R if $A \otimes_R k(P) = k(P)^{[1]}$ for all $P \in \operatorname{Spec} R$.

A *retraction* Φ from an R-algebra A to R is a ring homomorphism $\Phi : A \to R$ such that $\Phi|_R = 1_R$, i.e., it is an R-algebra homomorphism from A to R. If a retraction exists, R is said to be a *retract* of A.

Let k be a field and let $\overline{k}$ denote the algebraic closure of k. A k-algebra B is said to be *geometrically integral over* k if $B \otimes_k \overline{k}$ is an integral domain, and an $\mathbf{A}^1$*-form over* k if $B \otimes_k \overline{k} = \overline{k}^{[1]}$.

The following result on $\mathbf{A}^1$-fibration was proved in [**4**, 3.4, 3.5]:

Theorem 1.1. *Let R be a Noetherian domain with quotient field K and A a faithfully flat finitely generated R-algebra such that $A \otimes_R K =$*

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$K^{[1]}$ and $A \otimes_R k(P)$ is geometrically integral over $k(P)$ for each height one prime ideal P of R. Under these hypotheses, we have the following results:

(i) *If R is normal, then $A \cong \mathrm{Sym}_R(I)$ for an invertible ideal I of R.*

(ii) *If R contains $\mathbf{Q}$, then A is an $\mathbf{A}^1$-fibration over R.*

A striking feature of this result is that conditions on merely the generic and codimension-one fibers imply that all fibers are $\mathbf{A}^1$. Analogous results were proved for subalgebras of polynomial algebras ([**2**, 3.10, 3.12]) without the hypothesis "A is finitely generated over R". In this paper we investigate whether the condition "A is finitely generated" in Theorem 1.1 can be replaced by a weaker hypothesis like "A is Noetherian" when the R algebra A is known to have a retraction to R. Recently, in [**3**], Bhatwadekar, Dutta and Onoda have shown the following:

Theorem 1.2. *Let R be a Noetherian normal domain with field of fractions K and A a Noetherian flat R-algebra such that $A_P = {R_P}^{[1]}$ for each prime ideal P of R of height one. Suppose that there exists a retraction $\Phi : A \twoheadrightarrow R$. Then $A \cong \mathrm{Sym}_R(I)$ for an invertible ideal I in R.*

The above theorem occurs in [**3**] as a consequence of a general structure theorem for any faithfully flat algebra over a Noetherian normal domain R which is locally $\mathbf{A}^1$ in codimension-one. The statements and proofs in [**3**] are quite technical. In this paper, we will first prove (see Theorem 3.9) an analogue of Theorem 1.1 (i). Our approach, which is more in the spirit of the proof in [**4**, 3.4], will provide a short and direct proof of Theorem 1.2. Next we will prove the following analogue of Theorem 1.1 (ii) (see Theorem 3.13):

Theorem A. *Let $\mathbf{Q} \hookrightarrow R$ be a Noetherian domain with quotient field K and A be a Noetherian flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) *$A \otimes_R K = K^{[1]}$.*

(2) *$A \otimes_R k(P)$ is an integral domain for each height one prime ideal P of R.*

Then A is an $\mathbf{A}^1$-fibration over R. Thus, if R is seminormal, then $A \cong \mathrm{Sym}_R(I)$ for some invertible ideal I of R.

As an intermediate step, we shall prove the following result (see Proposition 3.11) which gives a generalization of Theorem 1.2 over an arbitrary Noetherian domain:

Proposition A. *Let R be a Noetherian domain with quotient field K, and let A be a Noetherian flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) $A \otimes_R K = K^{[1]}$.

(2) $A \otimes_R k(P)$ *is geometrically integral over* $k(P)$ *for each height one prime ideal P of R.*

Then A is finitely generated over R and there exists a finite birational extension R' of R and an invertible ideal I of R' such that $A \otimes_R R' \cong \mathrm{Sym}_{R'}(I)$.

In fact, in our results, the hypothesis "A is Noetherian" can be replaced by "$\operatorname{Ker} \Phi$ is finitely generated."

2. A version of the Russell-Sathaye criterion for an algebra to be a polynomial algebra. In this section we present a version of the Russell-Sathaye criterion ([**11**, 2.3.1]) for an algebra to be a polynomial algebra. Our version is an extension of the version given by Dutta and Onoda ([**5**, 2.4]) and suitable for algebras which are known to have retractions to the base ring. For convenience, we first record a few preliminary results. The first two results are easy.

Lemma 2.1. *Let $B \subset A$ be integral domains. Suppose that there exists a non-zero element p in B such that $B[1/p] = A[1/p]$ and $pA \cap B = pB$. Then $B = A$.*

Lemma 2.2. *Let C be a D-algebra such that D is a retract of C. Then the following hold*:

(I) *$pC \cap D = pD$ for all $p \in D$.*

(II) *If $D \subset C$ are domains, then D is algebraically closed in C.*

Lemma 2.3. *Let R be a ring, and let A be an R-algebra with a generating set $S = \{x_i : i \in \Lambda\}$ where Λ is some indexing set. Suppose that there is a retraction $\Phi : A \twoheadrightarrow R$. Then* $\operatorname{Ker} \Phi = (\{x_i - r_i : i \in \Lambda\})A$ *where $r_i = \Phi(x_i)$ for each $i \in \Lambda$.*

Proof. Let $\widetilde{S} = \{x_i - r_i : i \in \Lambda\}$, and let I be the ideal of A generated by $\widetilde{S}$. Note that $R[S] = R[\widetilde{S}]$. It is easy to see that $A = R \oplus \operatorname{Ker}\Phi = R \oplus I$. Since $I \subseteq \operatorname{Ker}\Phi$, it follows that $\operatorname{Ker}\Phi = I$. □

Lemma 2.4. *Let $R \subset A$ be integral domains, and let $\Phi : A \twoheadrightarrow R$ be a retraction with finitely generated kernel. Suppose that there exists an element p which is a non-zero non-unit in R such that $A[1/p] = R[1/p]^{[1]}$. Then there exists an $x \in \operatorname{Ker}\Phi$ such that $x \notin pA$ and $A[1/p] = R[1/p][x]$.*

Proof. Suppose, if possible, that $x \in pA$ for every $x \in \operatorname{Ker}\Phi$ for which $A[1/p] = R[1/p][x]$.

Let $\operatorname{Ker}\Phi = (a_1, a_2, \dots, a_m)A$. Choose $x_0 \in \operatorname{Ker}\Phi$ such that $A[1/p] = R[1/p][x_0]$. Note that Φ extends to a retraction $\Phi_p : A[1/p] \twoheadrightarrow R[1/p]$ with kernel $x_0(A[1/p])$. By our assumption, $x_0 = px_1$ for some $x_1 \in A$. Obviously, $x_1 \in \operatorname{Ker}\Phi$ and $A[1/p] = R[1/p][x_1]$, and hence $x_1 \in pA$. Arguing in a similar manner, we get $x_2 \in \operatorname{Ker}\Phi$ such that $x_1 = px_2$, $A[1/p] = R[1/p][x_2]$ and $x_2 \in pA$. Continuing this process we get a sequence $\{x_n\}_{n \geq 0}$ such that $x_n \in \operatorname{Ker}\Phi$, $A[1/p] = R[1/p][x_n]$ and $x_n = px_{n+1}$. Thus $x_0 = p^n x_n$ for all $n \geq 1$.

Note that $(x_0, x_1, \dots, x_n, \dots)A \subseteq (a_1, a_2, \dots, a_m)A$. But since $a_i \in A \subset A[1/p] = R[1/p][x_0]$, there exist $n_i \in \mathbf{N}$ and $\alpha_{ij} \in R[1/p]$ such that $a_i = \sum_{j=0}^{n_i} \alpha_{ij} {x_0}^j$. Choose $N \in \mathbf{N}$ such that $\alpha_{ij} p^{jN} \in R$ for all i, j, and set $\lambda_{ij} := \alpha_{ij} p^{jN}$. Now since $x_0, a_i \in \operatorname{Ker}\Phi_p$, we have $\alpha_{i0} = 0$ for all i, and hence $a_i = \sum_{j=1}^{n_i} \alpha_{ij} {x_0}^j$. Thus $a_i = \sum_{j=1}^{n_i} \lambda_{ij} {x_N}^j \in x_N R[x_N] \subseteq x_N A$ for all i, $1 \leq i \leq m$. So, we have $\operatorname{Ker}\Phi = (a_1, a_2, \dots, a_m)A = x_N A$. Now $x_{N+1} \in \operatorname{Ker}\Phi = x_N A$, which implies that $x_{N+1} = \alpha x_N$ for some $\alpha \in A$. Since $x_N = px_{N+1}$, it follows that $\alpha p = 1$, which is a contradiction to the fact that p is not a unit in A.

Thus there exists an $x \in \operatorname{Ker}\Phi$ such that $x \notin pA$ and $A[1/p] = R[1/p][x]$. □

We now present a version of the Russell-Sathaye criterion when there exists a retraction.

Proposition 2.5. *Let $R \subset A$ be integral domains such that there exists a retraction $\Phi : A \twoheadrightarrow R$. Suppose that there exists a prime p in R*

such that

(1) p *is a prime in* A.

(2) $A[1/p] = R[1/p]^{[1]}$.

Then $pA \cap R = pR$, R/pR *is algebraically closed in* A/pA, *and there exists an increasing chain* $A_0 \subseteq A_1 \subseteq A_2 \cdots \subseteq A_n \subseteq \cdots$ *of subrings of* A *and a sequence of elements* $\{x_n\}_{n\geq 0}$ *in* $\operatorname{Ker}\Phi$ *with* $x_0A \subseteq x_1A \subseteq \cdots \subseteq x_nA \subseteq \cdots$ *such that*

(a) $A_n = R[x_n] = R^{[1]}$ *for all* $n \geq 0$.

(b) $A[1/p] = A_n[1/p]$ *for all* $n \geq 0$.

(c) $pA \cap A_n \subseteq pA_{n+1}$ *for all* $n \geq 0$.

(d) $A = \cup_{n\geq 0} A_n = R[x_1, x_2, \ldots, x_n, \ldots]$.

(e) $\operatorname{Ker}\Phi = (x_0, x_1, x_2, \ldots, x_n, \ldots)A$.

Moreover the following are equivalent:

(i) $\operatorname{Ker}\Phi$ *is finitely generated.*

(ii) $\operatorname{Ker}\Phi = x_NA$ *for some* $N \geq 0$.

(iii) A *is finitely generated over* R.

(iv) $A = R[x_N]$ *for some* $N \geq 0$.

(v) *There exists an* $x \in \operatorname{Ker}\Phi \backslash pA$ *such that* $A = R[x] = R^{[1]}$.

The conditions (i)–(v) *will be satisfied if* $\cap_{n\geq 0} p^nA = (0)$.

Proof. $pA \cap R = pR$ by Lemma 2.2. Since Φ induces a retraction $\Phi_p : A/pA \twoheadrightarrow R/pR$, R/pR is algebraically closed in A/pA by Lemma 2.2.

By condition (2), there exists $x'_0 \in A$ such that $A[1/p] = R[1/p][x'_0]$. Let $x_0 = x'_0 - \Phi(x'_0)$. Then $x_0 \in \operatorname{Ker}\Phi$ and $A[1/p] = R[1/p][x_0] = R[1/p]^{[1]}$. Set $A_0 := R[x_0](= R^{[1]})$. Then $A_0 \subseteq A$ and $A[1/p] = A_0[1/p] = R[1/p][x_0]$.

Now suppose that we have obtained elements $x_0, x_1, \ldots, x_n \in \operatorname{Ker}\Phi$ such that setting $A_m := R[x_m](= R^{[1]})$ for all m, $0 \leq m \leq n$, we have $A_0 \subseteq A_1 \subseteq A_2 \cdots \subseteq A_n \subseteq A$ and $A_m[1/p] = A[1/p]$; $0 \leq m \leq n$.

We now describe our choice of x_{n+1}:

Let $\overline{x_n}$ denote the image of x_n in A/pA. We consider separately the two possibilities:

(I) $\overline{x_n}$ is transcendental over R/pR.

(II) $\overline{x_n}$ is algebraic over R/pR.

Case I. $\overline{x_n}$ is transcendental over R/pR. In this case the map $A_n/pA_n(= R[x_n]/pR[x_n]) \to A/pA$ is injective, i.e., $pA_n = pA \cap A_n$. Since $A_n[1/p] = A[1/p]$, we get $A_n = A$ by Lemma 2.1. Now we set $x_{n+1} := x_n$ and $A_{n+1} := R[x_{n+1}](= A_n = A)$.

Case II. $\overline{x_n}$ is algebraic over R/pR. Since R/pR is algebraically closed in A/pA, we see that $\overline{x_n} \in R/pR$. Thus $x_n = pu_n + c_n$ for some $u_n \in A$ and $c_n \in R$. Applying Φ, we get $0 = \Phi(x_n) = p\Phi(u_n) + c_n$ showing that $c_n \in pR$ and hence $x_n \in pA$. Set $x_{n+1} := x_n/p(\in A)$. Clearly $x_{n+1} \in \operatorname{Ker}\Phi$. Now setting $A_{n+1} := R[x_{n+1}](= R^{[1]})$, we see that $A_0 \subseteq A_1 \subseteq A_2 \cdots \subseteq A_n \subseteq A_{n+1} \subseteq A$ and $A_{n+1}[1/p] = A_n[1/p] = A[1/p]$.

Thus we set $x_{n+1} := x_n$ or $x_{n+1} := x_n/p$ depending on whether the image of x_n in A/pA is transcendental or algebraic over R/pR. By construction, conditions (a) and (b) hold. We now verify (c).

If $x_n = x_{n+1}$, i.e., $A_{n+1} = A_n = A$, then $pA \cap A_n = pA = pA_{n+1}$. Now consider the case $x_n = px_{n+1} \in pA_{n+1}$. Let $a \in pA \cap A_n$. Then $a = r_0 + r_1(px_{n+1}) + \cdots + r_l(px_{n+1})^l$ for some $l \geq 0$ and $r_0, r_1, \ldots, r_l \in R$. Then $r_0 \in pA \cap R = pR \subset pA_{n+1}$. Therefore, $a \in pA_{n+1}$. Thus $pA \cap A_n \subseteq A_{n+1}$.

We now prove (d). Let $B = \cup_{n\geq 0} A_n$. Obviously, $B \subseteq A$ and $B[1/p] = A[1/p]$. Hence, by Lemma 2.1, it is enough to show that $pA \cap B = pB$.

Clearly, $pB \subseteq pA \cap B$. Now let $y \in pA \cap B$. Then there exists an $i \geq 0$ such that $y \in pA \cap A_i \subseteq pA_{i+1} \subseteq pB$. Thus $pA \cap B = pB$.

(e) follows from Lemma 2.3.

(i) $\Rightarrow$ (v) follows from Lemma 2.4. Our construction shows that (iii) $\Rightarrow$ (iv). The implications (v) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) are easy.

Note that our construction shows that the sequence $\{x_n\}_{n\geq 0}$ is eventually a constant sequence (i.e., there exists an $N \geq 0$ such that $x_{N+r} = x_N$ for all $r \geq 0$) if and only if there exists an $N \geq 0$ such that the image of x_N in A/pA is transcendental over R/pR. It is easy to see that each of the conditions (i)–(v) is equivalent to the above condition.

If the image of x_m in A/pA is algebraic over R/pR for $1 \leq m \leq n$, then $x_n = p^n x_0 \in p^n A$. Therefore, if $\cap_{n\geq 0} p^n A = (0)$, then the sequence

$\{x_n\}_{n\geq 0}$ must be eventually constant and hence the conditions (i)–(v) hold. □

Proposition 2.5 shows that we can extend the Dutta-Onoda version ([**5**, 2.4]) of Russell-Sathaye criterion for A to be $R^{[1]}$ as follows:

Corollary 2.6. *Let $R \subset A$ be integral domains. Suppose that there exists a prime p in R such that*

(1) *p is a prime in A.*

(2) *$pA \cap R = pR$.*

(3) *$A[1/p] = R[1/p]^{[1]}$.*

(4) *R/pR is algebraically closed in A/pA.*

Then the following are equivalent:

(i) *A is finitely generated over R.*

(ii) *A has a retraction to R with finitely generated kernel.*

(iii) $\operatorname{trdeg}_{R/pR}(A/pA) > 0$.

(iv) $A = R^{[1]}$.

Proof. Follows from [**5**, 2.4] and Proposition 2.5. □

By repeated application of Proposition 2.5 we get the following:

Corollary 2.7. *Let $R \subset A$ be integral domains with a retraction $\Phi : A \twoheadrightarrow R$. Suppose that there exist primes $p_1, p_2, \ldots, p_n$ in R such that*

(1) $\operatorname{Ker}\Phi$ *is finitely generated.*

(2) *$p_1, p_2, \ldots, p_n$ are primes in A.*

(3) $A[1/(p_1p_2\cdots p_n)] = R[1/(p_1p_2\cdots p_n)]^{[1]}$.

Then there exists an $x \in \operatorname{Ker}\Phi$ such that $A = R[x] = R^{[1]}$.

3. Codimension-one $\mathbf{A}^1$-fibration with retraction. In this section we will prove our main theorems (Theorems 3.9 and 3.13) and auxiliary results (Propositions 3.8 and 3.11).

We first state a few preliminary results. The first result occurs in [**2**, 3.4].

Lemma 3.1. *Let R be a Noetherian ring and R_1 a ring containing R which is finitely generated as an R-module. If A is a flat R-algebra such that $A \otimes_R R_1$ is a finitely generated R_1-algebra, then A is a finitely generated R-algebra.*

The following result follows from [**2**, 3.3 and 3.5].

Lemma 3.2. *Let R be a Noetherian ring and A a flat R-algebra such that, for every minimal prime ideal P of R, PA is a prime ideal of A, $PA \cap R = P$ and A/PA is finitely generated over R/P. Then A is finitely generated over R.*

The next result is easy to prove.

Lemma 3.3. *Let R be a ring and A an R-algebra. If R' is a faithfully flat algebra over R such that $A \otimes_R R'$ is finitely generated over R', then A is finitely generated over R.*

We now quote a theorem on finite generation due to Onoda [**10**, 2.20].

Theorem 3.4. *Let R be a Noetherian domain, and let A be an integral domain containing R such that*

(1) *There exists a non-zero element $t \in A$ for which $A[1/t]$ is a finitely generated R-algebra.*

(2) *$A_{\mathfrak{m}}$ is a finitely generated $R_{\mathfrak{m}}$-algebra for each maximal ideal $\mathfrak{m}$ of R.*

Then A is a finitely generated R-algebra.

The results on $\mathbf{A}^1$-fibrations in [**3, 4, 5**]) crucially involve certain patching techniques. We state below one such "patching lemma" ([**5**, 3.2]).

Lemma 3.5. *Let $R \subset A$ be integral domains with A being faithfully flat over R. Suppose that there exists a non-zero element $t \in R$ such that*

(1) $A[1/t] = R[1/t]^{[1]}$.

(2) $S^{-1}A = (S^{-1}R)^{[1]}$, *where* $S = \{r \in R \mid r$ *is not a zero-divisor in* $R/tR\}$.

Then there exists an invertible ideal I in R such that $A \cong \mathrm{Sym}_R(I)$.

We now observe a property of algebras with retractions.

Lemma 3.6. *Let R be an integral domain with quotient field K and A an integral domain containing R with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) $\mathrm{Ker}\,\Phi$ *is finitely generated.*

(2) $A \otimes_R K = K^{[1]}$.

Then there exists a $t \in R$ and an $F \in \mathrm{Ker}\,\Phi$ such that $A[1/t] = R[1/t][F]$.

Proof. Let $S = R \backslash \{0\}$. By (2), $S^{-1}A = K^{[1]}$. Since A has a retraction Φ, it is easy to see that there exists an $F \in \mathrm{Ker}\,\Phi$ such that $S^{-1}A = K[F](= K^{[1]})$ and hence $F(S^{-1}A) = (\mathrm{Ker}\,\Phi)S^{-1}A$. Therefore, by (1), there exists a $t \in S$ such that $FA[1/t] = (\mathrm{Ker}\,\Phi)A[1/t]$. Thus $FA[1/t]$ is the kernel of the induced retraction $\Phi_t : A[1/t] \twoheadrightarrow R[1/t]$. Hence we have

$$
\begin{aligned}
A[1/t] &= R[1/t] \oplus FA[1/t] \\
&= R[1/t] \oplus FR[1/t] \oplus F^2A[1/t] \\
&\cdots \\
&= R[1/t] \oplus FR[1/t] \oplus F^2R[1/t] \oplus \\
&\cdots \oplus F^nR[1/t] \oplus F^{n+1}A[1/t] \quad \forall n \in \mathbf{N}.
\end{aligned}
$$

As $S^{-1}A = \oplus_{n \geq 0} KF^n$, it follows that $A[1/t] = R[1/t][F]$. □

Remark 3.7. In Lemma 3.6 if we assume that $\mathrm{Ker}\,\Phi$ is principal, say, $\mathrm{Ker}\,\Phi = (G)$, then $A = R[G]$.

We now deduce a local-global result. Our approach gives a simpler proof of Theorem 1.2 which is obtained in [**3**] as a consequence of a highly technical structure theorem.

Proposition 3.8. *Let R be either a Noetherian domain or a Krull domain with quotient field K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) $\operatorname{Ker}\Phi$ *is finitely generated.*

(2) $A_P = {R_P}^{[1]}$ *for every prime ideal P of R satisfying* $\operatorname{depth}(R_P)=1$.

Then there exists an invertible ideal I of R such that $A \cong \operatorname{Sym}_R(I)$.

Proof. The case $\dim R = 0$ is trivial. So we assume that $\dim R \geq 1$. Note that A is a faithfully flat R-algebra and an integral domain. By Lemma 3.6, $A[1/t] = R[1/t][F]$. If t is a unit in R, then $A = R^{[1]}$ and we would be through. So we assume that t is a non-unit in R.

Let $P_1, P_2, \ldots, P_s$ be the associated prime ideals of tR. Let $S = R \backslash (\cup_{i=1}^{s} P_i) = \{r \in R \mid r \text{ is not a zero-divisor in } R/tR\}$. By (2), for each maximal ideal $\mathfrak{m}$ of $S^{-1}R$, $(S^{-1}A)_{\mathfrak{m}} = {(S^{-1}R)_{\mathfrak{m}}}^{[1]}$ and hence $S^{-1}A = (S^{-1}R)^{[1]}$, $S^{-1}R$ being a semilocal domain. Hence, by Lemma 3.5, $A \cong \operatorname{Sym}_R(I)$ for some invertible ideal I of R. □

We now prove Theorem A for the case R is a Krull domain.

Theorem 3.9. *Let R be a Krull domain with quotient field K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) $\operatorname{Ker}\Phi$ *is finitely generated.*

(2) $A \otimes_R K = K^{[1]}$.

(3) $A \otimes_R k(P)$ *is an integral domain for each height one prime ideal P of R.*

Then there exists an invertible ideal I of R such that $A \cong \operatorname{Sym}_R(I)$.

Proof. Let P be a prime ideal in R for which $\operatorname{depth}(R_P)(= \operatorname{ht} P) = 1$. Then R_P is a DVR. Let π_P be a uniformizing parameter of R_P. Note that the retraction $\Phi : A \twoheadrightarrow R$ induces a retraction $\Phi_P : A_P \twoheadrightarrow R_P$

with finitely generated kernel, condition (2) ensures that $A_P[1/\pi_P] = R_P[1/\pi_P]^{[1]} = K^{[1]}$, and condition (3) ensures that π_P is a prime in A_P. Hence, by Corollary 2.7, $A_P = {R_P}^{[1]}$. Therefore, by Proposition 3.8, $A \cong \mathrm{Sym}_R(I)$ for some invertible ideal I of R. □

As an immediate consequence we get the following variant of a Lüroth-type result over UFD (see [**11**, 3.4]):

Corollary 3.10. *Let R be a UFD with quotient field K and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) $\mathrm{Ker}\,\Phi$ *is finitely generated.*

(2) $A \otimes_R K = K^{[1]}$.

(3) $A \otimes_R k(P)$ *is an integral domain for each height one prime ideal P of R.*

Then there exists an $x \in \mathrm{Ker}\,\Phi$ such that $A = R[x] = R^{[1]}$.

We now prove Proposition A.

Proposition 3.11. *Let R be a Noetherian ring and A a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) $\mathrm{Ker}\,\Phi$ *is finitely generated.*

(2) $A \otimes_R k(P) = k(P)^{[1]}$ *for each minimal prime ideal P of R.*

(3) $A \otimes_R k(P)$ *is geometrically integral over $k(P)$ for each height one prime ideal P of R.*

Then:

(I) $A \otimes_R k(P)$ *is an $\mathbf{A}^1$-form over $k(P)$ for each prime ideal P of R.*

(II) *A is finitely generated over R.*

(III) *If R is an integral domain, then there exists a finite birational extension R' of R and an invertible ideal I of R' such that $A \otimes_R R' \cong \mathrm{Sym}_{R'}(I)$.*

Proof. (I) Note that, for any prime ideal P of R, $A \otimes_R k(P) = A_P \otimes_{R_P} k(P)$. So, to prove the fiber condition (I), we replace R by R_P

(and A by A_P) and assume that R is a local ring with maximal ideal $\mathfrak{m}$. We prove that $A \otimes_R k(\mathfrak{m})$ is an $\mathbf{A}^1$-form over $k(\mathfrak{m})$ by induction on height $\mathfrak{m}$, i.e., $\dim R$.

Case $\dim R = 0$. Trivial.

Case $\dim R = 1$. Replacing R by R/P_0 for some minimal prime ideal P_0, we may assume that R is a Noetherian one-dimensional local integral domain with quotient field K. Note that condition (3) implies that $A \otimes_R k(\mathfrak{m})$ is geometrically integral over $k(\mathfrak{m})$.

Let $\widetilde{R}$ be the normalization of R, and let $\widetilde{A} = A \otimes_R \widetilde{R}$. Then, by the Krull-Akizuki theorem, $\widetilde{R}$ is a Dedekind domain ([**9**, page 85]; and since R is local, $\widetilde{R}$ is semilocal and hence a PID. Let $\widetilde{\mathfrak{m}}_1, \widetilde{\mathfrak{m}}_2, \dots, \widetilde{\mathfrak{m}}_r$ be the maximal ideals of $\widetilde{R}$. Again, by Krull-Akizuki theorem ([**9**, page 85]), $k(\widetilde{\mathfrak{m}}_i)$ is a finite algebraic extension of $k(\mathfrak{m})$. Clearly, the retraction $\Phi : A \twoheadrightarrow R$ gives rise to a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$. From the split exact sequence $0 \to \operatorname{Ker}\Phi \to A \to R \to 0$, it follows that $\operatorname{Ker}\widetilde{\Phi} = \operatorname{Ker}\Phi \otimes_R \widetilde{R} = \operatorname{Ker}\Phi \otimes_A \widetilde{A} = (\operatorname{Ker}\Phi)\widetilde{A}$ and hence $\operatorname{Ker}\widetilde{\Phi}$ is finitely generated.

Thus, from (1), (2) and (3), we have:

(i) $\operatorname{Ker}\widetilde{\Phi}$ is finitely generated.

(ii) $\widetilde{A} \otimes_{\widetilde{R}} K = K^{[1]}$.

(iii) $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{\mathfrak{m}}_i)$ is geometrically integral over $k(\widetilde{\mathfrak{m}}_i)$ for every maximal ideal $\widetilde{\mathfrak{m}}_i$ of $\widetilde{R}$.

Hence, by Corollary 3.10, $\widetilde{A} = \widetilde{R}^{[1]}$. In particular, $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{\mathfrak{m}}_i) = k(\widetilde{\mathfrak{m}}_i)^{[1]}$ for each maximal ideal $\widetilde{\mathfrak{m}}_i$ of $\widetilde{R}$. This shows that $A \otimes_R k(\mathfrak{m})$ is an $\mathbf{A}^1$-form over $k(\mathfrak{m})$.

Case $\dim R \geq 2$. By the induction hypothesis we have that $A \otimes_R k(P)$ is an $\mathbf{A}^1$-form for every non-maximal prime ideal P of R. Let $\widehat{R}$ denote the completion of R, and let $\widehat{A} = A \otimes_R \widehat{R}$. Then $\widehat{R}$ is a complete local ring with maximal ideal $\widehat{\mathfrak{m}}$ and $\widehat{R}/\widehat{\mathfrak{m}} \cong R/\mathfrak{m}$. Since R is Noetherian, $\widehat{R}$ is Noetherian and faithfully flat over R and hence $\widehat{A}$ is faithfully flat over both A and $\widehat{R}$. The retraction $\Phi : A \twoheadrightarrow R$ gives rise to a retraction $\widehat{\Phi} : \widehat{A} \twoheadrightarrow \widehat{R}$. Note that $\operatorname{Ker}\widehat{\Phi} = (\operatorname{Ker}\Phi)\widehat{A}$ is finitely generated. Now, for any non-maximal prime ideal $\widehat{P}$ of $\widehat{R}$, $\widehat{P} \cap R \neq \mathfrak{m}$ and hence $\widehat{A} \otimes_{\widehat{R}} k(\widehat{P})$ is an $\mathbf{A}^1$-form over $k(\widehat{P})$.

Replacing R by $\widehat{R}$, we may assume R to be a complete local Noetherian ring. Further, replacing R by R/P_0, where P_0 is a minimal prime ideal of R, we may assume R to be a complete, local, Noetherian domain with maximal ideal $\mathfrak{m}$ and quotient field K such that

(a) $A \otimes_R K = K^{[1]}$.

(b) $A \otimes_R k(P)$ is an $\mathbf{A}^1$-form over $k(P)$ for each non-maximal prime ideal P of R.

Let $\widetilde{R}$ be the normalization of R. Since R is a complete local ring, $\widetilde{R}$ is a finite R-module ([**9**, page 263]) and hence is a Noetherian normal local domain. Let $\widetilde{A} = A \otimes_R \widetilde{R}$. As before, the retraction $\Phi : A \twoheadrightarrow R$ induces a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel $(\operatorname{Ker}\Phi)\widetilde{A}$. Now we have the following:

$\widetilde{R}$ is a Noetherian normal local domain with quotient field K and $\widetilde{A}$ is a faithfully flat $\widetilde{R}$-algebra such that

(1′) There is a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel.

(2′) $\widetilde{A} \otimes_{\widetilde{R}} K = A \otimes_R K = K^{[1]}$.

(3′) $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{P})$ is an $\mathbf{A}^1$-form over $k(\widetilde{P})$ for each height one prime ideal $\widetilde{P}$ of $\widetilde{R}$ (since, for any height one prime ideal $\widetilde{P}$ of $\widetilde{R}$, $\widetilde{P} \cap R \neq \mathfrak{m}$).

By Theorem 3.9, $\widetilde{A} = \widetilde{R}^{[1]}$; in particular, $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{\mathfrak{m}}) = k(\widetilde{\mathfrak{m}})^{[1]}$. This shows that $A \otimes_R k(\mathfrak{m})$ is an $\mathbf{A}^1$-form over $k(\mathfrak{m})$ and hence $A \otimes_R k(P)$ is an $\mathbf{A}^1$-form over $k(P)$ for every prime ideal P of R.

(II) We now show that A is finitely generated over R. By Lemma 3.2, it is enough to take R to be an integral domain; by Theorem 3.4 and Lemma 3.6, it is enough to assume R to be local and, by Lemma 3.3, it is enough to take R to be complete. Thus we assume that R is a Noetherian local complete integral domain. Let $\widetilde{R}$ be the normalization of R. Then the proof of (I) shows that $A \otimes_R \widetilde{R} = \widetilde{R}^{[1]}$; in particular, $A \otimes_R \widetilde{R}$ is finitely generated over $\widetilde{R}$. Since $\widetilde{R}$ is a finite module over R, by Lemma 3.1, A is finitely generated over R.

(III) Now R is given to be an integral domain. By (I), $A \otimes_R k(P)$ is an $\mathbf{A}^1$-form over $k(P)$ for every prime ideal P of R.

Let $\widetilde{R}$ be the normalization of R. Then $\widetilde{R}$ is a Krull domain ([**9**, page 91]). Let $\widetilde{A} = A \otimes_R \widetilde{R}$. As before, there is a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel. We now have the following:

$\widetilde{R}$ is a Krull domain with quotient field K and $\widetilde{A}$ is a faithfully flat $\widetilde{R}$-algebra such that

(1″) There is a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel.

(2″) $\widetilde{A} \otimes_{\widetilde{R}} K = K^{[1]}$.

(3″) $\widetilde{A} \otimes_{\widetilde{R}} k(\widetilde{P})$ is an $\mathbf{A}^1$-form over $k(\widetilde{P})$ for each prime ideal $\widetilde{P}$ of $\widetilde{R}$ (since $k(\widetilde{P})$ is algebraic over $k(\widetilde{P} \cap R)$).

Using Theorem 3.9, we get that $A \otimes_R \widetilde{R} = \widetilde{R}[\widetilde{I}T]$ for some invertible ideal $\widetilde{I}$ of $\widetilde{R}$. Let $\widetilde{I} = (a_1, a_2, \ldots, a_n)\widetilde{R}$, and let $\alpha_1, \ldots, \alpha_n \in \widetilde{I}^{-1}$ be such that $a_1\alpha_1 + \cdots a_n\alpha_n = 1$. Set $b_{ij} := a_i\alpha_j (\in \widetilde{R})$, $1 \leq i, j \leq n$. Let $a_pT = \sum_{q=1}^{s_p} u_{pq} \otimes c_{pq}$ where $c_{pq} \in \widetilde{R}$ and $u_{pq} \in A$.

By (II), A is finitely generated; let $A = R[y_1, y_2, \ldots, y_t]$ where each $y_\ell \in \operatorname{Ker}\Phi$. Then

$$y_\ell \otimes 1 = \sum_{m=0}^{r_\ell} \sum_{m_1+m_2+\cdots+m_n=m} d_{\ell\ m_1m_2\cdots m_n}\ {a_1}^{m_1}{a_2}^{m_2}\cdots {a_n}^{m_n}\ T^m$$

for some $d_{\ell\ m_1m_2\cdots m_n} \in \widetilde{R}$.

Now, let R' be the R-subalgebra of $\widetilde{R}$ generated by the elements $a_1, a_2, \ldots, a_n$; b_{ij} where $1 \leq i, j \leq n$; c_{pq} where $1 \leq q \leq s_p$, $1 \leq p \leq n$; and $d_{\ell\ m_1m_2\cdots m_n}$ where $m_1+m_2+\cdots+m_n = m$, $0 \leq m \leq r_\ell$, $1 \leq \ell \leq t$. Let I be the ideal $(a_1, a_2, \ldots, a_n)R'$. Then R' is a finite birational extension of R and I is an invertible ideal of R'.

Since A is faithfully flat over R, we have $A \otimes_R R' \subseteq A \otimes_R \widetilde{R} \subseteq A \otimes_R K = K[T]$. Now considering $A \otimes_R R'$ and $R'[IT]$ as subrings of $A \otimes_R K$, it is easy to see that $A \otimes_R R' = R'[IT]$.

This completes the proof. □

Remark 3.12. The above proof shows that in the statement of Proposition 3.11, it is enough to assume in (2) that the generic fibers are $\mathbf{A}^1$-forms. (In the proof take $\widetilde{R}$ to be the integral closure of R in L where L is a finite extension of K such that $A \otimes_R L = L^{[1]}$.)

We now prove Theorem A.

Theorem 3.13. *Let $\mathbf{Q} \hookrightarrow R$ be a Noetherian ring, and let A be a flat R-algebra with a retraction $\Phi : A \twoheadrightarrow R$ such that*

(1) $\operatorname{Ker}\Phi$ *is finitely generated.*

(2) $A \otimes_R k(P) = k(P)^{[1]}$ *at each minimal prime ideal P of R.*

(3) $A \otimes_R k(P)$ *is an integral domain at each height one prime ideal P of R.*

Then:

(I) *A is an $\mathbf{A}^1$-fibration over R.*

(II) *If R is an integral domain, then there exists a finite birational extension R' of R and an invertible ideal I of R' such that $A \otimes_R R' \cong \operatorname{Sym}_{R'}(I)$.*

(III) *If R_{red} is seminormal, then $A \cong \operatorname{Sym}_R(I)$ for some finitely generated rank one projective R-module I.*

Proof. (I) By Proposition 3.11, it is enough to show that $A \otimes_R k(P) = k(P)^{[1]}$ for each prime ideal P in R of height one.

Fix a prime ideal P in R of height one. Replacing R by R_P, we assume that R is a one-dimensional Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Moreover, replacing R by R/P_0 for some minimal prime ideal P_0, we may further assume that R is an integral domain with quotient field K. We show that $A \otimes_R k = k^{[1]}$.

Note that k is a field of characteristic 0. By the Krull-Akizuki theorem, there exists a discrete valuation ring $(\widetilde{R}, \pi, \widetilde{k})$ such that $R \subset \widetilde{R} \subset K$ and $\widetilde{k}$ is a finite separable extension of k. Let $\widetilde{A} = A \otimes_R \widetilde{R}$. Since separable $\mathbf{A}^1$-forms are $\mathbf{A}^1$, to show that $A \otimes_R k = k^{[1]}$, it is enough to show that $\widetilde{A}/\pi\widetilde{A} (= A \otimes_R \widetilde{k}) = \widetilde{k}^{[1]}$ and hence enough to show that $\widetilde{A} = \widetilde{R}^{[1]}$.

Now, the retraction $\Phi : A \twoheadrightarrow R$ with finitely generated kernel induces a retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ with finitely generated kernel. Also $\widetilde{A}[1/\pi] = K^{[1]}$. Using Lemma 2.4, we get an $x \in \operatorname{Ker}\widetilde{\Phi} \setminus \pi\widetilde{A}$ such that $\widetilde{A}[1/\pi] = K[x]$.

Let $B = \widetilde{R}[x] \subset \widetilde{A}$. We will show that $\widetilde{A} = B$. Since π is a non-zero divisor and since $\widetilde{A}_\pi = B_\pi$, by Lemma 2.1, it suffices to show that $\pi\widetilde{A} \cap B = \pi B$.

Let $D = A \otimes_R k$. Then $\widetilde{A}/\pi\widetilde{A} = \widetilde{A} \otimes_{\widetilde{R}} \widetilde{k} = (A \otimes_R k) \otimes_k \widetilde{k} = D \otimes_k \widetilde{k}$. By hypothesis, D is an integral domain and hence, as $\widetilde{k}|_k$ is separable, $\widetilde{A}/\pi\widetilde{A} = D \otimes_k \widetilde{k}$ is a reduced ring. Note that $\widetilde{A}/\pi\widetilde{A}$ is a finite flat module over D and hence $\widetilde{A}$ has only finitely many minimal prime ideals $P_1, P_2, \dots, P_n$ containing $\pi\widetilde{A}$. To show that $\pi\widetilde{A} \cap B = \pi B$, it is enough to show that $P_i \cap B = \pi B$ for some i.

Suppose, if possible, that $P_i \cap B \neq \pi B$ for all i. Let $P_i \cap B = Q_i$. Then $\operatorname{ht} Q_i > 1$, i.e., Q_is are maximal ideals of B (since $\dim B = 2$). Let t be the number of distinct ideals in the family $\{Q_1, Q_2, \dots, Q_n\}$. By reindexing, if necessary, we assume that $Q_1, Q_2, \dots, Q_t$ are all distinct. Let $I_i = \cap_{P_j \cap B = Q_i} P_j$. Since Q_is are pairwise comaximal, I_is are pairwise comaximal. Thus $\widetilde{A}/\pi\widetilde{A} = \widetilde{A}/I_1 \times \widetilde{A}/I_2 \times \cdots \times \widetilde{A}/I_t$.

Since $D = A \otimes_R k$, the retraction $\Phi : A \twoheadrightarrow R$ induces a retraction $\Phi_k : D \twoheadrightarrow k$. Let $\mathfrak{m}_0$ be a maximal ideal of D such that $D/\mathfrak{m}_0 = k$. Note that $D \hookrightarrow D_{\mathfrak{m}_0}$ and hence, due to flatness, $D \otimes_k \widetilde{k} \hookrightarrow D_{\mathfrak{m}_0} \otimes_k \widetilde{k}$. Since $D_{\mathfrak{m}_0}$ is local and since $\widetilde{k}|_k$ is a finite extension, $D_{\mathfrak{m}_0} \otimes_k \widetilde{k}$ is also local with maximal ideal $\mathfrak{m}_0(D_{\mathfrak{m}_0} \otimes_k \widetilde{k})$ and residue field $\widetilde{k}$. As the local ring $D_{\mathfrak{m}_0} \otimes_k \widetilde{k}$ is a localization of $D \otimes_k \widetilde{k} = \widetilde{A}/\pi\widetilde{A}$, it follows that there exists a prime ideal $\wp$ of $\widetilde{A}/\pi\widetilde{A}$ such that $D_{\mathfrak{m}_0} \otimes_k \widetilde{k} = (\widetilde{A}/\pi\widetilde{A})_{\wp}$.

Note that $\widetilde{A}/\pi\widetilde{A} = D \otimes_k \widetilde{k} \hookrightarrow D_{\mathfrak{m}_0} \otimes_k \widetilde{k} = (\widetilde{A}/\pi\widetilde{A})_{\wp}$. As the map $\widetilde{A}/\pi\widetilde{A} \longrightarrow (\widetilde{A}/\pi\widetilde{A})_{\wp}$ is one-to-one, it follows that the zero-divisors of $\widetilde{A}/\pi\widetilde{A}$ are contained in $\wp$. Consequently, $\overline{P_i} \subset \wp$ where $\overline{P_i}$ is the image of P_i in $\widetilde{A}/\pi\widetilde{A}$. But this would imply that the local ring $(\widetilde{A}/\pi\widetilde{A})_{\wp}$ is a product of t rings which is possible only if $t = 1$. So $P_i \cap B = Q$ for all i, which implies that $\pi\widetilde{A} \cap B = Q$. Note that the retraction $\widetilde{\Phi} : \widetilde{A} \twoheadrightarrow \widetilde{R}$ induces a retraction $\widetilde{\Phi}_\pi : \widetilde{A}/\pi\widetilde{A} \twoheadrightarrow \widetilde{R}/\pi\widetilde{R}$. Now since $\pi\widetilde{A} \cap B = Q$, the retraction $\widetilde{\Phi}_\pi$ induces a retraction $\widetilde{\Phi}'_\pi : B/Q \twoheadrightarrow \widetilde{k}$. But Q is a maximal ideal of B, i.e., B/Q is a field. Hence $\widetilde{\Phi}'_\pi$ is an isomorphism. As $x \in \operatorname{Ker} \widetilde{\Phi}$, it then follows that $x \in Q \subset \pi\widetilde{A}$ and hence $x \in \pi\widetilde{A}$, a contradiction.

Thus $\pi\widetilde{A} \cap \widetilde{R}[x] = \pi\widetilde{R}[x]$ and hence $\widetilde{A} = \widetilde{R}^{[1]}$ showing that $A \otimes_R k = k^{[1]}$.

(II) Follows from (III) of Proposition 3.11.

(III) Follows from (I) and the result ([**1**, 3.4]) of Asanuma, using results of Hamann ([**8**, 2.6 or 2.8]) and Swan ([**12**, 6.1]); also see ([**7**]). ◻

Remark 3.14. Examples from existing literature would show that the hypotheses in our results cannot be relaxed. For instance, the hypothesis that "$\operatorname{Ker}\Phi$ is finitely generated" is necessary in all the results as can be seen from the example: Let (R, π) be a DVR and $A = R[X, X/\pi, X/\pi^2, \dots, X/\pi^n, \dots]$.

An example of Eakin-Silver ([**6**, 3.15]) shows that the hypothesis "A has a retraction to R" is necessary in Proposition 3.8. Even if R is local and factorial and A Noetherian, the hypothesis "A has a retraction to R" would still be necessary in Theorem 3.9 even to conclude that A is finitely generated as has been shown recently in [**3**]. Even if A is finitely generated, the hypothesis "A has retraction to R" would still be necessary in Theorem 3.9 to conclude that A is a symmetric algebra (consider $R = k[[t_1, t_2]]$ where k is any field and $A = R[X, Y]/(t_1X + t_2Y - 1)$).

The following example of Yanik ([**13**, 4.1]) shows the necessity of the seminormality hypothesis in the passage from (I) to (III) in Theorem 3.13: Let k be a field of characteristic zero, $R = k[[t^2, t^3]]$ and $A = R[X + tX^2] + (t^2, t^3)R[X]$; also see [**7**].

For other examples (e.g., the necessity of "geometrically integral" in Proposition 3.11, the necessity of "$\mathbf{Q} \hookrightarrow R$" in Theorem 3.13 and the necessity of "flatness"), see [**2**, Section 4].

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